

ABSTRACT

SENGUL ORGUT, IREM. Modeling for the Equitable and Effective Distribution of Food Donations under Capacity Constraints. (Under the direction of Dr. Julie S. Ivy and Dr. Reha Uzsoy.)

Food insecurity is an increasing threat to health and quality of life. In the United States, local food banks serve the population at risk for hunger to reduce food insecurity in their service area. We present and analyze several mathematical models to facilitate the equitable and effective distribution of donated food by a large local food bank among the population at risk for hunger. Demand typically exceeds the donated food supply, and is proportional to the poverty population within the service area. The food bank is required to distribute food donations in an equitable manner such that each person in poverty receives the same amount of food in each period. This objective conflicts with the goal of effectively distributing donated food by minimizing the amount of undistributed food.

We first develop deterministic network-flow models to minimize the amount of undistributed food while maintaining a user-specified upper bound on the deviation from perfect equity and derive closed-form optimal solutions. We then extend this model to obtain optimal policies for the allocation of additional receiving capacity to counties in the service area. These deterministic models show that locations with low capacity-to-demand ratios, *bottlenecks*, constrain the amount of food distributed owing to the equity requirement. Therefore, counties' capacities, which in practice are uncertain, strongly influence the optimal solution.

To address stochastic capacities, we develop a single-period model under which food distribution decisions are made before capacities at the receiving locations are known. After the capacities of the counties are observed, shipment decisions made at the beginning of the

period can be corrected at additional cost. We prove that this model has a newsvendor-type closed-form optimal solution, which we use to develop a Myopic Heuristic for the multi-period problem. In the multi-period problem, supply is received at the beginning of each time period and shipments are made before observing the capacities for that time period. After capacities are observed, shipment decisions can be corrected by either shipping extra food from the branch to the counties or sending surplus food to waste from the counties. Any unshipped supply in the food bank is transferred to the following period as starting inventory. We use the structural properties of this problem to develop upper and lower bounds on the optimal shipment amounts and develop several heuristics that provide improvements on the Myopic Heuristic. An extensive numerical study demonstrates the promising performance of the heuristics.

Lastly, we develop a robust optimization model that allows the capacity parameters to vary within a range. We obtain conservative yet realistic solutions which focus on the capacity deviations at the bottleneck locations from their nominal values to achieve maximal influence on the objective. We also extend the deterministic food distribution model to obtain a different robust model that considers deviations in equity bounds while limiting the total level of inequity in the system. We develop algorithms to solve both of the robust models optimally and illustrate our results using historical data from our collaborating food bank.

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Modeling for the Equitable and Effective Distribution of Food Donations under Capacity
Constraints

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Industrial Engineering

Raleigh, North Carolina

2015

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DEDICATION

This dissertation is dedicated to

My husband, Resulali Emre Örgüt, for staying closest to me through sorrow and success;

My mother, Füsün Şengül, for teaching me to love unconditionally;

My father, Mehmet Şengül, for always believing in me;

My grandmother, Ayser Kalıpçı, for showing me that hard work is always rewarded;

and

My grandfathers, Orhan Hamdi Kalıpçı and Hamza Şengül, for protecting me under their

wings as angels in the heavens above.

BIOGRAPHY

Irem Sengul Orgut was born in Ankara, Turkey on July 7th, 1986. After graduating from Kadikoy Anatolian High School in Istanbul, Turkey in 2005, she received her Bachelor of Science degrees in Industrial Engineering and Mechanical Engineering from Bogazici University in Istanbul, Turkey in 2010. During her undergraduate studies, she was an ERASMUS Exchange Student during the spring semester of 2008 in the University of Southern Denmark in Odense, Denmark. After finishing her undergraduate studies, she joined the Edward P. Fitts Department of Industrial and Systems Engineering at North Carolina State University (NCSU) in Raleigh, NC, and began pursuing her Ph.D. degree under the guidance of Dr. Julie S. Ivy and Dr. Reha Uzsoy.

Her research interests are focused on the stochastic modeling of complex supply chains with multiple objectives and conflicting decision makers. Her research is motivated by the problems in humanitarian and public health systems and aims to find applicable policies for improving people's circumstances.

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She is currently the president of the INFORMS Student Chapter at NCSU. In the past years she served as the Vice President (2013-2014) and Secretary (2012-2013) of the same chapter. She is a member of INFORMS, the Institute of Industrial Engineers (IIE) and Alpha Pi Mu (Industrial Engineering Honor Society).

Upon graduation, she will be joining Lenovo Group Ltd. as the Corporate Quality Statistics Project Manager in Morrisville, NC.

ACKNOWLEDGMENTS

I feel overwhelmed with gratitude towards the following people, for I know that none of this would have been possible without them.

First and foremost, I would like to express my deepest appreciation to my advisors Dr. Julie S. Ivy and Dr. Reha Uzsoy for their guidance, support, and patience. It was with their motivation and encouragement that I became the researcher I am now and I am eternally grateful for all that they have done for me.

My sincere thanks also go to Dr. James R. Wilson, for all his help and invaluable guidance. He has been my mentor not only for research and teaching but also for life. I am also grateful to Dr. Peter Bloomfield for his feedback and support and for showing me the beauty of statistics. I would also like to thank Dr. Hong Luo for his time and agreeing to be my graduate school representative.

I would like to acknowledge the support for this research, which was funded by the National Science Foundation with the grant numbers #CMMI-1000018 and #CMMI-1000828. I would also like to thank the Food Bank of Central and Eastern North Carolina; specifically, Charlie Hale and Earline E. Middleton for their help and support.

I have been very lucky to be a part of the Department of Industrial and Systems Engineering at NC State. I would like to thank all of the faculty and staff members for making me feel at home and for keeping their doors open. I would like to specifically thank Dr. Richard Bernhard, Dr. Yahya Fathi, and Dr. Anita Vila-Parrish for providing me their support at every opportunity.

I am indebted to Dr. Ümit Bilge from Bogazici University for encouraging me to embark on this journey and helping me find my way. My thanks also go to Dr. Lauren Davis from North Carolina A&T State University for her support and interest in my work. I would also like to express my sincere thanks to Laura Laltrello from Lenovo for her help and guidance and to Dr. Erinç Albey, who has been my mentor, teacher, and dear friend throughout this journey.

I owe everything in my life to my loving family. Without the continuous support of my parents, Füsün and Mehmet Şengül, I couldn't have even taken the first step of this journey. Since my childhood, you taught me to be strong, independent and hungry for knowledge. You have been my greatest inspiration and your continuous love and support have always kept me going. I am also deeply grateful to my grandmother Ayser Kalıpçı and my late grandfathers Orhan Hamdi Kalıpçı and Hamza Şengül for surrounding me with their love wherever I go and whichever path I take. My thanks also go to Huriye Şengül, Gill Gur, Gizem İmram, Ahmet Kalıpçı, and Makbule Tekin for all that they have done for me.

I would like to thank my second family, Demet Örgüt, Melih Örgüt and Buket Erlat not only for welcoming me to their family but also for somehow making me feel like I had always been a part of it. Also, to Mehmet İhsan Örgüt; for being the little brother that I have always wanted. I would also like to thank Sebahat Erlat and the rest of the Örgüt Family, especially İbrahim and Nadire Örgüt, for giving me their endless love and support, I am very lucky to have all of you in my life.

I would like to thank my dear friends in the US, especially Dr. Müge Çapan, Dr. Gökçe Akın Aras, Korhan Aras, Elif Çar Albey, and Carl Pankok for sharing this journey

with me, through the good and the bad. I also would like to thank Gülcan and Enis Uysalol for their friendship. You made everything easier, fun and meaningful.

I am very fortunate to call the following four people my friends, although they have been much more than that. We grew together and I know we will age together. I would like to thank Hatice Çetin for magically sensing when I need her and being there no matter what; Memetcan Dalay for the fun and compassion he hides behind his brick wall; Zuhal Kavak for all her selflessness and support, all the years and all the fun; and Tuğçe Tokuş for her unique way of looking at life and always listening without judgement. I owe so much to you and thank you for making long distance just a technicality.

Lastly, I would like to thank Resulali Emre Örgüt; my best friend in the world, my confidant, my ally in many wars, my husband. There are no words to thank you for all of your support, kindness and understanding. I can only say that I could not have done any of this without you. I cherish all of our adventures around the world and I am very excited to see what life will bring our way in years to come.

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LIST OF ABBREVIATIONS

BH	Bounding Heuristic
CD	Capacity-to-Demand
C-RA	Robust Optimization Algorithm for Capacity Uncertainty
C-RM	Robust Optimization Model for Capacity Uncertainty
DAH	Deterministic Approximation Heuristic
DF	Demand Fraction
E-RA	Robust Optimization Algorithm for Equity Deviation
E-RM	Robust Optimization Model for Equity Deviation
FBCENC	Food Bank of Central and Eastern North Carolina
LAH	Look-ahead Heuristic
LP	Linear Program
MCD	Modified Capacity-to-Demand
MH	Myopic Heuristic
MOG	Multiple Optima Generation
MSM	Multi-stage Model
SPM	Single-period Model
SPM-A	Aggregated Single-period Model

Chapter 1

Introduction

According to the Food and Agriculture Organization of the United Nations (FAO, 2013) about 870 million people in the world suffered from food insufficiency and hunger between 2010 and 2012. In 2013, 49.1 million Americans, of whom 15.8 million were children, lived in food-insecure households (Feeding America, 2015) and these numbers increase each year. *Food insecurity* is defined as “a household-level economic and social condition of limited or uncertain access to adequate food” and *hunger* as “an individual-level physiological condition that may result from food insecurity” by the United States Department of Agriculture (USDA, 2011). Although the term “hunger” was more common in the 1980s, “food security or insecurity” is the term used recently (Campbell, 1991). The research described in this dissertation analyzes the operations of a large food bank to identify equitable and effective distribution strategies for donated food to the population in need under capacity constraints.

