ABSTRACT

CHEN, YOUHUA. Output Feedback Receding Horizon Control For Constrained Linear Systems. (Under the direction of Dr. Fen Wu).

To overcome the situations where we cannot access the full information of the plant states, advanced output feedback receding horizon control for constrained linear systems are studied in this thesis. We focus on three types of constrained linear systems: LTI systems without disturbances, LTI systems with energy-bounded disturbances and uncertain linear systems with magnitude-bounded disturbances. A stable observer introduced to estimate the plant states ensures that the errors between the original plant states and the estimated states will approach zero with time going on. Our studies are based on these estimated states instead of the original. The output feedback controllers are developed on the basis of measured output signals. The synthesis conditions are solved by LMI, including initial condition, stability condition and constrained input/output conditions.

For constrained LTI systems without disturbances, an offline output feedback controller is proposed by minimizing a quadratic performance of the controlled output signals. This work is extended to online infinite-horizon RHC by recalculate the same synthesis conditions at each time step. The resulted performance index is monotonically decreasing. The cost of online finite-horizon RHC consists of performance over finite horizon and a terminal cost. A terminal set determined from the offline output feedback control, is used as a terminal constraint in the finite-horizon RHC to ensure the states go into this region over finite steps. The finite-horizon RHC is solved by treating the predicted states and control forces as decision variables instead of controller parameters in the offline output feedback control and the infinite-horizon RHC.

For constrained LTI systems with energy-bounded disturbances, an offline robust output feedback controller is achieved by optimizing closed-loop $\mathbf{H}_\infty$ performance index from disturbances to controlled output. An dissipation constraint is added in the corresponding robust output feedback infinite-horizon RHC to guarantee the moving horizon stability. The dissipation terms are defined by a recursive equation.

In robust output feedback finite-horizon RHC, we allow the system has different controller parameters at each step. Then the synthesis conditions are calculated by choosing controller parameters as decision variables. Thus the finite-horizon RHC is less conservative than the infinite-horizon RHC and achieves a better performance index.

For constrained uncertain linear systems with magnitude-bounded disturbances, the systems matrices of the uncertain linear systems belongs to a convex hull. The synthesis conditions become a set of similar conditions in which the system parameters are chosen as vertex points of the convex hull. Our goal to design an offline robust output feedback controller for uncertain linear systems is to find a stability region such that once the plant states go into this region, they will not leave this region any more. Thus the offline robust output feedback control has a different stability condition instead of $\mathbf{H}_\infty$ or $\mathbf{H}_2$ criteria. The corresponding online robust output feedback infinite-horizon RHC obtains a optimal stability region at each time step. Based on this stability region, an online robust output feedback finite-horizon RHC is proposed by assuming that the cost will be zero once the plant states go into the stability region, the cost performance index is a quadratic function of controlled output signals without terminal cost. The finite-horizon RHC algorithm have two procedures. The first is to solve the robust output feedback infinite-horizon RHC problem to achieve a stability region. The second is to compute the robust output feedback finite-horizon RHC problem to obtain control force.

All of proposed output feedback control algorithms are demonstrated by simulations of single microcantilever system, two-mass/sping/damper system and single, non-isothermal continuous stirred-tank reactor. All control algorithms obtain satisfied dynamic response and performance index.

OUTPUT FEEDBACK RECEDING HORIZON CONTROL FOR CONSTRAINED LINEAR SYSTEMS

by

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Dedication

Dedicated to my wife
Jiling Xuan
and my mother
Wenzhi Wan,
with all my love.

Biography

Youhua Chen was born on October 30 1978, in Jiangxi, China. He received his BS and MS degrees in Mechanical Engineering from Shanghai Jiaotong University, China, in 2000 and 2003, respectively. He was enrolled in graduate school at NCSU in the spring of 2004.

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Mathematical Symbols

$\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$	the n by n diagonal matrix with elements a_1, a_2, …, a_n on the diagonal line
I_n	the n-dimensional identity matrix
M^T	the transpose of the matrix M
M^{-1}	the inverse of the invertible matrix M
$M > 0$ ($M \geq 0$)	the matrix M is positive definite (positive semi-definite)
$M < 0$ ($M \leq 0$)	the matrix M is negative definite (negative semi-definite)
$\mathbf{R}$	set of real numbers
$\mathbf{R}_+$	set of non-negative real numbers
$\mathbf{R}^n$	set of n-dimensional real vectors
$\mathbf{R}^{m\times n}$	set of real $m \times n$ matrices
$\mathbf{S}^{n\times n}$	set of symmetric matrices in $\mathbf{R}^{n\times n}$
$\mathbf{S}_+^{n\times n}$	set of positive definite matrices in $\mathbf{R}^{n\times n}$
$\mathbf{Z}_+$	positive integer
$\begin{bmatrix} a_{11} & \star \\ a_{21} & a_{22} \end{bmatrix}$	simplified description of symmetric matrix
$\Vert x\Vert$	$= \left[\sum_{k=0}^{\infty} x^T(k)x(k)\right]^{\frac{1}{2}}$

List of Acronyms

LMI	linear matrix inequality
LMIs	linear matrix inequlaities
LTI	linear time invariant
LTV	linear time varying
MPC	model predictive control
QP	quadratic programming
RHC	receding horizon control

Chapter 1

Introduction

In this chapter, we first provide the background knowledge of receding horizon control, then give the problem definition of robust RHC and related issues.

In §1.1, we state the classic definition of RHC problem. In §1.2, we talk about several important issues of RHC. §1.3 show that current research work on robust RHC. §1.4 is the proposed RHC schemes in this dissertation. §1.5 is the dissertation overview.

1.1 RHC Problem Formulation

Receding horizon control (RHC), also referred to as model predictive control (MPC) has been widely used in industry as an effective means to deal with multivariable control problem [66], [6] with hard input/output constraints.

Receding horizon control is obtained by solving an online optimal problem at each time step, using the current state of the plant as the initial state, rather than deriving offline a feedback policy that provides the optimal control for all states. The online optimization yields an optimal control sequence and the first control in this sequence is applied to the plant. This idea is illustrated in Fig. 1.1. The optimization problem is quadratic if the cost function is expressed through ℓ_2 norm of required output signal and control force, or linear if expressed through ℓ_1/ℓ_∞ norm. For those systems, which have constraints, and/or the system dynamics is nonlinear, it is vastly more difficult to solve Hamilton-Jacobi-Bellman equation. The strength of RHC lies in its unique ability to provide implicit nonlinear feedback for these systems, replacing a complex offline dynamic programming problem by an online optimal control problem that is tractable, at least in many applications.

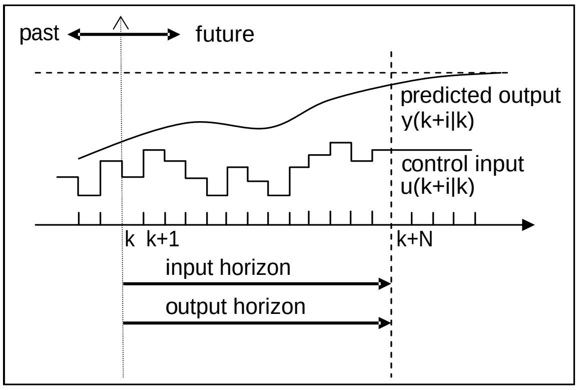

Figure 1.1: Receding horizon control scheme: only the first one of the computed control force $u(k|k)$ is implemented

In most literatures, RHC is formulated in the state space. The system to be controlled is usually described by an ordinary differential equation. However, since the control is normally piecewise constant, the system is often modeled by a difference equation as well.

Specially, for the linear system, the plant is described as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.1}$$

where matrix (A, B, C) are constant for LTI systems, while vary with time for LTV systems. $x(\cdot) \in \mathbf{R}^n, u(\cdot) \in \mathbf{R}^{n_u}$ and $y(\cdot) \in \mathbf{R}^{n_y}$ denote plant state trajectory, control input sequence and output signal sequence respectively. n, n_u, n_y are the dimensions of plant state, control input and output signal. The cost function is normally defined as a quadratic function,

$$J_N(k) = \sum_{i=0}^{N-1} \{x^T(k+i \mid k) Q_x x(k+i \mid k) + u^T(k+i \mid k) Q_u u(k+i \mid k)\} + x^T(k+N \mid k) X_N x(k+N \mid k), \qquad (1.2)$$

where suffix N is the prediction horizon length, k is current time step.

Correspondingly, for the nonlinear systems, the plant and the cost function are

$$\begin{aligned} x(k+1) &= f(x(k, u(k))) \\ y(k) &= h(x(k)) \qquad (1.3) \\ J_N(k) &= \sum_{i=0}^{N-1} \ell(x(k+i \mid k), u(k+i \mid k)) + F(x(k+N \mid k)), \qquad (1.4) \end{aligned}$$

where $f(\cdot), h(\cdot), \ell(\cdot), F(\cdot)$ are smooth nonlinear functions.

As a distinguish feature from other control methods, RHC always deals with the problem with state and control constraints. Generally, the state trajectory, control sequence and output sequence must satisfy $x(\cdot) \in \mathbb{X}, u(\cdot) \in \mathbb{U}, y(\cdot) \in \mathbb{Y}$. Usually, $\mathbb{X}$ a convex, closed subset of $\mathbf{R}^n$, $\mathbb{U}$ is a convex, compact subset of $\mathbf{R}^{n_u}$ and $\mathbb{Y}$ is a convex, compact subset of $\mathbf{R}^{n_y}$. A terminal constraint

$$x(k+N \mid k) \in \mathbb{X}_f \subset \mathbb{X} \qquad (1.5)$$

is sometimes imposed to guarantee the stability.

Problem 1 (RHC Problem) *At time k, the optimal RHC for the LTI system (1.1) is defined by*

$$\min \ J_N(k) \qquad (1.6)$$

$$s.t. \quad x(\cdot) \in \mathbb{X}, u(\cdot) \in \mathbb{U}, y(\cdot) \in \mathbb{Y}.$$

By solving the RHC problem, it yields the optimal control sequence

$$\mathbf{u}^\star(k) = \{u^\star(k \mid k), u^\star(k+1 \mid k), \ldots, u^\star(k+N-1 \mid k)\} \qquad (1.7)$$

and the optimal state trajectory

$$\mathbf{x}^\star(k) = \{x^\star(k \mid k), x^\star(k+1 \mid k), \ldots, x^\star(k+N-1 \mid k)\} \tag{1.8}$$

and the optimal cost function value

$$J_N^\star(k) = J_N(\mathbf{x}^\star(k), \mathbf{u}^\star(k)) \tag{1.9}$$

The first control force $u^\star(k \mid k)$ is applied to the plant. Then, the implicit control law is

$$\kappa_N(k) := u^\star(k \mid k) \tag{1.10}$$

In the sequel, it is convenient to refer to the infinite horizon problem $J_\infty^\star(k)$ and the associated infinite horizon control law $\kappa_\infty(k)$ for the problem with N replaced by ∞. Therefore, the basic RHC algorithm is described by

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Algorithm 1

1: Get the new state $x(k)$

2: Solve the optimization problem (1.6)

3: Apply $u^\star(k \mid k)$

4: $k = k + 1$, go to step 1

1.2 RHC Issues

The most challenges of RHC are to reduce the computations, and keep its feasibility and robust stability as most as possible at the same time. The following discussion is pertinent to linear systems only.

1.2.1 Feasibility

Feasibility of the problem (1.6) at each step k must be ensured. It is easy to be done for the nominal unconstrained systems. Assume the feasible optimal control sequence at time k is $\{u^\star(k \mid k), u^\star(k+1 \mid k), \ldots, u^\star(k+N-1 \mid k)\}$. Then a shifted optimal control sequence $\{u^\star(k+1 \mid k), u^\star(k+2 \mid k), \ldots, u^\star(k+N-1 \mid k), 0\}$ is a feasible solution at time $k+1$ definitely. But for the constrained systems with model uncertainty and/or disturbance, the state might transgress the previous state greatly and goes out of feasible region. Thus the feasibility of robust RHC is more complex. [20] proposed a algorithm to ensure robust feasibility at future time k if the optimal control problem has solution at time $k = 0$. By choosing a suitable stability condition, the inequality condition $J_N(k) \leq J_N(k-1) - \Gamma(\Delta)$ is proved, where $\Gamma(\Delta) > 0$ is the function of model uncertainty. [14] proposes soft constraints by adding a slack variable ϵ such that the state constraints are shown as $E_2x+Fu \leq G_2+\epsilon \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$. Thus relaxing the states constraints removes the feasibility problem at least for stable systems. From a practical point of view, keeping the state constraints tight makes no sense because of the existence of noise, disturbances, and numerical errors. While as the control inputs are generated by the optimization procedure, the control input constraints are regarded as hard.

1.2.2 Stability

The standard RHC controllers, such as dynamic matrix control (DMC), quadratic dynamic matrix control (QDMC), model algorithmic control (MAC), which are open-loop control and using finite prediction horizon, do not offer a guarantee of stability automatically, even in the nominal case. For robust stability, the analysis is even more complex and difficult. Research on nominal stability of RHC has now reached a relatively mature stage and it can be achieved by introducing

1. terminal equality/inequality constraint [107]. The stability constraint is $x(k+N \mid k) = 0$. This render the sequence $\{u^\star(k+1 \mid k), u^\star(k+2 \mid k), \ldots, u^\star(k+N-1 \mid k), 0\}$, feasible at step $k+1$, and therefore $V(k+1) \leq J_N(k+1) \leq J_N^\star(k) = V(k)$ is a Lyapunov function of the system [10]. The main drawback of using terminal constraints is that the control effort required to steer the states to the origin can be large, especially for the short horizon N, and therefore feasibility

is more critical because of the control input constraints. Then the domain of attraction is limited to the set of initial states $x(0)$ that can be steered to the origin, which can be considerably small.

2. terminal cost function over remaining infinite horizon. [120] proposes a Lyapunov function $V(x) = x^T X_N x$ as the upper bound of cost function, where $X_N \in \mathbf{S}_{+}^{n\times n}$. This Lyapunov function, which is obtained by solving an offline LMI condition, leads to the stability of the system. It is easy way to guarantee the stability by adding this Lyapunov function as the last term in the cost function. Thus the cost performance will be asymptotically decreasing for the nominal systems. Since the Lyapunov function is the upper bound of the cost over the remaining infinite horizon and is solved based on the initial states $x(0)$, it is conservative and may not achieve a good optimal results when the current states $x(k)$ is far away from $x(0)$.

3. terminal constraint set. [66] relaxes the terminal equality/inequality constraint into the set constraint $x(k+N \mid k) \in \mathbb{X}_f$ and $u(k+i \mid k) = Fx(k+i \mid k),\ i = 0, 1, \ldots, N-1$, where F is a state feedback control gain, The set $\mathbb{X}_f$ is invariant under control gain F and such that the constraints are fulfilled inside $\mathbb{X}_f$.

4. terminal cost and constraint set [65]. This method is combined with item 2 and item 3. It not only ensures the stability condition of the cost performance, but also enforce the terminal states $x(k+N \mid k)$ to lie in the constraint set $\mathbb{X}_f$. If the cost performance is quadratic, terminal set $\mathbb{X}_f$ is defined as $\mathbb{X}_f := \left\{x \in \mathbf{R}^{n\times n} : x^T X_N x \leq 1\right\}$, this method always can not be solved by linear programming methods directly.

1.2.3 Computation

The complexity of the solver for the optimization problem (1.6) depends on the choice of the cost function and constraints. When terminal set $\mathbb{X}_f$ is a polytope, the cost function is the form of (1.2), the optimization problem (1.6) can be formulated as a quadratic programming. It has $2N$ decision variables of states and control input over finite horizon N steps. If there are m equalites/inequalites at each step, the problem will have $m \times n$ constraints. Alternatively, one obtains a linear programming by formulating the cost by ℓ_1/ℓ_∞ norm [38]. For the technique proposed by [104], where a worst case quadratic performance criterion is minimized over a finite set of models subject to input/state constraints, the authors report that problems with more than 1000 variables and 5000 constraints can be solved in a few minutes on a workstation by using interior-point methods.

1.3 Robust RHC

Robust control is to design a controller by using the nominal model with a set descriptions of uncertainties that give acceptable performance despite of a specified range of model variation (model uncertainty) and a class of noise signals (disturbance). The statement about robustness of a particular control algorithm usually makes reference to a specific uncertainty range as well as specific stability and performance criteria [82]. When constraints on states and controls are present, it is necessary to ensure that model variation or disturbances do not cause transgression of the constraints. Therefore, constrained robust RHC is much more complex than unconstrained robust RHC.

1.3.1 Model Uncertainty Description

Because RHC is primarily a time-domain technique, frequency-domain description of uncertainty are not suitable for the formulation of robust RHC [14]. Under some specific uncertainly descriptions, most of the robust approaches share basic ideas in the nominal cases to achieve robust stability. In most literature, the uncertainty description are classified as

1. Impulse/Step-response uncertainty. Uncertainty is described as impulse-response function or step-response function. Earlier analysis and design of robust model

predictive control employed finite impulse response (FIR) model [26], [32]. In later papers, a typical robust RHC strategy consists of solving a min-max problem to optimize robust performance (the minimum over the control input of the maximum over the uncertainties and/or disturbances), which does not adopt impulse/step-response uncertainty description any longer.

2. Structured feedback uncertainty. It assigns a block-diagonal operator, like $\mathbf{\Delta} = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$, as uncertainties in the feedback loop, where each block always satisfies $\| \Delta_i \|_2 \leq 1$ [118].

3. Multi-model and multi-disturbance. Model uncertainty and disturbance are parameterized by a finite list of possible model and disturbance.

4. Polytopic set. Specially, a convex hull of model or disturbance is always given. This is a special case of unstructured uncertainty.

5. Bounded disturbance. It is a reasonable choice when assumption of knowing disturbance set seems restrictive. Typical types of such disturbances include magnitude and energy bounded disturbances.

1.3.2 Robust Stability

The complexity of robust stability guarantee is more heavily investigated than uncertainty description. The work of [99] is a milestone to be mentioned, where the infinite prediction horizon is used for stability and a procedure for the computation of a finite horizon problem is provided. Depending on the uncertainty description considered, the proposed robust methods are different and can be classified into different categories, such as the soft-bound approach and the hard-bound approach [25]. For example, if the model parameters are assumed to be random variables, the optimal control problem is usually formulated as minimization of the expectation of the future output error variance and stochastic control methods are used. If deterministic bounds are assigned to the parameters, the entire vertices must be considered for the worst-case minimization and the complexity of optimization could become a problem [102]. Under the assumption that the true plant is in a parameterized model set [6], [95], a robust algorithm is introduced by adding cost constraints that prevent the sequence of optimal controller costs from increasing for the true plant, where the uncertainty is reflected in the constraints. Various design procedures achieve robust stability basically in two different ways:

1. Indirectly by specifying the performance objective and uncertainty description, the optimal control computations lead to robust stability.

2. Directly by enforcing a type of robust contraction constraint, the state will shrink in some metric of state $x(k)$ for all uncertainties.

Min-max Optimization

Usually, the uncertainty description is in the model parameters and the worst-case scenario is addressed. It describes a robust constraint that requires the largest possible terminal state to be smaller than the initial state at the end of a user-defined finite horizon. In this approach, the control moves are computed by minimizing the worst case performance over all possible uncertainties, while meeting the constraints [79]. Since the work of [75], the min-max formulation has been widely adopted in solving robust RHC problems. However, the resulting controller may be conservative, because it is designed for the worst-case situation, which may have a low probability of occurring. Another disadvantage of the min-max approach is the computational complexity caused by the curse of dimensionality [77]. In addition, to simplify the online computations one must choose the simplest, albeit unrealistic, model as well as model uncertainty description. This will inevitably result in the loss of performance.

