The Relative Efficiency of the Approximate F-Tests Frequently Encountered in Unbalanced Designs

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1. INTRODUCTION

The subject of estimating variance components in unbalanced designs has generated considerable discussions in the literature. However, little research has been done in the area of testing variance components. The main problems concerning variance component testing in unbalanced designs are:

1. The mean squares in the analysis of variance table, with the exception of error mean square, do not have chi-square distributions in general.
2. The mean squares are not in general mutually independent.
3. It is unlikely to find a pair of mean squares which have the same expected values under the null hypotheses usually postulated.

Therefore, an exact F test is not usually available for the unbalanced designs. There are, however, several procedures which suggest approximate F tests. The commonly used procedures are based on the synthesis of mean square technique introduced by Satterthwaite [1941 and 1946]. In these procedures, the nonindependence and nonchi-squareness of the mean squares are ignored and a synthetic, approximate F test is performed by constructing a numerator mean square (MSN) and a denominator mean square (MSD) using single or linear combinations of the mean squares from the analysis of variance table so that both MSN and MSD have the same expected values under the null hypothesis being tested. A linear combination of mean squares is assumed to have approximately a chi-square distribution with its degrees of freedom computed by the Satterthwaite's formula. An approximate F statistic is then obtained as the ratio of MSN to MSD. It is assumed to follow approximately an F distribution with the two computed degrees of freedom. Numerous tests based on different linear combinations of mean squares from different types of analyses can be found in this fashion.

A conventional F test involving the ratio of two mean squares can be used as an alternative to the synthetic F test mentioned above, provided that their expected values under the null hypothesis are only slightly different.

The analysis of unweighted means provides another alternative to the synthetic F test. The cell mean of each subclass is considered as the only observation in the cell, and the mean squares are computed as if the data were balanced. The computations involved are very simple, and for the two-way crossed designs, there is no need to synthesize mean squares because the appropriate numerator and denominator mean squares have the same expected values under the null hypothesis.

The approximate F statistic does not usually follow the F distribution for obvious reasons. Attempts to find the exact distribution of this statistic were successful only under limited conditions.

The purpose of this research is to find a more generalized expression for the exact distribution of the approximate F statistic, to evaluate the performances of the various approximate F tests under different conditions, and to provide guidelines for choice among these procedures.

We will confine our efforts to the random, unbalanced, two-way crossed designs with no missing cell, and to avoid complications, to the linear combinations of mean squares with positive coefficients.

Chapter 3 reviews the sums of squares, the estimable functions and the hypotheses associated with three different types of analyses.

Chapter 4 describes Satterthwaite's formula and seven test procedures that are based on different linear combinations of mean squares. The performances of these test procedures are evaluated in Chapter 6.

Chapter 5 deals with the distributional aspects of the approximate F statistic. First, we derive the distribution of a quadratic form in normal variates. Then, the distribution of the ratio of two independent quadratic forms is obtained. Finally, the distribution of the approximate F statistic is derived.

The evaluation of the performances of the test procedures is treated in Chapter 6. Fifteen two-way designs with different dimensions and cell frequencies are selected, nine different combinations of values for the variance components are used, and the power functions of each test procedure is computed at three different significance levels.

Chapter 7 provides a summary of this study and some suggestions for future research.

2. REVIEW OF LITERATURE

When testing variance components in a random model with unbalanced data, the exact F test in the form of a ratio of two independent mean squares does not usually exist. Nevertheless, we can always find a ratio of two linear combinations of mean squares as an approximation to the exact F statistic using the synthesis of mean square technique. This technique was first introduced by Satterthwaite [1941 and 1946], who showed that a linear combination of mean squares can be approximated by a chi-square variate whose degrees of freedom is a function of the expected values and degrees of freedom of the component mean squares. This technique was later employed in computing synthetic or approximate F statistic by synthesizing numerator and denominator mean squares (see Searle [1971]). Howe and Myers [1970] suggested an approximate F test which involves a single mean square in the numerator and a linear combination of mean squares in the denominator. Gaylor and Hopper [1969] investigated the performance of the Satterthwaite's formula when some of the coefficients in the linear combination of mean squares are negative. Hudson and Krutchkoff [1968] studied the sizes and powers of approximate F tests which involve addition as well as subtraction of mean squares.

Tietjen [1974] showed that if the expected values of two mean squares from an unbalanced design are reasonably close to each other under the null hypothesis, then the conventional F test provided by the ratio of these two mean squares may be more favorable than the synthetic F test.

The analysis of unweighted means was first introduced by Yates [1934] for its simplicity in computation. Gosslee and Lucas [1965] used it in testing hypothesis for two way fixed model. Hirotsu [1968] and Webster [1968] extended it to testing varince components in random model. Their results indicated that the approximate F statistic obtained in this analysis follows the F distribution very closely.

All of the approximate F statistics in these test procedures involve ratios of correlated quadratic forms which are not distributed as chi-square variates in general. Cochran [1934] showed that these quadratic forms, by way of orthogonal transformations, can be expressed as linear combinations of independent chi-square random variates. Box [1954] showed that the ratio of two quadratic forms can be transformed into a ratio of two independent linear combinations of chi-square random variates, and that the exact distribution of this ratio is a finite series for some limited cases. It can be shown that the exact distribution function of this ratio is an infinite weighted sum of F distribution functions as suggested by Robbins and Pitman [1949] and Laha [1954].

3. THE ANALYSIS

3.1 The Model

Consider the random model:

$$Y = X_o\beta_o + \sum_{i=1}^{k} X_i\beta_i + e \qquad (3.1)$$

where

Y is an nx1 vector of observations,

X_o is an nx1 vector of 1's,

X_i $(i=1,2,\ldots,k)$ are nxm_i known matrices,

β_o is an unknown constant,

each β_i $(i=1,2,\ldots,k)$, representing the i^{th} random effect, is an $m_i x1$ vector of uncorrelated random variates assumed to be distributed as $N(0,\ \sigma_i^2\ I_{m_i})$,

β_i, β_j $(i\neq j)$ and e are independent of each other, and

e is an nx1 vector of uncorrelated random variates assumed to be distributed as $N(0,\ \sigma_e^2\ I_n)$.

Thus, Y is distributed as $N(X_o\beta_o,\ V)$, where

$$V = \sum_{i=1}^{k} X_i X_i'\ \sigma_i^2 + I\ \sigma_e^2\quad .$$

This model can also be represented in matrix form as:

$$Y = X\beta + e \qquad (3.2)$$

where

$$X = [X_o | X_1 | \ldots | X_k] \tag{3.3}$$

and

$$\beta = \begin{bmatrix} \beta_o \\ \hline \beta_1 \\ \hline \vdots \\ \hline \beta_k \end{bmatrix} . \tag{3.4}$$

3.2 The Sum of Squares and Its Expected Value

The usual way of computing the sum of squares in a random model is to treat β as if it were fixed, and compute the quadratic function of Y associated with testing a linear function $L\beta$, where

$$L = [L_o | L_1 | \ldots | L_k] . \tag{3.5}$$

The column dimension of L is the same as that of X in equation (3.3), and it has a row rank of n_L. Then the sum of squares corresponding to testing (H_o: $L\beta=0$) is:

$$SSL = (Lb)'(L(X'X)^{-}L')^{-1}(Lb) \tag{3.6}$$

where

$$b = (X'X)^{-}X'Y$$

and $(X'X)^{-}$ is a generalized inverse of X'X.

It is obvious that SSL is a quadratic form in Y:

$$\begin{aligned} SSL &= Y'X(X'X)^{-}L'(L(X'X)^{-}L')^{-1}L(X'X)^{-}X'Y \\ &= Y'QY , \end{aligned} \tag{3.7}$$

and the matrix of quadratic form is:

$$Q = X(X'X)^{-}L'(L(X'X)^{-}L')^{-1}L(X'X)^{-}X' \quad (3.8)$$

Thus the expected value of SSL is:

$$\begin{aligned} E(SSL) &= E(Y'QY) \\ &= \beta_o' X_o' Q X_o \beta_o + \sum_{i=1}^{k} tr(X_i' Q X_i)\sigma_i^2 + tr(Q)\sigma_e^2 . \quad (3.9) \end{aligned}$$

The corresponding mean square, MSL, and its expected value can then be obtained by dividing SSL and E(SSL) by the degrees of freedom n_L.

Goodnight and Speed [1978] developed the following theorem to compute the expected value of SSL when $L\beta$ is an estimable function.

<u>Theorem 3.2.1</u> If L is from the row space of X with full row rank n_L, and SSL is defined as in equation (3.7), then there exists a matrix $C = [C_o | C_1 | \ldots | C_k]$ of the same dimensions as L, such that

(a) $$C = (U')^{-1}L$$

(b) $$E(SSL) = \beta_o' C_o' C_o \beta_o + \sum_{i=1}^{k} SSQ(C_i)\sigma_i^2 + n_L \sigma_e^2 \quad (3.10)$$

where U is the upper triangle of the Cholesky Decomposition of $L(X'X)^{-}L'$, and $SSQ(C_i)$ is the sum of squares of the elements of the C_i submatrix.

The proof of the theorem is given below so that intermediate results may be used later:

If L is in the row space of X, then

$$L = L(X'X)^{-}X'X$$

or

$$L_i = L(X'X)^{-}X'X_i \quad . \tag{3.11}$$

Substituting equation (3.11) into equation (3.8), we have

$$X_i'QX_i = L_i'(L(X'X)^{-}L')^{-1}L_i \quad . \tag{3.12}$$

This is a submatrix of $L'(L(X'X)^{-}L')^{-1}L$.

If we form the matrix

$$[L(X'X)^{-}L'|L] \quad , \tag{3.13}$$

and perform the Cholesky Decomposition on the left hand matrix, then it becomes $[U|C]$ where U is the upper triangular matrix such that

$$U'U = L(X'X)^{-}L' \quad .$$

Thus

$$C = (U')^{-1}L$$

and

$$C'C = L'(L(X'X)^{-}L')^{-1}L \quad .$$

It follows from equation (3.12) that

$$X_i'QX_i = C_i'C_i \qquad \text{for } i = 1, 2, \ldots, k$$

and

$$E(SSL) = \beta_o'C_o'C_o\beta_o + \sum_{i=1}^{k} tr(C_i'C_i)\sigma_i^2 + tr(Q)\sigma_e^2$$

$$= \beta_o'C_o'C_o\beta_o + \sum_{i=1}^{k} SSQ(C_i)\sigma_i^2 + n_L\sigma_e^2 \quad .$$

As can be seen from inspection of equation (3.10), a direct consequence of this theorem is that if any submatrix L_i of L is zero, then the expected value of SSL will not involve the i^{th} effect.

3.3 The Type II Analysis

This Analysis is based on the type II estimable function of SAS-GLM procedure (see SAS User's Guide, 1979 edition).

For the two-way classification, the model in equation (3.1) can be rewritten as:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \qquad (3.14)$$

where $i=1,2,...,a$, $j=1,2,...,b$,

$k=1,2,...,n_{ij}$; and

$n_{ij}>0$ is the number of observations in the $(i,j)^{th}$ cell,

μ is a constant,

α_i, β_j, γ_{ij} and e_{ijk} are row, column, interaction effects and random error. They are independent random variates from normal populations with zero means and variances σ^2_A, σ^2_B, σ^2_{AB} and σ^2_e respectively.

The type II estimable function for row effect A is:

$$L_2\beta = \begin{bmatrix} \phi_1 \\ \hline \phi_2 \\ \hline \vdots \\ \hline \phi_{a-1} \end{bmatrix} \qquad (3.15)$$

where

$$\phi_i = \sum_{j=1}^{b} [n_{ij}(\alpha_i + \gamma_{ij}) - \sum_{k=1}^{a} \frac{n_{ij}n_{kj}}{n_{.j}} (\alpha_k + \gamma_{kj})]$$

and $\quad n_{.j} = \sum_{i=1}^{a} n_{ij} \quad$ (see Searle [1971] p.304).

L_2 can also be obtained from the associated rows of the Forward Doolittle of X'X for the model which has been rearranged so that all effects which do not contain A (i.e. column effect B) are put before A. The columns of L_2 are then rearranged back to the original order (see Goodnight [1980]).

The corresponding sum of squares

$$SSA_2 = SS(H_o:L_2\beta = 0)$$

is equivalent to $R(A|\mu,B)$, the reduction in total sum of squares due to fitting A after μ and B.

The type II estimable function for the interaction effect AB is:

$$L_{AB}\beta = \begin{bmatrix} \underline{\Psi_{11}} \\ \underline{\Psi_{12}} \\ \vdots \\ \underline{\Psi_{ij}} \\ \vdots \end{bmatrix} \quad \text{for} \quad \begin{array}{l} i = 1,2, \ldots, a-1 \\ j = 1,2, \ldots, b-1 \end{array} \qquad (3.16)$$

where

$$\Psi_{ij} = \gamma_{ij} - \gamma_{ib} - \gamma_{aj} + \gamma_{ab} \quad .$$

Notice that Ψ_{ij} involves only the interaction effects. The corresponding sum of squares

$$SSAB = SS(H_o:L_{AB}\beta = 0)$$

is equivalent to $R(AB|\mu, A, B)$, the reduction in total sum of squares due to fitting AB after μ, A, and B.

The matrices of quadratic form and the expected values of SSA_2 and SSAB can be computed using equations (3.8) and (3.10).

3.4 The Type III Analysis

This analysis is based on the type III estimable function of SAS-GLM procedure. It is the same as the type II estimable function for the balanced design. For two-way classification, the type III estimable function for row effect, A, is:

$$L_3\beta = \begin{bmatrix} \phi_1 \\ \hline \phi_2 \\ \hline \vdots \\ \hline \phi_{a-1} \end{bmatrix} \qquad (3.17)$$

where

$$\phi_i = \alpha_i + \bar{\gamma}_{i.} - (\alpha_a + \bar{\gamma}_{a.})$$

and

$$\bar{Y}_{i.} = \frac{1}{b} \sum_{i=1}^{b} Y_{ij} \quad .$$

Notice that Φ_i does not involve the column effect and it is independent of the cell frequencies, n_{ij}'s. The corresponding sum of squares

$$SSA_3 = SS(H_o : L_3\beta = 0)$$

and its expected value can be computed using equations (3.7) and (3.10).

The type III estimable function for the interaction effect, SSAB, is the same as that of type II analysis.

The error sum of squares, SSE, is:

$$Y'(I-X(X'X)^{-}X')Y$$

for both type II and type III analyses.

It can be shown that the type III estimable function for A in equation (3.17) is orthogonal to the type II estimable function for AB in equation (3.16).

SSA_2 and SSA_3, the sums of squares for A in type II and type III analyses, are usually different from each other. However, it can be shown that if every cell in each row has the same number of observations (the number may be different in different rows), then the hypothesis (H_o: $L_2\beta=0$) can be reduced to (H_o: $L_3\beta=0$), and the two types of sum of squares will be equal.

3.5 The Unweighted Mean Analysis

The Unweighted Mean Analysis was first introduced by Yates [1934] as a simple analysis. The mean of each cell is treated as the only observation in the cell, and a balanced analysis of variance is performed as if there was one observation per cell. Later Gosslee and Lucas [1965], Hirutsu [1968], Webster [1968], and Levy, Narula and Abrami [1975] extended its use to testing of hypothesis.

For the model in equation (3.3), we define the mean of the $(i,j)^{th}$ cell as:

$$x_{ij} = \bar{y}_{ij.} = \sum_{k=1}^{n_{ij}} \frac{y_{ijk}}{n_{ij}}$$

and the means of x_{ij} as:

$$\bar{x}_{i.} = \sum_{j=1}^{b} \frac{x_{ij}}{b}, \qquad \bar{x}_{.j} = \sum_{i=1}^{a} \frac{x_{ij}}{a},$$

$$\bar{x}_{..} = \sum_{j=1}^{b} \frac{\bar{x}_{.j}}{b} = \sum_{i=1}^{a} \frac{\bar{x}_{i.}}{a}$$

In addition, let H be the harmonic mean of all cell frequencies and

$$n_h = \frac{1}{H} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{1}{n_{ij}} , \qquad (3.18)$$

(see Searle [1971] p.366).

The sums of squares for row and interaction effects are:

$$SSA_u = b \sum_{i=1}^{a} (\bar{x}_{i.} - \bar{x}_{..})^2$$

$$= b \sum_{i=1}^{a} \bar{x}_{i.}^2 - ab\, \bar{x}_{..}^2$$

$$= \frac{1}{b} \sum_{i=1}^{a} \left(\sum_{j=1}^{b} \frac{y_{ij.}}{n_{ij}} \right)^2 - \frac{1}{ab} \left(\sum_{i=1}^{a} \sum_{j=1}^{b} \frac{y_{ij.}}{n_{ij}} \right)^2$$

$$= Y'(Q_1 - Q_4)Y$$

$$= Y'Q_AY$$

and

$$SSAB_u = \sum_{i=1}^{a} \sum_{j=1}^{b} (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2$$

$$= \sum_{i=1}^{a} \sum_{j=1}^{b} x_{ij} - a \sum_{j=1}^{b} \bar{x}_{.j}^2 - b \sum_{i=1}^{a} \bar{x}_{i.}^2 + ab\, \bar{x}_{..}^2$$

$$= Y'(Q_3 - Q_1 - Q_2 + Q_4)Y$$

$$= Y'Q_{AB}Y$$

such that

$$Y'Q_1Y = \frac{1}{b} \sum_{i=1}^{a} \left(\sum_{j=1}^{b} \frac{y_{ij.}}{n_{ij}} \right)^2$$

$$= \frac{1}{b} AA' \qquad (3.19)$$

where

$$A = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & \ddots & \\ & & & & A_a \end{bmatrix},$$

$$A_i = \begin{bmatrix} \frac{1}{n_{i1}} 1_{n_{i1}} \\ \hline \frac{1}{n_{i2}} 1_{n_{i2}} \\ \hline \vdots \\ \hline \frac{1}{n_{ib}} 1_{n_{ib}} \end{bmatrix} = \begin{bmatrix} A_{i1} \\ \hline A_{i2} \\ \hline \vdots \\ \hline A_{ib} \end{bmatrix}, \qquad A_{ij} = \frac{1}{n_{ij}} 1_{n_{ij}} ,$$

and $1_{n_{ij}}$ is an $(n_{ij} x 1)$ vector of 1's;

$$\begin{aligned} Y'Q_2Y &= \frac{1}{a} \sum_{j=1}^{b} \left(\sum_{i=1}^{a} \frac{y_{ij.}}{n_{ij}} \right)^2 \\ &= \frac{1}{a} BB' , \end{aligned} \qquad (3.20)$$

where

$$B = [B_1 | B_2 | \ldots | B_b] ,$$

$$B_k = \begin{bmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{1b} \\ B_{21} \\ \vdots \\ B_{ab} \end{bmatrix}, \qquad B_{ij} = \delta_{Kj} A_{ij} ,$$

$$\delta_{Kj} = \begin{cases} 1 & \text{if } j = K \\ 0 & \phantom{\text{if }} j \neq K \end{cases}$$

$$\begin{aligned} Y'Q_3Y &= \sum_{i=1}^{a} \sum_{j=1}^{b} \left(\frac{y_{ij.}}{n_{ij}} \right)^2 \\ &= CC' , \end{aligned} \qquad (3.21)$$

where

$$C = \begin{bmatrix} A_{11} & & & & \\ & A_{12} & & & \\ & & A_{13} & & \\ & & & \ddots & \\ & & & & A_{ab} \end{bmatrix},$$

and

$$Y'Q_4Y = \frac{1}{ab}\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y_{ij.}}{n_{ij}}\right)^2$$

$$= \frac{1}{ab}\,DD' \qquad (3.22)$$

where

$$D = \begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1b} \\ A_{21} \\ \vdots \\ A_{ab} \end{bmatrix}.$$

Notice that if we form the matrix

$$U = [D|A|B|C],$$

and the matrix X in equation (3.2) is partitioned as:

$$X = [X_o|X_1|X_2|X_3]$$

so that X_o, X_1, X_2, and X_3 correspond to the mean, row, column, and interaction effects, then U will be identical to X if we change the values of all its non-zero elements to 1.