The Food Bank of Central and Eastern North Carolina (FBCENC), located in Raleigh, NC, distributes food and other donations through its warehouse and four branches (located in Wilmington, Durham, Sandhills, and Greenville, NC) to partner agencies such as food pantries and soup kitchens in a 34-county service area as shown in the color coded map in Figure 1.1.

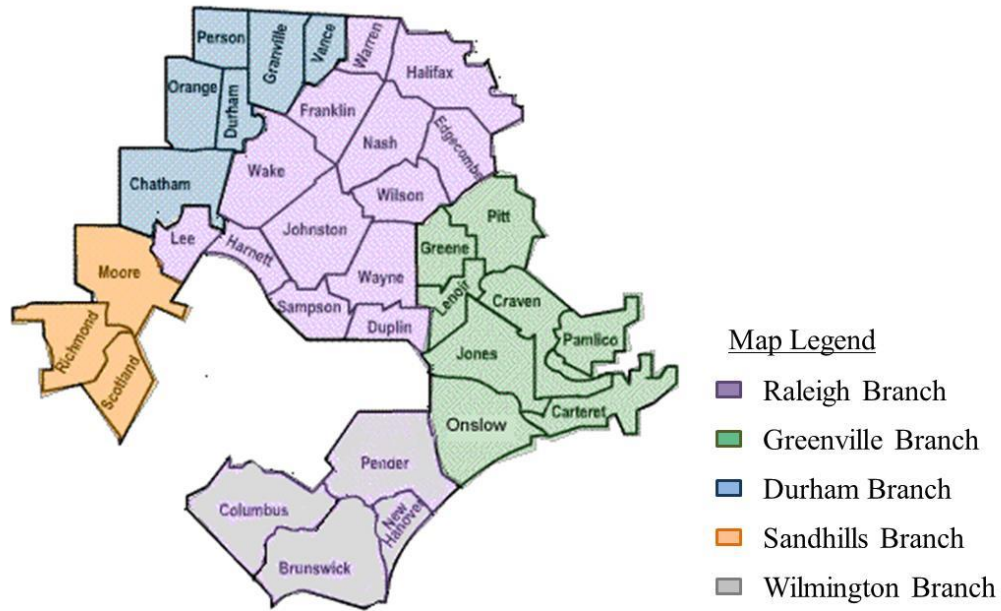


Figure 1.1: FBCENC’s Service Region (FBCENC, 2013).

The food is distributed to the food-insecure population in these counties by the partner agencies. Each branch is assigned a set of counties to serve, and a county may receive food from more than one branch. Food can be transferred from one branch to another before distribution to agencies. FBCENC’s food distribution network is illustrated in Figure 1.2.

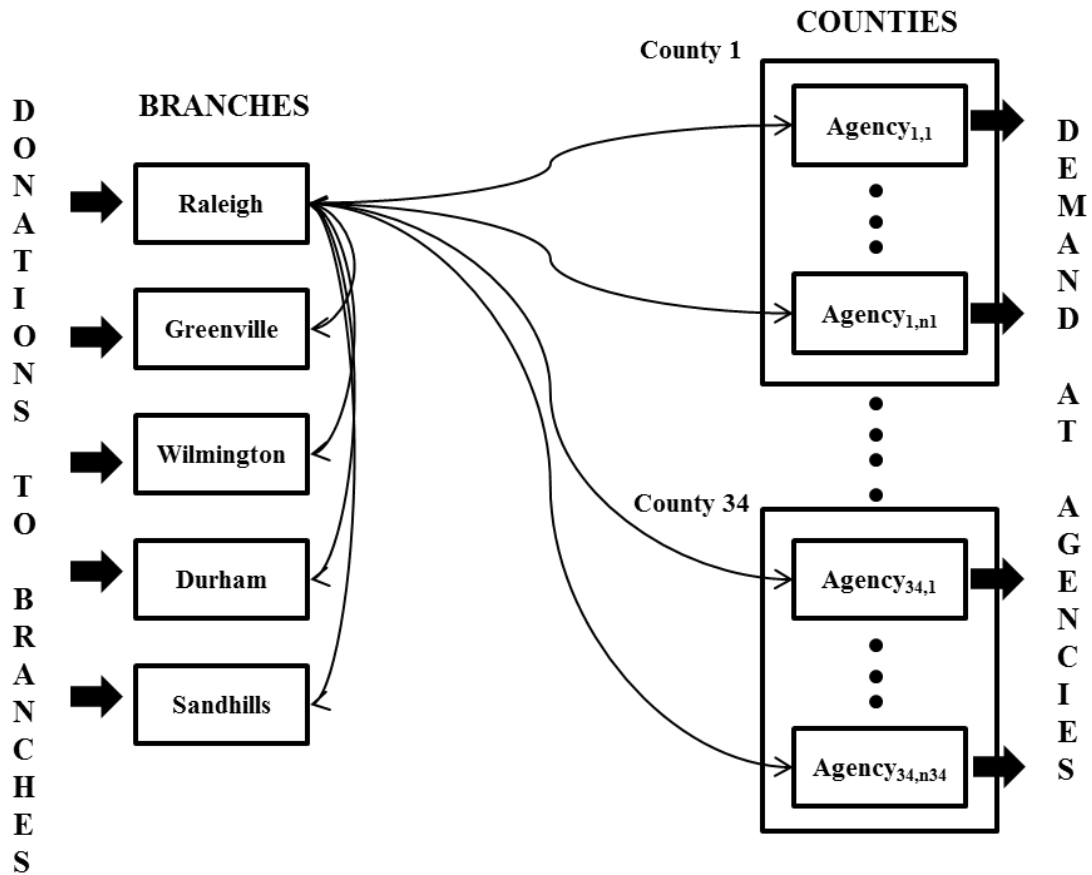


Figure 1.2: Food Bank of Central and Eastern North Carolina Supply Chain.

FBCENC receives in-kind donations from many sources: 59% of their food is from local donors who are grocers, growers, packers, and manufacturers; 21% is from state and federal governmental sources; 11% is from Feeding America; 5% is from other food banks; and the remaining 4% is from food drives (FBCENC, 2013). Although the mission of FBCENC is “No one goes hungry in central and eastern North Carolina”, the amount of food donated is much lower than the demand in this service region. For this reason, rather than trying to satisfy the demand and supply all of the food needed by the food-insecure

population, FBCENC tries to distribute the food that they receive as equitably as possible while minimizing food waste. FBCENC is required by its supervising agencies such as Feeding America to distribute food across counties in proportion to the population in poverty in each county so that, ideally, each person in poverty in FBCENC's service area receives exactly the same amount of donated food, by weight, over a specified reporting period. However, historic data suggests that some counties are at a disadvantage compared with others in terms of obtaining their share of the donated food. In some cases, the food donations per person in one county can be much less than those received by people in another county. In addition, food waste is observed in some places, whereas other agencies face stockouts.

In this dissertation, we develop mathematical models with the objective of maximizing the amount of distributed food while maintaining equity as defined by the fair-share measure when the capacity limitations of the counties are considered. We first address this problem by developing deterministic linear programming models to understand the structural properties of this system, and obtain easily implementable policies. The models developed can be used both for benchmarking the performance of FBCENC by exploring the trade-off between equity and the total distributed food, and for obtaining managerial insights into how capacity investments can be made in collaboration with local agencies to improve FBCENC's ability to meet its goals. We then extend these models by incorporating stochastic capacities and study the theoretical and practical results for those models. We provide insights into how capacity uncertainty affects the optimal solution of this problem when

capacities follow a known distribution and when capacities vary within pre-specified ranges. We illustrate the results through extensive numerical studies.