Robust Contraction Constraint

Additional constraints

$$\| x(k+1 \mid k) \|_P \leq \lambda \| x(k) \|_P, \lambda < 1 \tag{1.11}$$

is used to enforce the state to contract. When positive definite matrix P is chosen as the solution of Lyapunov equation, this constraint can always be met for some control force u. [6] proposes a special contraction constraint to enforce optimal cost function to decrease.

Robust Invariant Terminal Set

For standard RHC, one way to guarantee its stability is by enforcing terminal equality constraints. However, for uncertainty systems, this will not be possible anymore. Therefore, invariant ellipsoidal terminal sets are proposed to relax the terminal equality constraint [11]. Such technique can be extended to robust RHC formulations, for instant, by LMI [59]. Invariant terminal ellipsoid sets lead to quadratically constraint quadratic programming (QCQP). Alternatively, invariant terminal polyhedral sets lead to linear constraints and quadratic programming (QP), which is computational cheaper than QCQP. The terminal inequality conditions in [65], [62] seem to be quite general for the cost monotonicity. The results in [65] also apply to time-varying systems; however, no practical guidelines are proposed about how to find feasible terminal weighting matrices for cost monotonicity in the case of time-varying systems.

1.3.3 Synthesis Approach

There are several approaches to design robust RHC.

Open-loop Robust RHC

Typically, open-loop robust RHC treats control input sequence $\{u(k \mid k), u(k+1 \mid k), \ldots, u(k+N-1 \mid k)\}$ as optimal decision variables. Although open-loop min-max model predictive control is attractive, it may be conservative due to the open-loop nature. The trajectories satisfying may diverge, causing the domain of attraction to be small or even empty for the constraints horizon length N [79].

Closed-loop Robust RHC

Recursive Algorithm

[13] proposes a closed-loop robust RHC scheme as a recursive algorithm. It is a

Algorithm 2

1: Initialize $J_N^\star(k+N \mid k) = x^T(k+N \mid k) X_N x(k+N \mid k)$

2: Set $i = N-1$

3: At time $k+i$, calculate $J_N^\star(k+i \mid k) = \underset{u(\cdot)}{min}\ \underset{w(\cdot)}{max}\{x^T(k+i \mid k)Q_x x(k+i \mid k) + u^T(k+i \mid k)Q_u u(k+i \mid k) + J_N^\star(k+i+1 \mid k)\}$

4: When $i = 0$, go to next, otherwise $i = i-1$, go back to 2

5: Apply $u(k \mid k)$

6: $k = k+1$, go to 1

dynamic programming problem. It uses the future optimal result in current instant optimization, which requires less computations because it inherits the commutated results of last recursive step and does not search all admissible uncertainties any longer. Nevertheless, the computational cost of dynamic programming increases very quickly as horizon size N. Therefore, it is prohibitive in solving large size problems.

State Feedback Robust RHC

Another important approach is feedback robust RHC. Feedback could prevent the trajectories from diverging excessively. The scenario generated in solving (1.6) does not model accurately the uncertain control problem because it ignores feedback by searching over open-loop control input $u(\cdot)$ in minimizing $J_N(k)$. To address this deficiency, the feedback RHC was proposed in [59], [64], [102]. The decision variables are replaced by a sequence of control law $\pi(k)$, which is defined by

$$\pi(k) := \{u(k \mid k), \kappa_N(k+1 \mid k), \ldots, \kappa_N(k+N-1 \mid k)\} \tag{1.12}$$

where, $\kappa_N(\cdot) : \mathbb{X} \to \mathbb{U}$ is a control law whereas $u(0)$ is a control action. The feedback version of receding horizon control appears attractive but more complex.

Robust constrained RHC with state feedback has been studied extensively in literature [14]. Additional constraints is imposed to guarantee robust stability. For a finite input horizon, the strategy is to use the previous optimal input sequence at time k as a feasible input sequence at time k+1, and force the feasible cost at time k+1 to be less than the optimal cost at time k in the presence of model uncertainty [6], [119] [90]. For an infinite input and output prediction horizon, a state feedback control law is adopted to facilitate a finite dimensional formulation [59]. At each time step, an optimal upper bound on the worst case performance cost over the infinite horizon is obtained by forcing a quadratic function of the state to decrease with time going on. If the optimal feedback law computed at time k is applied at time k+1, the achieved upper bound at time k+1 must be less than the optimal upper bound at time k. This approach was extended to LPV systems, in which the current system is known exactly, but the future systems are uncertain [19] due to unavailability of uncertainty information for future steps.

Output Feedback Robust RHC

For an output feedback system, RHC design involves separate design of a controller assuming all state elements are available and design of an estimator to reconstruct the state given a partial measurement [16]. Thus at each time step, RHC computes optimal control moves based on the prediction of the estimated state over a fixed time horizon. It is well known that the combination of the separately designed controller and estimator has no guaranteed stability margins. Specifically, for the system with polytopic uncertainty or norm-bound uncertainty, supposing a robust constrained state feedback RHC [59] and an estimator based on a nominal model are designed separately, the error dynamics of the estimator are dependent on the system dynamics, and the optimality argument is no longer valid. Since the optimal cost based on the estimated state may not be the optimal cost based on the true state, monotonic decrease of the performance cost based on the true state is not guaranteed. As a result, controller designers have to analyze the robust stability of the combined

RHC and estimator. But unlike H_2 and H_∞ linear optimal control which have an easily implementable linear feedback law, robust RHC requires online optimization and has no explicit offline form of feedback control law. Furthermore, because of the nonlinearities of input and output constraints, the implicit RHC control law is nonlinear in nature. All of these factors make it very difficult to analyze the robust stability of the closed-loop system with output feedback RHC [14]. To avoid the interdependency of the controller and estimator, [120] choose the finite impulse response (FIR) model, where the output prediction in the controller is only based on the current measurement and the past inputs, and no estimator is involved in the problem formulation.

State feedback and output feedback problems can be solved by LMI. Thus when formulate the problem, we need to derive the initial, stability and constraint conditions by LMI form. Especially, rich stability theories exist for robust feedback control of linear systems, such as H_2, H_∞ theory. There have been efforts to extend H_∞ synthesis to handle input/output constraints.

Dual-mode Robust RHC

Computationally efficient extension to the uncertain and/or nonlinear case are harder to develop. [79] proposes a dual-mode control strategy, according to which a fixed control law is used as soon as the system state enters a neighborhood of the origin. The fixed law can be selected as to the methodology in [59] so that the current state belongs to an invariant set, thereby providing a guarantee of stability even for uncertain linear time varying systems.

1.4 Proposed RHC

In this dissertation, we make an effort to work on the output feedback RHC for constrained systems. For LTI systems, we design a dynamic output feedback controller for measured output signals based on a pre-designed observer, while current research work always focus on static output feedback controller, which is combined with the state feedback controller and the state observer. To obtain the stability of the system, ℓ_2 norm of estimated states and control force is introduced as the cost function. Then it is upper bounded by a Lyapunov function, which is asymptotically decreasing. We mainly propose three output feedback RHC schemes: offline output feedback, infinite horizon output feedback RHC and finite horizon feedback RHC. Especially, in the finite horizon feedback RHC, a terminal term is added to the cost function to play as the upper bound of the cost performance from step $k+N$ to ∞. Both of the terminal and stability set are obtained by solving LMI conditions.

We also extend the output feedback RHC to robust output feedback RHC for constrained systems. First it handles the constrained LTI system with energy bounded disturbances. By H_∞ criteria from required output signals to disturbances, the robust output feedback RHC is designed and its stability and feasibility are achieved. Then our work is extended to deal with the constrained system with model uncertainty and magnitude bounded disturbances. The model uncertainty is described by a convex hull. The optimal control is solved by min-max scheme over model uncertainty and disturbances.

1.5 Dissertation Overview

The dissertation is organized in five chapters. §1 is the global review of robust RHC and this dissertation. §2 discusses the output feedback RHC for the constrained nominal LTI system. We propose a dynamic output feedback controller and then extend it to infinite horizon RHC and finite horizon RHC. §3 shows our work on the robust output feedback RHC for the constrained LTI system with energy bounded disturbances. §4 is the design of the robust output feedback RHC for the constrained system with model uncertainty and magnitude disturbances. The last chapter §5 is the conclusion of our work in this dissertation.

Chapter 2

Output Feedback RHC for Constrained Linear Systems

Since lack of signal information, the design of output feedback RHC is a major difficult issue. It is well known that the estimator degrades the stability of the whole control system and it is hard to handle it while the estimator existing. Rich research on output feedback RHC focuses on separating the design work into two individual procedures: state feedback RHC and observer. Nevertheless, the output feedback control designs with hard input/output constraints have not been well addressed. While, in this chapter, we design the output feedback RHC from the measured output signals.

In §2.1, we define the constrained LTI systems in the form of state space. §2.2 shows the output feedback problem formulation and the controller synthesis. §2.3 and §2.4 are the proposed infinite-horizon and finite-horizon RHC schemes. §2.5 provides the simulation results to verify the performance of control algorithms.

2.1 Controlled System Description

We first define an open-loop LTI system for control synthesis and output controller. Consider a constrained discrete-time LTI system, which is governed by

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ z_2(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_2 \\ C_{11} & D_{121} \\ C_{12} & D_{122} \\ C_2 & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \tag{2.1}$$

where state $x \in \mathbf{R}^n$, $z_1 \in \mathbf{R}^{n_{z1}}$ is controlled output and $z_2 \in \mathbf{R}^{n_{z2}}$ is constrained signal, $u \in \mathbf{R}^{n_u}$ is control input and $y \in \mathbf{R}^{n_y}$ is measured output for control. All of state-space matrices have compatible dimensions. It is also assumed that (A, B_2) pair is stabilizable and (A, C_2) pair is detectable. For simplicity, we assume $D_{22} = 0$.

We impose a constraint set as

$$\mathbf{Z}_2 = \left\{ z_2 \in \mathbf{R}^{n_{z2}} : \ |z_2(k)| \leq \bar{z}_2, \ \bar{z}_2 = [\bar{z}_{2,1} \ \ \bar{z}_{2,2} \cdots \bar{z}_{2,n_{z2}}]^T \right\} \tag{2.2}$$

$$= \left\{ |e_j^T C_{cl,2} x_{cl}(k)| \leq \bar{z}_{2,j}, j = 1, 2, \cdots, n_{z2} \right\}. \tag{2.3}$$

where e_j is the jth vector from unity matrix. Note that the above constraint set includes both input and output constraints.

The output feedback controller that we are interested in is in the form of

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix}, \tag{2.4}$$

where $x_c \in \mathbf{R}^n$ has the same dimension with that of plant states.

Therefore, the closed-loop system is obtained by combining eqn. (2.1) and (2.4)

$$\begin{bmatrix} x_{cl}(k+1) \\ z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} A_{cl} \\ C_{cl,1} \\ C_{cl,2} \end{bmatrix} \begin{bmatrix} x_{cl}(k) \end{bmatrix}, \tag{2.5}$$

where

$$\begin{bmatrix} A_{cl} \\ C_{cl,1} \\ C_{cl,2} \end{bmatrix} = \left[\begin{array}{cc} A & 0 \\ 0 & 0 \\ \hline C_{11} & 0 \\ C_{12} & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{121} \\ 0 & D_{122} \end{array}\right] \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}. \tag{2.6}$$

To estimate the information of plant states, we introduce an observer as

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_2u(k) - L[y(k) - C_2\hat{x}(k)] \\ \hat{z}_1(k) &= C_{11}\hat{x}(k) + D_{121}u(k). \end{aligned} \tag{2.7}$$

where L is observer gain. State estimation error is defined as $\tilde{x} = x - \hat{x}$ and controlled output error $\tilde{z}_1 = z_1 - \hat{z}_1$, then their dynamics are governed by

$$\tilde{x}(k+1) = (A + LC_2)\tilde{x}(k) \tag{2.8}$$

$$\tilde{z}_1(k) = C_{11}\tilde{x}(k). \tag{2.9}$$

When the observer gain L is predetermined such that $A + LC_2$ is asymptotically stable, future plant states will be approximated using equation (2.7).

2.2 Constrained Output Feedback Synthesis

We will denote estimated closed-loop states as $\hat{x}_{cl} = \begin{bmatrix} \hat{x}^T & x_c^T \end{bmatrix}^T$. Given initial conditions $\hat{x}(0), x_c(0)$, the objective of constrained control design is to find a feasible output feedback control law (2.4) that optimizes the quadratic performance of $\hat{z}_1$ while enforcing input/output constraints. Using a modified Lyapunov function $V(\hat{x}_{cl}) = \hat{x}_{cl}^T X \hat{x}_{cl}$, where $X \in \mathbf{S}_+^{2n\times 2n}$, this problem can be formulated as

Problem 2 (Constrained Output Feedback Control)

$$\begin{aligned} \min_{\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c} \quad & \gamma \\ s.t. \quad & (2.1),\ (2.4),\ (2.7), \\ & V(\hat{x}_{cl}(0)) \le \gamma \\ & V(\hat{x}_{cl}(k+1)) - V(\hat{x}_{cl}(k)) + \hat{z}_1^T(k)\hat{z}_1(k) < 0 \\ & z_2(k) \in \mathbf{Z}_2. \end{aligned} \tag{2.10}$$

By summing eqn.(2.10) from $k = 0$ to ∞, the constrained quadratic performance is like

$$J = \sum_{k=0}^{\infty} \hat{z}_1^T(k)\hat{z}_1(k) < V(\hat{x}_{cl}(0)) \leq \gamma. \tag{2.11}$$

Then we can see the Lyapunov function $V(\hat{x}_{cl})$ is asymptotically decreasing and is also the upper bound of the quadratic performance J. By minimize the initial Lyapunov function value $V(\hat{x}_{cl}(0))$, we obtain the optimal performance. The controlled output $\hat{z}_1(k)$ always satisfies

$$\hat{z}_1^T(k)\hat{z}_1(k) = \hat{x}^T(k)Q_x\hat{x}(k) + u^T(k)Q_u u(k),$$

where $Q_x \in \mathbf{S}_+^{n\times n}$, $Q_u \in \mathbf{S}_+^{n_u\times n_u}$. Then the performance J is the quadratic function of estimated plant states, and control input.

The following theorem provides an output feedback synthesis condition that solves the constrained control problem in LMI formulation.

Theorem 1 *Given the initial condition $\hat{x}(0), x_c(0)$, the system (2.1) is stabilized by an output-feedback controller (2.4) and its quadratic performance less than γ if there exist matrices $R, S \in \mathbf{S}_+^{n\times n}$, and $\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c \in (\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n_y} \times \mathbf{R}^{n_u\times n} \times \mathbf{R}^{n_u\times n_y})$ such that*

$$\begin{bmatrix} \gamma & (\hat{x}(0) - x_c(0))^T S & x_c^T(0) & 0 \\ S(\hat{x}(0) - x_c(0)) & S & 0 & 0 \\ x_c(0) & 0 & R & I \\ 0 & 0 & I & S \end{bmatrix} \geq 0, \tag{2.12}$$

$$\begin{bmatrix} R & \star & \star & \star & \star & \star \\ I & S & \star & \star & \star & \star \\ -I & -S & S & \star & \star & \star \\ AR + B_2\hat{C}_c & A + B_2\hat{D}_cC_2 & -(A + LC_2) & R & \star & \star \\ \hat{A}_c & SA + \hat{B}_cC_2 & -S(A + LC_2) & I & S & \star \\ C_{11}R + D_{121}\hat{C}_c & C_{11} + D_{121}\hat{D}_cC_2 & -C_{11} & 0 & 0 & I \end{bmatrix} > 0, \tag{2.13}$$

$$\begin{bmatrix} \bar{z}_{2,j}^2 R & \star & \star & \star \\ \bar{z}_{2,j}^2 I & \bar{z}_{2,j}^2 S & \star & \star \\ -\bar{z}_{2,j}^2 I & -\bar{z}_{2,j}^2 S & \bar{z}_{2,j}^2 S & \star \\ e_j^T(C_{12}R + D_{122}\hat{C}_c) & e_j^T(C_{12} + D_{122}\hat{D}_cC_2) & 0 & 1 \end{bmatrix} \geq 0, \ \ j = 1, 2, \cdots, n_{z2} \tag{2.14}$$

Moreover, the output feedback controller gain is given by

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} -S & SB_2 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c - SAR & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ C_2R & I \end{bmatrix}^{-1}. \tag{2.15}$$

Proof:

We first define transform matrices

$$Z_1 = \begin{bmatrix} R & I \\ R - S^{-1} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & -S \end{bmatrix},$$

and specify $X = Z_2 Z_1^{-1}$. Note Z_1, Z_2 are both non-singular.

Then $X = \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix}$ is positive definite with $XZ_1 = Z_2$ and

$$
\begin{aligned}
Z_1^T X Z_1 &= \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \\
Z_1^T \left(X A_{cl} \right) Z_1 &= \begin{bmatrix} AR + B_2 \hat{C}_c & A + B_2 \hat{D}_c C_2 \\ \hat{A}_c & SA + \hat{B}_c C_2 \end{bmatrix}, \\
Z_1^T \left(X B_{cl} \right) &= \begin{bmatrix} B_1 + B_2 \hat{D}_c D_{21} \\ SB_1 + \hat{B}_c D_{21} \end{bmatrix}, \\
C_{cl,i} Z_1 &= \begin{bmatrix} C_{1i} R + D_{12i} \hat{C}_c & C_{1i} + D_{12i} \hat{D}_c C_2 \end{bmatrix}, \qquad i = 1, 2,
\end{aligned}
$$

where the transformed controller data relates to the original (A_c, B_c, C_c, D_c) by

$$
\begin{bmatrix} \hat{A}_c & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} = \begin{bmatrix} SAR & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S & SB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ C_2 R & I \end{bmatrix}. \tag{2.16}
$$

Using Schur complement, eqn.(2.12) becomes

$$
\begin{aligned}
\gamma &\geq \begin{bmatrix} \hat{x}^T(0) - x_c^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) - x_c(0) \\ x_c(0) \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ x_c(0) \end{bmatrix} \\
&= \hat{x}_{cl}^T(0) X \hat{x}_{cl}(0) = V(\hat{x}_{cl}(0)).
\end{aligned}
$$

Next, multiplying diag $\left\{ Z_1^{-T}, I, X^{-1} Z_1^{-T}, I \right\}$ to the left and its transpose from the right of eqn.(2.13), we obtain

$$
\begin{bmatrix}
X & -X \begin{bmatrix} I \\ 0 \end{bmatrix} & A_{cl}^T & C_{cl,1}^T \\
-\begin{bmatrix} I & 0 \end{bmatrix} X & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} & -\begin{bmatrix} (A + LC_2)^T & 0 \end{bmatrix} & -C_{11}^T \\
A_{cl} & -\begin{bmatrix} A + LC_2 \\ 0 \end{bmatrix} & X^{-1} & 0 \\
C_{cl,1} & -C_{11} & 0 & I
\end{bmatrix} > 0. \tag{2.17}
$$

Using Schur complement, eqn.(2.17) can be rewritten as

$$\begin{bmatrix} X & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} A_{cl}^T & C_{cl,1}^T \\ -\begin{bmatrix} (A+LC_2)^T & 0 \end{bmatrix} & -C_{11}^T \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}$$
$$\times \begin{bmatrix} A_{cl} & -\begin{bmatrix} A+LC_2 \\ 0 \end{bmatrix} \\ C_{cl,1} & -C_{11} \end{bmatrix} > 0.$$