The expected values of SSA_u and $SSAB_u$ can be computed using equation (3.9). But, before we do that, we need to compute

$$
\begin{aligned}
tr(X_i'Q_AX_i) &= tr[X_i'(\frac{AA'}{b} - \frac{DD'}{ab})\ X_i] \\
&= \frac{1}{b}\ tr(X_i'AA'X_i) - \frac{1}{ab}\ tr(X_i'DD'X_i) \\
&= \frac{1}{b}\ SSQ(X_i'A) - \frac{1}{ab}\ SSQ(X_i'D)\ . \qquad (3.23)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
tr(X_i'Q_{AB}X_i) = SSQ(X_i'C) &- \frac{1}{b}\ SSQ(X_i'A) \\
&- \frac{1}{a}\ SSQ(X_i'B) + \frac{1}{ab}\ SSQ(X_i'D)\ . \qquad (3.24)
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
&X_o'A = b\cdot 1_a'\ ,\quad X_1'A = b\cdot I_a\ , \\
&X_o'B = a\cdot 1_b'\ ,\quad X_1'B = J_{ab}\ , \\
&X_o'C = 1_{ab}'\ ,\quad X_1'C = \sum_{i=1}^{a} {}^{+}\, 1_b'\ , \\
&X_o'D = ab\ ,\quad X_1'D = b\cdot 1_a\ , \qquad (3.25) \\
&X_2'A = J_{ab}\ ,\quad X_3'A = \sum_{i=1}^{a} {}^{+}\, I_b\ , \\
&X_2'B = a\cdot I_b\ ,\quad X_3'B = 1_a \otimes I_b\ ,
\end{aligned}
$$

$$X_2'C = 1_a' \otimes I_b\,,\quad X_3'C = I_{ab}\,,$$

$$X_2'D = a \cdot 1_b\,,\quad X_3'D = 1_{ab}$$

where

$$\sum_{i=1}^{a} + 1_b = \begin{bmatrix} 1_b & & & \\ & 1_b & & \\ & & \ddots & \\ & & & 1_b \end{bmatrix}, \qquad 1_A \otimes I_b = \begin{bmatrix} I_b \\ \hline I_b \\ \hline \vdots \\ \hline I_b \end{bmatrix},$$

1_b is a (bx1) vector of 1's,

I_b is an identity matrix of order b, and

$J_{a,b}$ is an (axb) matrix of 1's.

Substituting equation (3.25) into equations (3.23) and (3.9), we get

$$E(SSA_u) = (a-1)n_h\,\sigma_e^2 + (a-1)\,\sigma_{AB}^2 + (a-1)b\sigma_A^2 .$$

Similarly,

$$E(SSAB_u) = (a-1)(b-1)n_h\,\sigma_e^2 + (a-1)(b-1)\,\sigma_{AB}^2 .$$

The expected mean squares can be obtained by dividing the expected sums of squares by their respective degrees of freedom.

The fact that the expected mean squares of A and AB are identical when σ_A^2 is equal to zero makes this analysis even more attractive when the design is unbalanced.

4. THE TEST PROCEDURES

4.1 Introduction

In this Chapter, we will discuss seven test procedures which will be evaluated by their performances on a selected set of designs based on the model in equation (3.14). The null hypothesis, which we will refer to hereafter, is (H_0: $\sigma_A^2=0$).

In each procedure, the numerator and denominator mean squares (MSN and MSD) are constructed using one or two mean squares in the form of

$$MS = T\cdot MS1 + (1-T)\cdot MS2 \qquad (4.1)$$

where $0 \leq T \leq 1$. The corresponding degrees of freedom (DFN and DFD) are computed by the Satterthwaite's formula (see Satterthwaite [1941 and 1946]).

Thus the degrees of freedom for the MS in equation (4.1) is:

$$f = \frac{(EMS)^2}{\frac{(T\cdot EMS1)^2}{d_1} + \frac{[(1-T)\cdot EMS2]^2}{d_2}} \qquad (4.2)$$

where f, d_1, d_2, and EMS, EMS1, EMS2 are the degrees of freedom and the expected values of MS, MS1, MS2, respectively. Notice that EMS, EMS1 and EMS2 are unknown quantities in practice; therefore, they will be replaced by their observed values.

An approximate F statistic is then computed as

$$F' = \frac{MSN}{MSD}$$

This statistic is assumed to have approximately an F distribution with DFN and DFD degrees of freedom. Once the statistics F', DFN, and DFD are known, we will follow the usual procedure. That is, the null hypothesis will be rejected if F' is greater than $F_{DFN,DFD,\alpha}$ at a given significance level α.

The degrees of freedom and the expected values of mean squares are shown in Table 4.1 where K1, K2, K3, K4, and K5 are the sums of squares of different submatrices C_i's defined in equation (3.10), and n_h is the reciprocal of the harmonic mean of the cell frequencies defined in equation (3.18). Only the mean squares needed for testing the null hypothesis are shown here.

Table 4.1 The Degrees of Freedom and Expected Values of Mean Squares for model (3.14).

(a) The Type II Analysis

Mean Square	D.F.	Expected Mean Square
MSA_2	a-1	$K1\ \sigma_A^2 + K2\ \sigma_{AB}^2 + \sigma_e^2$
MSAB	(a-1)(b-1)	$K3\ \sigma_{AB}^2 + \sigma_e^2$
MSE	n-ab	σ_e^2

(b) The Type III Analysis

Mean Square	D.F.	Expected Mean Square
MSA_3	a-1	$K4\ \sigma_A^2 + K5\ \sigma_{AB}^2 + \sigma_e^2$
MSAB	(a-1)(b-1)	$K3\ \sigma_{AB}^2 + \sigma_e^2$
MSE	n-ab	σ_e^2

(c) The Unweighted Mean Analysis

Mean Square	D.F.	Expected Mean Square
MSA_u	a-1	$b\ \sigma_A^2 + \sigma_{AB}^2 + n_h\ \sigma_e^2$
$MSAB_u$	(a-1)(b-1)	$\sigma_{AB}^2 + n_h\ \sigma_e^2$

4.2 Procedure A

This procedure is based on the Type II Analysis.

(1). Let $T=\frac{K2}{K3}$, $S=\frac{1}{T}$.

(2). If T>1 then

$MSN=S\cdot MSA_2 + (1-S)\cdot MSE$, $MSD=MSAB$,

DFN is given by (4.2) , $DFD=(a-1)(b-1)$;

otherwise

$MSN=MSA_2$, $MSD=T\cdot MSAB + (1-T)\cdot MSE$,

$DFN=(a-1)$, DFD is given by (4.2).

4.3 Procedure B

This procedure involves the conventional F test based on the mean squares from the Type II Analysis.

$MSN=MSA_2$, $MSD=MSAB$,

$DFN=(a-1)$, $DFD=(a-1)(b-1)$.

4.4 Procedure C

This procedure is based on the Type III Analysis.

(1). Let $T=\frac{K5}{K3}$, $S=\frac{1}{T}$.

(2). If T>1 then

$MSN = S \cdot MSA_3 + (1-S) \cdot MSE$, $MSD = MSAB$,
DFN is given by (4.2) , $DFD = (a-1)(b-1)$;

otherwise

$MSN = MSA_3$, $MSD = T \cdot MSAB + (1-T) \cdot MSE$,
$DFN = (a-1)$, DFD is given by (4.2).

4.5 Procedure D

This procedure involves the conventional F test based on the Mean squares from the Type III Analysis.

$MSN = MSA_3$, $MSD = MSAB$,
$DFN = (a-1)$, $DFD = (a-1)(b-1)$.

4.6 Procedure E

This procedure involves both type II and type III analyses. MSN is constructed as a linear combination of MSA_2 and MSA_3. However, it is meaningless to do so unless K2 is not equal to K5 and the value of K3 lies between that of K2 and K5.

(1). Let $T = \frac{K3-K5}{K2-K5}$.

(2). $MSN = T \cdot MSA_2 + (1-T) \cdot MSA_3$, $MSD = MSAB$,
DFN is given by (4.2) , $DFD = (a-1)(b-1)$.

4.7 Procedure F

This procedure is a modified version of procedure E. MSN is constructed the same way as it is done in procedure E. But instead of computing DFN from equation (4.2), we simply use the degrees of freedom for A.

(1). Let $T=\frac{K3-K5}{K2-K5}$.

(2). $MSN=T\cdot MSA_2 + (1-T)\cdot MSA_3$, $MSD=MSAB$,

$DFN=(a-1)$, $DFD=(a-1)(b-1)$.

4.8 Procedure G

This procedure is based on the Unweighted Mean Analysis.

$MSN=MSA_u$, $MSD=MSAB_u$,

$DFN=(a-1)$, $DFD=(a-1)(b-1)$.

5. DISTRIBUTION OF APPROXIMATE F STATISTICS

In Chapter 4, we have discussed the various ways to construct MSN and MSD using linear combinations of the mean squares from three types of analyses. Since these mean squares are quadratic forms in Y, MSN and MSD, being linear combinations of mean squares, are also quadratic forms in Y. However, in the unbalanced designs, MSN and MSD are not usually independent, nor do they have chi-square distributions. Therefore, the test statistic F', the ratio of two correlated quadratic forms, could not follow an F distribution.

In this Chapter, we will show that the exact distribution of F' is actually an infinite weighted sum of F distribution functions.

5.1 Distribution of Quadratic Forms

We know that the mean squares from the unbalanced designs do not, in general, have chi-square distributions. The following theorems will show the exact distribution of the individual or linear combinations of the mean squares.

Definition 5.1.1 If Y is distributed as $N(X_o\mu, V)$, then the quadratic form $Y'AY$ is said to be translation invariant if

$$Y'AY = (Y - X_o\mu^*)'A\,(Y-X_o\mu^*)$$

for any arbitrary μ^*. This is equivalent to the restriction on A that $X_o'AX_o=0$.

Theorem 5.1.1 If Y is distributed as N(0,V), and A is any real symmetric matrix of rank r, then Y'AY is distributed as $W = \sum_{i=1}^{r} \lambda_i z_i$, where z_i $(i = 1, 2, \ldots, r)$ are independent, central chi-square random variates each with 1 degree of freedom, and λ_i $(i = 1, 2, \ldots, r)$ are the real non-zero eigenvalues of the matrix product VA (see Cochran [1934] and Ruben [1962]).

Proof: This is achieved by the linear transformation

$$Y=LPX$$

where L is the lower triangular matrix such that

$$V=LL',$$

and P is the orthogonal matrix of the eigenvectors of L'AL such that

$$P'L'ALP=D$$

where D is the diagonal matrix of eigenvalues of L'AL, or equivalently of VA, and λ_i's are the diagonal elements of D.

Thus Y'AY is distributed like $W = \sum_{i=1}^{r} \lambda_i x_i^2$ where x_i are independent unit normal (N(0,1)) variates. This implies that Y'AY is distributed like $W = \sum_{i=1}^{r} \lambda_i z_i$ where z_i are independent chi-square variates each with 1 degree of freedom.

Corollary 5.1.1 If Y is distributed as $N(X_o\mu, V)$, and A is any real symmetric matrix of rank r such that $Y'AY$ is translation invariant, then $Y'AY$ is distributed as $W = \sum_{i=1}^{r} \lambda_i z_i$ where z_i $(i = 1, 2, \ldots, r)$ are independent, central chi-square random variates each with 1 degree of freedom, and λ_i $(i = 1, 2, \ldots, r)$ are the real non-zero eigenvalues of AV.

Corollary 5.1.2 If the matrix product AV in the above corollary has m distinct eigenvalues each with multiplicity r_j such that $r_1+r_2+\ldots+r_m=r$, then $Y'AY$ is distributed as $W = \sum_{j=1}^{m} \lambda_j x_j$ where x_j are independent, central chi-square random variates each with r_j degrees of freedom.

Theorem 5.1.2 If Y is distributed as $N(X_o\mu, V)$, then any mean square from the type II or type III analysis is distributed as $W = \sum_{i=1}^{r} \lambda_i z_i$ where z_i $(i = 1, 2, \ldots, r)$ are independent, central chi-square random variates each with 1 degree of freedom, and λ_i $(i = 1, 2, \ldots, r)$ are the eigenvalues of the matrix product VA for some matrix A.

Proof: Since any mean square from the type II or type III analysis is a quadratic form $Y'QY$ with Q defined by (3.8), and it is associated with an estimable function $L\beta$ where L is in the row space of X with row rank n_L, all we need to show is that $Y'QY$ is translation invariant, i.e.,$X_o'AX_o=0$. From equation (3.11), we have

$$L(X'X)^{-}X'X_o = L_o = 0 \qquad (5.1)$$

since the estimable functions from type II or type III analysis do not involve the mean μ. Combining equations (3.8) and (5.1), we get

$$\begin{aligned} QX_o &= X(X'X)^{-}L'(L(X'X)^{-}L')^{-1}L(X'X)^{-}X'X_o \\ &= X(X'X)^{-}L'(L(X'X)^{-}L')^{-1}L_o \\ &= 0 \quad . \end{aligned}$$

Thus $X_o'AX_o=0$. This is also true when MS is MSE, the error mean square. Therefore, type II and III mean squares are not only translation invariant, but are also distributed as $W = \sum_{i=1}^{r} \lambda_i z_i$ according to Corollary 5.1.1.

<u>Theorem 5.1.3</u> If Y is distributed as $N(X_o\mu, V)$, then any mean square from the unweighted mean analysis with r degrees of freedom is distributed as $W = \sum_{i=1}^{r} \lambda_i z_i$ where z_i ($i = 1, 2, \ldots, r$) are independent, central chi-square random variates with 1 degree of freedom, and λ_i ($i = 1, 2, \ldots, r$) are non-zero eigenvalues of the matrix product VQ for some matrix Q.

<u>Proof</u>: Here, mean squares of interest are MSA_u and $MSAB_u$. Recall that

$$\begin{aligned} MSA_u &= \frac{1}{a-1} SSAu \\ &= \frac{1}{a-1} Y'(Q_1 - Q_2)Y \\ &= \frac{1}{a-1} Y'Q_AY \end{aligned}$$

and

$$MSAB_u = \frac{1}{(a-1)(b-1)} SSAB_u$$
$$= \frac{1}{(a-1)(b-1)} Y'(Q_3 - Q_1 - Q_2 + Q_4)Y$$
$$= \frac{1}{(a-1)(b-1)} Y'Q_{AB}Y$$

where Q_1, Q_2, Q_3 and Q_4 are defined in equations (3.19), (3.20), (3.21) and (3.22), respectively. We only have to show that $Y'Q_AY$ and $Y'Q_{AB}Y$ are translation invariant, i.e., $X_o'Q_{AB}X_o=X_o'Q_AX_o=0$. It can be shown that if X_o is an nx1 vector of 1's, then

$$Q_1X_o = Q_2X_o = Q_3X_o = Q_4X_o = D = \begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1b} \\ A_{21} \\ \vdots \\ A_{ab} \end{bmatrix}$$

where D is defined in equation (3.22). Thus

$$Q_AX_o = (Q_1 - Q_4)X_o$$
$$= 0$$

and

$$Q_{AB}X_o = (Q_3 - Q_1 - Q_2 + Q_4)X_o$$
$$= 0 .$$

Therefore, $Y'Q_AY$ and $Y'Q_{AB}Y$ are translation invariant and, according to Corollary 5.1.2, MSA_u and $MSAB_u$ are distributed as linear combination of independent, central chi-square random variates.

Theorems 5.1.2 and 5.1.3 have shown that any mean square in Table 4.1.1 is distributed as a linear combination of independent, central chi-square random variates each with 1 degree of freedom. Since the MSN and MSD constructed in Chapter 4 are linear combinations of these mean squares, they are also distributed as linear combinations of independent, central chi-square random variates.

We now proceed to find the distributions of linear combinations of independent, central chi-square random variates. Robbins and Pitman [1949] used the characteristic functions of chi-square variates to show that the distribution function of a linear combination of independent, central chi-square random variates is an infinite weighted sum of chi-square distribution functions. A similar result was obtained by Laha [1954] using the distribution of Bessel functions.

<u>Theorem 5.1.4</u> The distribution function of a linear combination of independent, central chi-square random variates with positive coefficients is an infinite weighted sum of chi-square distribution functions.