The remainder of this dissertation is structured as follows. In Chapter 2, we review the related literature. Chapter 3 examines deterministic linear network flow models for optimally distributing food donations. The models developed in this chapter have the objective of maximizing the amount of distributed food while keeping the deviation from equity under a pre-specified level. We study these models to develop policies for obtaining optimal food distributions and capacity allocations, and use data obtained from FBCENC to support our findings. We also perform a probabilistic sensitivity analysis to generalize our conclusions. Chapter 4 develops a two-stage stochastic model to address the equitable and effective food distribution problem in a single period under capacity uncertainty. In the first stage, the decision maker is required to make shipments to the counties without knowing the realized capacities. The only restrictions imposed on the first stage shipments are the following (i) they should be *perfectly equitable*, i.e., each person in the service region should receive exactly the same amount of food; and (ii) the total shipment cannot exceed the supply. Recourse actions are taken in the second stage, after capacities are realized, and can be in the form of additional shipments from the Food Bank or sending surplus food to waste at the counties, while incurring additional cost on the objective. We prove that this model has a newsvendor-type closed-form optimal solution. We present numerical results using data from FBCENC, and perform a sensitivity analysis on the cost coefficients. Chapter 5 extends the single-period problem of Chapter 4 to multiple periods. In the multi-period problem, supply is received at the beginning of each period and any unshipped food in the Food Bank

is transferred to the following period as inventory. We use the results from Chapter 4 and the structural properties of the multi-period problem to develop easily implementable heuristics. We test the performance of these by using a Multi-Stage Stochastic Programming Model as a benchmark through extensive computational experiments. Chapter 6 develops two robust optimization models. The first robust model addresses the capacity uncertainty by obtaining solutions that are feasible when capacities deviate within a given range. The second model extends the deterministic food shipment model from Chapter 3 by preventing the overall inequity in the system from exceeding a specified threshold. For both these models, we develop algorithms that obtain the optimal solutions and illustrate their behavior using data from FBCENC.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, we review the literature that is related to our study. In Section 2.2, we assess the literature on food insecurity. Rather than studying methodological articles, this section seeks to motivate our research by examining studies that provide evidence of the impact of food insecurity on individuals' quality of life and health status. Section 2.3 discusses the literature related to humanitarian logistics, focusing on resource allocation problems. In Section 2.4, we discuss the literature related to food distribution. We provide a framework for reviewing the literature in this area and show how our study addresses a hitherto unstudied issue in this broad field. Section 2.5 presents a discussion of studies which consider equity as an objective, and Section 2.6 provides a brief review of the literature on production planning under uncertainty as it relates to our work. Section 2.7 summarizes our contribution to the existing literature.

We would like to note that some of the articles discussed in this chapter apply to many of the above sections. We discuss these in the section most relevant to this work, and also mention them in other sections to which they are relevant.

2.2 Food Insecurity

In 2000, United Nations published eight goals for addressing the problems facing humanity: the first of these was “ending poverty and hunger” (United Nations, 2010). In this

section, we examine studies that support the importance of food insecurity as a global problem and motivate our work. Studies that use mathematical models to address this problem will be studied in the following sections.

Campbell (1991) develops definitions, conceptual frameworks and measures for food insecurity. She explains that food insecurity leads to a poor nutritional state which leads to deterioration in one's physical, social and mental well-being and a decrease in one's quality of life. In this study, measurement of food insecurity is based on the quantity, quality, psychological acceptability and social acceptability of the received food. This study motivates our research by showing that there is a strong correlation between food insecurity and the quality of life and health status of an individual.

There is an extensive literature supporting the correlation between food insecurity and individual health status. Some examples are Vailas, Nitzke, Becker, and Gast (1998), Stuff *et al.* (2004), Vozoris and Tarasuk (2003), Nelson, Cunningham, Andersen, Harrison, and Gelberg (2001) and Olson (1999). Vailas *et al.* (1998) focus on elderly people who are enrolled in a meal program in a specific area, and explore the nature and strength of the relationship between the factors related to malnutrition and quality of life. They propose that food insecurity is negatively correlated with quality of life and they perform statistical tests to prove this correlation. Stuff *et al.* (2004) examine the same issue in a more quantitative way. They focus on a population in a specific region and examine the relationship between household food insecurity (measured by the U.S. Food Security Survey Module) and self-reported physical and mental health (measured by the Short Form 12-item Health Survey

(SF-12)). Using logistic regression, they demonstrate that health status is associated with age, income, sex and the interaction between race and food security status.

Vozoris and Tarasuk (2003) also use multiple logistic regression to study the relationship between food insufficiency and the physical, mental and social health of people living in a specific region in Canada. They find that individuals in food-insufficient households are more likely to rate their health state as poor or fair, have restricted activity and poor functional health or suffer from multiple chronic conditions compared with those in food-sufficient households.

Nelson *et al.* (2001) focus their study on people with diabetes. They also use regression techniques to determine that adults with diabetes who are food-insecure are more likely to report fair or poor health states than those who were food secure. Olson (1999) focuses on the women of child-bearing age and school-age children in United States. Her results show that the body mass index is significantly higher, which is an indication of an unhealthy body, for the food-insecure women compared to the food secure women. They also find that food insecurity has a negative association with the psychosocial status of school-age children.

All of the discussed studies support a strong correlation between household food insecurity and health status. There are other studies that also focus on specific groups within the general population and show how food insecurity affects health status and quality of life (Lee and Frongillo, 2001; Cook *et al.*, 2004; Casey, Szeto, Lensing, Bogle and Weber, 2001).

In this section, we examined various social studies that provide evidence regarding the importance of food insecurity and its effect on quality of life and health status. Sidel (1997)

propose five steps for eliminating hunger and poverty in the United States and emphasize the importance of nutrition and food stamp programs for achieving this goal. This highlights the potential contribution of our research to optimize the operations of a food bank system using operations research and statistics tools on reducing the level of food insecurity in society.

2.3 Humanitarian Logistics

The Fritz Institute defines humanitarian logistics as the “process of planning, implementing and controlling the efficient, cost-effective flow and storage of goods and materials, as well as related information, from the point of origin to the point of consumption for the purpose of alleviating the suffering of vulnerable people” (Fritz Institute, 2008). Ergun, Keskinocak and Swann (2011) generally describe humanitarian applications as “research that is primarily directed toward promoting human welfare.” Food insecurity and poverty are among the important humanitarian issues in recent years (Celik *et al.*, 2012).

Celik *et al.* provide a detailed tutorial on humanitarian logistics. They separate humanitarian issues into two general areas: disasters and long-term humanitarian development issues. We follow this classification for reviewing the literature related to our study with a focus on resource allocation as illustrated in Figure 2.1. Celik *et al.* summarize the similarities between long-term issues and disasters, such as multiple and conflicting objectives, high level of uncertainty, and their dynamic and resource-constrained nature. They extend the generally accepted definition of humanitarian logistics to include “logistics activities related to preventing, reducing, preparing for, responding to or recovering from human suffering and environmental and financial effects due to disaster or a long-term

development issue”. They discuss the differences between commercial and humanitarian supply chains, including the different sources of uncertainty, the types of products that flow in the chain, potential network disruptions, and the characteristics of the objectives. They group the long-term humanitarian development topics addressed in humanitarian literature into the following categories: food and supply distribution, infrastructure network planning, health care and supply chain optimization.

Many areas of humanitarian logistics can be studied using the tools of operations research. Kovács and Spens (2009) explain the difficulties related to humanitarian logistics that a researcher may face, including lack of collaboration between parties, problems with policy support and legislation supporting humanitarian organizations, and scarcity of resources.

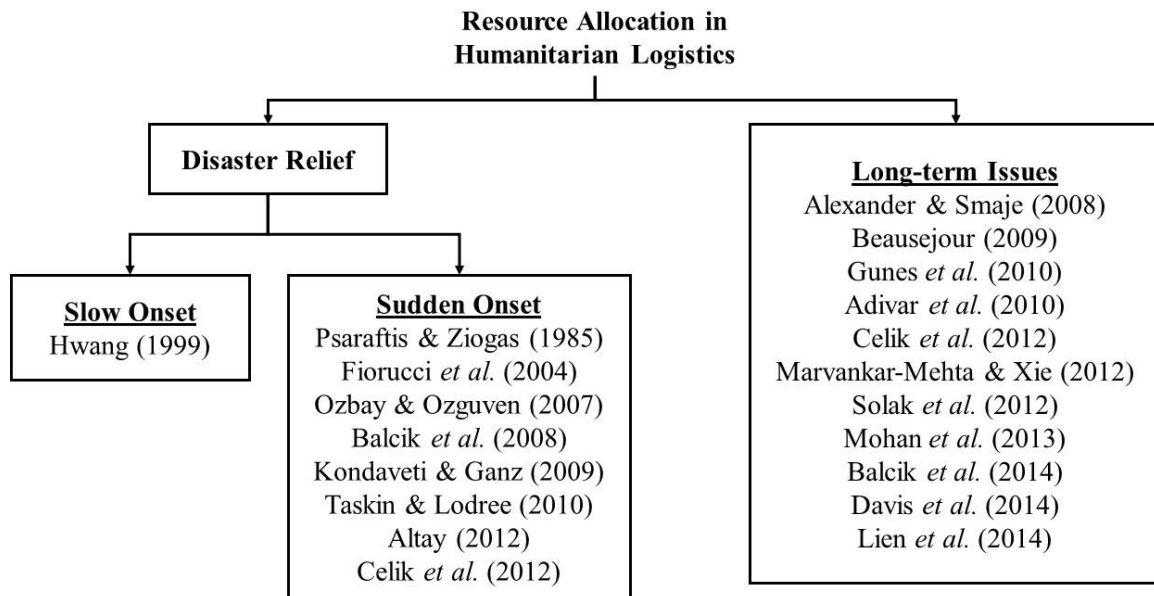


Figure 2.1: Framework for Humanitarian Logistics Literature.

Studies in the humanitarian logistics area can be classified according to the types of decisions (strategic, tactical, and/or operational), types of objectives and performance measures (cost, equity, sustainability, lead times, effectiveness, etc.), types of constraints (budget constrained, capacity constrained, time constrained etc.), and reasons for occurrence (natural versus man-made). For further discussion of the different performance measures to be studied in this context, we refer the reader to Beamon and Balcik (2008), who develop a performance measurement framework for humanitarian relief chains operated by non-governmental organizations for large-scale emergencies caused by sudden onset disasters. They list equity and effectiveness, which are the main objectives considered in our work, among the important performance measures for disaster relief chains. In addition, Vitoriano, Ortuño, Tirado and Montero (2011) suggest that there are many objectives to be considered in humanitarian aid systems. They formulate network flow models and include “equity in distribution” as one of the objectives in their goal programming model. Other objectives they consider are cost, time, reliability, priority and security.