Further multiplying the above inequality from the left by $\begin{bmatrix} x_{cl}^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose from the right side, we get

$$\begin{aligned} -\left(x_{cl}^T(k)A_{cl}^T - \begin{bmatrix} \tilde{x}^T(k)(A+LC_2)^T & 0 \end{bmatrix}\right) X \left(A_{cl}x_{cl}(k) - \begin{bmatrix} (A+LC_2)\tilde{x}(k) \\ 0 \end{bmatrix}\right) \\ + \left(x_{cl}^T(k) - \begin{bmatrix} \tilde{x}^T(k) & 0 \end{bmatrix}\right) X \left(x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix}\right) \\ - \left(x_{cl}^T(k)C_{cl,1}^T - \tilde{x}^T(k)C_{11}^T\right)\left(C_{cl,1}x_{cl}(k) - C_{11}\tilde{x}(k)\right) > 0. \end{aligned} \tag{2.18}$$

Since

$$\begin{aligned} \hat{x}_{cl}(k+1) &= A_{cl}x_{cl}(k) - \begin{bmatrix} (A+LC_2)\tilde{x}(k) \\ 0 \end{bmatrix}, \\ \hat{x}_{cl}(k) &= x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix}, \\ \hat{z}_1(k) &= C_{cl,1}x_{cl}(k) - C_{11}\tilde{x}(k), \end{aligned}$$

eqn.(2.18) is equivalent to the required condition eqn.(2.10),

$$V(\hat{x}_{cl}(k+1)) - V(\hat{x}_{cl}(k)) + \hat{z}_1^T(k)\hat{z}_1(k) < 0.$$

Also, pre-multiplying diag $\left\{Z_1^{-T}, I, 1\right\}$ and post-multiplying its transpose at both sides of eqn.(2.14), we have

$$\begin{bmatrix} \bar{z}_{2,j}^2 \begin{bmatrix} X & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} C_{cl,2}^T \\ 0 \end{bmatrix} e_j \\ e_j^T \begin{bmatrix} C_{cl,2} & 0 \end{bmatrix} & 1 \end{bmatrix} \geq 0,$$

which is equivalent to

$$\bar{z}_{2,j}^2 \begin{bmatrix} X & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} C_{cl,2}^T \\ 0 \end{bmatrix} e_j e_j^T \begin{bmatrix} C_{cl,2} & 0 \end{bmatrix} \geq 0.$$

Multiplying $\begin{bmatrix} x_{cl}^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose on both sides and we obtain

$$\bar{z}_{2,j}^2 \left(x_{cl}^T(k) - \begin{bmatrix} \tilde{x}^T(k) & 0 \end{bmatrix} \right) X \left(x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix} \right) - z_{2,j}^T(k) z_{2,j}(k) \geq 0.$$

That is

$$\bar{z}_{2,j}^2 \hat{x}_{cl}^T(k) X \hat{x}_{cl}(k) - z_{2,j}^T(k) z_{2,j}(k) \geq 0. \tag{2.19}$$

Since $\hat{x}_{cl}^T(k) X \hat{x}_{cl}(k) < \hat{x}_{cl}^T(0) X \hat{x}_{cl}(0) \leq \gamma \leq 1$, the constrained input/output satisfy

$$|z_{2,j}(k)| \leq \bar{z}_{2,j}, \qquad j = 1, 2, \cdots, n_{z2}$$

from eqn.(2.19). *Q.E.D.*

The general type of Lyapunov function used in congruent transformation is in the form of

$$X = \begin{bmatrix} S & N \\ N^T & ? \end{bmatrix}, \qquad X^{-1} = \begin{bmatrix} R & M \\ M^T & ? \end{bmatrix}$$

where M, N^T can be chosen as any matrices satisfying $MN^T = I - RS$. However, in the proof of Theorem 1, we specified $N = -S$ and $M = R - S^{-1}$ to render convex optimization conditions.

On the other hand, eqn.(2.13) also implies

$$S - (A + LC_2)^T S(A + LC_2) > 0.$$

This can be satisfied given the matrix $A + LC_2$ is asymptotically stable.

As can be seen from Theorem 1, the resulting Lyapunov function will be affected by initial condition $\hat{x}(0)$ and $x_c(0)$. Generally speaking, the stability required performance will be conservative. This motivates us to use online optimization to improve controlled performance.

2.3 Infinite-horizon RHC

To improve the control performance, we would solve the optimization problem using receding horizon control strategy. In RHC scheme, the current estimated plant states $\hat{x}(k)$ and controller states $x_c(k)$ will provide the information of the closed-loop systems, and the choice of feedback control gains. By minimizing the performance level γ online, we obtains the best possible performance, while keeping the time-domain constraints satisfied.

Let $\hat{x}(k+i|k), \hat{z}_1(k+i|k)$ denote the predicted states and controlled output of the estimation based on the measurement at step k. $u(k+i|k)$ is the predicted control action for step $(k+i)$. In particular, $\hat{x}(k|k) = \hat{x}(k), \hat{z}_1(k|k) = \hat{z}_1(k)$ and $u(k|k) = u(k)$. Given current states $\hat{x}(k)$ and $x_c(k)$, we will solve an optimal control problem over an output feedback controller on the remaining infinite horizon. The infinite-horizon RHC problem will be formulated as

Problem 3 (Output Feedback Infinite-horizon RHC)

$$\min_{\hat{A}_c(k),\hat{B}_c(k),\hat{C}_c(k),\hat{D}_c(k)} \quad \gamma_k \tag{2.20}$$

$$\begin{aligned}
s.t. \qquad & (2.1),\ (2.4),\ (2.7), \\
& V(\hat{x}_{cl}(k)) \leq \gamma_k \\
& V(\hat{x}_{cl}(k+i+1|k)) - V(\hat{x}_{cl}(k+i|k)) + \hat{z}_1^T(k+i|k)\hat{z}_1(k+i|k) < 0 \\
& z_2(k+i|k) \in \mathbf{Z}_2.
\end{aligned}$$

The following theorem provides the output feedback synthesis conditions of infinite-horizon RHC.

Theorem 2 *At time step k, given the constrained linear system (2.1) with current states $\hat{x}(k), x_c(k)$ and the RHC solution from last step as R_{k-1}, S_{k-1} and γ_{k-1}.*

1. *There always exist a scalar $\gamma_k \leq \gamma_{k-1}$, matrices $R_k, S_k \in \mathbf{S}_+^{n\times n}$, and $\hat{A}_c^k, \hat{B}_c^k, \hat{C}_c^k, \hat{D}_c^k$ of appropriate dimensions satisfying (2.12) - (2.14).*

2. *The constrained linear system is stabilized by the output-feedback RHC control law*

$$\begin{bmatrix} A_c^k & B_c^k \\ C_c^k & D_c^k \end{bmatrix} = \begin{bmatrix} -S_k & S_k B_2 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c^k - S_k A R_k & \hat{B}_c^k \\ \hat{C}_c^k & \hat{D}_c^k \end{bmatrix} \begin{bmatrix} R_k - S_k^{-1} & 0 \\ C_2 R_k & I \end{bmatrix}^{-1}, \tag{2.21}$$

over the remaining horizon. Moreover, we have

$$\sum_{i=0}^{\infty} \hat{z}_1^T(k+i|k)\hat{z}_1(k+i|k) < \hat{x}_{cl}^T(k) X_k \hat{x}_{cl}(k) \leq \gamma_k.$$

Proof:

We denote the optimal solution from online LMI optimization at time step $(k-1)$ as

$$\mathbf{O}_{k-1} = \left\{ R^*_{k-1}, S^*_{k-1}, \gamma^*_{k-1}, (\hat{A}_c^{k-1})^*, (\hat{B}_c^{k-1})^*, (\hat{C}_c^{k-1})^*, (\hat{D}_c^{k-1})^* \right\}.$$

Since $\hat{x}^T_{cl}(k)X_{k-1}\hat{x}_{cl}(k) < \hat{x}^T_{cl}(k-1)X_{k-1}\hat{x}_{cl}(k-1)$, the optimal solution at time step $(k-1)$ satisfies conditions (2.12) - (2.14) for any state $\hat{x}_{cl}(k)$. That means $\mathbf{O}_{k-1}$ is a feasible solution for time step k. If we have optimal solution $\mathbf{O}_k$ at time step k, it should render a better performance, i.e. $\gamma^*_k \leq \gamma^*_{k-1}$. *Q.E.D.*

Algorithm 3 Infinite-horizon RHC

1: Choose a suitable observer L to ensure $A + LC_2$ asymptotically stable.

2: Set time $k = 0$.

3: Given $\hat{x}(k)$, $x_c(k)$, solve conditions (2.12)-(2.14).

4: Construct a RHC controller according to (2.21).

Calculate and implement the first control force $u(k|k)$

5: Set k=k+1, return to Step 3.

The monotonicity property of infinite-horizon quadratic performance indicates the controlled performance will be improved gradually through online optimizations. While infinite-horizon RHC improves controlled performance through online optimization techniques, it may not be able to expand the stability region for the constrained systems.

2.4 Finite-horizon RHC

Instead of infinite-horizon RHC updating one controller for the remaining infinite horizon, finite-horizon RHC is desired to develop a sequence of controllers $\pi(k) =$

$\{\kappa_0, \kappa_1, \ldots, \kappa_{N-1}\}$ over N steps. To ensure its stability, both terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{X}_f$ will be introduced as

$$\begin{aligned} V_f(\hat{x}_{cl}) &= \hat{x}_{cl}^T X^\star \hat{x}_{cl} \\ \mathbb{X}_f &= \left\{\hat{x}_{cl} \in \mathbf{R}^2 n : \ V_f(\hat{x}_{cl}) \leq \gamma^\star\right\} \end{aligned} \tag{2.22}$$

where $X^\star$, $\gamma^\star$ are offline optimization results, which determines a stability ellipsoid. The finite-horizon RHC is to steer states to terminal set $\mathbb{X}_f$ in N steps. Within $\mathbf{X}_f$, the stabilizing output feedback controller $\kappa^\star$ derived from offline synthesis will be employed.

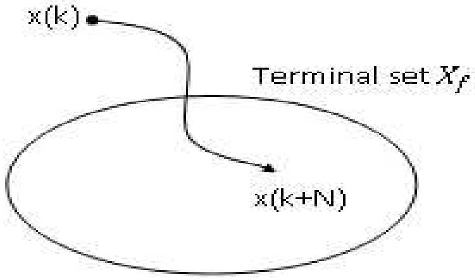


Figure 2.1: Finite-horizon receding horizon control scheme

Problem 4 (Output Feedback Finite-horizon RHC) *Given the system (2.1)-(2.4) with current state $\hat{x}(k)$, terminal matrix $X^\star$ and offline optimal performance $\gamma\star$, the finite-horizon RHC problem can be solved online by LMI optimization scheme*

$$\min_{u(\cdot),\hat{x}(\cdot)} J(k) = \sum_{i=0}^{N-1} \hat{z}_1^T(k+i \mid k)\hat{z}_1(k+i \mid k) + \hat{x}_{cl}^T(k+N \mid k) X^\star \hat{x}_{cl}(k+N \mid k)$$

$$\begin{aligned} s.t. \quad & \hat{x}(k+i+1 \mid k) = A\hat{x}(k+i \mid k) + B_2 u(k+i \mid k), \\ & z_2(k+i \mid k) \in \mathbf{Z}_2, \\ & \hat{x}_{cl}(k+N \mid k) \in \mathbb{X}_f. \end{aligned}$$

The quadratic performance $J(k)$ includes two parts. One is the cost over N steps. The other is called terminal cost, which is the upper bound of the performance over the remaining time from $k+N$ to ∞.

The LMI formulation of Prob.4 involves $(n_u+n)N$ number of optimization variables and $(nN+2+n_{z2})$ number of equality/inequality constraints. It can be solved efficiently for a reasonable size plant.

Algorithm 4 Finite-horizon RHC

1: Choose a suitable observer L to ensure $A+LC_2$ asymptotically stable.

2: Given initial conditions $\hat{x}(0)$, $x_c(0)$, solve LMI (2.12)-(2.14), then obtain terminal matrix $X^\star$ and optimal performance $\gamma\star$.

3: Set time $k=0$.

4: Given $\hat{x}(k)$, $X^\star$, $\gamma\star$, solve Prob.4.

Achieve a sequence of predicted state $\{\hat{x}(k+1 \mid k), \hat{x}(k+2 \mid k), \ldots, \hat{x}(k+N \mid k)\}$ and control force $\{u(k \mid k), u(k+1 \mid k), \ldots, u(k+N-1 \mid k)\}$.

Implement the first control force $u(k \mid k)$.

5: Set k=k+1, return to Step 4.

2.5 Example

In this section, we would like to compare the performance of the control algorithms that we have proposed in the previous sections.

Consider the single microcantilever system as Fig.2.2, which has the dynamic

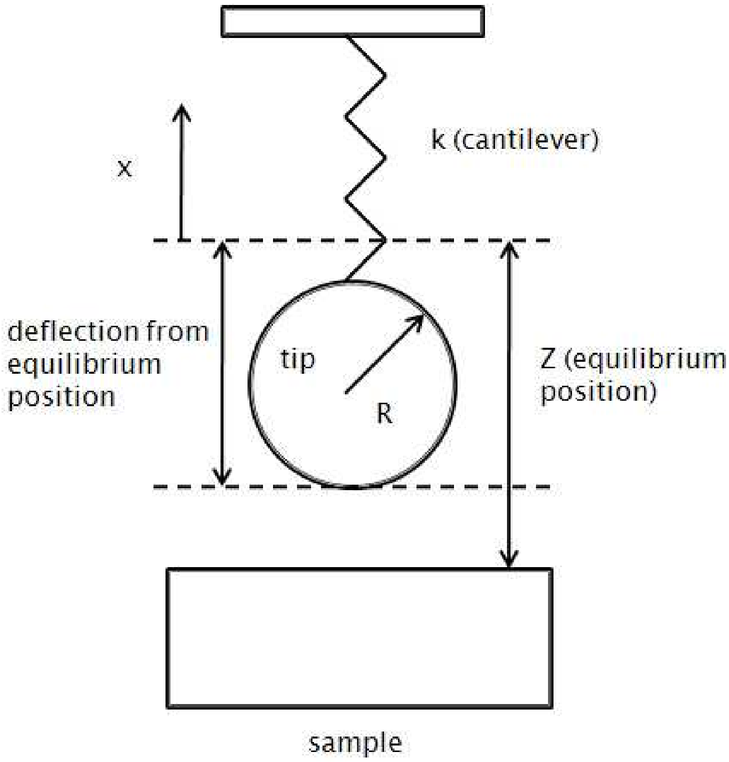


Figure 2.2: Single microcantilever system

equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_1^2 x_1 - \frac{D\omega_1^2}{(Z + x_1)^2} \tag{2.23}$$

where, $\omega_1 = \sqrt{k/m}$ is the first modal frequency of the system, $D = AR/6k$ and $Z = 1.5(2D)^{1/3}$ is the position of the equilibrium position. x_1 is position of cantilever. x_2 is velocity of cantilever. A is the Hamaker constant. R is the radius of the tip. k is the spring constant. Typical values found are $A = 10^{-19}J$, $R = 1500\mathring{A}$, and $k = 0.0167N/m$.

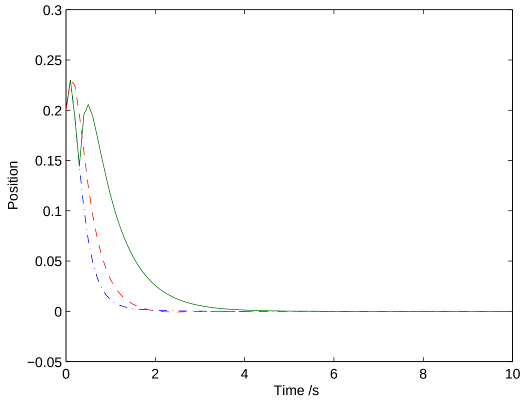
In order to study the qualitative behavior of the system, it is convenient to perform the following change of variables. By setting $T = \omega_1 t$, and dividing the left and right hand sides of (2.23) by Z, we get

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = -\xi_1 - \frac{4}{27(1+\xi_1)^2} \tag{2.24}$$

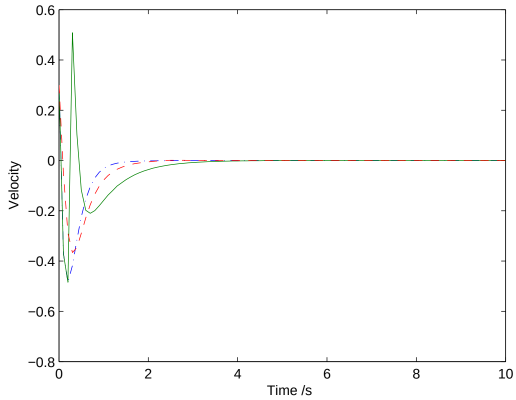
The system (2.24) has three equilibrium points at $(-\frac{4}{3}, 0)$, $(-\frac{1}{3}, 0)$ and $(-\frac{1}{3}, 0)$. In this study, we will choose one of equilibrium points for RHC design. By linearizing the model at $(-\frac{4}{3}, 0)$, the linear model is show as

$$\left[\begin{array}{c} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \\ \hline z_1(t) \\ z_2(t) \\ y(t) \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -9 & 0 & 1 \\ \hline 1.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \eta_1(t) \\ \eta_2(t) \\ \hline u(t) \end{array}\right] \tag{2.25}$$

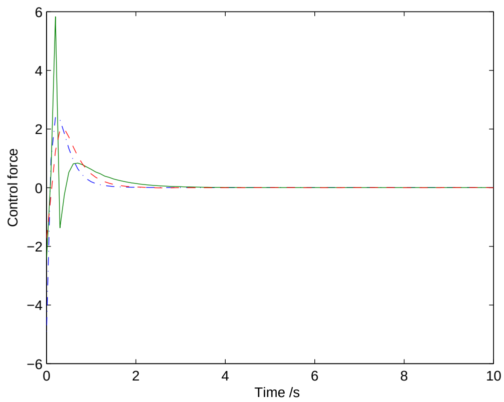
Starting from the initial positions $(0.2, 0.3)$, the simulation results are shown in Fig. 2.3. Specially, the dash line is the result of the typical output feedback control, which designs a state feedback controller $F(k)$ working on the estimated states $\hat{x}(k)$. Thus at step k, the control input is calculated by $u(k) = F(k)\hat{x}(k)$.



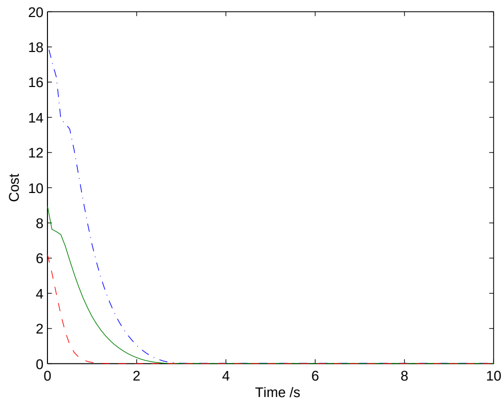
(a). position



(b). velocity



(c). control force



(d). cost

Figure 2.3: Comparison of proposed finite-horizon RHC (solid), infinite-horizon RHC (dash dot) and typical output feedback control (dash).

From the results figures, we can see that the dynamic response of infinite-horizon RHC is more smooth than that of finite-horizon RHC, while its cost performance is worse. The control scheme choice is determined by which aspect of control performance we are concerning about.

2.6 Conclusion

In this chapter, we first proposed an output feedback control based on the estimator and measured output signals. Then we extended this controller to two kinds of output feedback RHC schemes: infinite-horizon and finite-horizon. The infinite-horizon RHC is to update the controller online by solving a sequel of LMI. The obtained controller is to control the system over the remaining infinite horizon. The finite-horizon RHC is to optimize a quadratic cost function, which consists of performance over a finite horizon and a terminal cost. Simulation results show that both of infinite-horizon and finite-horizon output feedback RHC have good dynamic response and asymptotic stability.