Proof: Let

$$u = \lambda(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m)$$

where x_j ($j = 1, 2, \ldots m$) are independent, central chi-square random variates each with f_i degrees of freedom such that $\sum_{j=1}^{m} f_j = M$, and without loss of generality, we can assume that

$$1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_m .$$

Let

$$p_j = \frac{f_i}{2} \quad \text{and} \quad \sum_{j=1}^{m} p_j = \frac{M}{2} .$$

Since x_j is a chi-square variate, its characteristic function is:

$$\phi_{x_j}(t) = (1 - 2it)^{-p_j} .$$

Thus the characteristic function of $\frac{u}{\lambda}$ is:

$$\phi_{u/\lambda}(t) = \prod_{j=1}^{m} (1 - 2i\lambda_j t)^{-p_j} \tag{5.2}$$

$$= \prod_{j=1}^{m} [\lambda_j(1 - 2it) - (\lambda_j - 1)]^{-p_j}$$

$$= \prod_{j=1}^{m} [\lambda_j^{-p_j} (1 - 2it)^{-p_j} \sum_{k_j=0}^{\infty} \frac{p_j(p_j+1)\ldots(p_j+k_j-1)}{k_j!}$$

$$(1 - \frac{1}{\lambda_j})^{k_j} (1 - 2it)^{-k_j}]$$

$$= \prod_{j=1}^{m} [\sum_{k_j=0}^{\infty} \lambda_j^{-p_j} \frac{p_j(p_j+1)\ldots(p_j+k_j-1)}{k_j!}$$

$$(1 - \frac{1}{\lambda_j})^{k_j} (1 - 2it)^{-(p_j+k_j)}] .$$

Letting

$$a_{j,k_j} = \lambda_j^{-p_j} \frac{p_j(p_j+1)\ldots(p_j+k_j-1)}{k_j!} (1 - \frac{1}{\lambda_j})^{k_j} \qquad (5.3)$$

and

$$a_K = \sum_{k_1+k_2+\ldots+k_m = K} [\prod_{j=1}^{m} a_{j,k_j}] , \qquad (5.4)$$

we have

$$\phi_{u/\lambda}(t) = \prod_{j=1}^{m} [\sum_{k_j=0}^{\infty} a_{j,k_j} (1 - 2it)^{-(p_j+k_j)}]$$

$$= \sum_{K=0}^{\infty} a_K (1 - 2it)^{-\sum_{j=1}^{m} (p_j+k_j)}$$

$$= \sum_{K=0}^{\infty} a_K(1 - 2it)^{-(\frac{M}{2} + K)} \quad . \qquad (5.5)$$

This is a linear combination of the characteristic functions of chi-square variates whose degrees of freedom are M+2K where K=0,1,..., respectively. It follows that

$$Pr(\frac{u}{\lambda} \leq w) = \sum_{K=0}^{\infty} a_K \, F_{M+2K}(w) \qquad (5.6)$$

where $F_{M+2K}(\ )$ is the distribution function of a chi-square variate with M+2K degrees of freedom. Letting $v=\lambda w$, we have

$$Pr(u < v) = \sum_{k=0}^{\infty} a_K F_{M+2K}(\frac{v}{\lambda}) \quad . \qquad (5.7)$$

If we let t=0 in equations (5.2) and (5.5) , we get

$$\sum_{K=0}^{\infty} a_K = 1 \quad . \qquad (5.8)$$

It can also be shown that

$$\sum_{k_j=0}^{\infty} a_{j,k_j} = 1 \qquad \text{for } j = 1, 2, \ldots, n \quad .$$

This implies that the infinite sum in equation (5.7) is less or equal to 1. Thus, the distribution function of a linear combination of independent, central chi-square random variates is an infinite weighted sum of chi-square distribution functions as given in equation (5.7).

5.2 Distribution of The Ratio of Two Independent Quadratic Forms

In this section, we are interested in finding the distribution of the ratio of two independent quadratic forms which do not have chi-square distributions. We will limit our discussions to quadratic forms that are non-negative definite and translation invariant. It has been shown, in the previous section, that a quadratic form is distributed as a linear combination of independent chi-square variates. Therefore, the ratio of two independent quadratic forms can be considered as the ratio of two linear combinations of independent chi-square variates.

Theorem 5.2.1 The distribution function of the ratio of two linear combinations of independent, central chi-square random variates with positive coefficients is an infinite weighted sum of F distribution functions.

Proof: Let

$$u = \frac{x}{z}$$

$$= \frac{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m}{\xi_1 z_1 + \xi_2 z_2 + \ldots + \xi_n z_n}$$

where x_i $(i=1,2,\ldots,m)$ and z_j $(j=1,2,\ldots,n)$ are independent, central chi-square random variates each with p_i and q_j degrees of freedom, respectively, such that

$$\sum_{i=1}^{m} p_i = M , \qquad \sum_{j=1}^{n} q_j = N ,$$

and without loss of generality, we can assume that $\lambda_i^{'}$s and $\xi_j^{'}$s are all greater than or equal to unity. Then, from Theorem 5.1.4, the distribution functions of x and z are:

$$G(x) = \sum_{K=0}^{\infty} a_K F_{M+2K}(x)$$

and

$$H(z) = \sum_{L=0}^{\infty} b_L F_{N+2L}(z)$$

where a_K and b_L are defined as in equation (5.4). Let

$$e^w = u$$

Thus $e^{iwt} = x^{it} \cdot z^{-it} = u^{it}$.

We will first derive the distribution of w. Its characteristic function is:

$$\phi_w(t) = \int_0^\infty \int_0^\infty e^{iwt} \, dG(x) \cdot dH(z)$$

$$= \sum_{K=0}^{\infty} \frac{a_K}{2^{\frac{M}{2}+K} \Gamma(\frac{M}{2}+K)} \int_0^\infty x^{it} x^{\frac{M}{2}+K-1} e^{-\frac{1}{2}x} dx$$

$$\cdot \sum_{L=0}^{\infty} \frac{b_L}{2^{\frac{N}{2}+L} \Gamma(\frac{N}{2}+L)} \int_0^\infty z^{-it} z^{\frac{N}{2}+L-1} e^{-\frac{1}{2}z} dz$$

$$= \sum_{K=0}^{\infty} \frac{a_K \, \Gamma(\frac{M}{2} + K + it)}{\Gamma(\frac{M}{2} + K)} \cdot \sum_{L=0}^{\infty} \frac{b_L \, \Gamma(\frac{N}{2} + L - it)}{\Gamma(\frac{N}{2} + L)} \quad .$$

Applying Fourier Transformation, we get the density function of w as:

$$f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwt} \phi_w(t) \, dt$$

$$= \sum_{K,L=0}^{\infty} \frac{a_K b_L}{\Gamma(\frac{M}{2} + K) \cdot \Gamma(\frac{N}{2} + L)}$$

$$\cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwt} \, \Gamma(\tfrac{M}{2} + K + it) \cdot \Gamma(\tfrac{N}{2} + L - it) \, dt \quad . \qquad (5.9)$$

Let

$$\frac{N}{2} + L - it = -h \quad .$$

Thus

$$it = \frac{N}{2} + L + h \quad .$$

The integral in equation (5.9) can be written as:

$$\frac{1}{i} \, e^{-w(\frac{N}{2} + L)} \int_{-\frac{N}{2} - L - i\infty}^{-\frac{N}{2} - L + i\infty} e^{-wh} \, \Gamma(\tfrac{M}{2} + K + \tfrac{N}{2} + L + h) \cdot \Gamma(-h) \, dh$$

$$= e^{-w(\frac{N}{2} + L)} (1 + e^{-w})^{-(\frac{M}{2} + K + \frac{N}{2} + L)} \, \Gamma(\tfrac{M}{2} + K + \tfrac{N}{2} + L) \cdot 2\pi$$

(see Whittaker and Watson [1927] p.289).

Substituting this into equation (5.9), we get

$$f(w) = \sum_{K,L=0}^{\infty} \frac{a_K b_L \; e^{-w(\frac{N}{2}+L)} \; (1+e^{-w})^{-(\frac{M}{2}+K+\frac{N}{2}+L)} \; \Gamma(\frac{M}{2}+K+\frac{N}{2}+L)}{\Gamma(\frac{M}{2}+K) \cdot \Gamma(\frac{N}{2}+L)} .$$

Since $e^w = u$ and $\frac{dw}{du} = \frac{1}{u}$,

the distribution of u is:

$$f(u) = \sum_{K,L=0}^{\infty} \frac{a_K b_L \; (\frac{1}{u})^{\frac{N}{2}+L+1}}{(1+\frac{1}{u})^{\frac{M}{2}+K+\frac{N}{2}+L} \cdot B(\frac{M}{2}+K, \frac{N}{2}+L)}$$

where B(.,.) is the Beta function. Let f_1=M+2K and f_2=N+2L, then

$$f(u) = \sum_{K,L=0}^{\infty} a_K b_L \frac{u^{\frac{f_1}{2}-1}}{(1+u)^{\frac{f_1+f_2}{2}} \cdot B(\frac{f_1}{2}, \frac{f_2}{2})} .$$

The distribution function of u becomes

$$F(u) = \sum_{K,L=0}^{\infty} a_K b_L \int_0^u \frac{u^{\frac{f_1}{2}-1}}{(1+u)^{\frac{f_1+f_2}{2}} B(\frac{f_1}{2}, \frac{f_2}{2})} du$$

Making the transformation $u=\frac{f_1}{f_2} v$, we have

$$F(u) = \sum_{K,L=0}^{\infty} a_K b_L \int_0^{\frac{f_2}{f_1}u} \frac{(\frac{f_1}{f_2})^{\frac{f_1}{2}} v^{\frac{f_1}{2}-1}}{(1+\frac{f_1}{f_2}v)^{\frac{f_1+f_2}{2}} B(\frac{f_1}{2}, \frac{f_2}{2})} dv$$

$$= \sum_{K,L=0}^{\infty} a_K b_L \cdot F_{f_1,f_2} (\frac{f_2}{f_1} u)$$

$$= \sum_{K,L=0}^{\infty} a_K b_L \cdot F_{M+2K,N+2L} (\frac{N+2L}{M+2K} u) \qquad (5.10)$$

where $F_{M+2K,N+2L}(\ )$ is the F distribution function with M+2K and N+2L degrees of freedom.

Thus the distribution function of the ratio of two linear combinations of independent, central chi-square random variates with positive coefficients is an infinite weighted sum of F distribution functions as given in equation (5.10). This implies that the distribution function of the ratio of two independent, non-negative definite and translation invariant quadratic forms is also an infinite weighted sum of F distribution functions.

5.3 The Exact Distribution of Approximate F Statistics

In Chapter 4, we have shown different ways to construct the numerator and denominator mean squares, MSN and MSD, which are quadratic forms in Y. Let

$$MSN=Y'AY$$

and

$$MSD=Y'BY.$$

The approximate F statistic is the ratio

$$F' = \frac{MSN}{MSD}$$

From the discussions in Chapter 3, we know that both MSN and MSD, being linear combinations of mean squares with positive coefficients, are non-negative definite and translation invariant quadratic forms. In an unbalanced design, MSN and MSD are not usually independent, nor do they have chi-square distributions. Therefore, F' does not follow an F distribution. Box [1954] showed that the distribution function of F' is a finite series if the multiplicity of every distinct eigenvalue of VA and VB is even, (V is the variance-covariance matrix of Y). This condition is seldom satisfied in a typical unbalanced design. A more generalized distribution function is needed.

Given any value $f>0$,

$$\Pr(F' \leq f) = \Pr\left[\frac{MSN}{MSD} \leq f\right]$$

$$= \Pr\left[\frac{Y'AY}{Y'BY} \leq f\right]$$

$$= \Pr\left[Y'(A - f \cdot B)Y \leq 0\right]$$

$$= \Pr\left[Y'QY \leq 0\right] \quad (5.11)$$

Since MSD is a linear combination of mean squares with positive coefficients, we are sure that Y'BY is non-negative definite; otherwise, equation (5.11) will not hold. Furthermore, since MSN and MSD are both translation invariant, so is Y'QY. Applying Corollary 5.1.2, equation (5.11) can be written as:

$$\Pr(F' \leq f) = \Pr\left[\sum_{i=1}^{m+n} \lambda_i' z_i \leq 0\right]$$

$$= \Pr\left[\sum_{i=1}^{m} \lambda_i' z_i - \sum_{j=1}^{n} \xi_j' y_j \leq 0\right]$$

$$= \Pr\left[\frac{\sum_{i=1}^{m} \lambda_i' z_i}{\sum_{j=1}^{n} \xi_j' y_j} \leq 1\right] \quad (5.12)$$

where z_i $(i = 1, 2, \ldots, m + n)$ are independent, central chi-square random variates each with p_i degrees of freedom, λ_i' $(i = 1, 2, \ldots, m + n)$ are the non-zero, distinct eigenvalues of VQ such that

$$\lambda_1' > \lambda_2' > \ldots > \lambda_m' > 0 > \lambda_{m+1}' > \lambda_{m+2}' > \ldots > \lambda_{m+n}' ,$$

$$p_1 + p_2 + \dots + p_m = M, \quad p_{m+1} + p_{m+2} + \dots + p_{m+n} = N$$

and

$$\xi_j' = -\lambda_{m+j}', \quad y_j = z_{m+j} \quad \text{for} \quad j = 1, 2, \dots, n \ .$$

Letting

$$\lambda = \lambda_m', \qquad \xi = \xi_1' \ ,$$

$$\lambda_i = \frac{\lambda_i'}{\lambda} \geq 1 \qquad \text{for } i = 1, 2, \dots, m \ , \tag{5.13}$$

and

$$\xi_j = \frac{\xi_j'}{\xi} \geq 1 \qquad \text{for } j = 1, 2, \dots, n \ , \tag{5.14}$$

equation (5.12) becomes

$$\Pr(F' \leq f) = \Pr\left[\frac{\lambda \sum_{i=1}^{m} \lambda_i z_i}{\xi \sum_{j=1}^{n} \xi_j y_j} \leq 1\right]$$

$$= \Pr\left[\frac{\sum_{i=1}^{m} \lambda_i z_i}{\sum_{j=1}^{n} \xi_j y_j} \leq \frac{\xi}{\lambda}\right] \ .$$

Applying theorem 5.2.1, we get

$$\Pr(F' \leq f) = \sum_{K,L=0}^{\infty} a_K b_L \cdot F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right] \qquad (5.15)$$

where $F_{M+2K,N+2L}(\ )$ is the F distribution function with M+2K and N+2L degrees of freedom, and a_K, b_L are defined as in equation (5.4).

Thus the distribution function of F' is an infinite weighted sum of F distribution function given in equation (5.15). This result provides us with a powerful tool to evaluate the test procedures under different conditions.

5.4 Computation of The Distribution Function

It is obvious that we cannot compute the probability in equation (5.15) because it is a double infinite sum. If this infinite sum is replaced by a finite sum with K_1 and L_1 as the upper bound of K and L, respectively, then the truncation error is:

$$E = \Pr(F' \leq f) - \sum_{K=0}^{K_1} \sum_{L=0}^{L_1} a_K b_L \cdot F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right]$$

$$= \sum_{K=K_1+1}^{\infty} \sum_{L=0}^{L_1} a_K b_L \cdot F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right] +$$

$$\sum_{K=0}^{\infty} \sum_{L=L_1+1}^{\infty} a_K b_L \cdot F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right] \quad . \qquad (5.16)$$

Since $\sum_{K=0}^{\infty} a_K = 1$ and

$$F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right] \leq 1 \qquad \text{for } K, L = 0, 1, 2, \ldots ,$$

equation (5.16) becomes

$$E \leq \sum_{K=K_1+1}^{\infty} \sum_{L=0}^{L_1} a_K b_L + \sum_{K=0}^{\infty} \sum_{L=L_1+1}^{\infty} a_K b_L \tag{5.17}$$

$$= E_a (1 - E_b) + E_b$$

$$= E_a + E_b - E_a \cdot E_b$$

where

$$E_a = \sum_{K=K_1+1}^{\infty} a_K = 1 - \sum_{K=0}^{K_1} a_K$$

and

$$E_b = \sum_{L=L_1+1}^{\infty} b_L = 1 - \sum_{L=0}^{L_1} b_L \quad .$$

Equation (5.17) provides the upper limit of the truncation error due to replacing the double infinite sum in equation (5.15) with a finite sum. This equation is most helpful when we want to find the upper bounds of K and L.

If we choose K_1 and L_1 large enough such that the right hand side of equation (5.17) is less than the predetermined upper limit of truncation error. Then equation (5.15) becomes

$$Pr(F' \leq f) \doteq \sum_{K=0}^{K_1} \sum_{L=0}^{L_1} a_K b_L \cdot F_{M+2K,N+2L}\left[\frac{(N+2L)\xi}{(M+2K)\lambda}\right] \quad . \tag{5.18}$$

Notice that an F distribution function needs to be computed for each term of the double sum. This can be very time consuming. It can be shown that by making the transformation

$$x = \frac{\frac{\xi}{\lambda}}{1 + \frac{\xi}{\lambda}} \quad ,$$

the F distribution function in equation (5.18) can be replaced by an Incomplete Beta function, and this equation becomes

$$Pr(F' \leq f) \doteq \sum_{K=0}^{K_1} \sum_{L=0}^{L_1} a_K b_L I_x(f_1, f_2) \tag{5.19}$$

where $I_x(.,.)$ is the Incomplete Beta function,

$$f_1 = \frac{M+2K}{2} \text{ and } f_2 = \frac{N+2L}{2} \quad .$$

In addition, the recurrence formula

$$I_x(f_1,f_2) = x \cdot I_x(f_1-1,\ f_2) + (1-x) \cdot I_x(f_1,\ f_2-1)$$

can be used to generate most of the Incomplete Beta functions needed in equation (5.19).

As for the coefficients a_K and b_L, they can be computed using equations (5.3) and (5.4). Since the only difference between a_K and b_L is that a_K depends on the positive eigenvalues of VQ while b_L depends on the negative eigenvalues of VQ (Q defined as in equation (5.11)), we will only discuss the computation of a_K here.

We recall in equation (5.5) that

$$\sum_{K=0}^{\infty} a_K (1 - 2it)^{-(\frac{M}{2} + K)} = \prod_{j=1}^{m} \left[\sum_{k_j=0}^{\infty} a_{j,k_j} (1 - 2it)^{-(p_j+k_j)} \right] \tag{5.20}$$

where

$$K = \sum_{j=1}^{m} k_j \quad , \quad \frac{M}{2} = \sum_{j=1}^{m} p_j$$

and a_K, a_{j,k_j} are defined in equation (5.4) and (5.3), respectively. If we let $x=(1-2it)^{-1}$, then equation (5.20) becomes

$$\sum_{K=0}^{\infty} a_K x^K = \prod_{j=1}^{m} \left(\sum_{k_j=0}^{\infty} a_{j,k_j} x^{k_j} \right) \quad . \tag{5.21}$$

If the upper bound of K is chosen to be n, then equation

(5.21) becomes

$$\sum_{K=0}^{n} a_K x^K = \prod_{j=1}^{m} \left(\sum_{k_j=0}^{n} a_{j,k_j} x^{k_j} \right) \quad . \tag{5.22}$$

This is equivalent to multiplying m polynomials of degree n such that a_{j,k_j} is the k_jth coefficient of the jth polynomial and a_K is the Kth coefficient of the product polynomial which is truncated to degree n.