Our discussion of the humanitarian logistics literature focuses on studies considering tactical resource allocation in a humanitarian context. The studies that we examine, classified according to the source of the problem, are shown in Figure 2.1. Although our study falls in the category of long-term humanitarian issues, we first discuss studies related to disasters. While disaster-related humanitarian issues may have different constraints, such as urgency of solution or extremely dynamic behavior of donors, there are similarities with our study, such as the types of supply chains (flow of material from donors to beneficiaries) and objective functions (e.g. equity). Altay (2012) compares resource management challenges in the face

of disasters to those associated with routine operations. The interested reader can refer to Apte (2009), Altay and Green III (2006) and Caunhye, Nie and Pokharel (2012) for an extensive review of the literature on humanitarian logistics.

We first consider resource allocation applications in the face of sudden-onset disasters such as earthquakes, floods and hurricanes. Altay (2012) classifies the capabilities of the available resources for effective disaster response based on several criteria. He first models the case where the supply on hand exceeds the demand in the disaster region. In this case, since there is no need for equitable allocation of resources, he focuses on minimizing the response time. In the opposite case, when demand exceeds supply, he considers the problem of equity and formulates a multi-objective optimization problem, minimizing both the total deployment time and the total capability deficit. This idea is similar to our problem since we also aim to distribute resources equitably among people. However, there are two major differences which also constitute the core distinction between disaster chains and long-term development chains. First, we do not consider urgency as a constraint in our models since our system requires a continuous flow of goods. Second, in our case, the donated supply is always much lower than the total demand so satisfying demand is never treated as an objective.

Balcik, Beamon and Smilowitz (2008) investigate the “last mile” distribution problem from local distribution centers to the beneficiaries affected by disasters. They characterize emergency relief items according to the urgency of their need by the beneficiaries. Since, as in our case, demand is usually not satisfied, they seek an equitable distribution of resources that balance the unsatisfied or late-satisfied demand over a time horizon. Their approach

involves solving the routing problem for the delivery vehicles, as well as identifying the optimal times to visit demand sites, the amount to deliver to each site and the vehicle loads.

Fiorucci *et al.* (2004) find that after a sudden-onset disaster, the allocation of the resources on hand can vastly affect the efficiency of the system, especially if there is the risk of a hazardous situation. They formulate a real-time resource allocation problem with the objective of minimizing estimated damage, inadequate assignment of resources, and transfer costs between the nodes. Kondaveti and Ganz (2009) propose a decision support system for identifying, managing and allocating emergency response resources after a disaster. They develop the decision support algorithm with three phases: clustering the victims affected by the disaster according to location, allocation of resources to the clusters in order to minimize overall risk, and dispatching of resources from the warehouses into the disaster sites. They formulate mathematical models for each of these phases, generate a hypothetical disaster scenario and apply their proposed methodology to this scenario to illustrate its usefulness.

Psaraftis and Ziogas (1985) develop a mathematical model for allocating resources for cleaning up after an oil spill with the objective of minimizing the sum of damage and cleanup costs and present deterministic dynamic programming algorithms for solving this problem. They apply their methodology to an actual oil spill disaster. Taskin and Lodree (2010) approach this problem from a stochastic modeling perspective. They use discrete-time Markov chains to model the goods inventory allocation problem in case of an emergency, e.g., a hurricane. Ozbay and Ozguven (2007) also work on stochastic modeling of post-disaster inventory control.

Slow onset disasters are disasters that happen over a prolonged period of time, such as drought or famine. Hwang (1999) models the optimal allocation of food and inventory in a famine relief area and presents a case study of a North Korean region. He integrates inventory allocation and vehicle routing problems in a three level model with the objectives of minimizing total travel distance and maximizing the amount of famine relief. Although some studies include food insecurity and poverty in the scope of slow onset disasters (Altay & Green III, 2006), we make the following distinction between slow onset disasters and long-term issues: disasters have a specific cause whereas it is usually not possible to link long-term issues to a specific cause or event.

In their extensive chapter on community-based operations research, Johnson and Smilowitz (2012) include poverty and food insecurity among important long-term humanitarian development issues. They develop their definition of community-based operations research on important characteristics of the problems in this area such as multiple stakeholders and decision makers who are localized and often disadvantaged, the importance of stakeholder participation in the study, the accountability of the results and the conflict between uniqueness and generalizability of the applications. They also stress the importance of addressing the “tradeoffs between efficiency, effectiveness and equity”. Our study shares these characteristics and falls within the area of community-based operations research. They list the application areas of community-based operations research as human services, community development, non-profit management, public health and safety and they provide a thorough review of the literature in these areas. They also highlight the fact that there is little work on non-profit distribution of food donations.

We will now consider literature related to long-term humanitarian issues with a focus on optimal resource allocation. Section 2.4 includes some studies in this area considering food distribution but not necessarily with the objective of optimal resource allocation. In Section 2.5, we examine literature with equity as an objective.

Adivar, Atan, Oflac and Orten (2010) coined the term “Social Welfare Chain” which they define as “the processes of designing, planning and implementing a wide range of social development and improvement programs involving all the logistics activities in meeting the needs, managing social problems and maximizing the opportunities for the purpose of improved social welfare.” The authors provide a thorough analysis and comparison of social welfare chains, humanitarian relief chains and commercial supply chains. Adivar *et al.* point out that social welfare and humanitarian relief chains have similarities in terms of the reasons for their existence but also have structural differences. Social welfare chains are different from commercial supply chains because donors and the government are the suppliers and the beneficiaries are the recipients. Furthermore, they compare commercial organizations to nongovernmental organizations (NGOs) which play the major role in a social welfare chain. The authors point out that operations research has been widely used for sudden-onset disaster relief operations but “very little, if any, research focuses on optimizing relief efforts for slow-onset disasters or social welfare operations.” The authors work with an NGO in Turkey and their optimal transportation distribution planning model focuses on optimizing coal distribution by choosing the optimal transfer sites and corresponding amounts of shipments.

Beauséjour (2009) considers the sanitation and wastewater management problem for poor areas with a case study for a small village in Vietnam. In their extensive tutorial, Celik

et al. (2012) develop two decision support tools that consider long-term development issues. The first problem they consider is a strategic decision problem based on planning the expansion of a donated human breast milk banking supply chain in South Africa in order to ensure fair distribution of milk to hospital Neonatal Intensive Care Units. More specifically, their objective is to ensure that every needy infant had equal probability of access to donated breast milk. Their objective is similar to our objective; however they focus on a strategic problem (network design) rather than a tactical one (operations). The second decision problem focuses on tactical and operational problems for a food aid program in East Africa with various distribution and transportation constraints.

Malvankar-Mehta and Xie (2012) examine the optimal allocation of HIV/AIDS prevention resources considering the complex hierarchical structure of this allocation system. They consider a specific hierarchy with three levels. They assume a distribution of these resources to be proportional to the size of the infected population in a given region. This is similar to our study where we estimate the demands based on the poverty population in FBCENC's service region as will be explained in Chapter 3.

Some other studies that consider long-term humanitarian issues are Solak, Scherrer, and Ghoniem (2012), Davis, Sengul, Ivy, Brock III and Miles (2014), Gunes, van Hove and Tayur (2010), Alexander and Smaje (2008), Balcik, Iravani and Smilowitz (2014), Lien, Iravani and Smilowitz (2014) and Mohan, Gopalakrishnan and Mizzi (2013). We provide a detailed review of these studies in Sections 2.4 and 2.5.

We reviewed applications of tactical resource allocation problems in the humanitarian context. The supply chains associated with disaster relief are mostly response chains and

therefore may have different objectives than our problem. Although many studies consider the allocation of different resources for humanitarian relief, to the best of our knowledge, no study focuses on obtaining optimal policies for equitable and effective allocation of food donations considering the capacities of the distribution locations in a food bank system.

2.4 Food Distribution

The extensive literature related to food distribution can be classified in various ways. Figure 2.2 presents a classification scheme for studies on this topic and highlights where our work falls in this literature.

Some of the most commonly studied objectives are (sudden-onset) disaster relief, food security and hunger relief, improving food quality, improving food safety, obtaining environmental sustainability and other supply chain related objectives such as cost reduction and efficiency maximization. It is also important to note that most studies consider multiple objectives. Ahumada and Villalobos (2009) emphasize the importance of models that focus on multiple issues simultaneously and claim that “potential benefits of those models outweigh the added complexity”. While they make this statement in the context of the agri-food supply chain, this idea is valid for most of studies in this field. Although the focus of our study is to reduce food insecurity, we do so by addressing the trade-off between equitable and effective distribution of food donations.