Chapter 3

Robust Output Feedback RHC

A fundamental problem of RHC is its robustness to modeling uncertainties and external disturbances. In this chapter, we extend the work of Chapter 2 to the robust output feedback RHC for the constrained LTI systems with energy bounded disturbances. To combine practical advantages of RHC with the robustness properties of $\mathbf{H}_\infty$ control, a moving horizon $\mathbf{H}_\infty$ control problem is considered. Our objective is to minimize the effect of bounded disturbance on the system output in terms of $\mathbf{H}_\infty$ performance index. The introduction of invariant ellipsoidal sets provide efficient means to handle inequality constraints in RHC algorithms and to guarantee closed-loop robust stability. A Lyapunov function derived from offline robust $\mathbf{H}_\infty$ control will be used as the terminal cost function. Then robust stability and feasibility of the proposed RHC algorithm are achieved.

In §3.1, we define the constrained LTI systems in the form of state space with energy bounded disturbances. §3.2 shows the output feedback problem formulation and the controller synthesis. §3.3 and §3.4 are infinite-horizon and finite-horizon RHC schemes that we propose. §3.5 is the simulation results of the two-mass-spring system to demonstrate the performance of proposed robust RHC.

3.1 Constrained LTI System Modeling

The open-loop LTI system that we handle is described by

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ z_2(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_{11} & D_{111} & D_{121} \\ C_{12} & D_{112} & D_{122} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{3.1}$$

where state $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{n_w}$ is disturbance, $z_1 \in \mathbf{R}^{n_{z1}}$ is controlled output and $z_2 \in \mathbf{R}^{n_{z2}}$ is constrained signal, $u \in \mathbf{R}^{n_u}$ is control input and $y \in \mathbf{R}^{n_y}$ is measured output for control. All of state-space matrices have compatible dimensions. It is also assumed that (A, B_2) pair is stabilizable and (A, C_2) pair is detectable. For simplicity, we assume $D_{22} = 0$.

we would like to design an output feedback control law in the form of

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix}, \tag{3.2}$$

where $x_c \in \mathbf{R}^n$.

Then the closed-loop system becomes

$$\begin{bmatrix} x_{cl}(k+1) \\ z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl,1} & D_{cl,1} \\ C_{cl,2} & D_{cl,2} \end{bmatrix} \begin{bmatrix} x_{cl}(k) \\ w(k) \end{bmatrix}, \tag{3.3}$$

where

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl,1} & D_{cl,1} \\ C_{cl,2} & D_{cl,2} \end{bmatrix} = \left[\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_{11} & 0 & D_{111} \\ C_{12} & 0 & D_{112} \end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{121} \\ 0 & D_{122} \end{array}\right] \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \left[\begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array}\right].$$

For constrained control, we impose a constraint set as

$$\begin{aligned} \mathbf{Z}_2 &= \left\{ z_2 \in \mathbf{R}^{n_{z2}} : \ |z_2(k)| \le \bar{z}_2, \ \bar{z}_2 = [\bar{z}_{2,1} \ \ \bar{z}_{2,2} \cdots \bar{z}_{2,n_{z2}}]^T \right\} \\ &= \left\{ |e_j^T (C_{cl,2} x_{cl}(k) + D_{cl,2} w(k))| \le \bar{z}_{2,j}, j = 1, 2, \cdots, n_{z2} \right\}. \end{aligned}$$

where e_j is the jth vector of unity matrix. Note that the above constraint set includes both input and output constraints. In addition, the disturbance w is also assumed energy-bounded, therefore it belongs to a bounded set determined by a scalar $\beta > 0$,

$$\mathbf{W}_\beta = \left\{ w \in \mathbf{R}^{n_w} : \sum_{k=0}^{\infty} w^T(k)w(k) \le \beta \right\}.$$

Like in Ch.2, an observer is introduced to estimate plant states. Its dynamic equations are shown as

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_2u(k) - L[y(k) - C_2\hat{x}(k)] \\ \hat{z}_1(k) &= C_{11}\hat{x}(k) + D_{121}u(k). \end{aligned} \tag{3.4}$$

where L is chosen such that $A + LC_2$ is asymptotically stable. Given any initial estimated state $\hat{x}(0)$, the future state values will be approximated using equation (3.4).

Define the estimation errors of states and controlled output as $\tilde{x} = x - \hat{x}$ and $\tilde{z}_1 = z_1 - \hat{z}_1$, then their dynamics are governed by

$$\begin{aligned} \tilde{x}(k+1) &= (A + LC_2)\tilde{x}(k) + (B_1 + LD_{21})w(k) \\ \tilde{z}_1(k) &= C_{11}\tilde{x}(k) + D_{111}w(k). \end{aligned}$$

As can be seen easily, for energy bounded disturbance, the estimation errors are also bounded.

3.2 Robust Output Feedback Problem

After a suitable observer is applied, it is clear that if the estimated states $\hat{x}$ is controlled from the initial point to a stability region, the real plant states will approach the same region. Therefore, in the problem formulation, the states we handle will be the estimated states $\hat{x}$ instead of the real plant states x. We denote the estimated closed-loop states as $\hat{x}_{cl} = \begin{bmatrix} \hat{x}^T & x_c^T \end{bmatrix}^T$. Given the initial condition $\hat{x}(0), x_c(0)$, the objective of constrained control design is to find a feasible output feedback control law (3.2) that optimizes the closed-loop $\mathbf{H}_\infty$ performance from w to z_1 while enforcing input/output constraints for all possible energy bounded disturbances. Using a modified Lyapunov function $V(\hat{x}_{cl}) = \hat{x}_{cl}^T X \hat{x}_{cl}$, this problem can be formulated as

Problem 5 (Robust Constrained Output Feedback Control)

$$
\begin{aligned}
&\min \gamma \\
&s.t. \quad (3.1),\ (3.2),\ (3.4), \\
&\qquad V(\hat{x}_{cl}(0)) \le 1 \\
&\qquad V(\hat{x}_{cl}(k+1)) - V(\hat{x}_{cl}(k)) + \frac{1}{\gamma^2}\hat{z}_1^T(k)\hat{z}_1(k) - w^T(k)w(k) < 0 \qquad (3.5) \\
&\qquad z_2(k) \in \mathbf{Z}_2.
\end{aligned}
$$

Therefore, the constrained $\mathbf{H}_\infty$ performance will be less than γ.

To derive the control law for the constrained linear system, we often need the following lemma.

Lemma 1 *If $V(\hat{x}_{cl}(0)) \le 1$ and $w \in \mathbf{W}_\beta$, then $V(\hat{x}_{cl}(N)) < 1 + \beta$, $\forall N \in \mathbf{Z}_+$.*

Proof:

Summing eqn.(3.5) for $k = 0, 1, \ldots, N-1$, we get

$$
\begin{aligned}
V(x(N)) &< V(x(0)) - \frac{1}{\gamma^2}\sum_{k=0}^{N-1}\hat{z}_1^T(k)\hat{z}_1(k) + \sum_{k=0}^{N-1} w^T(k)w(k) \\
&\le 1 + \sum_{k=0}^{N-1} w^T(k)w(k) \\
&\le 1 + \sum_{k=0}^{\infty} w^T(k)w(k) = 1 + \beta.
\end{aligned}
$$

Q.E.D.

The following theorem provides an output feedback synthesis condition that solves Prob.5 in LMI formulation.

Theorem 3 *Given initial conditions $\hat{x}(0)$, $x_c(0)$, the constrained LTI system (3.1) is stabilized by an output-feedback controller (3.2) and its $\mathbf{H}_\infty$ performance less than γ if there exist matrices $R, S \in \mathbf{S}_+^{n\times n}$, $\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c \in (\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n_y} \times \mathbf{R}^{n_u\times n} \times \mathbf{R}^{n_u\times n_y})$ such that*

$$\begin{bmatrix} 1 & (\hat{x}(0)-x_c(0))^T S & x_c^T(0) & 0 \\ S(\hat{x}(0)-x_c(0)) & S & 0 & 0 \\ x_c(0) & 0 & R & I \\ 0 & 0 & I & S \end{bmatrix} \geq 0, \tag{3.6}$$

$$\begin{bmatrix} R & \star & \star & \star & \star & \star & \star \\ I & S & \star & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star & \star \\ -I & -S & 0 & S & \star & \star & \star \\ AR+B_2\hat{C}_c & A+B_2\hat{D}_cC_2 & (B_2\hat{D}_c-L)D_{21} & -(A+LC_2) & R & \star & \star \\ \hat{A}_c & SA+\hat{B}_cC_2 & (\hat{B}_c-SL)D_{21} & -S(A+LC_2) & I & S & \star \\ C_{11}R+D_{121}\hat{C}_c & C_{11}+D_{121}\hat{D}_cC_2 & D_{111}+D_{121}\hat{D}_cD_{21} & 0 & 0 & 0 & \gamma^2 I \end{bmatrix} > 0, \tag{3.7}$$

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\beta}R & \star & \star & \star & \star \\ \frac{\bar{z}_{2,j}^2}{1+\beta}I & \frac{\bar{z}_{2,j}^2}{1+\beta}S & \star & \star & \star \\ 0 & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}I & \star & \star \\ -\frac{\bar{z}_{2,j}^2}{1+\beta}I & -\frac{\bar{z}_{2,j}^2}{1+\beta}S & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}S & \star \\ e_j^T(C_{12}R + D_{122}\hat{C}_c) & e_j^T(C_{12} + D_{122}\hat{D}_c C_2) & e_j^T(D_{112} + D_{122}\hat{D}_c D_{21}) & 0 & 1 \end{bmatrix} \geq 0,$$

$$j = 1, 2, \cdots, n_{z2} \tag{3.8}$$

Moreover, the output feedback controller gain is given by

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} -S & SB_2 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c - SAR & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ C_2 R & I \end{bmatrix}^{-1}. \tag{3.9}$$

Proof:

We first define two non-singular matrices

$$Z_1 = \begin{bmatrix} R & I \\ R - S^{-1} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & -S \end{bmatrix},$$

and specify $X = Z_2 Z_1^{-1}$. Then $X = \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix} > 0$, $XZ_1 = Z_2$ and

$$\begin{aligned} Z_1^T X Z_1 &= \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \\ Z_1^T (XA_{cl}) Z_1 &= \begin{bmatrix} AR + B_2\hat{C}_c & A + B_2\hat{D}_c C_2 \\ \hat{A}_c & SA + \hat{B}_c C_2 \end{bmatrix}, \\ Z_1^T (XB_{cl}) &= \begin{bmatrix} B_1 + B_2\hat{D}_c D_{21} \\ SB_1 + \hat{B}_c D_{21} \end{bmatrix}, \\ C_{cl,i} Z_1 &= \begin{bmatrix} C_{1i}R + D_{12i}\hat{C}_c & C_{1i} + D_{12i}\hat{D}_c C_2 \end{bmatrix}, \qquad i = 1, 2 \\ D_{cl,i} &= D_{11i} + D_{12i}\hat{D}_c D_{21}, \qquad i = 1, 2 \end{aligned}$$

where the transformed controller data relates to the original (A_c, B_c, C_c, D_c) by

$$\begin{bmatrix} \hat{A}_c & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} = \begin{bmatrix} SAR & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S & SB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ C_2 R & I \end{bmatrix}. \tag{3.10}$$

Using the Lyapunov function $V(\hat{x}_{cl}) = \hat{x}_{cl}^T X \hat{x}_{cl}$, we will show that the closed loop system obeys all the constraints and provides desired $\mathbf{H}_\infty$ performance in the rest of proof.

Using Schur complement, eqn.(3.6) becomes

$$\begin{aligned} 1 &\geq \begin{bmatrix} \hat{x}^T(0) - x_c^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) - x_c(0) \\ x_c(0) \end{bmatrix} \\ &= \begin{bmatrix} \hat{x}^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ x_c(0) \end{bmatrix} \\ &= \hat{x}_{cl}^T(0) X \hat{x}_{cl}(0) = V(\hat{x}_{cl}(0)). \end{aligned}$$

Next, multiplying diag $\left\{Z_1^{-T}, I, I, X^{-1} Z_1^{-T}, I\right\}$ to the left and its transpose from the right of eqn.(3.7), we obtain

$$\begin{bmatrix} X & \star & \star & \star & \star \\ 0 & I & \star & \star & \star \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & 0 & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} & \star & \star \\ A_{cl} & B_{cl} - \begin{bmatrix} (B_1 + LD_{21}) \\ 0 \end{bmatrix} & -\begin{bmatrix} (A + LC_2) \\ 0 \end{bmatrix} & X^{-1} & \star \\ C_{cl,1} & D_{cl,1} & 0 & 0 & \gamma^2 I \end{bmatrix} > 0. \tag{3.11}$$

If there is no disturbance, we get from eqn.(3.11) that

$$\begin{bmatrix} X & A_{cl}^T \\ A_{cl} & X^{-1} \end{bmatrix} > 0,$$

which leads to $A_{cl}^T X A_{cl} - X < 0$ and $V(x_{cl}(k+1)) - V(x_{cl}(k)) < 0$, therefore the closed-loop system is asymptotically stable.

Also using Schur complement, eqn.(3.11) can be rewritten as

$$\begin{bmatrix} X & 0 & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & 0 & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} A_{cl}^T & C_{cl,1}^T \\ B_{cl}^T - \begin{bmatrix} (B_1 + LD_{21})^T & 0 \end{bmatrix} & D_{cl,1}^T \\ -\begin{bmatrix} (A + LC_2)^T & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & \frac{1}{\gamma^2} I \end{bmatrix}$$
$$\times \begin{bmatrix} A_{cl} & B_{cl} - \begin{bmatrix} (B_1 + LD_{21}) \\ 0 \end{bmatrix} & -\begin{bmatrix} (A + LC_2) \\ 0 \end{bmatrix} \\ C_{cl,1} & D_{cl,1} & 0 \end{bmatrix} > 0.$$

Further multiplying the above inequality from the left by $\begin{bmatrix} x_{cl}^T(k) & w^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose from the right side, we get

$$\begin{aligned} &-\Bigg[\Big(\left(x_{cl}^T(k)A_{cl}^T + w^T(k)B_{cl}^T\right) - \begin{bmatrix} \tilde{x}^T(k)(A + LC_2)^T + w^T(k)(B_1 + LD_{21})^T & 0 \end{bmatrix}\Big) X \\ &\quad \times \left((A_{cl}x_{cl}(k) + B_{cl}w(k)) - \begin{bmatrix} (A + LC_2)\tilde{x}(k) + (B_1 + LD_{21})w(k) \\ 0 \end{bmatrix} \right)\Bigg] \\ &\quad + \left(x_{cl}^T(k) - \begin{bmatrix} \tilde{x}^T(k) & 0 \end{bmatrix} \right) X \left(x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix} \right) \\ &\quad - \frac{1}{\gamma^2}\left(x_{cl}^T(k)C_{cl}^T + w^T(k)D_{cl}^T \right)\left(C_{cl}x_{cl}(k) + D_{cl}w(k) \right) + w^T(k)w(k) > 0, \end{aligned}$$

which is equivalent to eqn.(3.5). When $w(k) \neq 0$ and $\hat{x}_{cl}(0) = 0$, summing the above inequality from $k = 0$ to $N - 1$, then

$$V(\hat{x}_{cl}(N)) - V(\hat{x}_{cl}(0)) + \sum_{k=0}^{N-1} \left[\frac{1}{\gamma^2}\hat{z}_1^T(k)\hat{z}_1(k) - w^T(k)w(k) \right] < 0.$$

As $N \to \infty$, we confirm that $\|z_1\|_2^2 < \gamma^2\|w\|_2^2$. Thus γ is a $\mathbf{H}_\infty$ type performance index.

Also, pre-multiplying diag $\left\{Z_1^{-T}, I, I, 1\right\}$ and post-multiplying its transpose at both sides of eqn.(3.8), we have

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\beta} \begin{bmatrix} X & 0 & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & 0 & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} C_{cl,2}^T \\ D_{cl,2}^T \\ 0 \end{bmatrix} e_j \\ e_j^T \begin{bmatrix} C_{cl,2} & D_{cl,2} & 0 \end{bmatrix} & 1 \end{bmatrix} \geq 0,$$

which is equivalent to

$$\frac{\bar{z}_{2,j}^2}{1+\beta} \begin{bmatrix} X & 0 & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -\begin{bmatrix} I & 0 \end{bmatrix} X & 0 & \begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} C_{cl,2}^T \\ D_{cl,2}^T \\ 0 \end{bmatrix} e_j e_j^T \begin{bmatrix} C_{cl,2} & D_{cl,2} & 0 \end{bmatrix} \geq 0.$$

Multiplying $\begin{bmatrix} x_{cl}^T(k) & w^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose on both sides and we obtain

$$\frac{\bar{z}_{2,j}^2}{1+\beta} \left[\left(x_{cl}^T(k) - \begin{bmatrix} \tilde{x}^T(k) & 0 \end{bmatrix} \right) X \left(x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix} \right) + w^T(k)w(k) \right] - z_{2,j}^T(k) z_{2,j}(k) \geq 0,$$

which is equivalent to

$$\frac{\bar{z}_{2,j}^2}{1+\beta} \left[\hat{x}_{cl}^T(k) X \hat{x}_{cl}(k) + w^T(k)w(k) \right] - z_{2,j}^T(k) z_{2,j}(k) \geq 0. \tag{3.12}$$

Applying Lemma 1, it is clear that $\hat{x}_{cl}^T(k) X \hat{x}_{cl}(k) + w^T(k)w(k) < 1 + \beta$. Then, the constrained input/output satisfy

$$|z_{2,j}(k)| \leq \bar{z}_{2,j}, \qquad j = 1, 2, \cdots, n_{z2}$$

from eqn.(3.12). *Q.E.D.*

3.3 Robust Infinite-horizon RHC

Based on the offline robust output feedback control, we will design online infinite-horizon robust RHC controller. At each time step k, given current states $\hat{x}(k), x_c(k)$, we will solve an optimal control problem to obtain an output feedback controller κ_k and Lyapunov matrix X_k for the remaining infinite horizon. The online optimal problem is formulated as

Problem 6 (Robust Output Feedback Infinite-horizon RHC)

$$\begin{aligned}
&\min \ \gamma_k \\
&s.t. \ \ (3.1),\ (3.2),\ (3.4), \\
&\qquad V(\hat{x}_{cl}(k)) \leq 1 \\
&\qquad V(\hat{x}_{cl}(k+i+1 \mid k)) - V(\hat{x}_{cl}(k+i \mid k)) + \frac{1}{\gamma_k^2} z_1^T(k+i \mid k) z_1(k+i \mid k) \\
&\qquad\qquad - w^T(k+i \mid k) w(k+i \mid k) < 0 \quad (3.13) \\
&\qquad z_2(k+i \mid k) \in \mathbf{Z}_2 \\
&\qquad p_0 - p_{k-1} + \hat{x}_{cl}^T(k) X_{k-1} \hat{x}_{cl}(k) - \hat{x}_{cl}^T(k) X_k \hat{x}_{cl}(k) \geq 0. \quad (3.14)
\end{aligned}$$

An dissipation constraint (3.24) is added to guarantee the moving horizon stability [24]. In the dissipation constraint, we let p_k to be defined recursively as

$$p_k = p_{k-1} - \left[\hat{x}_{cl}^T(k) X_{k-1} \hat{x}_{cl}(k) - \hat{x}_{cl}^T(k) X_k \hat{x}_{cl}(k)\right].$$

The recursion is initialized by setting X_0 as the matrix from offline computation and $p_0 = \hat{x}_{cl}^T(0) X_0 \hat{x}_{cl}(0)$.