When n in equation (5.22) is small, the following algorithm can be used to compute the coefficients a_K's.

Let S be a (2x(n+1)) matrix, C be a vector of n+1 elements and I1, I2 be integers that are either 1 or 2 such that the I1th row of S contains the coefficients of the product of the first j-1 polynomials, and the I2th row of S contains the coefficients of the product of the first j polynomials, and the vector C contains the coefficients of the jth polynomial.

Algorithm 1:

step 1: Set I1=1, I2=2 and j=1.

step 2: Set $S(I1,K)=a_{j,K}$ for each K=0,1,...,n.

step 3: Set j=j+1.

step 4: Set $C(K)=a_{j,K}$ for each K=0,1,...,n.

step 5: Set $S(I2,K)=\sum_{i=0}^{K} S(I1,K-i)\cdot C(i)$ for each $K=0,1,\ldots,n$.

step 6: Interchange I1 and I2.

step 7: If $j=m$ then go to step 8; otherwise go to step 3.

step 8: Set $a_K=S(I1,K)$ for each $K=0,1,\ldots,n$.

As we can see from equation (5.3) that the following recurrence relation can be used to compute $a_{j,K}$:

$$a_{j,K} = a_{j,K-1}\,\frac{P_j+K-1}{K}\left(1-\frac{1}{\lambda_j}\right) \qquad \text{for } K = 1, 2, \ldots$$

where

$$a_{j,0} = \lambda_j^{-P_j} .$$

The computing time for this algorithm is in the order of $O(n^2)$. This can be quite costly when n is large. To find a faster algorithm, we acknowledge the fact that the coefficients of the product of two polynomials are identical to the convolution of the two coefficient vectors of the original polynomials (see Aho, Hopcroft and Ullman [1974] p.255).

Let

$$p(x) = \sum_{j=0}^{n} b(j)\cdot x^j, \qquad q(x) = \sum_{j=0}^{n} c(j)\cdot x^j$$

be the two original polynomials, and let b and c, both of length n+1, be the coefficient vector of p(x) and q(x) respectively, then the elements of the coefficient vector of the product polynomial, d of length 2n+1, can be computed as:

$$d(K) = \sum_{j=0}^{K} b(j) \cdot c(K-j) \qquad \text{for } K = 0, 1, \ldots, 2n .$$

These are exactly the components of the convolution of the two vectors b and c, if we ignore d(2n+1) which is zero.

Let B be the Fourier Transform of the vector b such that

$$B(K) = \sum_{j=0}^{n} b(j) \cdot w^{Kj} \qquad \text{for } K = 0, 1, \ldots, n \qquad (5.23)$$

where $w = e^{\frac{2\pi i}{n+1}}$ and i is the square root of -1. The elements of b can be recovered from that of B by the Inverse Fourier Transformation

$$b(j) = \frac{1}{n+1} \sum_{K=0}^{n} B(K) \cdot w^{-Kj} \qquad \text{for } j = 0, 1, \ldots, n .$$

Similarly, let C and D be the Fourier Transforms of c and d, respectively.

We observe from equation (5.23) that B(K), the K^{th} coefficient of the Fourier Transform of b, is the value of the polynomial p(x) at the point $x=w^K$. This implies that

$B(K)\cdot C(K)$, the product of the values of $p(x)$ and $q(x)$ at $x=w^K$, should be equal to $D(K)$, the value of the product polynomial at the same point. The only problem is that the product of two polynomials of degree n is a polynomial of degree 2n. This can be solved by expanding the two coefficient vectors, b and c, to length 2(n+1) and letting their last n+1 elements be zeroes.

Theorem 5.4.1 (Convolution Theorem). The convolution of two vectors is the Inverse Fourier Transform of the elementwise product of the Fourier Transforms of the two vectors.

Proof: Let

$$b = [b(0),\ldots,b(n),0,\ldots,0]'$$

and

$$c = [c(0),\ldots,c(n),0,\ldots,0]'$$

be vectors of length 2(n+1), and let

$$d = [d(0),d(1),\ldots,d(2n+1)]'$$

be the convolution of b and c such that

$$d(K) = \sum_{j=0}^{2n+1} b(j)\cdot c(K-j) \qquad \text{for } K = 0,\ 1,\ \ldots,\ 2n\ . \quad (5.24)$$

Notice that $d(2n+1)=0$ and $c(K)=0$ if $K<0$.

Let B,C and D, all vectors of length 2(n+1), be the

Fourier Transforms of b,c and d, respectively. We need to prove that

$$D(L) = B(L)\cdot C(L) \qquad \text{for } L = 0, 1, \ldots, 2n+1 .$$

Applying Fourier Transform to equation (5.24), we get

$$D(L) = \sum_{K=0}^{2n+1} \sum_{j=0}^{2n+1} b(j)\cdot c(K-j)\cdot w^{KL}$$

$$\text{for } L = 0, 1, \ldots, 2n+1 .$$

Interchanging the order of summation and substituting s for K-j yields

$$D(L) = \sum_{j=0}^{2n+1} \sum_{s=-j}^{2n+1-j} b(j)\cdot c(s)\cdot w^{L(s+j)}$$

$$\text{for } L = 0, 1, \ldots, 2n+1 .$$

Since $b(j)=0$ if $j>n$, we can lower the upper limit of j to n, and since $c(s)=0$ if $s<0$ or $s>n$ and the upper limit of s is at least n+1 regardless of how large j is, we can replace the lower and upper limits of s with 0 and n, respectively. Thus

$$D(L) = \sum_{j=0}^{n} \sum_{s=0}^{n} b(j)\cdot c(s)\cdot w^{Lj}\cdot w^{Ls}$$

$$= B(L)\cdot C(L)$$

$$\text{for } L = 0, 1, \ldots, 2n+1 .$$

We have shown that the product of the L^{th} coefficient of the Fourier Transforms of b and c is the L^{th} coefficient of the Fourier Transform of d. It follows that the convolution of two vectors is the Inverse Fourier Transform of the elementwise product of the Fourier Transforms of the two vectors.

The importance of this theorem is that when multiplying two polynomials, the coefficient vector of the product polynomial can be obtained by taking the Fourier Transforms of the coefficient vectors of the two polynomials, obtaining the elementwise product of the two transforms, and then performing the Inverse Fourier Transformation on the product vector.

The following algorithm is obtained by applying theorem 5.4.1 to the polynomial multiplications in equation (5.22). It can be used to compute a_K's when n is large.

Algorithm 2:

step 1 : Set $j=1$.

step 2 : Set $S(k)=a_{j,K}$ for each $K=0,1,\ldots,n$,

and $S(K)=0$ for each $K=n+1,\ldots,2n+1$.

step 3 : Obtain Fourier Transform of vector S in place.

step 4 : Set $j=j+1$.

step 5 : Set $C(K)=a_{j,K}$ for each $K=0,1,\ldots,n$,

and $C(K)=0$ for each $K=n+1,\ldots,2n+1$.

step 6 : Obtain Fourier Transform of vector C in place.

step 7 : Set S(K)=S(K)·C(K) for each K=0,1,...,2n+1.

step 8 : If j<m then go to step 4; otherwise go to step 9.

step 9 : Perform Inverse Fourier Transformation on vector S in place.

step 10: Set a_K=S(K) for each K=0,1,...,n.

The Fast Fourier Transform algorithm (FFT) introduced by Cooley and Tukey [1965] can be used to compute the Fourier Transform and its inverse in the above algorithm. The computing time for FFT is in the order of O(nlogn). Therefore, time efficiency can be realized when n is large.

Notice that the above algorithm is an approximation to the correct algorithm, which can be obtained by deleting step 9, and inserting the following three steps between steps 7 and 8:

step 7.1: Perform Inverse Fourier Transformation on vector S in place.

step 7.2: Set S(K)=0 for K=n+1,...,2n+1.

step 7.3: Obtain Fourier Transform of vector S in place.

What this does is to eliminate the contributions of the higher order coefficients of the product polynomials in the intermediate steps. We know that if n is large enough and the coefficients of the product polynomial is truncated at the n^{th} term, the remainder is insignificant. Therefore, the above three steps can be ignored without loss of accuracy.

6. EVALUATION OF THE TEST PROCEDURES

6.1 Evaluation Process

The purpose of this chapter is to evaluate the performances of the seven test procedures using different designs and different values of variance components. For a given two-way crossed design, a given set of values of the variance components and the null hypothesis (H_0: $\sigma_A^2=0$), the following steps will be taken to compute the probability of rejecting the null hypothesis for each of the test procedures:

1. Construct the X matrix as in equation (3.2), and the variance-covariance matrix V.

2. Construct the matrices of quadratic forms for the mean squares in table 4.1.

3. Compute the coefficients of the variance components in the expected mean squares using either equation (3.10) or equation (3.18).

4. Depending on the particular test procedure being investigated, construct the matrices A and B using linear combinations of the matrices of quadratic forms constructed in step 2 such that MSN=Y'AY and MSD=Y'BY.

5. Compute the degrees of freedom DFN and DFD for MSN and MSD, respectively.

6. For a given significance level α, compute the critical value of the approximate F statistic

$$f = F_{DFN, DFD, \alpha} \quad .$$

7. Construct the matrix $Q = A - f \cdot B$ as in equation (5.11).

8. Obtain λ_i' $(i=1,2,...,m)$, the m distinct, positive eigenvalues of the VQ, and ξ_j' $(j=1,2,...,n)$, the absolute values of the n distinct, negative eigenvalues of VQ.

9. Divide the values of λ_i' $(i=1,2,...,m)$ by their smallest value, , and assign the results to λ_i $(i=1,2,...,m)$ as in equation (5.13).

Divide the values of ξ_j' $(j=1,2,...,n)$ by their smallest value, , and assign the results to ξ_j $(j=1,2,...,n)$ as in equation (5.14). Thus, λ_i and ξ_j are all greater than or equal to unity.

10. Given K_1 and L_1, compute a_K $(K=0,1,...,K_1)$ based on the values of λ_i's and their multiplicities, and compute b_L $(L=0,1,...,L_1)$ based on the values of ξ_j's and their multiplicities using either of the two algorithms discussed in Section 5.4.

11. Compute $Pr(F' < f)$ using equation (5.18) or, equivalently, equation (5.19).

12. Compute the probability of rejecting the null hypothesis at the significance level α:

$$Pr(F' > f) = 1 - Pr(F' < f).$$

A computer program has been written to implement the computations in the above twelve steps. The probability of rejecting the null hypothesis, or the power function of the test, is a measure of the performance of this test procedure under the given conditions. It is compared against the power functions of the other test procedures under the same conditions, and appropriately ranked. Since we want the power function to be as small as possible when the null hypothesis is true and as large as possible when the null hypothesis is false, the test procedure which has the largest power function will have a rank of 7 if the null hypothesis is true or a rank of 1 if the null hypothesis is false.

The probability in step 12 was computed so that its truncation error defined in equation (5.19) is less than 10^{-5}. In a few cases, where the infinite series in equation (5.8) did not converge fast enough, the upper limit of truncation error has been relaxed to 10^{-3} without disturbing the ranks of the procedures.

6.2 Designs and Values of Variance Components Investigated

Fifteen two-way, unbalanced designs were selected for the evaluation of the seven test procedures. Their dimensions range from (3x3) to (12x3). The cell frequencies of these designs are listed in table 6.1. where the number in the i^{th} row and j^{th} column of each design is the cell frequency or the number of observations in its $(i,j)^{th}$ cell. The number in parentheses on top of the design is the total number of observations. For each of these designs, the following nine sets of values were assigned to the variance components σ_A^2, σ_{AB}^2 and σ_e^2.

σ_A^2	σ_{AB}^2	σ_e^2
0	.25	1
0	1	1
0	4	1
.25	.25	1
.25	1	1
.25	4	1
5	.25	1
5	1	1
5	4	1

Table 6.1 The Cell Frequencies of Two-Way Designs

1. (15)	2. (17)	3. (20)	4. (24)	5. (24)
1 1 1	1 1 1	1 1 1	4 1 2	1 1 1
2 2 2	1 1 4	1 3 4	1 3 1	3 3 3
2 2 2	1 1 6	1 4 4	1 2 5	2 2 2
			2 1 1	2 2 2

6. (24)	7. (24)	8. (28)	9. (32)	10. (35)
5 1	1 1 1	1 3 3	1 1 4	1 2 4
1 4	4 1 1	1 3 1	1 2 3	1 1 3
2 3	1 1 2	4 1 2	1 1 3	1 5 1
4 1	1 1 3	1 1 2	1 1 5	1 2 3
1 2	1 2 3	2 2 1	5 2 1	4 1 1
				1 1 2

11. (42)	12. (42)	13. (48)	14. (50)	15. (50)
2 1 2	4 2	1 1 2	1 1 2	3 1 1
1 1 3	3 4	1 1 3	1 4 1	1 1 2
3 1 2	3 2	1 2 1	1 2 2	1 1 1
1 1 1	3 2	1 1 1	1 3 1	1 2 4
1 5 2	4 1	1 1 4	2 1 1	1 1 1
3 1 1	3 1	1 2 1	1 2 1	2 1 1
1 2 1	2 4	3 1 1	1 2 1	1 3 1
1 1 4	3 1	2 1 1	1 1 1	1 1 1
		3 2 1	2 1 4	2 1 1
		2 1 4	1 4 3	1 1 1
				1 4 1
				1 1 1

Furthermore, for each given set of values of the variance components, the probability that the null hypothesis is rejected by each of the seven test procedures was computed at three significance levels (.10, .05, .01).

The values of the coefficients of the variance components in the expected mean squares shown in table 4.1 are listed in table 6.2. It was found in all fifteen designs that the inequality

$$K2 \geq K3 \geq K5$$

held. Therefore, the ratio of K2 to K3 was always greater than or equal to unity, and we only synthesized MSN in procedure A. On the other hand, the ratio of K5 to K3 was always less than or equal to unity, which means we only synthesized MSD in procedure C.

The values of K2, K3 and K5 are all equal when the cell frequencies within each row are the same, i.e.

$$n_{ij} = n_i \qquad \text{for } i=1,2,\ldots,a.$$

When this condition is satisfied, as in the case of designs number 1 and 5, all procedures except procedure G are identical.

We also noticed, in designs number 6 and 12, that the values of K3 and K5 are equal when there are only two columns in the design. In that case, procedures C, D, E and F are identical.

Table 6.2 Coefficients of The Expected Mean Squares

DESIGN	K1	K2	K3	K4	K5	n_h
1	4.8000	1.6000	1.6000	4.8000	1.6000	0.6667
2	5.0909	2.4242	1.3333	3.6828	1.2276	0.8241
3	6.0417	2.3262	1.8577	4.7097	1.5699	0.6759
4	5.2209	2.0303	1.5953	4.4313	1.4771	0.6903
5	5.7500	1.9167	1.9167	5.7500	1.9167	0.5833
6	4.3916	2.6903	1.7013	3.4026	1.7013	0.6033
7	4.4417	1.7247	1.3585	3.8484	1.2828	0.7944
8	5.2333	2.0093	1.6120	4.4728	1.4909	0.6833
9	5.8641	2.6201	1.6220	4.4121	1.4707	0.6878
10	5.3619	2.2472	1.5573	4.2519	1.4173	0.7148
11	4.9615	1.9963	1.4826	4.1106	1.3702	0.7479
12	5.1422	2.8981	2.2441	4.4882	2.2441	0.4792
13	4.6505	1.8619	1.3943	3.9060	1.3020	0.7833
14	4.8509	1.9289	1.4610	4.0419	1.3473	0.7639
15	4.0354	1.5544	1.2405	3.5453	1.1818	0.8657

When two or more procedures are identical, the rank assigned to each of them is the mean of the ranks that would have been assigned to them had they not been identical.

6.3 Results

The power functions of the tests, in percent, for different designs and different values of variance components are listed, along with the other statistics, in tables 9.1 through 9.15, where VA and VAB are the values of σ_A^2 and σ_{AB}^2, respectively, R is the ratio of E(MSN) to E(MSD), and P10, p05, P01 are the power functions of the test at 10, 5, and 1 percent significance levels. Notice that we have introduced a new variable R, the ratio of the expected values of MSN to that of MSD. Although it is not the expected value of F', it gives us a good indication of how large the power function of the test will be. When the null hypothesis is true, R is equal to unity in all procedures except procedures B and D. When the null hypothesis is false, R will increase as σ_A^2 increases and decrease as σ_{AB}^2 increases.

For a better view of the results, the ranks of the test procedures are shown in tables 6.3 through 6.5 where VA and VAB are the values of σ_A^2 and σ_{AB}^2, respectively. We can see that these ranks are not random patterns. While the power functions change with the designs, the ranks of the test procedures remain very stable.