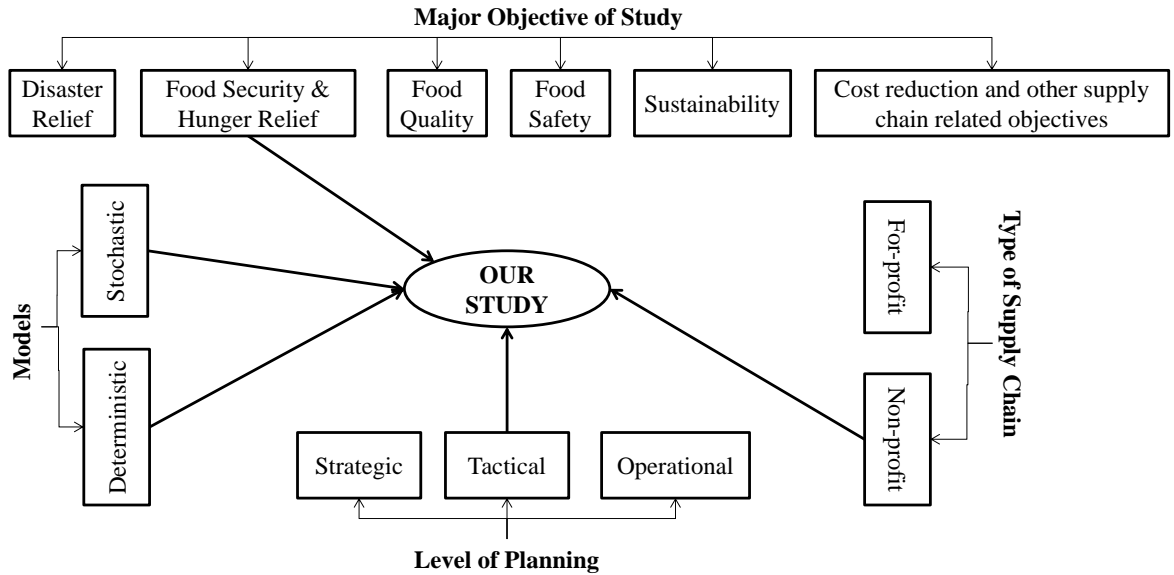


Figure 2.2: Categorization Schemes for Studies on Food Distribution.

Operations research has been used for solving problems and addressing issues related to food supply chains for a long time. We classify these methods as deterministic and stochastic methods. As shown in Figure 2.2, we link our study to both deterministic and stochastic methods since we start by developing deterministic models in Chapter 3 and then extend them to stochastic models in Chapters 4, 5 and 6.

The food supply chain literature focuses on three levels of planning: strategic (e.g., facility location), tactical (e.g., shipment decisions) and operational (e.g., vehicle routing). Our work can be classified as a tactical planning problem since it involves achieving distribution policies for the food donations in order to obtain equitable and effective food distribution.

The majority of the studies in food distribution literature consider for-profit food supply chains. We refer the reader to Akkerman, Farahani, and Grunow (2010) and Ahumada and Villalobos (2009) for an extensive review of studies on for-profit food distribution with a focus on food quality, food safety and sustainability.

Rong, Akkerman, and Grunow (2011) consider the production and distribution planning for a food supply chain and integrate the notion of food quality, which degrades over time and with storage temperature. The authors point out the difficulties associated with a food supply chain due to food-specific characteristics that are different from traditional commodities. Manzini and Accorsi (2013) develop a conceptual framework for identifying issues, problems and decisions related to a food supply chain “from farm to fork”. They focus on four objectives: food quality, safety, cost efficiency and environmental sustainability. Although they do not consider non-profit food distribution, they highlight the importance of “waste management” and “sustainable network design and management”. Dabbene, Gay, and Sacco (2008) address the trade-off between logistics costs and food quality perceived by customers in a fresh-food supply chain.

One of the objectives presented in Figure 2.2 is the reduction of food insecurity and hunger. Krejci and Beamon (2010) develop supply chain designs to address the problem of maintaining food security worldwide. The authors identify the critical characteristics of environmentally sustainable food systems and develop supply chain structures that are environmentally sustainable and that provide food security for varying customer demand.

Although there are numerous studies on for-profit food supply chains, studies that use operations research tools to address non-profit food distribution supply chains are limited. A

comparison of for-profit and non-profit food chains highlights the different dynamics that serve as the underpinning of each system. In a for-profit food chain demand is usually the critical source of uncertainty whereas in a non-profit food chain uncertainty usually originates from supply side, which is commonly in the form of food donations. In for-profit food supply chains, objectives like cost minimization or food quality are more common, while for non-profit food supply chains, typical objectives are finding optimal ways to collect donations, determining optimal locations for food distribution agencies and determining the equitable and effective distribution of food.

As mentioned in Section 2.3, Hwang (1999) focuses on developing a distribution model to determine the optimal allocation of food and inventory in a famine relief area by using an integrated approach for inventory allocation and vehicle routing problems. Schweigman, Bakker, and Snijders (1990) also use operations research tools to address food insecurity in developing countries by determining the optimal amount of land that should be cultivated to ensure that no food shortage occurs.

Solak *et al.* (2012) study the tactical decision problems arising in non-profit food distribution networks. They examine a network that is very similar to our network with food flow from warehouses to various sites to be picked up by agencies and later distributed to people in need. They seek to simultaneously optimize three decisions: site selection for food delivery, assignment of agencies to these delivery sites and routing for delivery vehicles with the objective of minimizing total transportation costs. Davis *et al.* (2014) consider the problem of collecting and distributing donated food by a food bank through a periodic vehicle pickup-and-delivery model with backhauls that incorporates food safety and other

constraints arising in this context. Although not incorporated in our work, the transportation constraints considered in these papers are relevant for our network.

Gunes *et al.* (2010) also consider a similar food bank food distribution network and focus on the vehicle routing problem for a perishable food rescue program to pick up food donations from various sites. Alexander and Smaje (2008) consider “surplus” food coming from retailers as donations to a charitable organization and identify problematic areas in this supply chain where reduction of waste is possible in order to minimize “food poverty”. They highlight the fact that in this redistribution process, the objectives of different actors (the retail suppliers, franchise-holders and recipients) can conflict with each other. For example, a food retailer may want to minimize food waste or excess supply whereas the charitable organization relies on donations from this excess food and would want to obtain food that is as fresh as possible. The food distribution networks for these studies are very similar to our network in terms of the problems and constraints; however, to our knowledge, there are no studies whose objective is to obtain equitable and effective food distribution policies for a non-profit food distribution network under capacity constraints.

2.5 Equity and Fairness

Sen (1973) states that economic inequality can be described either in an objective sense (using statistical measures) or in terms of some normative notion, i.e., a higher degree of inequality corresponds to a lower level of social welfare for a given total income. In this research we use the maximum absolute deviation as a measure of inequality. Since fairness and equity are abstract socio-political and subjective concepts, it is not possible to determine

an equity measure that is applicable to all problems (Ogryczak, 2007). Therefore, it is important to select the equity measure to be used according to the structure of the specific problem (Sen, 1973; Marsh and Schilling, 1994; Leclerc, McLay and Mayorga, 2012; Balcik, Iravani, and Smilowitz, 2010). In this section, we discuss various studies that consider equity as an objective and analyze their different approaches to these problems. We will see that the notion of equity can be an objective in many different problem contexts leading to a variety of interesting applications.

Marsh and Schilling (1994) give a brief overview of the many areas where equity is used as an objective. Some examples of these are geographers' concerns regarding equitable distribution of water rights in Western states, political scientists' discussions on equal representation in Congress and economists' studies on public welfare distribution and equitable distribution of income. Marsh and Schilling focus on facility siting decisions and they suggest that equity is obtained if each group that is affected by the facility siting decision receive their fair-share of the total effect. The authors' objective is to minimize inequity using 20 different measures proposed in the literature and to compare and analyze the performance of these measures for different situations.

Meng and Yang (2002) formulate the continuous network design problem which is the problem of allocating a capacity increase among existing roads under a budget constraint. They use a bilevel programming approach and incorporate equity as a constraint where different network users receive equal benefit from a capacity increase in terms of their average origin-destination travel costs. Ogryczak (2007) addresses the trade-off between equity and effectiveness in resource allocation models. He introduces different performance

measures to achieve Pareto optimality and suggested that the “max-min” type of objectives give both equitable and effective solutions.

Another area where equitable distribution of resources is important is the allocation of emergency medical resources. Chanta, Mayorga, Kurz, and McLay (2011) consider the problem of determining locations of ambulances such that the total “envy” over all the demand zones is minimized where their definition of envy is based on the patients’ dissatisfaction due to their distance from nearby ambulance facility locations. They formulate the problem as an integer programming model called the minimum p -Envy Location Model and used tabu search to obtain approximate solutions. They find that the proposed method performs well satisfying both equitable distribution (minimizing envy) and coverage of the service area.

The objective of equity is also used in assignment problems. Mazumdar, Mason, and Douligeris (1991) study a multiuser telecommunications network in which each user has the objective of optimizing its performance while being fair to the other users. They find that the Nash arbitration scheme from game theory gives a desirable and fair solution for individual users and different performance criteria. In their study, Vossen, Ball, Hoffman, and Wambsganss (2003) focus on an interesting equity problem. Their objective is to allocate the national air space equitably such that the amount of possible delay is distributed equitably among flights. They find that the ration-by-schedule approach gives a nearly equitable allocation of resources and minimizes total delay. In the previous sections, we encountered studies that consider equity as an objective, such as Altay (2012), Balcik *et al.* (2008),

Malvankar and Xie (2012), Vitorino *et al.* (2011), and Celik *et al.* (2012) in Section 2.3 and Krejci and Beamon (2010) in Section 2.4.