The optimization results $X_k^\star$ and $\kappa_k^\star$ will be updated at each step. The following theorem show the robust output feedback synthesis conditions of Prob.7 by LMI.

Theorem 4 *At time step k, given the constrained linear system (3.1) with current states $\hat{x}(k), x_c(k)$ and the robust RHC solution from last step as R_{k-1}, S_{k-1} and γ_{k-1}.*

1. If $w^T(k-1)w(k-1) \leq 1 - \hat{x}_{cl}^T(k-1)X_{k-1}\hat{x}_{cl}(k-1) + \frac{1}{\gamma_{k-1}^2}z_1^T(k-1)z_1(k-1)$*,*

then we have

$$\begin{bmatrix} 1 & (\hat{x}(k)-x_c(k))^T S_k & x_c^T(k) & 0 \\ S_k(\hat{x}(k)-x_c(k)) & S_k & 0 & 0 \\ x_c(k) & 0 & R_k & I \\ 0 & 0 & I & S_k \end{bmatrix} \geq 0, \tag{3.15}$$

$$\begin{bmatrix} R_k & \star & \star & \star & \star & \star & \star \\ I & S_k & \star & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star & \star \\ -I & -S_k & 0 & S_k & \star & \star & \star \\ AR_k + B_2\hat{C}_c^k & A + B_2\hat{D}_c^k C_2 & (B_2\hat{D}_c^k - L)D_{21} & -(A+LC_2) & R_k & \star & \star \\ \hat{A}_c^k & S_kA + \hat{B}_c^k C_2 & (\hat{B}_c^k - S_kL)D_{21} & -S_k(A+LC_2) & I & S_k & \star \\ C_{11}R_k + D_{121}\hat{C}_c^k & C_{11} + D_{121}\hat{D}_c^k C_2 & D_{111} + D_{121}\hat{D}_c^k D_{21} & 0 & 0 & 0 & \gamma_k^2 I \end{bmatrix} > 0, \tag{3.16}$$

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\beta}R_k & \star & \star & \star & \star \\ \frac{\bar{z}_{2,j}^2}{1+\beta}I & \frac{\bar{z}_{2,j}^2}{1+\beta}S_k & \star & \star & \star \\ 0 & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}I & \star & \star \\ -\frac{\bar{z}_{2,j}^2}{1+\beta}I & -\frac{\bar{z}_{2,j}^2}{1+\beta}S_k & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}S_k & \star \\ e_j^T(C_{12}R_k + D_{122}\hat{C}_c^k) & e_j^T(C_{12} + D_{122}\hat{D}_c^k C_2) & e_j^T(D_{112} + D_{122}\hat{D}_c^k D_{21}) & 0 & 1 \end{bmatrix} \geq 0, \quad j = 1, 2, \cdots, n_{z2} \tag{3.17}$$

$$\begin{bmatrix} p_0 - p_{k-1} + \hat{x}_{cl}^T(k)X_{k-1}\hat{x}_{cl}(k) & (\hat{x}(k) - x_c(k))^T S_k & x_c^T(k) & 0 \\ S_k(\hat{x}(k) - x_c(k)) & S_k & 0 & 0 \\ x_c(k) & 0 & R_k & I \\ 0 & 0 & I & S_k \end{bmatrix} \geq 0. \tag{3.18}$$

2. The constrained linear system is stabilized by an output-feedback RHC controller

$$\begin{bmatrix} A_c^k & B_c^k \\ C_c^k & D_c^k \end{bmatrix} = \begin{bmatrix} -S_k & S_k B_2 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c^k - S_k A R_k & \hat{B}_c^k \\ \hat{C}_c^k & \hat{D}_c^k \end{bmatrix} \begin{bmatrix} R_k - S_k^{-1} & 0 \\ C_2 R_k & I \end{bmatrix}^{-1} \tag{3.19}$$

over the remaining horizon. Moreover, we have

$$\sum_{i=0}^{k} z_1^T(i)z_1(i) < \gamma_{max}^2 \sum_{i=0}^{k} w^T(i)w(i) + \gamma_{max}^2 \hat{x}_{cl}^T(0)X_0\hat{x}_{cl}(0), \quad \gamma_{max} = \max_{0 \leq i \leq k} \gamma_i.$$

Proof:

We denote the optimal solution of solving (3.15)-(3.18) at time step $(k-1)$ as

$$\left\{R_{k-1}^*, S_{k-1}^*, \gamma_{k-1}^*, (\hat{A}_c^{k-1})^*, (\hat{B}_c^{k-1})^*, (\hat{C}_c^{k-1})^*, (\hat{D}_c^{k-1})^*\right\}.$$

If $w^T(k-1)w(k-1) \leq 1 - \hat{x}_{cl}^T(k-1)X_{k-1}\hat{x}_{cl}(k-1) + \frac{1}{\gamma_{k-1}^2} z_1^T(k-1)z_1(k-1)$, it can be shown that

$$\begin{aligned} V(\hat{x}_{cl}(k)) &\leq V(\hat{x}_{cl}(k-1)) - \frac{1}{\gamma_{k-1}^2} z_1^T(k-1)z_1(k-1) + w^T(k-1)w(k-1) \\ &\leq 1. \end{aligned}$$

Thus the solution R_{k-1}^*, S_{k-1}^* satisfies condition (3.15) for any state $\hat{x}_{cl}(k)$. The optimal solution at step $k-1$ is also feasible for time step k. If we have optimal solution $\left\{R_k^*, S_k^*, \gamma_k^*, (\hat{A}_c^k)^*, (\hat{B}_c^k)^*, (\hat{C}_c^k)^*, (\hat{D}_c^k)^*\right\}$ at step k, it should render a better performance than γ_k. That is

$$\gamma_k^* \leq \gamma_{k-1}^*.$$

The proof of eqn.(3.16) and eqn.(3.17) is similar to that of Theorem 3. Finally, we get from (3.16)

$$\hat{x}_{cl}^T(i+1)X_i\hat{x}_{cl}(i+1) - \hat{x}_{cl}^T(i)X_i\hat{x}_{cl}(i) + \frac{1}{\gamma_i^2} z_1^T(i)z_1(i) - w^T(i)w(i) < 0. \qquad (3.20)$$

for $i = 0, 1, \cdots, k$. Also from eqn.(3.18), we conclude that

$$\sum_{i=0}^{k-1} \left[\hat{x}_{cl}^T(i+1)X_i\hat{x}_{cl}(i+1) - \hat{x}_{cl}^T(i+1)X_{i+1}\hat{x}_{cl}(i+1)\right] \geq 0. \qquad (3.21)$$

Combining (3.20) and (3.21) leads to

$$\sum_{i=0}^{k} \left[\frac{1}{\gamma_i^2} z_1^T(i)z_1(i) - w^T(i)w(i)\right] \leq -\hat{x}_{cl}^T(k+1)X_k\hat{x}_{cl}(k+1) + \hat{x}_{cl}^T(0)X_0\hat{x}_{cl}(0).$$

or,

$$\sum_{i=0}^{k} z_1^T(i)z_1(i) \leq \gamma_{max}^2 \sum_{i=0}^{k} w^T(i)w(i) + \gamma_{max}^2 \hat{x}_{cl}^T(0)X_0\hat{x}_{cl}(0)$$

as desired. Note that $\gamma_{max} = \max_{0 \leq i \leq k} \gamma_i$. *Q.E.D.*

The monotonicity property of finite-horizon $\mathbf{H}_\infty$ norm indicates that the controlled performance will be improved gradually through online optimizations. The infinite-horizon robust output feedback RHC algorithm for constrained LTI systems is proposed as

Algorithm 5 Robust Output Feedback Infinite-horizon RHC

1: Choose a suitable observer gain L to ensure $A + LC_2$ asymptotically stable.

2: Given initial condition $\hat{x}(0)$, $x_c(0)$, solve LMIs (3.6)-(3.8).

Construct the controller according to (3.9).

Compute p_0, and control force $u(0)$.

3: Set time $k = 1$.

4: Given $\hat{x}(k)$, $x_c(k)$, p_{k-1}, solve LMIs (3.15)-(3.18).

Construct an output feedback controller κ_k according to (3.19).

Update p_k, and control force $u(k)$.

5: Set k=k+1, return to Step 4.

3.4 Robust Finite-horizon RHC

Different from robust infinite-horizon RHC case, robust finite-horizon RHC is to obtain a sequence of output feedback controllers to manipulate the plant over future N time steps, steering the states to the terminal set $\mathbb{X}_f$. Within $\mathbb{X}_f$, the stabilizing output feedback controller $\kappa^\star$ derived from offline robust control will be employed to control the plant over the remaining time. Generally speaking, robust finite-horizon RHC is more complex and requires more computation than that of robust infinite-horizon RHC. To guarantee the stability, the terminal cost function $V_f(\cdot)$ and terminal constraint set $\mathbb{X}_f$ will be defined first,

$$\begin{aligned} V_f(\hat{x}_{cl}) &= (1+\beta)\hat{x}_{cl}^T X^\star \hat{x}_{cl} \\ \mathbb{X}_f &= \left\{\hat{x}_{cl} \in \mathbf{R}^{2n} : \; V_f(\hat{x}_{cl}) \le 1+\beta\right\} \end{aligned} \tag{3.22}$$

Robust finite-horizon RHC uses different controller over finite-horizon time $\{k, k+1, \cdots, k+N-1\}$. Subsequently, there exist a sequence of Lyapunov functions, which denote as $\{V_k^0, V_k^1, \cdots, V_k^N\}$. The online optimization problem is formulated as

Problem 7 (Robust Output Feedback Finite-horizon RHC)

$$\min \gamma_k$$

$$s.t. \;\; (3.1), \; (3.2), \; (3.4),$$

$$V_k^0(\hat{x}_{cl}(k)) \le 1$$

$$\begin{aligned} V_k^{i+1}(\hat{x}_{cl}(k+i+1 \mid k)) - V_k^i(\hat{x}_{cl}(k+i \mid k)) + \frac{1}{\gamma_k^2} z_1^T(k+i \mid k) z_1(k+i \mid k) \\ - w^T(k+i \mid k) w(k+i \mid k) < 0 \end{aligned} \tag{3.23}$$

$$z_2(k+i \mid k) \in \mathbf{Z}_2$$

$$p_0 - p_{k-1} + \hat{x}_{cl}^T(k) X_{k-1}^0 \hat{x}_{cl}(k) - \hat{x}_{cl}^T(k) X_k^0 \hat{x}_{cl}(k) \ge 0. \tag{3.24}$$

where, Lyapunov function $V_k^i(\hat{x}_{cl}(k+i \mid k)) = \hat{x}_{cl}^T(k+i \mid k)X_k^i\hat{x}_{cl}(k+i \mid k)$. The following theorem is the synthesis conditions of robust finite-horizon RHC controller.

Theorem 5 *At time step k, given the constrained linear system (3.1) with current states $\hat{x}(k), x_c(k)$, terminal matrices $R_k^N = \frac{1}{1+\beta}R^\star$, $S_k^N = (1+\beta)S^\star$ and the RHC solution from last step as R_{k-1}, S_{k-1} and γ_{k-1}.*

1. *If $w^T(k-1)w(k-1) \leq 1 - \hat{x}_{cl}^T(k-1)X_{k-1}\hat{x}_{cl}(k-1) + \frac{1}{\gamma_{k-1}^2}z_1^T(k-1)z_1(k-1)$, then there always exist a scalar $\gamma_k \leq \gamma_{k-1}$, matrices $R_k^i, S_k^i \in \mathbf{S}_+^{n\times n}$, and $\hat{A}_c^{k,i}, \hat{B}_c^{k,i}, \hat{C}_c^{k,i}, \hat{D}_c^{k,i}$, $i = 0, 1, \ldots, N-1$ of appropriate dimensions satisfying*

$$\begin{bmatrix} 1 & (\hat{x}(k)-x_c(k))^T S_k^0 & x_c^T(k) & 0 \\ S_k^0(\hat{x}(k)-x_c(k)) & S_k^0 & 0 & 0 \\ x_c(k) & 0 & R_k^0 & I \\ 0 & 0 & I & S_k^0 \end{bmatrix} \geq 0 \tag{3.25}$$

$$\begin{bmatrix} R_k^i & \star & \star & \star & \star & \star & \star \\ I & S_k^i & \star & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star & \star \\ -I & -S_k^i & 0 & S_k^i & \star & \star & \star \\ AR_k^i + B_2\hat{C}_c^{k,i} & A + B_2\hat{D}_c^{k,i}C_2 & (B_2\hat{D}_c^{k,i} - L)D_{21} & -(A+LC_2) & R_k^{i+1} & \star & \star \\ \hat{A}_c^{k,i} & S_k^{i+1}A + \hat{B}_c^{k,i}C_2 & (\hat{B}_c^{k,i} - S_k^{i+1}L)D_{21} & -S_k^{i+1}(A+LC_2) & I & S_k^{i+1} & \star \\ C_{11}R_k^i + D_{121}\hat{C}_c^{k,i} & C_{11} + D_{121}\hat{D}_c^{k,i}C_2 & D_{111} + D_{121}\hat{D}_c^{k,i}D_{21} & 0 & 0 & 0 & \gamma_k^2 I \end{bmatrix} > 0, \tag{3.26}$$

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\beta}R_k^i & \star & \star & \star & \star \\ \frac{\bar{z}_{2,j}^2}{1+\beta}I & \frac{\bar{z}_{2,j}^2}{1+\beta}S_k^i & \star & \star & \star \\ 0 & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}I & \star & \star \\ -\frac{\bar{z}_{2,j}^2}{1+\beta}I & -\frac{\bar{z}_{2,j}^2}{1+\beta}S_k^i & 0 & \frac{\bar{z}_{2,j}^2}{1+\beta}S_k^i & \star \\ e_j^T(C_{12}R_k^i + D_{122}\hat{C}_c^{k,i}) & e_j^T(C_{12} + D_{122}\hat{D}_c^{k,i}C_2) & e_j^T(D_{112} + D_{122}\hat{D}_c^{k,i}D_{21}) & 0 & 1 \end{bmatrix} \geq 0, \quad j = 1, 2, \cdots, n_{z2} \tag{3.27}$$

$$\begin{bmatrix} p_0 - p_{k-1} + \hat{x}_{cl}^T(k)X_{k-1}\hat{x}_{cl}(k) & (\hat{x}(k) - x_c(k))^T S_k^0 & x_c^T(k) & 0 \\ S_k^0(\hat{x}(k) - x_c(k)) & S_k^0 & 0 & 0 \\ x_c(k) & 0 & R_k^0 & I \\ 0 & 0 & I & S_k^0 \end{bmatrix} \geq 0. \tag{3.28}$$

2. The RHC control law over finite horizon is

$$\begin{bmatrix} A_c^{k,i} & B_c^{k,i} \\ C_c^{k,i} & D_c^{k,i} \end{bmatrix} = \begin{bmatrix} -S_k^{i+1} & S_k^{i+1}B_2 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c^{k,i} - S_k^{i+1}AR_k^i & \hat{B}_c^{k,i} \\ \hat{C}_c^{k,i} & \hat{D}_c^{k,i} \end{bmatrix} \times \begin{bmatrix} R_k^i - (S_k^i)^{-1} & 0 \\ C_2R_k^i & I \end{bmatrix}^{-1}. \tag{3.29}$$

Proof:

The proof of eqn.(3.26) and eqn.(3.27) is similar to that of Theorem 3. From (3.26), we have

$$\hat{x}_{cl}^T(k+i+1)X_k^{i+1}\hat{x}_{cl}(k+i+1) - \hat{x}_{cl}^T(k+i)X_k^i\hat{x}_{cl}(k+i) \\ + \frac{1}{\gamma_k^2}z_1^T(k+i)z_1(k+i) - w^T(k+i)w(k+i) < 0. \quad (3.30)$$

for $i = 0, 1, \cdots, N-1$. Consequently, we conclude that

$$\sum_{i=0}^{N} z_1^T(k+i)z_1(k+i) \leq \gamma_k^2 \sum_{i=0}^{N} w^T(k+i)w(k+i) + \gamma_k^2\hat{x}_{cl}^T(k)X_k^0\hat{x}_{cl}(k)$$

as desired. *Q.E.D.*

Since finite-horizon RHC problem has more decision variables and more flexable than infinite-horizon RHC, it has larger feasibility region than that of infinite-horizon RHC. In finite-horizon RHC, the initial states $x(k)$ are able to be outside of the terminal set $\mathbb{X}_f$, while in infinite-horizon RHC, the initial states are always inside in the terminal set. The algorithm of finite-horizon RHC is summarized as

Algorithm 6 Robust Output Feedback Finite-horizon RHC

1: Choose a suitable observer gain L to ensure $A + LC_2$ asymptotically stable.

2: Given initial condition $\hat{x}(0)$, $x_c(0)$, solve LMIs (3.6)-(3.8). Results are denoted as $R^\star$, $S^\star$, etc.

3: Solve LMIs (3.25)-(3.27).

Construct a sequence of controllers $\pi(0) = \{\kappa_0, \kappa_1, \ldots, \kappa_{N-1}\}$ according to (3.29).

Update p_0, and implement the first control force $u^0(0)$.

4: Set time $k = 1$.

5: Given $\hat{x}(k)$, $x_c(k)$, p_{k-1}, $R^\star$, $S^\star$, solve LMIs (3.25)-(3.28).

Construct a sequence of RHC controllers $\pi(k) = \{\kappa_0, \kappa_1, \ldots, \kappa_{N-1}\}$ according to (3.29).

Update p_k, and implement the first control force $u^0(k)$.

6: Set k=k+1, return to Step 5.

3.5 Example

In this section, we would like to demonstrate advantages of the proposed robust RHC schemes. Consider the two-mass/spring/damper system as Fig.3.1, which is a generic model of a dynamic system with non-collocated sensor and actuator.

A control force acts on body 1, and the position of body 2 is measured resulting a non-collocated control problem. This system can be represented in the state-space form as

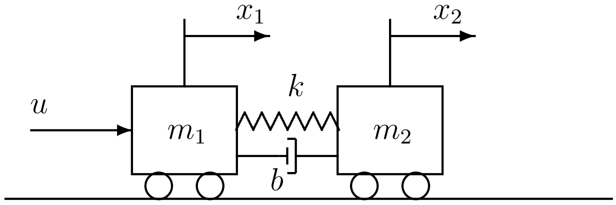


Figure 3.1: Two mass spring damper system

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \hline z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{cccc|ccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{k_s}{m_1} & \frac{k_s}{m_1} & -\frac{b}{m_1} & \frac{b}{m_1} & \frac{1}{m_1} & 0 & \frac{1}{m_1} \\
\frac{k_s}{m_2} & -\frac{k_s}{m_2} & \frac{b}{m_2} & -\frac{b}{m_2} & 0 & \frac{1}{m_2} & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \hline w_1(t) \\ w_2(t) \\ \hline u(t) \end{bmatrix}. \tag{3.31}
$$

where x_1, x_2 are positions of body $1, 2$ respectively. x_3, x_4 are the velocities of body $1, 2$. w_1, w_2 are the disturbance of body $1, 2$.

For simulation purpose, the control horizon is chosen as $N = 5$, the upper bound of disturbance energy $\beta = 0.1$, the hard input constraint is $\|u\| \leq 5.0$, the hard output constraint is $\|y\| \leq 5.0$. Fig.3.2 shows the feasibility region mapped on the plane of x_1

and $x2$. We can see that finite-horizon RHC has a larger feasibility region than that of infinite-horizon RHC. Starting from initial positions $(0, 1)$ with initial velocities as zeros, the results of robust control and finite-horizon RHC control are shown in Fig.3.3. It shows that the finite-horizon RHC scheme proposed in this paper can achieve better robust performance, at the mean time it still guarantees the stability and feasibility. Fig.3.4 shows the simulation results of proposed robust finite-horizon RHC under different initial condition.