Table 6.3 Ranks Of The Test Procedures. (VA=0)

VAB	DESIGN	RANK A	B	C	D	E	F	G
0.25	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	6	7	1	2	4	3	5
	3	5	7	1	2	6	3	4
	4	5	7	1	2	6	3	4
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	6	7	2.5	2.5	2.5	2.5	5
	7	5	7	2	1	6	3	4
	8	5	7	2	1	6	3	4
	9	5	7	2	1	6	3	4
	10	5	7	2	1	6	3	4
	11	5	7	2	1	6	3	4
	12	6	7	3.5	3.5	3.5	3.5	1
	13	5	7	2	1	6	3	4
	14	5	7	2	1	6	3	4
	15	5	7	3	1	6	2	4
1.00	1	4.5	4.5	4.5	4.5	4.5	4.5	1
	2	1	7	3	2	6	4	5
	3	5	7	2	1	6	4	3
	4	2	7	3	1	6	4	5
	5	4.5	4.5	4.5	4.5	4.5	4.5	1
	6	2	7	4.5	4.5	4.5	4.5	1
	7	2	7	4	1	6	3	5
	8	2	7	4	1	6	3	5
	9	2	7	4	1	6	3	5
	10	2	7	4	1	6	3	5
	11	2	7	5	1	6	3	4
	12	1	7	4.5	4.5	4.5	4.5	2
	13	2	7	5	1	6	3	4
	14	3	7	5	1	6	2	4
	15	3	7	5	1	6	2	4
4.00	1	4.5	4.5	4.5	4.5	4.5	4.5	1
	2	1	7	5	2	6	3	4
	3	3	7	4	1	6	5	2
	4	2	7	5	1	6	4	3
	5	4.5	4.5	4.5	4.5	4.5	4.5	1
	6	1	7	4.5	4.5	4.5	4.5	2
	7	1	7	5	2	6	4	3
	8	2	7	5	1	6	4	3
	9	1	7	5	2	6	3	4
	10	2	7	5	1	6	3	4
	11	2	7	5	1	6	4	3
	12	1	7	4.5	4.5	4.5	4.5	2
	13	2	7	5	1	6	3	4
	14	2	7	5	1	6	4	3
	15	2	7	5	1	6	4	3

Table 6.4 Ranks Of The Test Procedures. (VA=.25)

VAB	DESIGN	RANK A	B	C	D	E	F	G
0.25	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	5	6	2	4	3
	3	5	1	3	7	2	4	6
	4	6	1	3	7	2	4	5
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	6	1	3	7	2	4	5
	8	6	1	3	7	2	4	5
	9	7	1	3	6	2	5	4
	10	6	1	3	7	2	5	4
	11	7	1	3	6	2	4	5
	12	7	1	3.5	3.5	3.5	3.5	6
	13	7	1	3	5	2	4	5
	14	6	1	3	7	2	4	5
	15	7	1	3	6	2	4	5
1.00	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	5	6	2	4	3
	3	5	1	4	7	2	3	6
	4	6	1	3	7	2	5	4
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	6	1	3	7	2	5	4
	8	6	1	3	7	2	5	4
	9	7	1	3	6	2	5	4
	10	6	1	3	7	2	5	4
	11	6	1	3	7	2	5	4
	12	7	1	3.5	3.5	3.5	3.5	6
	13	7	1	3	6	2	5	4
	14	6	1	3	7	2	4	5
	15	7	1	3	6	2	4	5
4.00	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	3	6	2	5	4
	3	5	1	3	7	2	4	6
	4	6	1	3	7	2	4	5
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	7	1	3	6	2	4	5
	8	6	1	3	7	2	4	5
	9	7	1	3	6	2	5	4
	10	7	1	3	6	2	5	4
	11	6	1	3	7	2	5	4
	12	7	1	3.5	3.5	3.5	3.5	6
	13	6	1	3	7	2	5	4
	14	6	1	3	7	2	4	5
	15	6	1	3	7	2	4	5

Table 6.5 Ranks Of The Test Procedures. (VA=5)

VAB	DESIGN	RANK A	B	C	D	E	F	G
0.25	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	2	6	3	4	5
	3	6	1	2	7	3	4	5
	4	6	1	3	7	2	4	5
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	7	1	3	6	2	4	5
	8	7	1	3	6	2	4	5
	9	7	1	3	6	2	4	5
	10	7	1	3	6	2	4	5
	11	7	1	3	6	2	4	5
	12	7	1	3.5	3.5	3.5	3.5	6
	13	7	1	3	6	2	4	5
	14	7	1	3	6	2	4	5
	15	7	1	3	5	2	4	6
1.00	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	2	6	3	5	4
	3	6	1	2	7	3	4	5
	4	7	1	3	6	2	5	4
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	7	1	3	6	2	5	4
	8	7	1	3	6	2	5	4
	9	7	1	3	6	2	4	5
	10	7	1	3	6	2	5	4
	11	7	1	3	6	2	5	4
	12	7	1	3.5	3.5	3.5	3.5	6
	13	7	1	3	6	2	5	4
	14	7	1	3	6	2	5	4
	15	7	1	3	6	2	5	4
4.00	1	3.5	3.5	3.5	3.5	3.5	3.5	7
	2	7	1	2	6	3	5	4
	3	6	1	3	7	2	4	5
	4	7	1	3	6	2	5	4
	5	3.5	3.5	3.5	3.5	3.5	3.5	7
	6	7	1	3.5	3.5	3.5	3.5	6
	7	7	1	3	6	2	5	4
	8	7	1	3	6	2	5	4
	9	7	1	3	6	2	5	4
	10	7	1	3	6	2	5	4
	11	7	1	3	6	2	5	4
	12	7	1	3.5	3.5	3.5	3.5	6
	13	7	1	3	6	2	5	4
	14	7	1	3	6	2	5	4
	15	7	1	3	6	2	5	4

These ranks show that the performance of procedure B is the best when the null hypothesis is false and the worst when the null hypothesis is true. On the contrary, the performance of procedure D is just the opposite. This is expected because, when the null hypothesis is true, the value of R is always greater than or equal to unity in procedure B , and it is always less than or equal to unity in procedure D. A procedure with a large R value tends to reject the null hypothesis more often than it should. This can be verified by observing that the power function of procedure B can sometimes be twice as large as the significance level even when the null hypothesis is true. A procedure with a small R value tends not to reject the null hypothesis as often as it should. This explains why the performance of procedure D is so good when the null hypothesis is true and so poor when it is false.

The performance of procedure A is the worst of the seven procedures when the null hypothesis is false. Its R value is generally smaller than that of the other procedures, and it is less likely to reject the null hypothesis. When the null hypothesis is true, however, its performance seems to improve as the ratio of σ^2_{AB} to σ^2_e is increased. This is believed to be caused by the change of DFN, which decreases as the ratio of σ^2_{AB} to σ^2_e is increased. A smaller DFN results in a larger critical value for the approximate F statistic, and makes it less likely for procedure A to reject the null hypothesis. This effect is reflected in the ranks of procedure A.

Procedure E seems to perform very well when the null hypothesis is false. Unfortunately, it does poorly when the null hypothesis is true. To find out the reason, we compare its performance with that of procedure F since both of them have the same approximate F statistics. The difference is in the critical values for these approximate statistics. In procedure F, the DFN is the same as the degrees of freedom for A effect. In procedure E, the value of DFN is larger, and is computed using the Satterthwaite's formula. This results in a smaller critical value for the approximate F statistic, and makes it more likely for procedure E to reject the null hypothesis.

From the above discussions, it is believed that procedures A, B, D and E produce poor results, and they are not suitable for testing the null hypothesis.

Of the remaining three procedures, when the null hypothesis is false, procedure C generally has a larger power function than the other two. Although the differences are mostly less than one percent probability. When the null hypothesis is true, procedure C performs well when the ratio of σ^2_{AB} to σ^2_e is small, but procedures F and G are better when that ratio is large. Again, the differences in power functions are mostly less than one percent probability.

Unlike procedure A, procedure C synthesizes the MSD using linear combination of MSAB and MSE. Therefore, the computed DFD is independent of σ_A^2, and remains the same whether the null hypothesis is true or not.

It is worth mentioning that the distribution of the approximate F statistic obtained in procedure G is amazingly closed to an F distribution. The probabilities that this statistic is greater than the deciles of the corresponding F distribution have been computed for a few of the designs, and they are all within one percent probability difference from the actual probabilties.

7. SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

7.1 Summary

We have shown seven different procedures to construct an alternative test of variance component when the exact test is unavailable as in the case of an unbalanced design. Before choosing the right procedure, it is essential to understand the distribution of the approximate F statistic computed in the procedure. This research has provided an algorithm to compute the distribution function of the approximate F statistic and the power function of the test, i.e., the probability that the approximate F statistic is greater than the critical value of the test when the values of the variance components are given. It can be used to judge the performances of the test procedures.

The results clearly show that procedures A, B, D and E are inferior to the other three procedures. When the null hypothesis is false, procedure A produces an R value which is generally smaller than that of the other procedures, and it is the least powerful procedure. Procedures B and D are equivalent to the conventional F test. They are the only two procedures in which the expected values of MSN and MSD are not equal. Our results indicate that this inequality cannot be ignored, and both procedures should be abandoned.

Procedure E synthesizes MSN using a linear combination of type II and type III mean squares for factor A. The corresponding value of DFN, computed from Satterthwaite's formula, is larger than the degrees of freedom for factor A. A larger value of DFN reduces the critical value of the approximate F statistic, and worsens the type I error committed by this procedure. Therefore, these four procedures should not be used to test the variance components.

Although the performances of the remaining three procedures are all acceptable, procedure C appears to be the best with respect to the overall performance. Since it synthesizes the denominator, its degrees of freedom are independent of the value of the variance component being tested. When the null hypothesis is false, its chance of rejecting the null hypothesis is greater than that of procedures F and G for the majority of the designs and variance component combinations studied. Therefore, procedure C is recommended over procedures F and G.

Procedure F synthesizes the numerator using the linear combination of MSA from both type II and type III analyses with DFD equal to the degrees of freedom for factor A. Its performance is close to that of procedure C and slightly better than that of procedure G.

Procedure G, the test from the unweighted mean analysis, is easy to compute. There is no need to compute the degrees of freedom, and the approximate F statistic follows an F distribution very closely.

In addition to the comparison of the procedures, the results also indicate that, while the power function of each procedure can be affected by changing the size of the design, the relative performances of the procedures appear to be unaffected.

7.2 Suggestions for Future Research

The idea of obtaining approximate F statistics by constructing quadratic forms which are linear combinations of mean squares can be extended to three-way or multi-way designs. Our results have shown that it is better to leave the numerator alone and construct the denominator using a linear combination of mean squares associated with the type III estimable functions.

For the two-way design, since procedure F performs very well and since the inequality K2 $\geq$ K3 $\geq$ K5 holds in all fifteen designs as shown in Table 6.2, it is suggested that there might be a linear combination of type II and type III estimable functions (L's) such that the corresponding mean square has the same expected value as that of MSAB when the null hypothesis is true, and that the ratio of these two mean squares may prove to be a better approximate F statistic in testing the null hypothesis than any procedure examined here.

Most of the estimates of variance components are based on equating quadratic forms to their expected values and solving the system of linear equations. These estimates are naturally quadratic forms. The work presented in Chapter 5 may be modified to give the distribution of these estimates when the true values of the variance components are known.

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9. APPENDIX

Table 9.1 Power Functions In Percent For Design No. 1.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		B	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		C	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		D	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		E	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		F	2.00	4.00	1.000	10.025	5.019	1.006	3.5
		G	2.00	4.00	1.000	10.162	5.122	1.038	7
	1.00	A	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		B	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		C	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		D	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		E	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		F	2.00	4.00	1.000	10.116	5.087	1.027	4.5
		G	2.00	4.00	1.000	10.049	5.037	1.011	1
	4.00	A	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		B	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		C	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		D	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		E	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		F	2.00	4.00	1.000	10.230	5.174	1.054	4.5
		G	2.00	4.00	1.000	10.006	5.005	1.001	1
0.25	0.25	A	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		B	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		C	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		D	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		E	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		F	2.00	4.00	1.857	21.328	12.166	2.950	3.5
		G	2.00	4.00	1.818	21.049	11.981	2.897	7
	1.00	A	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		B	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		C	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		D	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		E	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		F	2.00	4.00	1.462	16.369	8.891	2.003	3.5
		G	2.00	4.00	1.450	16.166	8.723	1.942	7
	4.00	A	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		B	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		C	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		D	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		E	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		F	2.00	4.00	1.162	12.455	6.486	1.377	3.5
		G	2.00	4.00	1.161	12.207	6.282	1.307	7
5.00	0.25	A	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		B	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		C	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		D	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		E	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		F	2.00	4.00	18.143	79.416	69.955	44.272	3.5
		G	2.00	4.00	17.363	79.110	69.511	43.545	7
	1.00	A	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		B	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		C	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		D	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		E	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		F	2.00	4.00	10.231	67.680	55.315	28.221	3.5
		G	2.00	4.00	10.000	67.624	55.133	27.763	7
	4.00	A	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		B	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		C	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		D	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		E	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		F	2.00	4.00	4.243	43.687	30.275	10.498	3.5
		G	2.00	4.00	4.214	43.685	30.368	10.177	7

Table 9.2 Power Functions In Percent For Design No. 2.

VA	VAB	EBCC	DFN	DFC	R	P10	P05	P01	RANK
0.00	0.25	A	4.28	4.00	1.000	9.653	4.809	0.958	6
		B	2.00	4.00	1.205	12.522	6.437	1.333	7
		C	2.00	4.52	1.000	8.689	3.917	0.532	1
		D	2.00	4.00	0.980	8.915	4.292	0.812	2
		E	2.47	4.00	1.000	9.374	4.644	0.916	4
		F	2.00	4.00	1.000	9.153	4.438	0.949	3
		G	2.00	4.00	1.000	9.601	4.669	0.902	5
	1.00	A	3.03	4.00	1.000	8.988	4.373	0.841	1
		B	2.00	4.00	1.468	15.199	7.925	1.656	7
		C	2.00	4.30	1.000	9.585	4.574	0.730	3
		D	2.00	4.00	0.955	9.298	4.588	0.901	2
		E	2.58	4.00	1.000	9.964	5.070	1.041	6
		F	2.00	4.00	1.000	9.692	4.808	0.952	4
		G	2.00	4.00	1.000	9.881	4.909	0.971	5
	4.00	A	2.31	4.00	1.000	8.273	3.697	0.715	1
		B	2.00	4.00	1.689	17.154	8.950	1.847	7
		C	2.00	4.11	1.000	10.189	5.055	0.934	5
		D	2.00	4.00	0.933	9.405	4.687	0.935	2
		E	2.68	4.00	1.000	10.226	5.263	1.100	6
		F	2.00	4.00	1.000	9.912	4.959	0.995	3
		G	2.00	4.00	1.000	9.985	4.988	0.996	4
0.25	0.25	A	3.23	4.00	1.525	17.302	9.355	2.093	7
		B	2.00	4.00	2.159	24.671	14.532	3.707	1
		C	2.00	4.52	1.704	19.208	10.398	2.067	5
		D	2.00	4.00	1.671	18.520	10.092	2.260	6
		E	2.49	4.00	1.714	19.397	10.864	2.548	2
		F	2.00	4.00	1.714	19.015	10.435	2.368	4
		G	2.00	4.00	1.698	19.289	10.654	2.434	3
	1.00	A	2.74	4.00	1.300	13.428	6.926	1.441	7
		B	2.00	4.00	2.013	22.380	12.734	3.039	1
		C	2.00	4.30	1.413	15.563	6.174	1.581	5
		D	2.00	4.00	1.349	14.769	7.821	1.692	6
		E	2.57	4.00	1.408	15.714	8.570	1.943	2
		F	2.00	4.00	1.408	15.343	8.175	1.788	4
		G	2.00	4.00	1.411	15.540	8.289	1.810	3
	4.00	A	2.28	4.00	1.111	9.937	4.813	0.914	7
		B	2.00	4.00	1.890	19.940	10.808	2.359	1
		C	2.00	4.11	1.156	12.398	6.354	1.242	3
		D	2.00	4.00	1.079	11.431	5.855	1.212	6
		E	2.66	4.00	1.150	12.374	6.541	1.420	2
		F	2.00	4.00	1.150	12.019	6.184	1.289	5
		G	2.00	4.00	1.155	12.117	6.224	1.291	4
5.00	0.25	A	2.21	4.00	11.500	71.519	59.242	31.457	7
		B	2.00	4.00	20.296	80.898	71.976	47.140	1
		C	2.00	4.52	15.090	77.303	68.338	43.601	2
		D	2.00	4.00	14.791	76.089	65.555	38.611	6
		E	2.52	4.00	15.277	77.091	67.108	40.841	3
		F	2.00	4.00	15.277	76.744	66.346	39.515	4
		G	2.00	4.00	14.965	76.424	66.008	39.225	5
	1.00	A	2.11	4.00	7.000	58.527	44.669	19.229	7
		B	2.00	4.00	12.377	71.592	60.078	33.064	1
		C	2.00	4.30	9.266	66.721	54.456	27.679	2
		D	2.00	4.00	8.847	64.596	51.647	24.747	6
		E	2.53	4.00	9.158	65.961	53.522	26.720	3
		F	2.00	4.00	9.158	65.497	52.616	25.555	5
		G	2.00	4.00	9.223	65.672	52.846	25.692	4
	4.00	A	2.09	4.00	3.211	35.387	22.791	6.795	7
		B	2.00	4.00	5.708	51.660	38.015	14.760	1
		C	2.00	4.11	4.116	43.834	30.216	10.383	2
		D	2.00	4.00	3.841	41.288	28.030	9.284	6
		E	2.56	4.00	4.006	42.999	29.365	10.358	3
		F	2.00	4.00	4.006	42.421	28.382	9.734	5
		G	2.00	4.00	4.109	42.943	29.388	9.830	4

Table 9.3 Power Functions In Percent For Design No. 3.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	2.68	4.00	1.000	9.934	4.993	1.009	5
		B	2.00	4.00	1.080	11.161	5.679	1.164	7
		C	2.00	5.04	1.000	8.827	3.918	0.481	1
		D	2.00	4.00	0.951	8.864	4.282	0.813	2
		E	3.88	4.00	1.000	10.284	5.375	1.146	6
		F	2.00	4.00	1.000	9.602	4.741	0.932	3
		G	2.00	4.00	1.000	9.809	4.882	0.973	4
	1.00	A	2.31	4.00	1.000	10.192	5.143	1.042	5
		B	2.00	4.00	1.164	12.522	6.503	1.367	7
		C	2.00	4.52	1.000	9.869	4.684	0.697	2
		D	2.00	4.00	0.899	9.001	4.452	0.878	1
		E	3.95	4.00	1.000	10.898	5.814	1.283	6
		F	2.00	4.00	1.000	10.181	5.132	1.042	4
		G	2.00	4.00	1.000	9.937	4.957	0.988	3
	4.00	A	2.10	4.00	1.000	10.443	5.276	1.065	3
		B	2.00	4.00	1.222	13.565	7.140	1.520	7
		C	2.00	4.18	1.000	10.778	5.414	0.988	4
		D	2.00	4.00	0.863	9.074	4.553	0.920	1
		E	3.98	4.00	1.000	11.345	6.127	1.378	6
		F	2.00	4.00	1.000	10.607	5.412	1.118	5
		G	2.00	4.00	1.000	9.992	4.994	0.998	2
0.25	0.25	A	2.34	4.00	1.824	21.135	12.042	2.928	5
		B	2.00	4.00	2.111	24.337	14.381	3.707	1
		C	2.00	5.04	1.846	21.851	12.336	2.621	3
		D	2.00	4.00	1.755	19.923	11.097	2.571	7
		E	3.91	4.00	1.891	22.724	13.615	3.590	2
		F	2.00	4.00	1.891	21.546	12.270	2.969	4
		G	2.00	4.00	1.810	20.397	11.824	2.832	6
	1.00	A	2.21	4.00	1.422	16.152	8.797	1.991	5
		B	2.00	4.00	1.692	19.623	11.112	2.669	1
		C	2.00	4.52	1.458	16.719	8.938	1.726	4
		D	2.00	4.00	1.311	14.802	7.910	1.739	7
		E	3.95	4.00	1.456	17.539	10.097	2.506	2
		F	2.00	4.00	1.456	16.537	9.027	2.057	3
		G	2.00	4.00	1.448	16.078	8.547	1.914	6
	4.00	A	2.09	4.00	1.143	12.535	6.534	1.377	5
		B	2.00	4.00	1.401	16.689	8.758	1.964	1
		C	2.00	4.18	1.162	13.152	6.653	1.343	3
		D	2.00	4.00	1.003	11.082	5.719	1.202	7
		E	3.98	4.00	1.155	13.682	7.604	1.787	2
		F	2.00	4.00	1.155	12.838	6.750	1.457	4
		G	2.00	4.00	1.160	12.191	6.270	1.303	6
5.00	0.25	A	2.03	4.00	17.474	78.595	68.732	42.842	6
		B	2.00	4.00	21.708	81.681	73.079	48.861	1
		C	2.00	5.04	17.911	81.379	73.488	52.024	2
		D	2.00	4.00	17.031	78.394	68.617	42.591	7
		E	3.92	4.00	18.811	80.803	72.437	48.639	3
		F	2.00	4.00	18.811	79.959	70.637	45.230	4
		G	2.00	4.00	17.200	79.607	69.792	43.470	5
	1.00	A	2.03	4.00	9.442	65.491	52.907	26.547	6
		B	2.00	4.00	11.735	70.200	58.579	32.038	1
		C	2.00	4.52	10.163	69.331	57.942	32.196	2
		D	2.00	4.00	9.140	65.252	52.596	25.988	7
		E	3.95	4.00	10.127	68.715	57.470	31.517	3
		F	2.00	4.00	10.127	67.523	55.206	28.428	4
		G	2.00	4.00	9.950	67.542	55.041	27.694	5
	4.00	A	2.02	4.00	3.861	41.335	28.529	10.282	6
		B	2.00	4.00	4.805	47.393	34.162	13.070	1
		C	2.00	4.18	4.235	45.071	31.885	11.605	3
		D	2.00	4.00	3.657	40.404	27.536	9.308	7
		E	3.96	4.00	4.094	44.615	32.400	12.448	2
		F	2.00	4.00	4.094	43.330	30.209	10.711	4
		G	2.00	4.00	4.208	43.647	30.033	10.158	5