The trade-off between equity and other objectives has also been studied. Balcik *et al.* (2014) aim to maximize the minimum fill rate among the food distribution agencies of a food bank while implicitly maximizing the amount of distributed donations. Their model requires sequential decisions at the operational level whereas we consider the more tactical problem of obtaining food distribution policies before the food leaves the food bank. Lien *et al.* (2014) seek to maximize the minimum fill rate over all locations to address the objectives of equity (fair distribution) and effectiveness (minimizing waste) in distributing scarce resources by a non-profit organization under demand uncertainty. They use dynamic programming to obtain the optimal allocation policy to a sequence of customers for a given continuous demand distribution. They use this optimal policy to generate a heuristic when demand follows a discrete distribution. Mohan *et al.* (2013) address the objectives of efficiency and effectiveness in non-profit food distribution supply chains through a discrete-event simulation model. Their aim is to minimize the operational costs and maximize the quantity of food distributed by focusing on the operations of a global hunger relief organization.

Mandell (1991) addresses the equity versus effectiveness trade-off in public service delivery systems by using the Gini coefficient as the measure of equity. Leclerc *et al.* (2012) define effectiveness as “the degree to which a resource allocation causes needs to be met and the extent to which unintended negative impacts of an allocation are avoided.” In our study,

we define effectiveness as achieving maximum food distribution, which is parallel to this definition.

As seen above, there are many studies with the objective of satisfying equity over some measure. The contribution of our study over the presented literature is the incorporation of capacity constraints on the distribution agencies and the effect of these capacity limitations on the objectives of equity and effectiveness in a donated food distribution context.

2.6 Production Planning under Uncertainty

In this section, we provide a brief review of the literature on production planning with a focus on multi-echelon stochastic inventory control models. Production systems deal with uncertainty on a daily basis due to various sources of environmental uncertainty, e.g., demand or supply uncertainty, and system uncertainty, e.g., production lead time uncertainty (Ho, 1989). For this reason, the problem of obtaining optimal ordering policies has been studied extensively in the literature. Although the system which we consider in this work is different from a production planning environment, we will see in Chapters 4 and 5 that there are parallels in terms of the structures of the optimal solutions for the single-period and multi-period stochastic food shipment problems to stochastic production-inventory systems. Hence, our aim here is not to provide an extensive review of production planning literature but to give the reader an understanding of different methodologies used. For an extensive review of production planning literature under uncertainty, the reader is referred to Mula, Poler, Garcia-Sabater and Lario (2006) and Dolgui and Proudhon (2007).

We focus on multi-echelon serial stochastic inventory systems (Snyder and Shen, 2011). For those systems, it is known that an echelon base-stock policy is optimal for each stage under a no fixed costs assumption. This result holds for both finite-horizon problems (Clark and Scarf, 1960) and infinite-horizon problems (Federgruen and Zipkin, 1984). Clark and Scarf (1960) also show that exact solution for the optimal base stock levels for each stage can be obtained by solving a set of recursive equations. However, solving these equations requires substantial computational effort which may not always be practical. Furthermore, if additional constraints are added to the system such as capacity limitations (Levi, Roundy, Shmoys, & Truong, 2008) or workload dependent lead times (Aouam and Uzsoy, 2012), dynamic programming approaches become impractical due to the curse of dimensionality, i.e., the necessity to solve too many subproblems (Levi *et al.* 2008). This results in a need for easily implementable heuristic approaches that provide near-optimal solutions in reasonable computation times. Shang (2011) reviews heuristics developed for solving periodic and continuous review inventory planning problems.

In his pioneering paper, Veinott (1965) considers a multi-period, multi-product problem with linear backorder and holding costs in a periodic review environment with the objective of minimizing the expected discounted cost over the infinite horizon. He examines the conditions which result in the myopic policy that optimizes each period's ordering decisions separately giving the optimal solution for the entire system.

Shang and Song (2003) consider a multi-echelon, single product inventory system and develop a simple heuristic that uses the structure of the newsvendor solution and takes the simple average of the upper and lower bounds on the optimal order quantities in each

period to obtain a base stock level. Their heuristic performs very well with an average relative error of 0.24% and a maximum error of less than 1.5%. We will use some of the ideas by Shang and Song (2003) in Chapter 5 to develop heuristics for the multi-period food shipment problem.

Another heuristic approach for solving continuous-review inventory systems is to use the deterministic EOQ formula where the stochastic demand is replaced by its mean to determine the order quantity Q (Zheng, 1992; Axsäter, 1996). Then, the optimal reorder point r is calculated in the second step given Q yielding a tight bound for the relative cost increase compared to the optimal stochastic solution under certain conditions.

Levi, Pál, Roundy and Shmoys (2007) consider a finite horizon, periodic-review, capacitated inventory system with nonstationary and correlated demands evolving over time. They develop an ordering policy which equalizes the expected marginal holding and backorder costs in a period and provide bounds on the worst case performance of this policy. Levi *et al.* (2008) extend their previous work by considering capacitated systems. They argue that myopic policies perform worse in capacitated systems as compared to uncapacitated systems. They develop bounds on the optimal base stock levels by considering the effect of one period's backlogging cost over the entire horizon. In Chapter 5, we will also develop a myopic heuristic for addressing the optimal food shipment problem over multiple periods. We will improve the performance of this heuristic by developing bounds on the optimal shipments at each period.

2.7 Contribution of This Work

This proposed research contributes to the existing literature in the following ways: the proposed models

- Simultaneously consider the objectives of equity and effectiveness;
- Mathematically model the donations management problem while considering the capacities affecting distribution;
- Address the constraints associated with a non-traditional supply chain in which supply is usually more uncertain than demand;
- Obtain optimal food shipment policies for food distribution and capacity allocation in a non-profit food chain when capacities are known deterministically;
- Derive the closed-form optimal solution for the single-period food distribution problem when capacity is uncertain but follows a known distribution;
- Develop heuristic approaches for the multi-period food distribution problem under capacity uncertainty and perform an extensive numerical study to examine their performance;
- Develop algorithms that obtain single-period food shipment decisions that are robust to capacity deviations in pre-specified ranges; and
- Use real data from a food bank to illustrate the results.

Chapter 3

Modeling for the Equitable and Effective Distribution of Donated Food under Capacity Constraints

3.1 Introduction

According to the Food and Agriculture Organization of the United Nations (FAO, 2013), about 870 million people worldwide suffered from food insufficiency and hunger between 2010 and 2012. In 2013, 49.1 million Americans, 15.8 million of whom were children, were exposed to food insecurity, “a household-level economic and social condition of limited or uncertain access to adequate food” (Feeding America, 2015). In this chapter, we formulate and analyze mathematical models to identify equitable and effective strategies by which a food bank can distribute donated food to the population in need. The food bank is assumed to distribute donated food separately to each county (demand point) in its service area. In this situation, a perfectly equitable distribution of food requires that each county should receive a food allocation in exact proportion to the county’s demand. We focus on the tactical problem of distributing food to the counties in the food bank’s service area so as to minimize the amount of food that is undistributed, which may be wasted as explained below, while satisfying a user-specified upper bound on the magnitude of each county’s deviation from a perfectly equitable distribution.

Feeding America is the largest national non-profit hunger-relief organization in the United States. It serves 46.5 million people in the United States from 200 food banks, which

are the main food distribution hubs where food donations are collected before being distributed to the beneficiaries. Food banks are autonomous in their operations, except that they are required to report back to Feeding America regarding not only the amount of donated food they distribute in their service areas but also the extent to which they achieve an equitable distribution of that food across the counties in their service areas. For this reason, distributing food donations in an equitable and effective manner is of utmost importance for the continued success of food bank operations.

The Food Bank of Central and Eastern North Carolina (FBCENC), located in Raleigh, NC, is an affiliate of the Feeding America network and exemplifies the type of food bank considered in this study. FBCENC distributes food and other goods through its central warehouse and four branches (located in Wilmington, Durham, Sandhills, and Greenville, NC) to partner agencies such as food pantries and soup kitchens in a 34-county service area, who then distribute the food to the local population in need. The Raleigh branch serves as the headquarters of FBCENC and manages the distribution of food to each county in the service region. Each branch serves a set of counties, and a county may receive food from more than one branch. Food is usually received at the branch location that is most convenient for the donors, but may be transferred from one branch to another prior to distribution to the agencies. FBCENC's food distribution supply chain is illustrated in Figure 3.1. The figure shows arcs that are representative of a single branch (Raleigh), but similar flows are possible from all five branches.

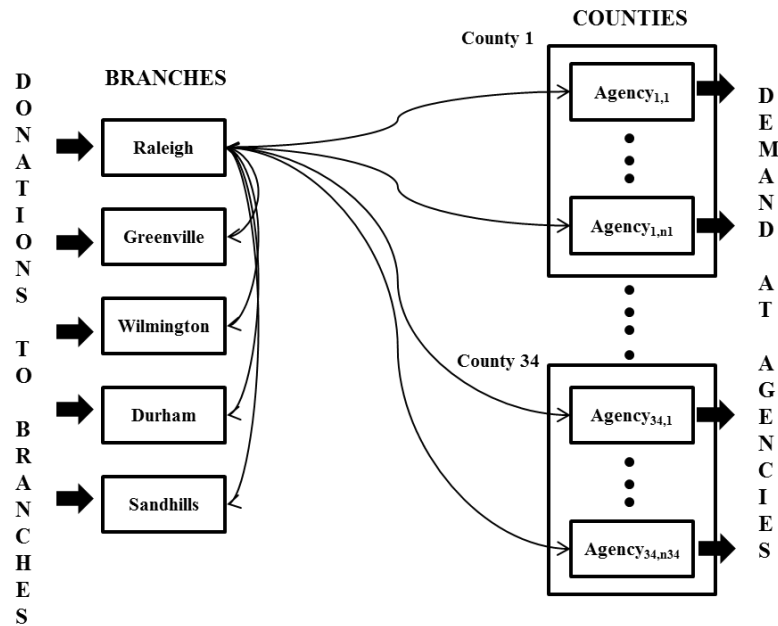


Figure 3.1: Supply chain for the Food Bank of Central and Eastern North Carolina.