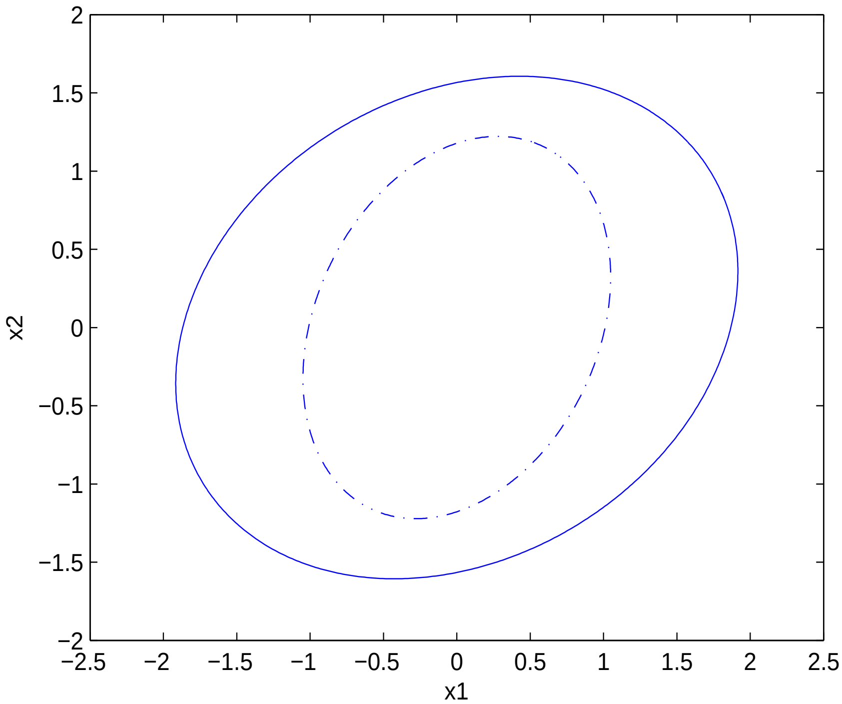
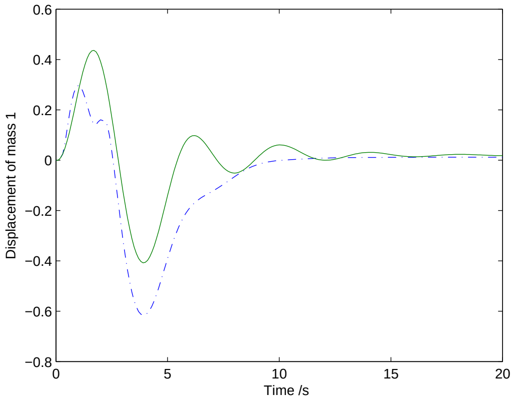


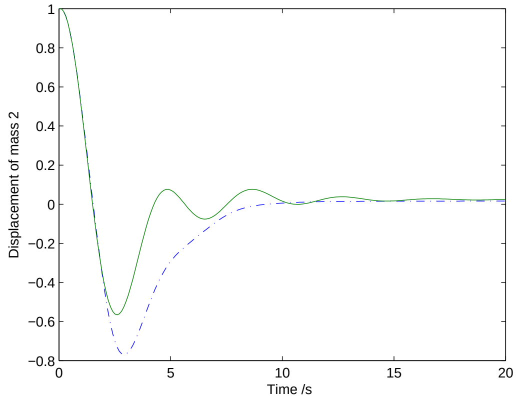
Figure 3.2: Feasibility region comparison of proposed finite-horizon RHC (solid), infinite-horizon RHC (dash dot).

3.6 Conclusion

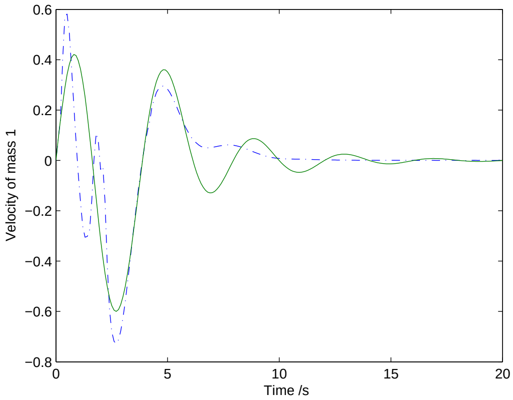
In this chapter, we propose three robust output feedback control methods for the LTI systems with energy bounded disturbances. The first scheme is offline output feedback control. It designs the controller by offline computation and use the result for the whole online running of the plant. Since it does not need the online optimization, it runs fastest. The second scheme, infinite-horizon robust RHC, updates the controller online for the remaining horizon time. It should get better and better controller and performance while time goes on. The last scheme, finite-horizon robust RHC is more relax than that of infinite-horizon RHC because it allows the system has different controller over next finite time steps. Thus it achieves better control performance while its dynamic response may be worse than that of infinite-horizon RHC.



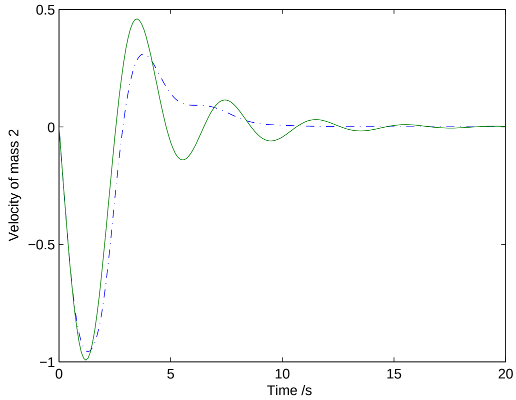
(a). displacement of mass1



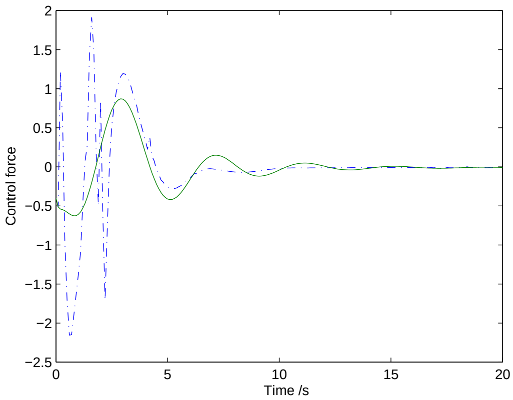
(b). displacement of mass2



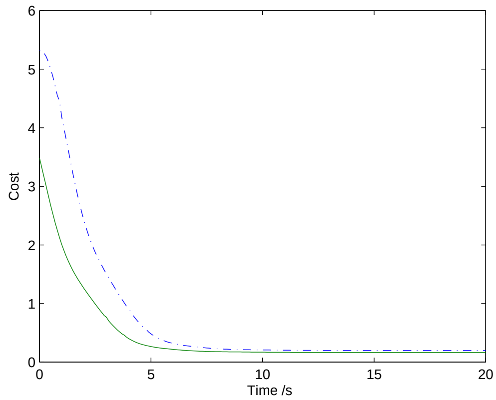
(c). velocity of mass1



(d). velocity of mass2

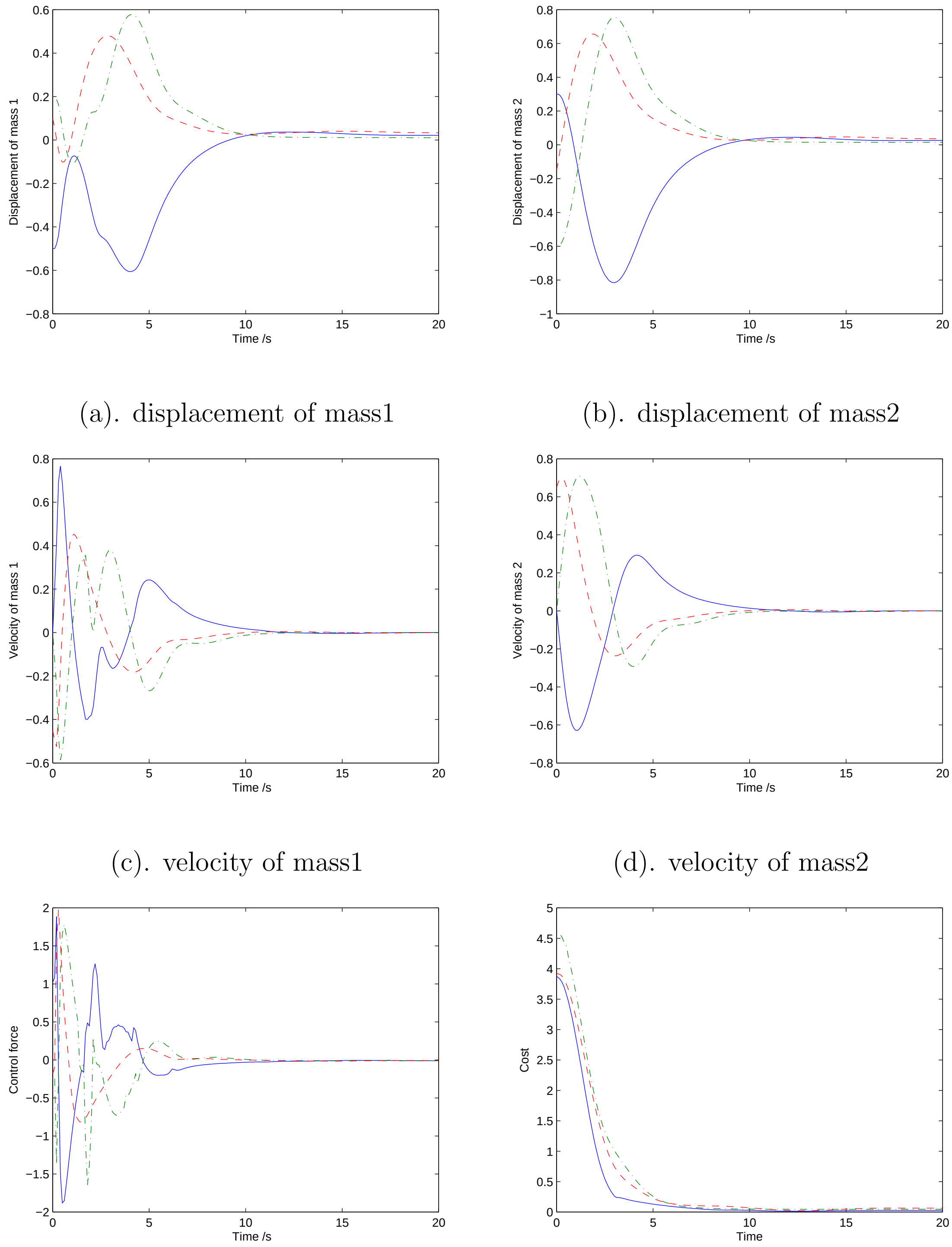


(e). control force



(f). cost

Figure 3.3: Dynamic response comparison of proposed finite-horizon RHC (solid), infinite-horizon RHC (dash dot).



(a). displacement of mass1 (b). displacement of mass2

(c). velocity of mass1 (d). velocity of mass2

(e). control force (f). cost

Figure 3.4: Dynamic response comparison of proposed finite-horizon RHC under different initial condition: (-0.5;0.3;0;0) (solid), (0.2;-0.69;0;0)(dash dot), (0.1;-0.15;-0.45;0.65)(dash).

Chapter 4

Robust Output Feedback RHC for Uncertain Linear systems

In Chapter 3, we proposed output feedback RHC for LTI systems with energy bounded disturbances. Nevertheless, it is desirable to develop robust RHC to hand model uncertainties and hard input/output constraints. A lot of RHC algorithms were developed in the past, which are robust against model uncertainties and guarantee a certain control performance. Specially, these RHC algorithms are usually based on explicit model uncertainty descriptions but assuming no disturbances. As one of such descriptions, polytopic system model is considered as an effective one for the uncertainty modeling. In this chapter, we introduce a new robust output feedback RHC for uncertain linear systems with magnitude bounded disturbances.

4.1 Constrained Uncertain Linear Systems

Consider a linear discrete-time uncertain system represents as following,

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ z_2(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\theta(k)) & B_1 & B_2(\theta(k)) \\ C_{11} & D_{111} & D_{121} \\ C_{12} & D_{112} & D_{122} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{4.1}$$

subject to vector constraints

$$\begin{aligned} \mathbf{Z}_2 &= \left\{ z_2 \in \mathbf{R}^{n_{z2}} : \ |z_2(k)| \le \bar{z}_2, \ \bar{z}_2 = [\bar{z}_{2,1} \ \ \bar{z}_{2,2} \cdots \bar{z}_{2,n_{z2}}]^T \right\} \\ &= \left\{ |e_j^T (C_{cl,2} x_{cl}(k) + D_{cl,2} w(k))| \le \bar{z}_{2,j}, j = 1, 2, \cdots, n_{z2} \right\}. \end{aligned} \tag{4.2}$$

where $x \in \mathbf{R}^n$ denotes the state, $w \in \mathbf{R}^{n_w}$ is disturbance, $u \in \mathbf{R}^{n_u}$ is control input, $z_1 \in \mathbf{R}^{n_{z1}}$ is controlled output, $z_2 \in \mathbf{R}^{n_{z2}}$ is constrained signal, $y \in \mathbf{R}^{n_y}$ is measured output and e_j is the jth vector of unity matrix. The constraint set (4.2) includes both input and output constraints. The disturbance w is magnitude bounded, which belongs to a set determined by a scalar $\bar{w} > 0$,

$$\mathbf{W}_{\bar{w}} = \left\{ w \in \mathbf{R}^{n_w} : \ \|w\|^2 \le \bar{w}^2 \right\}. \tag{4.3}$$

The system matrices $A(\theta(k))$ and $B_2(\theta(k))$ are affine functions of $\theta(k)$ and satisfy

$$\begin{aligned} &[A(\theta(k)) \mid B_2(\theta(k))] \in \Omega, \\ &\Omega = \mathbf{Co}\left\{ [A_1 \mid B_{2,1}], [A_2 \mid B_{2,2}], \cdots, [A_{n_v} \mid B_{2,n_v}] \right\} \end{aligned} \tag{4.4}$$

where A_j and $B_{2,j}$, $j = 0, \cdots, n_v$ are given constant matrices. $\mathbf{Co}(\cdot)$ denotes the convex hull, which means that there exist n_v nonnegative coefficients $\theta_j(k)$, such that

$$\begin{aligned} &A(\theta(k)) = \sum_{j=1}^{n_v} \theta_j(k) A_j, \qquad B_2(\theta(k)) = \sum_{j=1}^{n_v} \theta_j(k) B_{2,j}, \\ &\sum_{j=1}^{n_v} \theta_j(k) = 1, \ 0 \le \theta_j(k) \le 1. \end{aligned} \tag{4.5}$$

Therefore, the time-varying uncertain vector $\theta(k)$ belongs to the set

$$\Theta = \left\{ \theta \in \mathbf{R}^{n_v} : \sum_{j=1}^{n_v} \theta_j = 1,\ 0 \le \theta_j \le 1 \right\}. \tag{4.6}$$

In this chapter, we assume that $\theta(k)$ can be measurable at each time step k. Therefore, the current system matrices $[A(\theta(k)) \mid B_2(\theta(k))]$ are known exactly. However, the future ones $[A(\theta(k+i)) \mid B_2(\theta(k+i))]$, $i \ge 1$, are uncertain, but continued inside a prescribed polytope Ω.

We assume the polytopic model (4.1) is robustly stabilized by an output feedback controller

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix}, \tag{4.7}$$

where $x_c \in \mathbf{R}^n$ is the controller states.

Thus the closed-loop system is described as

$$\begin{bmatrix} x_{cl}(k+1) \\ z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl,1} & D_{cl,1} \\ C_{cl,2} & D_{cl,2} \end{bmatrix} \begin{bmatrix} x_{cl}(k) \\ w(k) \end{bmatrix}, \tag{4.8}$$

where

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl,1} & D_{cl,1} \\ C_{cl,2} & D_{cl,2} \end{bmatrix} = \left[\begin{array}{cc|c} A(\theta(k)) & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_{11} & 0 & D_{111} \\ C_{12} & 0 & D_{112} \end{array}\right] + \left[\begin{array}{cc} 0 & B_2(\theta(k)) \\ I & 0 \\ \hline 0 & D_{121} \\ 0 & D_{122} \end{array}\right] \times \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \left[\begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array}\right]. \tag{4.9}$$

4.2 Observer Design

We design a state observer of the form,

$$\hat{x}(k+1) = A(\theta(k))\hat{x}(k) + B_2(\theta(k))u(k) - L[y(k) - C_2\hat{x}(k)], \tag{4.10}$$

$$\hat{z}_1(k) = C_{11}\hat{x}(k) + D_{121}u(k).$$

The estimation errors $\tilde{x} = x - \hat{x}$ and $\tilde{z}_1 = z_1 - \hat{z}_1$ between real plant and observer are governed by

$$\tilde{x}(k+1) = (A(\theta(k)) + LC_2)\tilde{x}(k) + (B_1 + LD_{21})w(k), \tag{4.11}$$

$$\tilde{z}_1(k) = C_{11}\tilde{x}(k) + D_{111}w(k). \tag{4.12}$$

In the following, we will determine the observer gain L in terms of LMI. First, we define the Lyapunov function $\tilde{V}(\tilde{x}(k))$ as

$$\tilde{V}(\tilde{x}(k)) = \tilde{x}^T(k)X_e\tilde{x}(k), \;\; X_e \in S_+^{n\times n}. \tag{4.13}$$

If $\tilde{V}(\tilde{x}(k))$ satisfies $\mathbf{H}_\infty$ condition,

$$\tilde{V}(\tilde{x}(k+1)) - \tilde{V}(\tilde{x}(k)) + \frac{1}{\gamma_e^2}\tilde{z}_1^T(k)\tilde{z}_1(k) - w^T(k)w(k) < 0, \tag{4.14}$$

then the estimation errors will be bounded and the estimated states will not go far away from the real plant states. The observer gain L satisfying (4.14) will be synthesized through the following theorem.

Theorem 6 *If there exist matrices $X_e \in \mathbf{S}_+^{n\times n}$ and $M_e \in \mathbf{R}^{n\times n_y}$ such that*

$$\begin{bmatrix} X_e & 0 & (X_eA_j + M_eC_2)^T & C_{11}^T \\ 0 & I & (X_eB_1 + M_eD_{21})^T & D_{111}^T \\ X_eA_j + M_eC_2 & X_eB_1 + M_eD_{21} & X_e & 0 \\ C_{11} & D_{111} & 0 & \gamma_e^2 I \end{bmatrix} > 0,$$

$$j = 1, 2, \cdots, n_v, \tag{4.15}$$

then the observer gain is obtained by $L = X_e^{-1}M_e$ and the stability of the estimation errors (4.11) is guaranteed for any $[A(\theta(k)) \mid B_2(\theta(k))] \in \Omega$.