Table 9.4 Power Functions In Percent For Design No. 4.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	4.15	6.00	1.000	9.814	4.916	0.989	5
		B	3.00	6.00	1.078	11.299	5.747	1.180	7
		C	3.00	6.70	1.000	9.229	4.346	0.706	1
		D	3.00	6.00	0.979	9.111	4.409	0.831	2
		E	4.65	6.00	1.000	10.320	5.343	1.132	6
		F	3.00	6.00	1.000	9.453	4.625	0.892	3
		G	3.00	6.00	1.000	9.705	4.771	0.924	4
	1.00	A	3.56	6.00	1.000	9.665	4.786	0.941	2
		B	3.00	6.00	1.168	12.692	6.536	1.362	7
		C	3.00	6.37	1.000	9.868	4.850	0.885	3
		D	3.00	6.00	0.954	9.250	4.565	0.895	1
		E	4.80	6.00	1.000	10.768	5.686	1.251	6
		F	3.00	6.00	1.000	9.832	4.695	0.975	4
		G	3.00	6.00	1.000	9.912	4.931	0.977	5
	4.00	A	3.18	6.00	1.000	9.491	4.631	0.888	2
		B	3.00	6.00	1.236	13.694	7.088	1.481	7
		C	3.00	6.13	1.000	10.297	5.197	1.037	5
		D	3.00	6.00	0.936	9.263	4.615	0.922	1
		E	4.91	6.00	1.000	11.013	5.871	1.316	6
		F	3.00	6.00	1.000	10.027	5.031	1.016	4
		G	3.00	6.00	1.000	9.989	4.991	0.997	3
0.25	0.25	A	4.15	6.00	1.733	23.982	14.243	3.815	6
		B	3.00	6.00	2.011	27.767	16.987	4.786	1
		C	3.00	6.70	1.809	24.738	14.600	3.717	3
		D	3.00	6.00	1.771	23.523	13.717	3.517	7
		E	4.69	6.00	1.822	25.947	16.026	4.613	2
		F	3.00	6.00	1.822	24.348	14.328	3.750	4
		G	3.00	6.00	1.798	24.251	14.287	3.740	5
	1.00	A	3.56	6.00	1.395	17.206	9.462	2.207	6
		B	3.00	6.00	1.671	21.909	12.624	3.181	1
		C	3.00	6.37	1.447	18.253	10.123	2.323	3
		D	3.00	6.00	1.381	16.964	9.312	2.167	7
		E	4.78	6.00	1.443	19.280	11.262	2.946	2
		F	3.00	6.00	1.443	17.890	9.911	2.348	5
		G	3.00	6.00	1.444	17.941	9.927	2.336	4
	4.00	A	3.18	6.00	1.139	12.089	6.158	1.259	6
		B	3.00	6.00	1.413	17.039	9.227	2.078	1
		C	3.00	6.13	1.160	13.201	6.952	1.494	3
		D	3.00	6.00	1.085	11.926	6.186	1.317	7
		E	4.88	6.00	1.156	14.005	7.760	1.856	2
		F	3.00	6.00	1.156	12.842	6.713	1.448	4
		G	3.00	6.00	1.160	12.827	6.678	1.423	5
5.00	0.25	A	4.15	6.00	15.663	89.440	83.528	64.976	6
		B	3.00	6.00	19.739	91.359	86.280	69.583	1
		C	3.00	6.70	17.181	90.444	85.172	68.621	3
		D	3.00	6.00	16.818	89.495	83.524	64.724	7
		E	4.74	6.00	17.442	90.668	85.589	69.069	2
		F	3.00	6.00	17.442	90.032	84.263	65.905	4
		G	3.00	6.00	16.952	89.691	83.812	65.227	5
	1.00	A	3.56	6.00	8.903	78.511	68.577	43.543	7
		B	3.00	6.00	11.226	82.750	74.098	50.528	1
		C	3.00	6.37	9.945	80.967	71.867	48.064	3
		D	3.00	6.00	9.492	79.226	69.423	44.422	6
		E	4.75	6.00	9.862	81.321	72.730	49.600	2
		F	3.00	6.00	9.862	80.192	70.614	45.825	5
		G	3.00	6.00	9.674	80.258	70.725	45.899	4
	4.00	A	3.18	6.00	3.779	51.132	37.517	15.361	7
		B	3.00	6.00	4.772	59.141	45.711	21.085	1
		C	3.00	6.13	4.207	55.130	41.576	18.201	3
		D	3.00	6.00	3.938	52.329	38.768	16.270	6
		E	4.78	6.00	4.116	55.860	43.087	19.966	2
		F	3.00	6.00	4.116	53.972	40.329	17.256	5
		G	3.00	6.00	4.198	54.529	40.615	17.400	4

Table 9.5 Power Functions In Percent For Design No. 5.

VA	VAB	ERCC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		B	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		C	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		D	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		E	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		F	3.00	6.00	1.000	10.061	5.047	1.016	3.5
		G	3.00	6.00	1.000	10.346	5.272	1.095	7
	1.00	A	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		B	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		C	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		D	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		E	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		F	3.00	6.00	1.000	10.254	5.198	1.068	4.5
		G	3.00	6.00	1.000	10.098	5.076	1.026	1
	4.00	A	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		B	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		C	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		D	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		E	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		F	3.00	6.00	1.000	10.467	5.366	1.127	4.5
		G	3.00	6.00	1.000	10.012	5.009	1.003	1
0.25	0.25	A	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		B	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		C	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		D	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		E	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		F	3.00	6.00	1.972	27.108	16.569	4.682	3.5
		G	3.00	6.00	1.900	26.291	15.960	4.456	7
	1.00	A	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		B	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		C	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		D	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		E	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		F	3.00	6.00	1.493	19.077	10.832	2.691	3.5
		G	3.00	6.00	1.474	18.607	10.418	2.510	7
	4.00	A	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		B	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		C	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		D	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		E	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		F	3.00	6.00	1.166	13.419	7.172	1.613	3.5
		G	3.00	6.00	1.164	12.915	6.736	1.441	7
5.00	0.25	A	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		B	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		C	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		D	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		E	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		F	3.00	6.00	20.437	91.427	86.416	70.008	3.5
		G	3.00	6.00	19.001	91.066	85.845	68.807	7
	1.00	A	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		B	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		C	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		D	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		E	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		F	3.00	6.00	10.857	81.524	72.558	48.812	3.5
		G	3.00	6.00	10.474	81.524	72.410	48.109	7
	4.00	A	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		B	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		C	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		D	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		E	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		F	3.00	6.00	4.317	54.951	41.630	18.545	3.5
		G	3.00	6.00	4.273	55.166	41.471	17.860	7

Table 9.6 Power Functions In Percent For Design No. 6.

VA	VAB	EBCC	DFN	DFC	R	P10	P05	P01	RANK
0.00	0.25	A	7.02	4.00	1.000	9.797	4.884	0.974	6
		B	4.00	4.00	1.173	12.424	6.342	1.301	7
		C	4.00	4.00	1.000	9.065	4.397	0.842	2.5
		D	4.00	4.00	1.000	9.065	4.397	0.842	2.5
		E	4.00	4.00	1.000	9.065	4.397	0.842	2.5
		F	4.00	4.00	1.000	9.065	4.397	0.842	2.5
		G	4.00	4.00	1.000	9.749	4.835	0.956	5
	1.00	A	5.32	4.00	1.000	9.332	4.579	0.893	2
		B	4.00	4.00	1.366	14.776	7.609	1.570	7
		C	4.00	4.00	1.000	9.806	4.873	0.966	4.5
		D	4.00	4.00	1.000	9.806	4.873	0.966	4.5
		E	4.00	4.00	1.000	9.806	4.873	0.966	4.5
		F	4.00	4.00	1.000	9.806	4.873	0.966	4.5
		G	4.00	4.00	1.000	9.096	4.416	0.847	1
	4.00	A	4.40	4.00	1.000	8.815	4.239	0.805	1
		B	4.00	4.00	1.507	16.295	8.370	1.708	7
		C	4.00	4.00	1.000	10.094	5.063	1.017	4.5
		D	4.00	4.00	1.000	10.094	5.063	1.017	4.5
		E	4.00	4.00	1.000	10.094	5.063	1.017	4.5
		F	4.00	4.00	1.000	10.094	5.063	1.017	4.5
		G	4.00	4.00	1.000	9.969	4.980	0.995	2
0.25	0.25	A	5.78	4.00	1.487	17.561	9.359	2.036	7
		B	4.00	4.00	1.944	24.186	13.678	3.235	1
		C	4.00	4.00	1.597	18.427	9.818	2.124	3.5
		D	4.00	4.00	1.597	18.427	9.818	2.124	3.5
		E	4.00	4.00	1.597	18.427	9.818	2.124	3.5
		F	4.00	4.00	1.597	18.427	9.818	2.124	3.5
		G	4.00	4.00	1.586	18.333	9.775	2.145	6
	1.00	A	5.01	4.00	1.257	13.395	6.644	1.404	7
		B	4.00	4.00	1.773	21.167	11.559	2.585	1
		C	4.00	4.00	1.315	14.638	7.634	1.612	3.5
		D	4.00	4.00	1.315	14.638	7.634	1.612	3.5
		E	4.00	4.00	1.315	14.638	7.634	1.612	3.5
		F	4.00	4.00	1.315	14.638	7.634	1.612	3.5
		G	4.00	4.00	1.312	14.533	7.563	1.598	6
	4.00	A	4.37	4.00	1.089	10.206	4.987	0.962	7
		B	4.00	4.00	1.648	18.582	9.764	2.053	1
		C	4.00	4.00	1.109	11.742	5.990	1.230	3.5
		D	4.00	4.00	1.109	11.742	5.990	1.230	3.5
		E	4.00	4.00	1.109	11.742	5.990	1.230	3.5
		F	4.00	4.00	1.109	11.742	5.990	1.230	3.5
		G	4.00	4.00	1.109	11.603	5.894	1.203	6
5.00	0.25	A	4.21	4.00	10.742	81.872	69.112	35.697	7
		B	4.00	4.00	16.579	89.507	80.784	51.135	1
		C	4.00	4.00	12.936	85.290	74.320	42.008	3.5
		D	4.00	4.00	12.936	85.290	74.320	42.008	3.5
		E	4.00	4.00	12.936	85.290	74.320	42.008	3.5
		F	4.00	4.00	12.936	85.290	74.320	42.008	3.5
		G	4.00	4.00	12.719	85.117	74.040	41.577	6
	1.00	A	4.18	4.00	6.140	65.072	48.758	18.896	7
		B	4.00	4.00	9.495	77.924	64.244	31.209	1
		C	4.00	4.00	7.298	70.288	54.872	23.349	3.5
		D	4.00	4.00	7.298	70.288	54.872	23.349	3.5
		E	4.00	4.00	7.298	70.288	54.872	23.349	3.5
		F	4.00	4.00	7.298	70.288	54.872	23.349	3.5
		G	4.00	4.00	7.237	70.269	54.694	23.083	6
	4.00	A	4.14	4.00	2.779	33.590	21.680	5.717	7
		B	4.00	4.00	4.320	51.752	35.373	11.324	1
		C	4.00	4.00	3.180	40.541	25.911	7.453	3.5
		D	4.00	4.00	3.180	40.541	25.911	7.453	3.5
		E	4.00	4.00	3.180	40.541	25.911	7.453	3.5
		F	4.00	4.00	3.180	40.541	25.911	7.453	3.5
		G	4.00	4.00	3.172	40.417	25.716	7.318	6

Table 9.7 Power Functions In Percent For Design No. 7.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	5.56	8.00	1.000	9.842	4.935	0.996	5
		B	4.00	8.00	1.068	11.352	5.787	1.194	7
		C	4.00	8.71	1.000	9.368	4.484	0.786	2
		D	4.00	8.00	0.986	9.212	4.465	0.840	1
		E	5.70	8.00	1.000	10.390	5.368	1.139	6
		F	4.00	8.00	1.000	9.464	4.625	0.885	3
		G	4.00	8.00	1.000	9.783	4.826	0.938	4
	1.00	A	4.81	8.00	1.000	9.650	4.778	0.939	2
		B	4.00	8.00	1.155	12.962	6.717	1.418	7
		C	4.00	8.40	1.000	9.900	4.891	0.930	4
		D	4.00	8.00	0.968	9.333	4.607	0.901	1
		E	5.86	8.00	1.000	10.300	5.694	1.258	6
		F	4.00	8.00	1.000	9.804	4.874	0.967	3
		G	4.00	8.00	1.000	9.931	4.945	0.980	5
	4.00	A	4.28	8.00	1.000	9.401	4.577	0.869	1
		B	4.00	8.00	1.228	14.234	7.437	1.582	7
		C	4.00	8.15	1.000	10.270	5.179	1.043	5
		D	4.00	8.00	0.953	9.343	4.655	0.930	2
		E	5.99	8.00	1.000	11.052	5.885	1.329	6
		F	4.00	8.00	1.000	9.990	5.005	1.011	4
		G	4.00	8.00	1.000	9.991	4.993	0.997	3
0.25	0.25	A	5.56	8.00	1.653	25.420	15.391	4.349	6
		B	4.00	8.00	1.897	29.441	16.376	5.480	1
		C	4.00	8.71	1.728	26.150	15.744	4.301	3
		D	4.00	8.00	1.704	25.147	14.979	4.053	7
		E	5.75	8.00	1.737	27.549	17.295	5.240	2
		F	4.00	8.00	1.737	25.807	15.474	4.254	4
		G	4.00	8.00	1.718	25.723	15.465	4.272	5
	1.00	A	4.81	8.00	1.371	18.218	10.184	2.471	6
		B	4.00	8.00	1.626	23.525	13.849	3.680	1
		C	4.00	8.40	1.421	19.330	10.919	2.664	3
		D	4.00	8.00	1.376	18.194	10.165	2.464	7
		E	5.84	8.00	1.419	20.512	12.151	3.314	2
		F	4.00	8.00	1.419	18.979	10.682	2.628	5
		G	4.00	8.00	1.418	19.060	10.731	2.632	4
	4.00	A	4.28	8.00	1.136	12.404	6.356	1.315	7
		B	4.00	8.00	1.400	18.196	10.027	2.345	1
		C	4.00	8.15	1.157	13.643	7.240	1.597	3
		D	4.00	8.00	1.102	12.473	6.527	1.416	6
		E	5.96	8.00	1.153	14.549	8.115	1.990	2
		F	4.00	8.00	1.153	13.269	6.991	1.536	4
		G	4.00	8.00	1.156	13.296	6.984	1.518	5
5.00	0.25	A	5.56	8.00	14.058	93.892	89.941	76.410	7
		B	4.00	8.00	17.646	95.211	91.956	80.276	1
		C	4.00	8.71	15.570	94.507	90.963	78.789	3
		D	4.00	8.00	15.350	94.030	90.091	76.473	6
		E	5.80	8.00	15.743	94.752	91.437	79.761	2
		F	4.00	8.00	15.743	94.321	90.516	77.248	4
		G	4.00	8.00	15.362	94.069	90.184	76.664	5
	1.00	A	4.81	8.00	8.417	85.330	77.743	56.023	7
		B	4.00	8.00	10.572	89.009	82.593	63.263	1
		C	4.00	8.40	9.429	87.417	80.431	60.251	3
		D	4.00	8.00	9.126	85.396	78.866	57.451	6
		E	5.81	8.00	9.374	87.898	81.388	62.163	2
		F	4.00	8.00	9.374	87.009	79.672	58.566	5
		G	4.00	8.00	9.359	87.021	79.715	58.614	4
	4.00	A	4.28	8.00	3.719	58.712	45.321	21.312	7
		B	4.00	8.00	4.679	67.239	54.662	29.097	1
		C	4.00	8.15	4.138	62.805	49.815	25.080	3
		D	4.00	8.00	3.944	60.641	47.412	23.022	6
		E	5.85	8.00	4.070	63.865	51.606	27.327	2
		F	4.00	8.00	4.070	61.977	48.764	24.031	5
		G	4.00	8.00	4.129	62.516	49.301	24.298	4

Table 9.8 Power Functions In Percent For Design No. 8.