FBCENC receives in-kind donations from many sources: 59% of its food is donated by local donors such as grocers, growers, packers, and manufacturers; 21% from state and federal government sources; 11% from Feeding America; 5% from other food banks; and the remaining 4% from food drives (FBCENC, 2013). However, the amount of donated food is persistently much less than the demand in FBCENC’s service area. For example, over the first six months of 2009, donations in FBCENC's service area averaged 3.5 pounds of food per month per person in poverty. According to Feeding America's “Map the Meal Gap” study, the number of additional meals required to meet food needs in North Carolina in 2014 was estimated to be 175.1 meals per food-insecure person annually (Feeding America, 2015).

FBCENC is required to distribute the food that it receives as equitably as possible while minimizing food waste. Food waste may be observed in this network when an agency

receives more food than it can store or distribute. If perishable food is not stored properly or is not consumed within its specified shelf-life, then such food becomes unusable and goes to waste. Hence, even though supply is much less than demand, the receiving capacities of the charitable agencies play a significant role in determining the amount of food waste in this network. The receiving capacity of an agency depends on its physical capacity (storage and transportation capabilities) and its financial capacity (budget and workforce) for timely storage and delivery of donated food to its beneficiaries.

In 2013, FBCENC was recognized by Feeding America for its “Fairshare” program, which “uses readily available poverty rates in each county to provide a blueprint of the areas in greatest need” (FBCENC, 2013). FBCENC uses the US Government’s estimate of the population in poverty in each county to guide its food-allocation decisions so that ideally, each county in the service region receives their exact relative need by weight from the total food distribution measured according to the demand residing in that county over a specified reporting period. To achieve this goal, FBCENC allocates food to each county according to the estimated size of the poverty population in that county.

In this chapter, we will also use the estimated size of each county’s poverty population both as an estimate of the county’s population of food-insecure individuals and as a measure of the county’s need for donated food relative to other counties in the food bank’s service area. In our models, we adopt the definition of an equitable distribution of donated food that has been stipulated by Feeding America and adopted by FBCENC so that for each county, the deviation from perfect equity is the magnitude of the difference between the following:

(a) the size of the county's poverty population expressed as a fraction of the total poverty population in the food bank's service area (i.e., the county's "fair-share" of donated food); and

(b) the amount (weight) of donated food allocated to that county expressed as a fraction of the total amount of food allocated to all counties in the food bank's service area.

Using the size of the poverty population as a measure of food insecurity is supported by statistical results. It is estimated that 72% of the households served by Feeding America fall at or below the government-defined poverty levels (Feeding America, 2015).

Although FBCENC is among the most successful food banks in achieving its equity goals, historical data suggests that the amount of donated food received per person in poverty can vary substantially across the counties in FBCENC's service area. In addition, food waste is observed in some locations, while others simultaneously face stockouts. The variation in the amount of donated food received per person in poverty at different locations is caused by limited receiving capacity in "bottleneck" locations. FBCENC is often faced with conflicts between its objectives of satisfying the fair-share criterion and not wasting food. For situations in which excess food is available but the counties lacking their fair-share of donated food also lack the capacity to receive additional food, FBCENC will send the excess food to selected counties with sufficient receiving capacity, even though this action may increase the absolute deviations of the selected counties from a perfect-equity allocation. If a county is consistently underserved with respect to that county's fair-share of donated food, FBCENC will work to identify the source of the problem and try to fix it by actions such as

recruiting new agencies in the county or expanding the receiving capacity of the county's existing agencies.

In this chapter we have two primary aims: (a) to obtain policies that help benchmark and improve the performance of food banks by exploring the trade-off between equity and effectiveness of the total food distribution in a capacitated network; and (b) to obtain managerial insights into how capacity investments can be made in collaboration with local agencies to improve a food bank's ability to meet these goals regarding equity and effectiveness. To address these goals, we develop and analyze mathematical models to minimize the amount of undistributed food while maintaining a user-specified upper bound on the absolute difference between each county's "fair-share" of donated food and the proportion of total donated food sent to that county. We use data obtained from FBCENC to illustrate our results.

The rest of this chapter is organized as follows. In the next section we review previous related work. Section 3.3 introduces the generalized food distribution model, discusses its structure, and derives a closed-form optimal solution of that model. Section 3.4 focuses on the problem of optimally allocating a fixed amount of additional receiving capacity among the counties in the food bank's service area. Section 3.5 presents a case study using data obtained from FBCENC that illustrates our results and examines the sensitivity of optimal solutions to uncertainties or errors in the receiving capacities of the counties in the food bank's service area. In Section 3.6, we discuss the conclusions and limitations of our study and we suggest some directions for future research. Sengul Orgut et al. (2015) is a forthcoming archival-journal paper documenting the developments presented in this chapter.

3.2 Previous Related Work

Celik *et al.* (2012) divide humanitarian issues into two general categories: disasters and long-term humanitarian development issues. Studies in humanitarian logistics can be classified by the types of decisions (strategic, tactical, and/or operational); objectives and performance measures (cost, equity, sustainability, lead times, effectiveness, etc.); constraints (budget constrained, capacity constrained, time constrained etc.); and reasons for occurrence (natural versus man-made) (Beamon and Balcik, 2008).

We focus on studies of tactical resource allocation in the humanitarian context. Most literature on humanitarian supply chains focuses on disaster-related problems rather than long-term or public policy problems. While disaster-related issues may have different constraints such as urgency or extremely dynamic donor behavior (Altay, 2012; Balcik *et al.*, 2008; Hwang, 1999), there are similarities with our study, such as the need to coordinate the flow of material from donors to beneficiaries and the objectives to be considered (e.g., equity). Equity, effectiveness, and efficiency are common objectives in this literature and are discussed in a wide variety of contexts. Although the precise meanings of these terms are subjective and depend on the problem and the decision makers (Stone, 1997), for the purposes of this study, we will use *equity* to mean “the condition of being equal in quantity, amount, value, intensity, etc.” (Simpson and Weiner, 1989). We use the term *effectiveness* to mean the degree to which something is “capable of being used to a purpose” (*Webster’s Third New International Dictionary of the English Language, Unabridged.*, 1993) and *efficiency* to mean “getting the most out of a given input” or “achieving an objective for the lowest cost” (Stone, 1997). This chapter focuses on the objectives of equity and

effectiveness. We will address the objective of equity by limiting the absolute deviation between the proportion of food sent to a county and that county's relative need, while effectiveness corresponds to maximizing the total amount of food distributed. Maximizing the total amount of food distributed will also reduce the risk of food being spoiled, increasing the amount of food distributed and reducing food insecurity since higher amounts of healthy, usable food is being distributed in a timely manner. Cost efficiency is not included as a primary objective for two reasons: (i) assigning transportation costs between branches and counties may result in policies that favor urban locations over rural locations; and (ii) there is limited data available regarding the complex cost structure of this network.

This chapter contributes to the literature on humanitarian logistics by addressing the equity-effectiveness trade-off at the tactical level for a non-profit food chain. We address the objective of equity by minimizing the maximum absolute deviation between a county's fair-share of total food donations and the fraction of the total distribution of donated food allocated to that county. This objective directly conflicts with the objective of minimizing the amount of unshipped food. Considering the receiving capacities of the counties, we develop policies that minimize the amount of undistributed food for a specified maximum deviation from perfect equity for all counties. We then develop an algorithm to optimally allocate additional capacity to the counties and give the closed-form optimal solution for this problem. We will now explore how the objectives of equity, effectiveness and efficiency are addressed in the literature.

Altay (2012) considers the capabilities of available resources for effective disaster response. Focusing on minimizing the response time, he models the case where the available

supply exceeds demand in the disaster region. He also considers the case of demand exceeding supply, raising the issues of equity and effectiveness, and formulates a multiobjective optimization problem for minimizing the total deployment time and the total capability deficit, the difference between the resource capability required and that assigned. The total capability deficit addresses the effectiveness objective as in our problem, but equity is not explicitly considered. There are two principal differences between disaster supply chains and the long-term supply chains considered in this study. First, we do not consider time in our models, since our system requires a continuous flow of goods. Second, in our case the donated supply is much less than the total demand, so satisfying demand is not a useful objective.