Proof:

Multiply $\theta_j(k)$ from $j = 1, 2, \cdots, n_v$ to the left and right side of (4.15), then sum them together, we get

$$\begin{bmatrix} X_e & 0 & (X_eA(\theta(k)) + M_eC_2)^T & C_{11}^T \\ 0 & I & (X_eB_1 + M_eD_{21})^T & D_{111}^T \\ X_eA(\theta(k)) + M_eC_2 & X_eB_1 + M_eD_{21} & X_e & 0 \\ C_{11} & D_{111} & 0 & \gamma_e^2 I \end{bmatrix} > 0. \quad (4.16)$$

Next, multiplying diag $\{I, I, X_e^{-1}, I\}$ to the left and its transpose from the right of (4.16), we obtain

$$\begin{bmatrix} X_e & 0 & (A(\theta(k)) + LC_2)^T & C_{11}^T \\ 0 & I & (B_1 + LD_{21})^T & D_{111}^T \\ A(\theta(k)) + LC_2 & B_1 + LD_{21} & X_e^{-1} & 0 \\ C_{11} & D_{111} & 0 & \gamma_e^2 I \end{bmatrix} > 0. \quad (4.17)$$

Using Schur complement, eqn.(4.17) can be rewritten as

$$\begin{bmatrix} X_e & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} (A(\theta(k)) + LC_2)^T & C_{11}^T \\ (B_1 + LD_{21})^T & D_{111}^T \end{bmatrix} \begin{bmatrix} X_e & 0 \\ 0 & \frac{1}{\gamma_e^2} I \end{bmatrix} \times \begin{bmatrix} A(\theta(k)) + LC_2 & B_1 + LD_{21} \\ C_{11} & D_{111} \end{bmatrix} > 0.$$

Further multiplying the above inequality from the left by $\begin{bmatrix} \tilde{x}^T(k) & w^T(k) \end{bmatrix}$ and its transpose from the right side, we get the required condition (4.14). *Q.E.D.*

When $w(k) = 0$, it can be seen that $\tilde{V}(\tilde{x}(k+1)) - \tilde{V}(\tilde{x}(k)) < 0$ by eqn.(4.14), so the observer is stable. Summing eqn.(4.14) from $k = 0$ to ∞, we have

$$\|\tilde{z}_1\|^2 < \gamma_e^2 \|w\|^2 + \gamma_e^2 \tilde{V}(\tilde{x}(0)). \quad (4.18)$$

So if $\tilde{V}(\tilde{x}(0)) = 0$, the $\mathbf{H}_\infty$ gain from w to $\tilde{z}_1$ is bounded by γ_e.

4.3 Robust Output Feedback Control For Uncertain Linear Systems

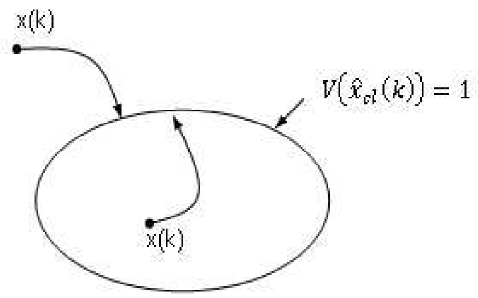


Figure 4.1: Robust Stability Region

Based on the estimated close-loop system $\hat{x}_{cl} = \begin{bmatrix} \hat{x}^T & x_c^T \end{bmatrix}^T$, our goal is to construct a new robust RHC method which solves the problem for the constrained uncertain linear systems,

Problem 8 (Robust Output Feedback For Uncertain Linear systems)

$$\min \gamma$$

$$s.t. \quad (4.1),\ (4.7),\ (4.10),$$

$$V(\hat{x}_{cl}(0)) \leq 1,$$

$$V(\hat{x}_{cl}(k+1)) - V(\hat{x}_{cl}(k)) < -\beta \bar{w}^2 V(\hat{x}_{cl}(k)) + \beta w^T(k) w(k), \tag{4.19}$$

$$z_2(k) \in \mathbf{Z}_2,$$

$$X \leq \gamma I. \tag{4.20}$$

where $V(\hat{x}_{cl}(k)) = \hat{x}_{cl}^T(k) X \hat{x}_{cl}(k)$ is Lyapunov function, β is a pre-specified positive scalar. Eqn.(4.20) shows that γ is the upper bound of the size of matrix X. The stability condition (4.19) implies that there exists a stability region (Fig.4.1) by

$$\mathbb{X}_f = \left\{\hat{x}_{cl} : \; \hat{x}_{cl}^T X \hat{x}_{cl} \leq 1\right\}. \tag{4.21}$$

If the states are outside of $\mathbb{X}_f$, the Lyapunov function $V(\hat{x}_{cl}(k))$ is decreasing. Once the states enter the region, they will not leave $\mathbb{X}_f$ any longer. The worst case is that the states approach the boundary of the stability region, where $V(\hat{x}_{cl}(k)) = 1$.

Theorem 7 *Given the initial condition $\hat{x}(0), x_c(0)$, the constrained uncertain linear system (4.1) is stabilized by an output feedback controller (4.7) if there exist matrices $R, S \in \mathbf{S}_+^{n\times n}$, $\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c \in (\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n_y} \times \mathbf{R}^{n_u\times n} \times \mathbf{R}^{n_u\times n_y})$ such that*

$$\begin{bmatrix} 1 & (\hat{x}(0) - x_c(0))^T S & x_c^T(0) & 0 \\ S(\hat{x}(0) - x_c(0)) & S & 0 & 0 \\ x_c(0) & 0 & R & I \\ 0 & 0 & I & S \end{bmatrix} \geq 0, \tag{4.22}$$

$$\begin{bmatrix} (1-\beta\bar{w}^2)R & \star & \star & \star & \star & \star \\ (1-\beta\bar{w}^2)I & (1-\beta\bar{w}^2)S & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star \\ -(1-\beta\bar{w}^2)I & -(1-\beta\bar{w}^2)S & 0 & (1-\beta\bar{w}^2)S & \star & \star \\ A_jR + B_{2,j}\hat{C}_c & A_j + B_{2,j}\hat{D}_c C_2 & (B_{2,j}\hat{D}_c - L)D_{21} & -(A_j + LC_2) & R & \star \\ \hat{A}_c & SA_j + \hat{B}_c C_2 & (\hat{B}_c - SL)D_{21} & -S(A_j + LC_2) & I & S \end{bmatrix} > 0,$$

$$j = 1, 2, \cdots, n_v \tag{4.23}$$

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}R & \star & \star & \star & \star \\ \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}I & \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}S & \star & \star & \star \\ 0 & 0 & \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}I & \star & \star \\ -\frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}I & -\frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}S & 0 & \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}S & \star \\ e_j^T(C_{12}R + D_{122}\hat{C}_c) & e_j^T(C_{12} + D_{122}\hat{D}_c C_2) & e_j^T(D_{112} + D_{122}\hat{D}_c D_{21}) & 0 & 1 \end{bmatrix} \geq 0,$$

$$j = 1, 2, \cdots, n_{z2} \tag{4.24}$$

$$\begin{bmatrix} \gamma I - S & S & 0 & 0 \\ S & \gamma I - S & I & 0 \\ 0 & I & R & I \\ 0 & 0 & I & S \end{bmatrix} \geq 0. \tag{4.25}$$

Moreover, the output feedback controller gain is given by

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} -S & SB_2(\theta(k)) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c(\theta(k)) - SA(\theta(k))R & \hat{B}_c(\theta(k)) \\ \hat{C}_c & \hat{D}_c \end{bmatrix} \times \begin{bmatrix} R - S^{-1} & 0 \\ C_2 R & I \end{bmatrix}^{-1}. \tag{4.26}$$

Proof: We define two non-singular matrices

$$Z_1 = \begin{bmatrix} R & I \\ R - S^{-1} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & -S \end{bmatrix},$$

and specify $X = Z_2 Z_1^{-1}$. Then $X = \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix} > 0$, $XZ_1 = Z_2$ and

$$
\begin{aligned}
Z_1^T X Z_1 &= \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \\
Z_1^T \left(X A_{cl} \right) Z_1 &= \begin{bmatrix} A(\theta(k))R + B_2(\theta(k))\hat{C}_c & A(\theta(k)) + B_2(\theta(k))\hat{D}_c C_2 \\ \hat{A}_c & SA(\theta(k)) + \hat{B}_c C_2 \end{bmatrix}, \\
Z_1^T \left(X B_{cl} \right) &= \begin{bmatrix} B_1 + B_2(\theta(k))\hat{D}_c D_{21} \\ SB_1 + \hat{B}_c D_{21} \end{bmatrix}, \\
C_{cl,2} Z_1 &= \begin{bmatrix} C_{12}R + D_{122}\hat{C}_c & C_{12} + D_{122}\hat{D}_c C_2 \end{bmatrix}, \\
D_{cl,2} &= D_{112} + D_{122}\hat{D}_c D_{21},
\end{aligned}
$$

where the transformed controller data relates to the original (A_c, B_c, C_c, D_c) by

$$
\begin{aligned}
\begin{bmatrix} \hat{A}_c(\theta(k)) & \hat{B}_c(\theta(k)) \\ \hat{C}_c & \hat{D}_c \end{bmatrix} = \begin{bmatrix} SA(\theta(k))R & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S & SB_2(\theta(k)) \\ 0 & I \end{bmatrix} \\
\times \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ C_2 R & I \end{bmatrix}.
\end{aligned}
\tag{4.27}
$$

Using Schur complement, eqn.(4.22) becomes

$$
\begin{aligned}
1 &\geq \begin{bmatrix} \hat{x}^T(0) - x_c^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) - x_c(0) \\ x_c(0) \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}^T(0) & x_c^T(0) \end{bmatrix} \begin{bmatrix} S & -S \\ -S & S + (R - S^{-1})^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ x_c(0) \end{bmatrix} \\
&= \hat{x}_{cl}^T(0) X \hat{x}_{cl}(0) = V(\hat{x}_{cl}(0)).
\end{aligned}
$$

Multiply $\theta_j(k)$ from $j = 1, 2, \cdots, n_v$ to the left and right side of (4.23), then sum together, we get

$$\begin{bmatrix} (1-\beta\bar{w}^2)R & \star & \star & \star & \star & \star \\ (1-\beta\bar{w}^2)I & (1-\beta\bar{w}^2)S & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star \\ -(1-\beta\bar{w}^2)I & -(1-\beta\bar{w}^2)S & 0 & (1-\beta\bar{w}^2)S & \star & \star \\ A(\theta(k))R + B_2(\theta(k))\hat{C}_c & A(\theta(k)) + B_2(\theta(k))\hat{D}_cC_2 & (B_2(\theta(k))\hat{D}_c - L)D_{21} & -(A(\theta(k)) + LC_2) & R & \star \\ \hat{A}_c & SA(\theta(k)) + \hat{B}_cC_2 & (\hat{B}_c - SL)D_{21} & -S(A(\theta(k)) + LC_2) & I & S \end{bmatrix} > 0 \qquad (4.28)$$

Next, multiplying diag $\left\{Z_1^{-T}, I, I, X^{-1}Z_1^{-T}\right\}$ to the left and its transpose from the right of the above inequality, we obtain

$$\begin{bmatrix} (1-\beta\bar{w}^2)X & \star & \star & \star \\ 0 & I & \star & \star \\ -(1-\beta\bar{w}^2)\begin{bmatrix} I & 0 \end{bmatrix} X & 0 & (1-\beta\bar{w}^2)\begin{bmatrix} I & 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} & \star \\ A_{cl} & B_{cl} - \begin{bmatrix} (B_1 + LD_{21}) \\ 0 \end{bmatrix} & -\begin{bmatrix} (A + LC_2) \\ 0 \end{bmatrix} & X^{-1} \end{bmatrix} > 0. \qquad (4.29)$$

Using Schur complement, eqn.(4.29) can be rewritten as

$$\begin{bmatrix} (1-\beta\bar{w}^2)X & 0 & -(1-\beta\bar{w}^2)X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -(1-\beta\bar{w}^2)\begin{bmatrix} I & 0 \end{bmatrix}X & 0 & (1-\beta\bar{w}^2)\begin{bmatrix} I & 0 \end{bmatrix}X\begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} A_{cl}^T \\ B_{cl}^T - \begin{bmatrix} (B_1+LD_{21})^T & 0 \end{bmatrix} \\ -\begin{bmatrix} (A+LC_2)^T & 0 \end{bmatrix} \end{bmatrix}$$

$$\times X \begin{bmatrix} A_{cl} & B_{cl} - \begin{bmatrix} (B_1+LD_{21}) \\ 0 \end{bmatrix} & -\begin{bmatrix} (A+LC_2) \\ 0 \end{bmatrix} \end{bmatrix} > 0.$$

Then multiply the above inequality from the left by $\begin{bmatrix} x_{cl}^T(k) & w^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose from the right side, we get

$$(1-\beta\bar{w}^2)V(\hat{x}_{cl}(k)) + \beta w^T(k)w(k) - V(\hat{x}_{cl}(k+1)) > 0 \qquad (4.30)$$

That is equivalent to the robust stability condition (4.19).

Multiplying diag $\{Z_1^{-T}, I, I, 1\}$ and its transpose at both sides of eqn.(4.24), we obtain

$$\begin{bmatrix} \frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}\begin{bmatrix} X & 0 & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -\begin{bmatrix} I & 0 \end{bmatrix}X & 0 & \begin{bmatrix} I & 0 \end{bmatrix}X\begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} C_{cl,2}^T \\ D_{cl,2}^T \\ 0 \end{bmatrix} e_j \\ e_j^T \begin{bmatrix} C_{cl,2} & D_{cl,2} & 0 \end{bmatrix} & 1 \end{bmatrix} \geq 0,$$

By Shur complement, the above equation is equivalent to

$$\frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}\begin{bmatrix} X & 0 & -X\begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 & I & 0 \\ -\begin{bmatrix} I & 0 \end{bmatrix}X & 0 & \begin{bmatrix} I & 0 \end{bmatrix}X\begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} C_{cl,2}^T \\ D_{cl,2}^T \\ 0 \end{bmatrix} e_j e_j^T \begin{bmatrix} C_{cl,2} & D_{cl,2} & 0 \end{bmatrix} \geq 0.$$

Also, multiplying $\begin{bmatrix} x_{cl}^T(k) & w^T(k) & \tilde{x}^T(k) \end{bmatrix}$ and its transpose on both sides, we have

$$\frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}\left[\left(x_{cl}^T(k) - \begin{bmatrix} \tilde{x}^T(k) & 0 \end{bmatrix}\right) X \left(x_{cl}(k) - \begin{bmatrix} \tilde{x}(k) \\ 0 \end{bmatrix}\right) + w^T(k)w(k)\right] \\ - z_{2,j}^T(k)z_{2,j}(k) \geq 0,$$

which is equivalent to

$$\frac{\bar{z}_{2,j}^2}{1+\bar{w}^2}\left[\hat{x}_{cl}^T(k)X\hat{x}_{cl}(k) + w^T(k)w(k)\right] - z_{2,j}^T(k)z_{2,j}(k) \geq 0. \tag{4.31}$$

Since the states are always inside of the stability region $\mathbb{X}_f$, it is clear that $\hat{x}_{cl}^T(k)X\hat{x}_{cl}(k)+ w^T(k)w(k) < 1+\bar{w}^2$. Thus, the constrained input/output satisfy

$$|z_{2,j}(k)| \leq \bar{z}_{2,j}, \qquad j = 1, 2, \cdots, n_{z2}.$$

By Schur complement, (4.25) is turned into

$$\gamma I - \begin{bmatrix} S & -S \\ -S & s + (R - S^{-1})^{-1} \end{bmatrix} \geq 0,$$

which is equivalent to (4.20). *Q.E.D.*

4.4 Robust Infinite-horizon RHC For Uncertain Linear Systems

Fig.4.1 shows that the stability region solved by offline robust output feedback control is relates to the initial states. The output feedback controller based on the initial states becomes conservative when the states go far away. So it is desirable to recompute the controller at each time step k for the following infinite horizon time. Then the robust infinite-horizon RHC is defined as

Problem 9 (Robust Infinite-horizon RHC for Uncertain Linear Systems)

$$
\begin{aligned}
&\min \gamma_k \\
&s.t. \quad (4.1),\ (4.7),\ (4.10), \\
&\qquad V(\hat{x}_{cl}(k)) \leq 1, \\
&\qquad V(\hat{x}_{cl}(k+i+1 \mid k)) - V(\hat{x}_{cl}(k+i \mid k)) < -\beta\bar{w}^2 V(\hat{x}_{cl}(k+i \mid k)) \\
&\qquad\qquad + \beta w^T(k+i \mid k) w(k+i \mid k), \qquad (4.32) \\
&\qquad z_2(k) \in \mathbf{Z}_2, \\
&\qquad X_k \leq \gamma_k.
\end{aligned}
$$

At time step k, given current states $\hat{x}(k), x_c(k)$, the robust infinite-horizon RHC is solved by a set of LMI synthesis conditions, including initial condition, stability condition and input/output constraint condition. When moving to next time step $k+1$, searching from the optimal result of step k, we can find a better optimal controller. The robust infinite-horizon RHC is summarized in the following theorem.

Theorem 8 *At time step k, given the constrained linear system (4.1) with current states $\hat{x}(k), x_c(k)$.*

1. *There always exist a scalar γ_k, matrices $R_k, S_k \in \mathbf{S}_+^{n\times n}$, and $\hat{A}_c^k, \hat{B}_c^k, \hat{C}_c^k, \hat{D}_c^k$ of appropriate dimensions satisfying (4.22) - (4.25).*

2. *The constrained linear system is stabilized by the output-feedback RHC control*

law

$$\begin{bmatrix} A_c^k & B_c^k \\ C_c^k & D_c^k \end{bmatrix} = \begin{bmatrix} -S_k & S_k B_2(\theta(k)) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c^k(\theta(k)) - S_k A(\theta(k)) R_k & \hat{B}_c^k(\theta(k)) \\ \hat{C}_c^k & \hat{D}_c^k \end{bmatrix} \times \begin{bmatrix} R_k - S_k^{-1} & 0 \\ C_2 R_k & I \end{bmatrix}^{-1}, \tag{4.33}$$

over the remaining horizon.

Proof:

First of all, we define the optimal solution of robust infinite-horizon RHC (4.22) - (4.25) at step $(k-1)$ as

$$\mathbf{O}_{k-1}^* = \left\{ R_{k-1}^*, S_{k-1}^*, \gamma_{k-1}^*, (\hat{A}_c^{k-1})^*, (\hat{B}_c^{k-1})^*, (\hat{C}_c^{k-1})^*, (\hat{D}_c^{k-1})^* \right\}.$$

Since the states are always inside of the stability region $\mathbb{X}_f$, the Lyapunov function at step k satisfies $V(\hat{x}_{cl}(k)) \leq 1$. Thus the solution R_{k-1}^*, S_{k-1}^* satisfies the initial condition (4.22) for any state $\hat{x}_{cl}(k)$. Therefore, $\mathbf{O}_{k-1}^*$ is a feasible solution for time step k. If we have optimal solution $\mathbf{O}_k^*$ at step k, it should render a better performance. *Q.E.D.*

The robust output feedback infinite-horizon RHC controller will be improved gradually through online optimizations. The control algorithm is shown as

Algorithm 7 Robust Output Feedback Infinite-horizon RHC for Uncertain Linear Systems

1: Solve (4.15) to obtain a stable observer gain L.

2: Set time $k = 0$.

3: Given current states $\hat{x}(k)$, $x_c(k)$, solve (4.22) - (4.25).

Construct the controller according to (4.33).

Calculate and implement the first control force $u(k \mid k)$

4: Set k=k+1, return to Step 3.

4.5 Robust Finite-horizon RHC For Uncertain Linear Systems

In most cases, we may not know the uncertainty parameter $\theta(k)$ exactly at each time step k. What we have is always the upper and lower bound of $\theta(k)$. Then the robust finite-horizon RHC becomes a min-max problem, which minimizes the cost function on the worst case of uncertainty parameter and disturbance over finite horizon time steps. The robust infinite-horizon RHC in §4.4 is used to determine a stability region $\mathbb{X}_f$ as the terminal constraint to ensure the stability. The finite-horizon RHC is to steer the states to the set $\mathbb{X}_f$ in N steps. Within $\mathbb{X}_f$, the stabilizing output feedback controller will be employed.