VA	VAB	ESCC	DFN	DFC	R	P 10	P05	P01	RANK
0.00	0.25	A	5.38	8.00	1.000	9.861	4.946	0.998	5
		B	4.00	8.00	1.071	11.420	5.829	1.205	7
		C	4.00	8.93	1.000	9.315	4.430	0.756	2
		D	4.00	8.00	0.978	9.045	4.364	0.814	1
		E	6.40	8.00	1.000	10.648	5.585	1.219	6
		F	4.00	8.00	1.000	9.444	4.614	0.884	3
		G	4.00	8.00	1.000	9.699	4.765	0.919	4
	1.00	A	4.67	8.00	1.000	9.748	4.837	0.953	2
		B	4.00	8.00	1.152	13.000	6.747	1.429	7
		C	4.00	8.50	1.000	9.943	4.910	0.927	4
		D	4.00	8.00	0.954	9.109	4.483	0.873	1
		E	6.59	8.00	1.000	11.114	5.498	1.354	6
		F	4.00	8.00	1.000	9.823	4.888	0.971	3
		G	4.00	8.00	1.000	9.910	4.929	0.975	5
	4.00	A	4.22	8.00	1.000	9.612	4.707	0.901	2
		B	4.00	8.00	1.213	14.169	7.418	1.585	7
		C	4.00	8.17	1.000	10.357	5.236	1.058	5
		D	4.00	8.00	0.935	9.072	4.507	0.897	1
		E	6.74	8.00	1.000	11.378	6.152	1.430	6
		F	4.00	8.00	1.000	10.028	5.032	1.018	4
		G	4.00	8.00	1.000	9.989	4.991	0.997	3
0.25	0.25	A	4.72	8.00	1.748	26.895	16.333	4.625	6
		B	4.00	8.00	2.003	31.645	20.123	6.223	1
		C	4.00	8.93	1.815	28.249	17.368	4.939	3
		D	4.00	8.00	1.775	26.714	16.150	4.494	7
		E	6.46	8.00	1.829	30.113	19.463	6.267	2
		F	4.00	8.00	1.829	27.785	16.968	4.834	4
		G	4.00	8.00	1.804	27.563	16.841	4.797	5
	1.00	A	4.47	8.00	1.402	18.810	10.542	2.567	6
		B	4.00	8.00	1.653	24.236	14.379	3.879	1
		C	4.00	8.50	1.449	20.066	11.437	2.832	3
		D	4.00	8.00	1.382	18.406	10.320	2.519	7
		E	6.57	8.00	1.445	21.619	13.066	3.712	2
		F	4.00	8.00	1.445	19.613	11.121	2.776	5
		G	4.00	8.00	1.446	19.665	11.141	2.764	4
	4.00	A	4.19	8.00	1.141	12.706	6.550	1.368	6
		B	4.00	8.00	1.389	18.200	10.055	2.365	1
		C	4.00	8.17	1.161	13.827	7.366	1.635	3
		D	4.00	8.00	1.085	12.203	6.372	1.379	7
		E	6.71	8.00	1.156	15.013	8.506	2.150	2
		F	4.00	8.00	1.156	13.380	7.062	1.557	4
		G	4.00	8.00	1.160	13.375	7.032	1.531	5
5.00	0.25	A	4.07	8.00	15.963	94.707	91.014	77.990	7
		B	4.00	8.00	19.721	96.066	93.321	83.152	1
		C	4.00	8.93	17.292	95.486	92.526	82.108	3
		D	4.00	8.00	16.919	94.930	91.504	79.305	6
		E	6.52	8.00	17.573	95.764	93.059	83.215	2
		F	4.00	8.00	17.573	95.303	92.061	80.374	4
		G	4.00	8.00	17.072	95.051	91.696	79.703	5
	1.00	A	4.07	8.00	9.037	86.528	78.925	57.342	7
		B	4.00	8.00	11.170	89.940	83.957	65.492	1
		C	4.00	8.50	9.978	88.529	82.044	62.899	3
		D	4.00	8.00	9.516	87.210	80.018	59.212	6
		E	6.54	8.00	9.902	89.114	83.213	65.301	2
		F	4.00	8.00	9.902	88.062	81.162	60.858	5
		G	4.00	8.00	9.911	88.097	81.244	60.971	4
	4.00	A	4.06	8.00	3.819	59.602	46.192	21.918	7
		B	4.00	8.00	4.727	67.717	55.230	29.663	1
		C	4.00	8.17	4.212	63.573	50.703	25.902	3
		D	4.00	8.00	3.938	60.625	47.425	23.084	6
		E	6.58	8.00	4.122	64.943	53.010	28.346	2
		F	4.00	8.00	4.122	62.303	49.413	24.594	5
		G	4.00	8.00	4.203	63.223	50.033	24.945	4

Table 9.9 Power Functions In Percent For Design No. 9.

VA	VAB	PROC	DFN	DFD	B	P10	P05	P01	RANK
0.00	0.25	A	7.29	8.00	1.000	9.682	4.833	0.968	5
		B	4.00	8.00	1.178	13.536	7.085	1.524	7
		C	4.00	9.19	1.000	8.652	4.088	0.648	2
		D	4.00	8.00	0.973	8.519	4.014	0.716	1
		E	5.42	8.00	1.000	9.746	4.892	0.982	6
		F	4.00	8.00	1.000	8.964	4.282	0.786	3
		G	4.00	8.00	1.000	9.433	4.540	0.835	4
	1.00	A	5.44	8.00	1.000	9.132	4.415	0.828	2
		B	4.00	8.00	1.381	17.403	9.377	2.086	7
		C	4.00	8.63	1.000	9.797	4.796	0.883	4
		D	4.00	8.00	0.942	8.767	4.275	0.819	1
		E	5.69	8.00	1.000	10.492	5.463	1.181	6
		F	4.00	8.00	1.000	9.572	4.719	0.923	3
		G	4.00	8.00	1.000	9.838	4.870	0.953	5
	4.00	A	4.44	8.00	1.000	8.553	3.972	0.637	1
		B	4.00	8.00	1.533	20.247	11.033	2.456	7
		C	4.00	8.22	1.000	10.329	5.212	1.046	5
		D	4.00	8.00	0.919	8.740	4.315	0.849	2
		E	5.90	8.00	1.000	10.815	5.711	1.271	6
		F	4.00	8.00	1.000	9.798	4.875	0.974	3
		G	4.00	8.00	1.000	9.980	4.984	0.994	4
0.25	0.25	A	5.68	8.00	1.646	25.076	14.855	4.007	7
		B	4.00	8.00	2.221	35.896	23.555	7.732	1
		C	4.00	9.19	1.806	28.055	17.124	4.763	3
		D	4.00	8.00	1.758	26.128	15.609	4.224	6
		E	5.48	8.00	1.819	28.928	18.188	5.501	2
		F	4.00	8.00	1.819	27.341	16.512	4.592	5
		G	4.00	8.00	1.800	27.451	16.639	4.672	4
	1.00	A	5.01	8.00	1.346	17.207	9.338	2.130	7
		B	4.00	8.00	1.940	30.051	18.577	5.353	1
		C	4.00	8.63	1.446	19.963	11.332	2.771	3
		D	4.00	8.00	1.363	17.697	9.952	2.390	6
		E	5.65	8.00	1.439	20.697	12.203	3.297	2
		F	4.00	8.00	1.439	19.270	10.838	2.662	5
		G	4.00	8.00	1.444	19.606	11.081	2.731	4
	4.00	A	4.39	8.00	1.121	11.281	5.524	1.042	7
		B	4.00	8.00	1.729	24.928	14.380	3.563	1
		C	4.00	8.22	1.160	13.801	7.341	1.621	3
		D	4.00	8.00	1.066	11.789	6.115	1.309	6
		E	5.85	8.00	1.154	14.325	7.928	1.916	2
		F	4.00	8.00	1.154	13.103	6.864	1.493	5
		G	4.00	8.00	1.160	13.364	7.023	1.527	4
5.00	0.25	A	4.16	8.00	13.914	93.808	89.428	74.425	7
		B	4.00	8.00	22.039	96.810	94.526	85.791	1
		C	4.00	9.19	17.130	95.530	92.624	82.449	3
		D	4.00	8.00	16.669	94.836	91.351	78.968	6
		E	5.54	8.00	17.376	95.610	92.724	82.240	2
		F	4.00	8.00	17.376	95.283	92.111	80.207	4
		G	4.00	8.00	16.995	95.014	91.639	79.602	5
	1.00	A	4.15	8.00	7.923	84.294	75.640	52.239	7
		B	4.00	8.00	12.563	91.791	86.664	70.048	1
		C	4.00	8.00	9.929	88.600	82.172	63.198	3
		D	4.00	8.00	9.356	86.938	79.623	58.574	6
		E	5.56	8.00	9.773	88.725	82.460	63.639	2
		F	4.00	8.00	9.773	87.974	80.987	60.467	4
		G	4.00	8.00	9.887	88.058	81.190	60.890	5
	4.00	A	4.12	8.00	3.424	55.492	41.627	18.156	7
		B	4.00	8.00	5.449	73.119	61.431	35.343	1
		C	4.00	8.22	4.203	63.644	50.777	25.964	3
		D	4.00	8.00	3.865	59.927	46.658	22.448	6
		E	5.63	8.00	4.074	63.974	51.576	27.127	2
		F	4.00	8.00	4.074	62.251	48.975	24.120	5
		G	4.00	8.00	4.200	63.199	50.051	24.920	4

Table 9.10 Power Functions In Percent For Design No. 10.

VA	VAB	SPCC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	8.05	10.00	1.000	9.783	4.900	0.980	5
		B	5.00	10.00	1.124	12.829	6.675	1.427	7
		C	5.00	11.44	1.000	9.066	4.252	0.707	2
		D	5.00	10.00	0.975	8.642	4.095	0.734	1
		E	7.22	10.00	1.000	10.220	5.242	1.098	6
		F	5.00	10.00	1.000	9.120	4.384	0.811	3
		G	5.00	13.00	1.000	9.552	4.642	0.870	4
	1.00	A	6.42	10.00	1.000	9.422	4.612	0.855	2
		B	5.00	10.00	1.270	16.087	8.627	1.928	7
		C	5.00	10.78	1.000	9.384	4.870	0.919	4
		D	5.00	10.00	0.945	8.741	4.261	0.815	1
		E	7.54	10.00	1.000	10.884	5.761	1.292	6
		F	5.00	10.00	1.000	9.643	4.764	0.935	3
		G	5.00	10.00	1.000	9.865	4.892	0.961	5
	4.00	A	5.45	10.00	1.000	9.018	4.290	0.774	2
		B	5.00	10.00	1.382	13.576	10.118	2.296	7
		C	5.00	10.27	1.000	10.386	5.259	1.070	5
		D	5.00	10.00	0.923	8.659	4.270	0.839	1
		E	7.78	10.00	1.000	11.218	6.023	1.392	6
		F	5.00	10.00	1.000	9.877	4.932	0.990	3
		G	5.00	10.00	1.000	9.982	4.986	0.995	4
0.25	0.25	A	8.05	10.00	1.669	29.194	16.339	5.635	6
		B	5.00	10.00	2.089	37.228	24.780	8.502	1
		C	5.00	11.44	1.785	30.597	19.211	5.764	3
		D	5.00	10.00	1.740	28.519	17.505	5.049	7
		E	7.30	10.00	1.799	32.171	21.005	6.969	2
		F	5.00	10.00	1.799	29.918	18.579	5.509	5
		G	5.00	10.00	1.777	29.806	18.587	5.552	4
	1.00	A	6.42	10.00	1.363	19.410	10.961	2.737	6
		B	5.00	10.00	1.794	29.989	18.694	5.576	1
		C	5.00	10.78	1.440	21.414	12.409	3.205	3
		D	5.00	10.00	1.361	19.106	10.808	2.699	7
		E	7.50	10.00	1.434	22.705	13.804	4.010	2
		F	5.00	10.00	1.434	20.713	11.878	3.049	5
		G	5.00	10.00	1.437	20.961	12.061	3.100	4
	4.00	A	5.45	10.00	1.128	12.313	6.228	1.251	7
		B	5.00	10.00	1.567	23.710	13.736	3.508	1
		C	5.00	10.27	1.159	14.327	7.702	1.749	3
		D	5.00	10.00	1.070	12.098	6.317	1.373	6
		E	7.73	10.00	1.154	15.287	8.657	2.207	2
		F	5.00	10.00	1.154	13.641	7.219	1.606	5
		G	5.00	10.00	1.159	13.824	7.322	1.621	4
5.00	0.25	A	8.05	10.00	14.373	97.298	95.256	87.374	7
		B	5.00	10.00	20.421	98.346	97.003	91.458	1
		C	5.00	11.44	16.697	97.741	96.056	89.590	3
		D	5.00	10.00	16.277	97.344	95.291	87.273	6
		E	7.39	10.00	16.976	97.869	96.292	90.040	2
		F	5.00	10.00	16.976	97.624	95.736	88.237	4
		G	5.00	10.00	16.547	97.442	95.460	87.687	5
	1.00	A	6.42	10.00	8.265	90.862	85.189	67.645	7
		B	5.00	10.00	11.753	94.752	91.074	78.246	1
		C	5.00	10.78	9.795	92.932	88.363	73.556	3
		D	5.00	10.00	9.259	91.717	86.397	69.519	6
		E	7.42	10.00	9.679	93.219	88.961	74.903	2
		F	5.00	10.00	9.679	92.522	87.537	71.374	5
		G	5.00	10.00	9.747	92.566	87.660	71.675	4
	4.00	A	5.45	10.00	3.570	64.121	51.102	26.195	7
		B	5.00	10.00	5.090	77.410	66.743	41.640	1
		C	5.00	10.27	4.188	70.420	58.422	33.235	3
		D	5.00	10.00	3.863	66.654	54.049	29.016	6
		E	7.50	10.00	4.070	71.151	59.906	35.579	2
		F	5.00	10.00	4.070	69.052	56.616	31.100	5
		G	5.00	10.00	4.181	69.976	57.704	32.094	4

Table 9.11 Power Functions In Percent For Design No. 11.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	10.39	14.00	1.000	9.871	4.958	1.006	5
		B	7.00	14.00	1.094	12.609	6.517	1.410	7
		C	7.00	15.68	1.000	9.301	4.443	0.783	2
		D	7.00	14.00	0.979	8.827	4.219	0.766	1
		E	10.23	14.00	1.000	10.642	5.552	1.209	6
		F	7.00	14.00	1.000	9.283	4.490	0.839	3
		G	7.00	14.00	1.000	9.739	4.792	0.924	4
	1.00	A	8.67	14.00	1.000	9.656	4.777	0.935	2
		B	7.00	14.00	1.207	15.834	8.551	1.955	7
		C	7.00	14.93	1.000	9.980	4.961	0.966	5
		D	7.00	14.00	0.955	8.813	4.309	0.827	1
		E	10.60	14.00	1.000	11.227	6.018	1.395	6
		F	7.00	14.00	1.000	9.731	4.823	0.951	3
		G	7.00	14.00	1.000	9.918	4.935	0.976	4
	4.00	A	7.55	14.00	1.000	9.401	4.562	0.852	2
		B	7.00	14.00	1.296	18.444	10.195	2.413	7
		C	7.00	14.33	1.000	10.443	5.305	1.101	5
		D	7.00	14.00	0.935	8.731	4.303	0.850	1
		E	10.89	14.00	1.000	11.576	6.298	1.510	6
		F	7.00	14.00	1.000	9.977	5.005	1.015	4
		G	7.00	14.00	1.000	9.989	4.991	0.997	3
0.25	0.25	A	8.83	14.00	1.672	32.797	21.022	6.786	7
		B	7.00	14.00	1.999	41.488	28.584	10.718	1
		C	7.00	15.68	1.765	35.159	23.030	7.698	3
		D	7.00	14.00	1.729	33.092	21.231	6.799	6
		E	10.33	14.00	1.778	37.422	25.599	9.542	2
		F	7.00	14.00	1.778	34.571	22.415	7.358	4
		G	7.00	14.00	1.752	34.082	22.131	7.295	5
	1.00	A	8.17	14.00	1.371	21.781	12.662	3.358	6
		B	7.00	14.00	1.707	32.432	20.802	6.692	1
		C	7.00	14.93	1.434	23.953	14.328	4.012	3
		D	7.00	14.00	1.369	21.555	12.574	3.366	7
		E	10.56	14.00	1.429	25.832	16.283	5.170	2
		F	7.00	14.00	1.429	23.302	13.782	3.794	5
		G	7.00	14.00	1.429	23.411	13.867	3.810	4
	4.00	A	7.48	14.00	1.133	13.441	7.000	1.492	6
		B	7.00	14.00	1.475	24.559	14.560	3.961	1
		C	7.00	14.33	1.159	15.172	8.288	1.966	3
		D	7.00	14.00	1.083	12.879	6.826	1.534	7
		E	10.83	14.00	1.154	16.582	9.617	2.604	2
		F	7.00	14.00	1.154	14.551	7.830	1.808	5
		G	7.00	14.00	1.158	14.633	7.852	1.788	4
5.00	0.25	A	7.19	14.00	14.442	99.196	98.381	94.615	7
		B	7.00	14.00	19.193	99.590	99.180	97.198	1
		C	7.00	15.68	16.309	99.419	98.881	96.439	3
		D	7.00	14.00	15.975	99.313	98.651	95.591	6
		E	10.44	14.00	16.552	99.465	99.018	96.863	2
		F	7.00	14.00	16.552	99.399	98.805	96.008	4
		G	7.00	14.00	16.032	99.330	98.682	95.681	5
	1.00	A	7.18	14.00	8.421	98.102	92.969	81.568	7
		B	7.00	14.00	11.199	98.024	96.325	89.449	1
		C	7.00	14.93	9.671	97.232	95.072	86.735	3
		D	7.00	14.00	9.234	96.741	94.123	84.299	6
		E	10.47	14.00	9.586	97.493	95.557	88.119	2
		F	7.00	14.00	9.586	97.120	94.725	85.541	5
		G	7.00	14.00	9.582	97.128	94.762	85.660	4
	4.00	A	7.14	14.00	3.658	74.830	63.416	38.229	7
		B	7.00	14.00	4.876	85.176	76.829	54.677	1
		C	7.00	14.33	4.171	80.125	70.221	46.360	3
		D	7.00	14.00	3.901	77.186	66.457	41.881	6
		E	10.56	14.00	4.076	81.148	72.149	49.869	2
		F	7.00	14.00	4.076	79.177	68.820	44.381	5
		G	7.00	14.00	4.159	79.998	69.851	45.405	4

Table 9.12 Power Functions In Percent For Design No. 12.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	9.49	7.00	1.000	9.886	4.937	0.987	6
		B	7.00	7.00	1.105	12.190	6.248	1.297	7
		C	7.00	7.00	1.000	9.473	4.639	0.893	3.5
		D	7.00	7.00	1.000	9.473	4.639	0.893	3.5
		E	7.00	7.00	1.000	9.473	4.639	0.893	3.5
		F	7.00	7.00	1.000	9.473	4.639	0.893	3.5
		G	7.00	7.00	1.000	9.401	4.589	0.879	1
	1.00	A	8.07	7.00	1.000	9.705	4.794	0.938	1
		B	7.00	7.00	1.202	14.186	7.381	1.362	7
		C	7.00	7.00	1.000	10.119	5.084	1.026	4.5
		D	7.00	7.00	1.000	10.119	5.084	1.026	4.5
		E	7.00	7.00	1.000	10.119	5.084	1.026	4.5
		F	7.00	7.00	1.000	10.119	5.084	1.026	4.5
		G	7.00	7.00	1.000	9.851	4.897	0.969	2
	4.00	A	7.33	7.00	1.000	9.525	4.656	0.905	1
		B	7.00	7.00	1.262	15.427	8.072	1.715	7
		C	7.00	7.00	1.000	10.424	5.301	1.094	4.5
		D	7.00	7.00	1.000	10.424	5.301	1.094	4.5
		E	7.00	7.00	1.000	10.424	5.301	1.094	4.5
		F	7.00	7.00	1.000	10.424	5.301	1.094	4.5
		G	7.00	7.00	1.000	9.983	4.989	0.967	2
0.25	0.25	A	8.40	7.00	1.638	25.433	14.782	3.778	7
		B	7.00	7.00	1.928	31.975	19.642	5.533	1
		C	7.00	7.00	1.719	26.685	15.662	4.060	3.5
		D	7.00	7.00	1.719	26.685	15.662	4.060	3.5
		E	7.00	7.00	1.719	26.685	15.662	4.060	3.5
		F	7.00	7.00	1.719	26.685	15.662	4.060	3.5
		G	7.00	7.00	1.686	25.924	15.070	3.833	6
	1.00	A	7.60	7.00	1.307	16.915	9.089	2.033	7
		B	7.00	7.00	1.598	23.928	13.717	3.410	1
		C	7.00	7.00	1.346	18.090	9.951	2.326	3.5
		D	7.00	7.00	1.346	18.090	9.951	2.326	3.5
		E	7.00	7.00	1.346	18.090	9.951	2.326	3.5
		F	7.00	7.00	1.346	18.090	9.951	2.326	3.5
		G	7.00	7.00	1.338	17.609	9.564	2.179	6
	4.00	A	7.30	7.00	1.100	11.763	5.938	1.231	7
		B	7.00	7.00	1.391	18.631	10.095	2.270	1
		C	7.00	7.00	1.112	12.912	6.771	1.468	3.5
		D	7.00	7.00	1.112	12.912	6.771	1.468	3.5
		E	7.00	7.00	1.112	12.912	6.771	1.468	3.5
		F	7.00	7.00	1.112	12.912	6.771	1.468	3.5
		G	7.00	7.00	1.112	12.419	6.398	1.341	6
5.00	0.25	A	7.15	7.00	13.754	97.521	94.602	80.541	7
		B	7.00	7.00	17.575	98.626	96.900	87.465	1
		C	7.00	7.00	15.376	97.959	95.553	83.519	3.5
		D	7.00	7.00	15.376	97.959	95.553	83.519	3.5
		E	7.00	7.00	15.376	97.959	95.553	83.519	3.5
		F	7.00	7.00	15.376	97.959	95.553	83.519	3.5
		G	7.00	7.00	14.714	97.839	95.297	82.664	6
	1.00	A	7.14	7.00	7.137	88.204	79.004	51.250	7
		B	7.00	7.00	9.127	92.379	86.423	63.240	1
		C	7.00	7.00	7.917	89.996	81.901	56.015	3.5
		D	7.00	7.00	7.917	89.996	81.901	56.015	3.5
		E	7.00	7.00	7.917	89.996	81.901	56.015	3.5
		F	7.00	7.00	7.917	89.996	81.901	56.015	3.5
		G	7.00	7.00	7.760	90.027	81.797	55.330	6
	4.00	A	7.11	7.00	2.996	53.297	37.852	14.220	7
		B	7.00	7.00	3.839	66.040	50.913	22.553	1
		C	7.00	7.00	3.249	57.574	42.383	17.167	3.5
		D	7.00	7.00	3.249	57.574	42.383	17.167	3.5
		E	7.00	7.00	3.249	57.574	42.383	17.167	3.5
		F	7.00	7.00	3.249	57.574	42.383	17.167	3.5
		G	7.00	7.00	3.233	57.546	41.999	16.510	6

Table 9.13 Power Functions In Percent For Design No. 13.

VA	VAB	ERCC	DFN	DFC	R	P10	P05	P01	RANK
0.00	0.25	A	13.24	18.00	1.000	9.897	4.792	1.010	5
		B	9.00	9.00	1.087	12.780	6.675	1.447	7
		C	9.00	19.89	1.000	9.358	4.494	0.807	2
		D	9.00	18.00	0.983	8.886	4.255	0.774	1
		E	12.75	18.00	1.000	10.696	5.582	1.220	6
		F	9.00	18.00	1.000	9.309	4.504	0.839	3
		G	9.00	18.00	1.000	9.761	4.809	0.930	4
	1.00	A	11.16	18.00	1.000	9.676	4.795	0.942	2
		B	9.00	18.00	1.195	16.412	8.951	2.093	7
		C	9.00	19.06	1.000	9.968	4.957	0.974	5
		D	9.00	18.00	0.961	8.826	4.312	0.826	1
		E	13.16	18.00	1.000	11.243	6.023	1.399	6
		F	9.00	18.00	1.000	9.719	4.812	0.946	3
		G	9.00	18.00	1.000	9.921	4.937	0.976	4
	4.00	A	9.72	18.00	1.000	9.403	4.571	0.857	2
		B	9.00	18.00	1.284	19.492	10.940	2.681	7
		C	9.00	18.39	1.000	10.386	5.279	1.094	5
		D	9.00	18.00	0.944	8.684	4.284	0.843	1
		E	13.51	18.00	1.000	11.579	6.294	1.513	6
		F	9.00	18.00	1.000	9.949	4.983	1.007	3
		G	9.00	18.00	1.000	9.989	4.991	0.997	4
0.25	0.25	A	11.35	18.00	1.646	35.999	23.676	8.113	7
		B	9.00	18.00	1.949	45.392	32.133	12.368	1
		C	9.00	19.89	1.737	38.526	25.890	9.214	3
		D	9.00	18.00	1.707	36.534	24.104	8.238	6
		E	12.86	18.00	1.747	40.974	28.714	11.361	2
		F	9.00	18.00	1.747	37.998	25.309	8.942	4
		G	9.00	18.00	1.726	37.475	24.994	8.774	5
	1.00	A	10.53	18.00	1.364	23.608	14.002	3.887	7
		B	9.00	18.00	1.681	35.536	23.391	8.002	1
		C	9.00	19.06	1.424	25.976	15.840	4.642	3
		D	9.00	18.00	1.369	23.534	14.013	3.924	6
		E	13.12	18.00	1.421	23.008	17.976	5.966	2
		F	9.00	18.00	1.421	25.325	15.280	4.396	5
		G	9.00	18.00	1.421	25.451	15.392	4.432	4
	4.00	A	9.63	18.00	1.132	14.021	7.380	1.609	6
		B	9.00	18.00	1.461	26.636	16.148	4.627	1
		C	9.00	18.39	1.157	15.823	8.718	2.108	3
		D	9.00	18.00	1.092	13.487	7.206	1.647	7
		E	13.44	18.00	1.153	17.331	10.132	2.804	2
		F	9.00	18.00	1.153	15.200	8.250	1.940	5
		G	9.00	18.00	1.157	15.324	8.306	1.934	4
5.00	0.25	A	9.25	18.00	13.912	99.767	99.484	97.960	7
		B	9.00	18.00	16.329	99.897	99.776	99.096	1
		C	9.00	19.89	15.734	99.844	99.669	98.753	3
		D	9.00	18.00	15.465	99.812	99.597	98.445	6
		E	12.99	18.00	15.937	99.865	99.719	98.952	2
		F	9.00	18.00	15.937	99.838	99.648	98.611	4
		G	9.00	18.00	15.517	99.817	99.606	98.478	5
	1.00	A	9.23	18.00	8.273	98.333	98.730	96.003	7
		B	9.00	18.00	10.907	99.279	98.542	95.115	1
		C	9.00	19.06	9.484	98.928	97.896	93.376	3
		D	9.00	18.00	9.118	98.698	97.449	92.045	6
		E	13.02	18.00	9.413	99.036	98.152	94.284	2
		F	9.00	18.00	9.413	98.868	97.746	92.778	5
		G	9.00	18.00	9.411	98.869	97.750	92.850	4
	4.00	A	9.19	18.00	3.647	82.041	72.400	48.676	7
		B	9.00	18.00	4.820	90.730	84.562	66.126	1
		C	9.00	18.39	4.146	85.612	76.720	57.435	3
		D	9.00	18.00	3.913	84.423	75.685	53.144	6
		E	13.13	18.00	4.063	87.499	80.443	60.992	2
		F	9.00	18.00	4.063	85.970	77.673	55.623	5
		G	9.00	18.00	4.136	86.617	78.576	56.768	4

Table 9.14 Power Functions In Percent For Design No. 14.

VA	VAB	PROC	DFN	DFD	R	P10	P05	P01	RANK
0.00	0.25	A	13.04	18.00	1.000	9.925	4.994	1.018	5
		B	9.00	18.00	1.086	12.776	6.679	1.451	7
		C	9.00	20.22	1.000	9.369	4.496	0.804	2
		D	9.00	18.00	0.979	8.792	4.203	0.762	1
		E	13.25	18.00	1.000	10.927	5.768	1.293	6
		F	9.00	18.00	1.000	9.318	4.513	0.843	3
		G	9.00	18.00	1.000	9.789	4.833	0.939	4
	1.00	A	11.02	18.00	1.000	9.816	4.986	0.976	3
		B	9.00	18.00	1.190	16.366	8.957	2.116	7
		C	9.00	19.23	1.000	10.049	5.013	0.988	5
		D	9.00	18.00	0.954	8.664	4.241	0.812	1
		E	14.00	18.00	1.000	11.537	6.262	1.498	6
		F	9.00	18.00	1.000	9.780	4.857	0.962	2
		G	9.00	18.00	1.000	9.931	4.945	0.980	4
	4.00	A	9.67	18.00	1.000	9.679	4.772	0.923	2
		B	9.00	18.00	1.273	19.362	10.935	2.730	7
		C	9.00	18.44	1.000	10.543	5.396	1.135	5
		D	9.00	18.00	0.934	8.532	4.216	0.834	1
		E	14.37	18.00	1.000	11.944	6.595	1.641	6
		F	9.00	18.00	1.000	10.075	5.076	1.041	4
		G	9.00	18.00	1.000	9.991	4.992	0.997	3
0.25	0.25	A	11.20	18.00	1.673	37.012	24.568	8.593	6
		B	9.00	18.00	1.974	46.217	32.934	13.400	1
		C	9.00	20.22	1.756	39.419	26.702	9.680	3
		D	9.00	18.00	1.719	37.019	24.534	8.470	7
		E	13.66	18.00	1.769	42.261	30.027	12.268	2
		F	9.00	18.00	1.769	38.845	26.052	8.247	4
		G	9.00	18.00	1.740	38.020	25.473	9.035	5
	1.00	A	10.42	18.00	1.373	24.091	14.414	4.083	6
		B	9.00	18.00	1.683	35.673	23.579	8.165	1
		C	9.00	19.23	1.430	26.360	16.165	4.801	3
		D	9.00	18.00	1.364	23.440	13.974	3.930	7
		E	13.95	18.00	1.427	28.720	18.668	6.382	2
		F	9.00	18.00	1.427	25.635	15.540	4.523	4
		G	9.00	18.00	1.425	25.643	15.539	4.494	5
	4.00	A	9.59	18.00	1.134	14.375	7.665	1.723	6
		B	9.00	18.00	1.451	26.458	16.120	4.694	1
		C	9.00	18.00	1.158	16.032	8.892	2.183	3
		D	9.00	18.00	1.081	13.275	7.101	1.631	7
		E	14.30	18.00	1.153	17.794	10.552	3.019	2
		F	9.00	18.00	1.153	15.357	8.362	1.999	4
		G	9.00	18.00	1.157	15.348	8.322	1.939	5
5.00	0.25	A	9.22	18.00	14.456	99.790	99.535	98.151	7
		B	9.00	18.00	16.851	99.905	99.792	99.159	1
		C	9.00	20.22	16.118	99.858	99.699	98.867	3
		D	9.00	18.00	15.782	99.823	99.620	98.530	6
		E	13.82	18.00	16.382	99.880	99.751	99.074	2
		F	9.00	18.00	16.382	99.852	99.677	98.718	4
		G	9.00	18.00	15.794	99.828	99.630	98.564	5
	1.00	A	9.21	18.00	8.465	98.402	96.867	90.408	7
		B	9.00	18.00	11.046	99.288	98.562	95.193	1
		C	9.00	19.23	9.610	98.967	97.973	93.622	3
		D	9.00	18.00	9.166	98.699	97.453	92.078	6
		E	13.85	18.00	9.533	99.083	98.251	94.614	2
		F	9.00	18.00	9.533	98.897	97.802	92.950	5
		G	9.00	18.00	9.504	98.906	97.830	93.042	4
	4.00	A	9.18	18.00	3.684	82.192	72.694	49.290	7
		B	9.00	18.00	4.817	90.532	84.324	65.928	1
		C	9.00	18.44	4.163	86.655	78.834	57.758	3
		D	9.00	18.00	3.866	84.053	75.241	52.705	6
		E	13.97	18.00	4.068	87.641	80.765	61.782	2
		F	9.00	18.00	4.063	85.961	77.625	55.598	5
		G	9.00	18.00	4.149	86.719	78.718	56.971	4

Table 9.15 Power Functions In Percent For Design No. 15.

VA	VAB	ERCC	DFH	DFE	R	P10	P05	P01	RANK
0.00	0.25	A	14.98	22.00	1.000	9.950	5.002	1.018	5
		B	11.00	22.00	1.060	12.135	6.288	1.345	7
		C	11.00	23.65	1.000	9.559	4.652	0.868	3
		D	11.00	22.00	0.989	9.202	4.464	0.834	1
		E	15.24	22.00	1.000	10.912	5.740	1.280	6
		F	11.00	22.00	1.000	9.508	4.644	0.882	2
		G	11.00	22.00	1.000	9.914	4.931	0.974	4
	1.00	A	13.19	22.00	1.000	9.844	4.920	0.986	3
		B	11.00	22.00	1.140	15.143	8.179	1.891	7
		C	11.00	22.97	1.000	10.000	4.996	0.992	5
		D	11.00	22.00	0.974	9.135	4.495	0.873	1
		E	15.60	22.00	1.000	11.330	6.080	1.421	6
		F	11.00	22.00	1.000	9.820	4.882	0.967	2
		G	11.00	22.00	1.000	9.970	4.976	0.991	4
	4.00	A	11.77	22.00	1.000	9.699	4.803	0.939	2
		B	11.00	22.00	1.211	17.893	9.974	2.442	7
		C	11.00	22.37	1.000	10.366	5.269	1.094	5
		D	11.00	22.00	0.961	9.033	4.485	0.893	1
		E	15.91	22.00	1.000	11.640	6.334	1.532	6
		F	11.00	22.00	1.000	10.041	5.050	1.029	4
		G	11.00	22.00	1.000	9.995	4.996	0.999	3
0.25	0.25	A	13.33	22.00	1.615	37.557	25.391	9.066	7
		B	11.00	22.00	1.830	45.497	32.254	13.034	1
		C	11.00	23.65	1.684	40.009	27.204	9.990	3
		D	11.00	22.00	1.665	38.545	25.571	9.229	6
		E	15.34	22.00	1.691	42.630	30.246	12.371	2
		F	11.00	22.00	1.691	39.644	26.791	9.709	4
		G	11.00	22.00	1.672	39.005	26.353	9.571	5
	1.00	A	12.57	22.00	1.359	25.324	15.344	4.477	7
		B	11.00	22.00	1.590	35.177	26.180	8.019	1
		C	11.00	22.97	1.406	27.250	16.827	5.094	3
		D	11.00	22.00	1.369	25.380	15.398	4.495	6
		E	15.56	22.00	1.404	29.501	19.179	6.569	2
		F	11.00	22.00	1.404	26.789	16.418	4.895	4
		G	11.00	22.00	1.402	26.775	16.414	4.881	5
	4.00	A	11.68	22.00	1.135	14.939	8.054	1.856	6
		B	11.00	22.00	1.380	25.477	15.425	4.464	1
		C	11.00	22.37	1.155	16.349	9.073	2.232	3
		D	11.00	22.00	1.109	14.483	7.849	1.846	7
		E	15.66	22.00	1.152	18.010	10.616	2.999	2
		F	11.00	22.00	1.152	15.875	8.714	2.101	4
		G	11.00	22.00	1.154	15.881	8.677	2.056	5
5.00	0.25	A	11.26	22.00	13.291	99.920	99.810	99.142	7
		B	11.00	22.00	16.461	99.962	99.911	99.595	1
		C	11.00	23.65	14.684	99.943	99.872	99.459	3
		D	11.00	22.00	14.519	99.937	99.855	99.360	5
		E	15.47	22.00	14.825	99.954	99.898	99.572	2
		F	11.00	22.00	14.825	99.945	99.870	99.418	4
		G	11.00	22.00	14.444	99.937	99.855	99.359	6
	1.00	A	11.25	22.00	8.187	99.263	98.451	94.607	7
		B	11.00	22.00	10.146	99.659	99.267	97.256	1
		C	11.00	22.97	9.125	99.499	98.969	96.398	3
		D	11.00	22.00	8.866	99.440	98.822	95.815	6
		E	15.49	22.00	9.084	99.582	99.145	97.029	2
		F	11.00	22.00	9.084	99.502	98.939	96.149	5
		G	11.00	22.00	9.040	99.501	98.942	96.167	4
	4.00	A	11.20	22.00	3.701	87.556	79.855	58.309	7
		B	11.00	22.00	4.595	93.262	88.353	72.570	1
		C	11.00	22.37	4.095	90.745	84.517	66.062	3
		D	11.00	22.00	3.934	89.459	82.611	62.932	6
		E	15.59	22.00	4.038	91.513	86.059	69.629	2
		F	11.00	22.00	4.038	90.571	83.867	64.787	5
		G	11.00	22.00	4.083	90.820	84.513	65.669	4