Problem 10 (Robust Finite-horizon RHC for Uncertain Linear Systems) *Given the system (4.1) with current state $\hat{x}(k)$, terminal set $\mathbb{X}_f$, the finite-horizon RHC problem for uncertain linear systems can be solved online by the optimization scheme*

$$\begin{aligned}\min_{u(\cdot),\hat{x}(\cdot)} \max_{\theta(\cdot),w(\cdot)} J(k) &= \sum_{i=0}^{N-1} \hat{z}_1^T(k+i \mid k)\hat{z}_1(k+i \mid k) \qquad (4.34)\\ s.t. \quad \hat{x}(k+i+1 \mid k) &= A(\theta(k+i \mid k))\hat{x}(k+i \mid k) + B_1 w(k+i \mid k)\\ &\quad + B_2(\theta(k+i \mid k))u(k+i \mid k),\\ z_2(k+i \mid k) &\in \mathbf{Z}_2,\\ \hat{x}_{cl}(k+N \mid k) &\in \mathbb{X}_f.\end{aligned}$$

In min-max problem, the uncertainty $\theta(k+i \mid k)$ and the disturbance $w(k+i \mid k)$ are chosen as vertex points to obtain the worst case. At each time step, the uncertainty has n_v vertex points and the disturbances has 2^{n_w} vertex points. Therefore, we will have n_v^N sequences of $\mathbf{\Theta}(k) = \{\theta(k \mid k), \theta(k+1 \mid k), \cdots, \theta(k+N-1 \mid k)\}$ and $(2^{n_w})^N$ sequences of $\mathbf{W}(k) = \{w(k \mid k), w(k+1 \mid k), \cdots, w(k+N-1 \mid k)\}$. Then there are $n_v^N \times (2^{n_w})^N$ sequences of $\{\mathbf{\Theta}(k)\ \mathbf{W}(k)\}$. Since the quadratic performance and quadratic constraints exist in Prob.10, the problem is solved by LMI instead of standard QP method. The robust finite-horizon RHC for uncertain linear systems is synthesized by the following theorem [13].

Theorem 9 *At time step k, given the constrained uncertain linear system (4.1) with current states $\hat{x}(k) = \hat{x}(k \mid k)$, terminal matrix $X^\star$, there exist a scalar γ_k, a sequences of states $\{\hat{x}(k+1 \mid k), \hat{x}(k+2 \mid k), \cdots, \hat{x}(k+N \mid k)\}$ and a sequences of control force $\{u(k \mid k), u(k+1 \mid k), \cdots, u(k+N-1 \mid k)\}$ of appropriate dimensions satisfying*

$$\begin{bmatrix} \gamma_k & \star & \star & \star & \star & \star \\ \hat{x}(k \mid k) & (Q_x)^{-1} & \star & \star & \star & \star \\ u(k \mid k) & 0 & (Q_u)^{-1} & \star & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \star & \star \\ \hat{x}(k+N-1 \mid k) & 0 & 0 & \cdots & (Q_x)^{-1} & \star \\ u(k+N-1 \mid k) & 0 & 0 & \cdots & 0 & (Q_u)^{-1} \end{bmatrix} > 0, \tag{4.35}$$

$$\begin{aligned} \hat{x}(k+i+1 \mid k) &= A(\theta(k+i \mid k))\hat{x}(k+i \mid k) + B_1 w(k+i \mid k) \\ &\quad + B_2(\theta(k+i \mid k))u(k+i \mid k), \ i = 0, 1, \cdots, N-1 \end{aligned} \tag{4.36}$$

$$\begin{bmatrix} \bar{z}_{2,j}^2 & \star \\ e_j^T \begin{bmatrix} C_{12} & D_{112} & D_{12} \end{bmatrix} \begin{bmatrix} \hat{x}(k+i \mid k) \\ w(k+i \mid k) \\ u(k+i \mid k) \end{bmatrix} & 1 \end{bmatrix} \geq 0, \ j = 0, 1, \cdots, n_{z2} \tag{4.37}$$

$$\begin{bmatrix} 1 & \star \\ \begin{bmatrix} \hat{x}(k+N \mid k) \\ x_c(k+N \mid k) \end{bmatrix} & (X^\star)^{-1} \end{bmatrix} \geq 0. \tag{4.38}$$

Note that $X^\star$ is coupled with $\hat{x}(k+N \mid k)$ and $x_c(k+N \mid k)$, while in the robust finite-horizon RHC, we do not need to get the information of x_c over finite N steps.

We often treat $x_c(k+N \mid k)$ as zero. Therefore, the terminal constraint is shown as

$$\begin{bmatrix} \hat{x}^T(k+N \mid k) & 0 \end{bmatrix} X^\star \begin{bmatrix} \hat{x}(k+N \mid k) \\ 0 \end{bmatrix} \leq 1 \tag{4.39}$$

, which leads eqn.(4.38) to be turned into

$$\begin{bmatrix} 1 & \star \\ \hat{x}(k+N \mid k) & \left(\begin{bmatrix} I & 0 \end{bmatrix} X^\star \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \end{bmatrix} \geq 0 \tag{4.40}$$

.

The robust output feedback infinite-horizon RHC algorithm is proposed as

Algorithm 8 Robust Output Feedback Finite-horizon RHC for Uncertain Linear Systems

1: Solve (4.15) and construct the observer gain by $L = X_e^{-1} M_e$.

2: Set time $k = 0$.

3: Given initial condition $\hat{x}(0)$, $x_c(0)$, solve (4.22) - (4.25).

Achieve $X^\star$ to construct the terminal set $\mathbb{X}_f$.

4: Given $\hat{x}(k)$, $\mathbb{X}_f$, solve (4.35) - (4.38).

Achieve a sequence of states $\{\hat{x}(k+1 \mid k), \hat{x}(k+2 \mid k), \cdots, \hat{x}(k+N \mid k)\}$ and control force $\{u(k \mid k), u(k+1 \mid k), \cdots, u(k+N-1 \mid k)\}$.

Implement the first control force $u(k \mid k)$.

5: Set k=k+1, return to Step 4.

4.6 Example

In this section, we present an example that illustrate the implementation of the proposed robust output feedback RHC for uncertain linear systems.

Consider the following linearized model derived for a single, non-isothermal continuous stirred-tank reactor (CSTR) [74]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{F}{V} - k_0 e^{-E/RT_s} & -\frac{E}{RT_s^2} k_0 e^{-E/RT_s} C_{AS} \\ \frac{-\Delta H_{rxn} k_0 e^{-E/RT_s}}{\rho C_p} & -\frac{F}{V} - \frac{UA}{V\rho C_p} - \Delta H_{rxn} \frac{E}{\rho C_p R T_s^2} k_0 e^{-E/RT_s} C_{AS} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -2.098 \times 10^5 \frac{T_s - 365}{V\rho C_p} \end{bmatrix} u(t). \tag{4.41}$$

where x_1 is reactor concentration, x_2 is temperature, u is coolant flow. $F = 1\ m^3/min$, $V = 1\ m^3$, $k_0 = 10^9 - 5 \times 10^9\ min^{-1}$, $E/R = 8330.1K$, $-\Delta H_{rxn} = 10^7 - 5 \times 10^7\ cal/kmol$, $\rho = 10^6\ g/m^3$, $UA = 5.34 \times 10^6\ cla/K$, and $C_p = 1\ cal/(gK)$. We focus on the linearized model at the steady state $T - s = 394\ K$ and $C_{AS} = 0.265\ kmol/m^3$ under the uncertainties k_0 and $-\Delta H_{rxn}$.

By substituting the given parameters and introducing disturbance $w(t)$ on the state x_2, the model is turned into

$$\left[\begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \hline z_1(t) \\ z_2(t) \\ y(t) \end{array}\right] = \left[\begin{array}{cc|cc} -1 - 0.6576\alpha(t) & -0.0098\alpha(t) & 0 & 0 \\ 6.5763\alpha(t)\beta(t) & -6.34 + 0.0935\alpha(t)\beta(t) & 1 & -6.0842 \\ \hline 0.8 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ \hline w(t) \\ u(t) \end{array}\right]. \tag{4.42}$$

where z_1, z_2 are the desired output. y is the measurable output. $1 \leq \alpha(t) = k_0/10^9 \leq 5$ and $1 \leq \beta(t) = -\Delta H_{rxn}/10^7 \leq 5$. Then the polytopic uncertainty set has four vertices

$$\Omega = \mathbf{Co}\left\{\begin{bmatrix}-1.6576 & -0.0098\\ 6.5763 & -6.2465\end{bmatrix}, \begin{bmatrix}-1.6576 & -0.0098\\ 32.8815 & -5.8725\end{bmatrix}, \right. \tag{4.43}$$
$$\left. \begin{bmatrix}-4.2880 & -0.0490\\ 32.8815 & -5.8725\end{bmatrix}, \begin{bmatrix}-4.2880 & -0.0490\\ 164.4075 & -4.0025\end{bmatrix}\right\}$$

In the simulation, the control horizon is chosen as $N = 3$, the upper bound of disturbance magnitude $\bar{w} = 0.05$, the hard input constraint is $\|u\| \leq 3.0$, the hard output constraint is $\|y\| \leq 10.0$. The disturbance $w(k)$ is generated by a random function (Fig.4.2). Fig.4.3 shows the simulation results of proposed robust finite-horizon RHC under different initial condition.

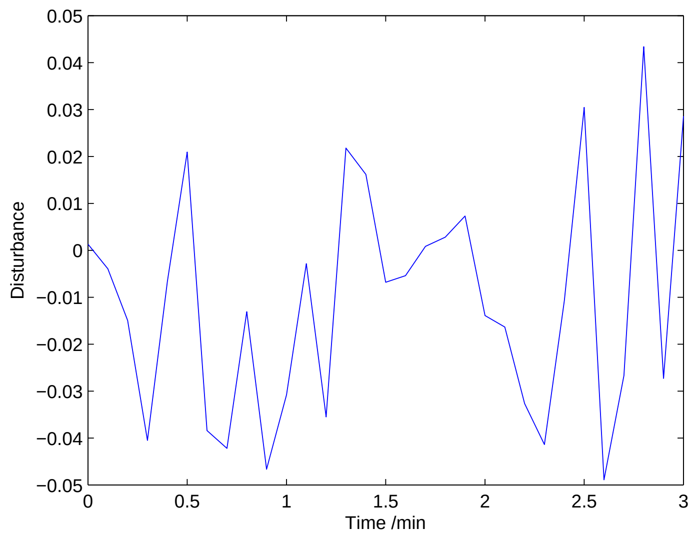
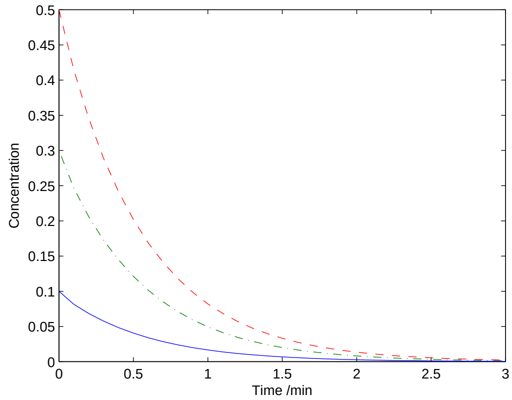
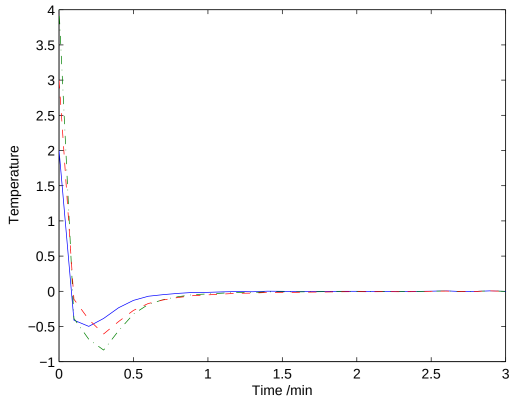


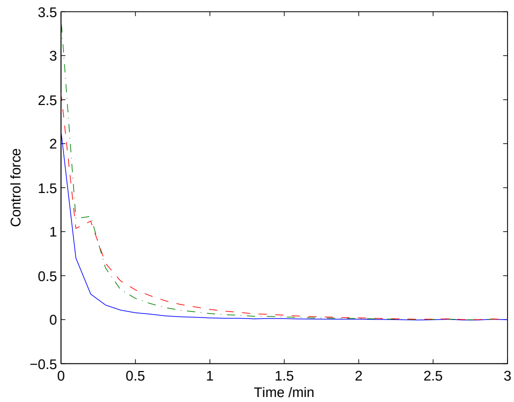
Figure 4.2: Disturbance.



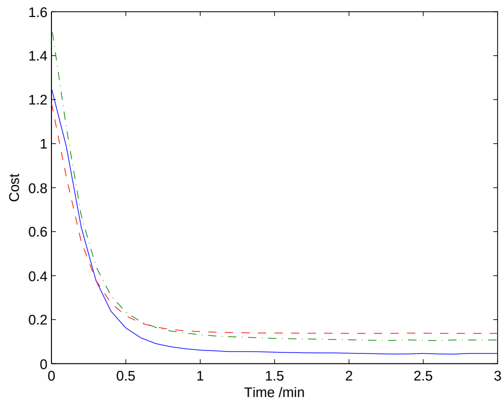
(a). concentration



(b). temperature



(c). control force



(d). cost

Figure 4.3: Dynamic response comparison of proposed finite-horizon RHC under different initial condition: (0.1;2) (solid), (0.3;4)(dash dot), (0.5;3)(dash).

4.7 Conclusion

In this chapter, we propose robust output feedback finite-horizon RHC for uncertain linear systems with magnitude-bounded disturbances. To guarantee the stability, a special stability region is introduced by solving offline robust output feedback control or robust output feedback infinite-horizon RHC online. Since it is necessary to search the solution for each vertex of the convex hull of model uncertainties and disturbances, the computation cost will increase exponentially when the dimension of the system and the moving horizon increase. Simulation results of single, non-isothermal CSTR show that the proposed robust finite-horizon RHC achieve satisfied dynamic response and performance index.

Chapter 5

Conclusion

In this thesis, output feedback RHC schemes were studied for constrained systems with/without disturbances or model uncertainties. The main goal of this research was to develop robust output feedback RHC methods on the basis of measured output signals and keep their feasibility and stability. Three RHC algorithms: offline output feedback control, infinite-horizon RHC and finite-horizon RHC, have been developed for three constrained systems: LTI systems without disturbances, LTI systems with energy-bounded disturbances and uncertain linear systems with magnitude-bounded disturbances. Their performances are demonstrated through extensive simulations, and comparison with existing output feedback RHC schemes.

5.1 Contributions

The first part of the thesis is to design output feedback RHC for the constrained LTI systems without disturbances. Since we can not access the full information of plant states, it is necessary to introduce an observer to estimate the states. Thus our following studies are based on these estimated states because we proved that the errors between the original plant states and the estimated states will approach zero with time going on if the proposed observer is asymptotically stable. Subsequently, we design an offline output feedback controller by minimizing a quadratic performance of the controlled output signals. The controller synthesis conditions are solved by LMI, including initial condition, stability condition and constrained input/output conditions. Then we extend this work to online infinite-horizon RHC. The infinite-horizon RHC is to redesign the output feedback controller at each time step. The resulted performance index is shown monotonically decreasing because the optimal solution at last time step is a feasible solution at current time step. This means that the output feedback controller is improved gradually through online optimizations. Finally, we propose an output feedback finite-horizon RHC. A terminal set determined from offline output feedback control, is used as a terminal constraint to ensure the states go into this region over finite steps. The cost function consists of performance over finite horizon and a terminal cost. Since the quadratic performance index and the quadratic constraints exist, we consider the states and control forces as decision variables and solve the finite-horizon RHC problem by LMI instead of QP programming. Simulations of the single micro cantilever showed that the proposed output feedback RHC schemes have good dynamic response and maintain asymptotic stability.

The second part of the thesis is to develop robust output feedback RHC for the constrained LTI systems with energy-bounded disturbances. Again, an observer is introduced as well. The original plant states and estimated states will approach the same stability region if the observer is stable. We first design an offline robust output feedback controller that optimizes the closed-loop $\mathbf{H}_\infty$ performance index from disturbances to controlled output. The synthesis conditions also include three parts: initial condition, stability condition and constrained input/output conditions. Then we propose an online robust output feedback infinite-horizon RHC. It basically solves the same synthesis conditions as those of the offline robust output feedback control at each time step. In addition, in order to guarantee the moving horizon stability, an dissipation constraint [24] is added to the optimization formulation. The dissipation terms are defined by a recursive equation. Thus the online optimal results are monotonically decreasing. In the end, an online robust output feedback finite-horizon RHC is developed. Different from the first part of the thesis, we choose the controller parameters as the decision variables while solving the synthesis conditions. In the finite-horizon RHC, we allow the system has different controller parameters at each step and the controller parameters of the last step are already known, which comes from the offline robust output feedback control. Therefore, the synthesis conditions of the finite-horizon RHC is less conservative than those of the infinite-horizon RHC and achieve a better performance index. Simulations of the two mass spring damper system show the feasibility and advantages of the proposed robust output feedback RHC approaches in both infinite and finite RHC settings.

The third part of the thesis is to develop robust output feedback RHC for the constrained uncertain linear systems with magnitude-bounded disturbances. The systems matrices of the uncertain linear systems belongs to a convex hull. An observer is obtained by solving a set of LMIs, in which the system parameters are chosen as vertex points of the convex hull. Then we propose an offline robust output feedback controller that has a different stability condition from that of the first part and the second part. The stability condition in the first part leads to a quadratic performance index, and it shows $\mathbf{H}_\infty$ performance in the second part. Our goal to design the offline robust output feedback controller for uncertain linear systems is to find a stability region such that once the plant states go into this region, they will not leave this region any more. The stability condition we propose implies this requirement. Secondly, we design an online robust output feedback infinite-horizon RHC, which recompute the same synthesis conditions at each step. By this method, we obtain a optimal stability region. Based on this stability region, we propose an online robust output feedback finite-horizon RHC. Because we assume that the cost will be zero once the plant states go into the stability region, the cost performance index is a quadratic function of controlled output signals without terminal cost. In other words, it is only defined over finite future horizon. To solve the robust finite-horizon RHC problem, we treat the states and control forces as decision variables. The synthesis conditions includes four parts: performance condition, dynamics condition, constrained signal condition and terminal condition. The problem is also solved by LMI. The finite-horizon RHC algorithm have two procedures at each time step. The first is to solve the robust output feedback infinite-horizon RHC problem to achieve a stability region. The second is to compute the robust output feedback finite-horizon RHC problem to obtain control force. The simulation results of a single, non-isothermal CSTR are satisfied, while computation is big.

5.2 Future Work

Besides the results presented in this thesis, there are some unsolved problems on robust output feedback RHC. For example, for uncertain linear systems, the conventional robust RHC utilizes the worst-case uncertainty bound, which often leads to conservative stability region estimate and high control effort. Moreover, searching its vertex points of uncertainty and disturbance sets leads to large computation cost. For LPV systems, if the model uncertainties are measured and predicted, the robust output feedback RHC controller can be formulated as an affine function of the uncertainty parameters.

Another direction of research is to extend the work on the single system to the distributed robust output feedback RHC. The distributed control problem is usually decoupled into a set of sub control problem to reduce the problem complexity and computation. Since distributed RHC handles the multi-agents, the problem formulation for each agent is augmented by extra signals from neighbor agents. How to deal with these signals is a challenge in the design of distributed robust output feedback RHC controller. Another important issue is how to guarantee the robust stability and achieve required performance index of the whole distributed system when those of each subsystem are obtained.

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