Adivar *et al.* (2010) point out that operations research has been widely used for sudden-onset disaster relief operations but “very little, if any, research focuses on optimizing relief efforts for slow-onset disasters or social welfare operations.” These authors work with an NGO in Turkey to optimize coal distribution by choosing optimal transfer sites and the corresponding shipment amounts using a mixed integer linear program to minimize total cost under various constraints. Celik *et al.* (2012) study a strategic decision problem based on planning the expansion of a supply chain for banking donated human breast milk in South Africa in order to ensure that every needy infant has equal probability of access to donated breast milk. Their equity objective is similar to ours, but they focus on a strategic problem (network design) rather than a tactical one (distribution). Malvankar-Mehta and Xie (2012) study the optimal allocation of HIV/AIDS prevention resources considering the complex hierarchical structure of this allocation system. They assume the distribution of these

resources is proportional to the size of the infected population in a given region. This is similar to our study, where we estimate demand based on the estimated size of the poverty population in FBCENC's service region.

The notion of equity is encountered in many different contexts. Sen (1973) states that economic inequality can be described either in an objective sense using statistical measures, or in terms of some normative notion such that a higher degree of inequality corresponds to a lower level of social welfare for a given total income. Although numerous equity measures have been proposed in the literature, there is no single generally accepted equity measure for all problems, making it necessary to select equity measures based on the specific problem (Sen, 1973; Marsh and Schilling, 1994; Leclerc *et al.*, 2012; Balcik *et al.*, 2010). Leclerc *et al.* (2012) emphasize that the nature of the resource to be allocated, the target group, and the time horizon are the key elements to be considered for the selection of an equity measure.

Marsh and Schilling (1994) discuss twenty equity measures from the literature and review the many areas where equity is used as an objective. Examples of these are geographers' concerns regarding equitable distribution of water rights in the western United States, political scientists' discussions of equal representation in Congress, economists' studies of public welfare distribution and equitable distribution of income. Some studies that consider equity as an objective are Chanta *et al.* (2011) (determining locations of ambulance dispatch facilities for equitable access by demand zones); Meng and Yang (2002) (optimizing road network expansion for equitable benefits received by people); Mazumdar *et al.* (1991) (equitable assignment of performance in a multiuser telecommunications network); Wang *et al.* (2007) (equitable water rights allocation between countries); and Vossen *et al.* (2003)

(equitable allocation of national air space). Marsh and Schilling (1994) focus on the equity objective for facility location problems, defining equity as the case where “each group receives its fair-share of the effect of the facility siting decisions.” This definition is similar to that used in this study, where we define equity to be the case that each county (group) receives its *fair-share* of the total food donations.

The trade-off between equity and other conflicting objectives has also been studied. Balcik *et al.* (2014) address the problem of equitable and efficient (low-waste) allocation of food donations through routing and allocation decisions for the donors and agencies of a food bank. Their objective is to maximize the minimum fill rate among all agencies as a measure of equity while implicitly maximizing the amount of distributed donations. The allocation decisions in their problem are made sequentially at the operational level (the driver decides how much food to allocate to an agency upon arrival), whereas in our study the decisions are made at the tactical level.

Mandell (1991) considers the equity versus effectiveness trade-off in public service delivery systems such as libraries, and develops mathematical models to address this trade-off while using the Gini coefficient as the measure of equity. Leclerc *et al.* (2012) define effectiveness as “the degree to which a resource allocation causes needs to be met and the extent to which unintended negative impacts of an allocation are avoided.” The second part of this definition supports our usage of the term *effectiveness* in this study—namely, with reference to minimizing the amount of undistributed food donations. Minimizing the maximum or maximizing the minimum of an equity measure is also common in the equity literature. These approaches seek to improve the overall equity level by improving the

condition of the worst-served group (Luss, 1999; Coluccia *et al.*, 2012; Balcik *et al.*, 2014). Although we model equity as a constraint rather than as an objective, we follow this minimax approach to ensure that the maximum deviation from a perfectly equitable food distribution remains below a specified level.

Although there are many studies of for-profit food supply chains (Akkerman *et al.*, 2010; Ahumada and Villalobos, 2009), few studies address non-profit food distribution supply chains. Cost minimization and food quality are more common objectives in for-profit supply chains, while some typical objectives for non-profit food supply chains are determining optimal food collection and distribution locations (Davis *et al.*, 2014) and the equitable and effective distribution of food.

Krejci and Beamon (2010) develop supply chain designs to address the problem of maintaining food security worldwide. The authors identify critical characteristics of environmentally sustainable food systems and use these to develop supply chain structures that are environmentally sustainable and provide food security for low, medium, and high customer demand. Schweigman *et al.* (1990) use operations research tools to address food insecurity in developing countries by determining the optimal amount of land that should be cultivated to prevent food shortage.

Solak *et al.* (2012) study tactical decision problems arising in non-profit food distribution networks. They examine a network very similar to ours with food flow from warehouses to distribution sites where they are picked up by agencies for distribution to people in need. They seek to simultaneously optimize three decisions: site selection for food delivery, assignment of agencies to these delivery sites, and routing of delivery vehicles to

minimize total transportation costs. None of the mentioned studies considers the equitable and effective allocation of food donations in a non-profit food chain under capacity constraints.

3.3 Food Distribution Model

We present a deterministic linear programming model to achieve optimal allocation of donated food considering objectives of both equity and effectiveness. The distribution of donated food is defined to be *perfectly equitable* if food donations are distributed to the counties such that the fraction of total donated food allocated to a county is exactly equal to the fraction of the total poverty population residing in that county. On the other hand, the distribution is *effective* if the amount of undistributed supply is minimized, minimizing the amount of wasted food by ensuring timely delivery of healthy, usable food to the beneficiaries.

Limited receiving capacity at the partner agencies who distribute the food creates conflict between the objectives of equity and effectiveness. To illustrate the issue, consider a simple supply chain with a single supply node and three demand nodes. The amount of supply at the supply node and the demand and capacities at each demand node are shown in Figure 3.2. A trivial solution with perfect equity is to send out no food at all, resulting in an equitable but ineffective solution with high waste. Although a zero – allocation solution is not realistic, it is an optimal solution if our sole objective is to achieve perfect equity by sending an equal amount of food to each person in poverty in the service region with no regard for effectiveness. On the other hand, in order to maximize effectiveness we should

distribute as much supply as the demand nodes can receive, which is equal to their capacities, for a total of 85 units. However, the amount of supply received per person in poverty, given by the capacity-to-demand ratio for each node, is 0.5 units at Node 1, 1.2 units at Node 2, and 0.35 units at Node 3; increased effectiveness results in a less equitable solution. In this chapter we consider both equity and effectiveness, formulating models to maximize distribution effectiveness with a specified maximum deviation from equity.

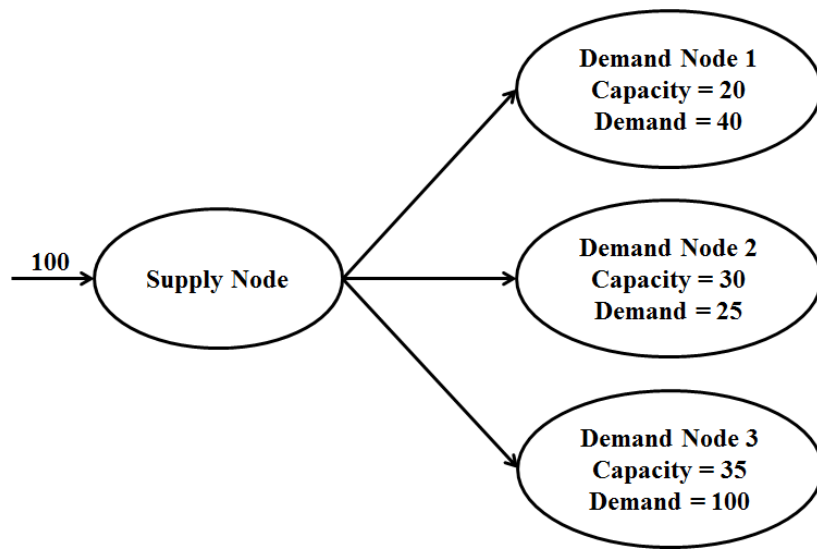


Figure 3.2: Example distribution network to illustrate the trade-off between equity and effectiveness.

Although agencies serve as food distribution points for the food-insecure population in FBCENC’s food distribution chain, our models use counties as the smallest distribution points. Therefore, any agency level data is aggregated in terms of the county in which the agency is located, and distribution is assumed to occur at the county level. Furthermore, the

models consider the allocation decisions faced by FBCENC in a single time period, such as a month. This is consistent with FBCENC's practice of distributing available food to the agencies as soon as possible given the perishable nature of the food and the imbalance between demand and supply.

The amount of donated food a county can receive from FBCENC depends on the available storage space, budget, and availability of transportation to the agencies in that county, since the agencies are responsible for picking up food from the food bank. If the donated food is perishable, additional issues such as the availability of refrigerated storage arise. Since FBCENC considers the receiving capacities of the counties to be the major constraint in their supply chain, we consider capacities at the county level. Since we make no distinction between branches in terms of capacity and no-cost interbranch flows are allowed, all five FBCENC branches are aggregated into a single supply node. The assumption of no-cost interbranch flows reflects the actual operations of the food bank since FBCENC collects donations at the branch location that is the most convenient for the donor. If additional food is required for a branch to serve its counties, the hub location (i.e., the Raleigh branch) supplies the extra food to that branch. Our models do not assess any cost for interbranch flows since this would penalize counties served from the rural branches over those served from the hub, conflicting with both the equity objective and the actual operation of the food bank.

Given a set $J = \{j: j = 1, \dots, n\}$ of n counties, we define the decision variables X_j to represent the total pounds of food to be shipped to county j , and P the pounds of undistributed food remaining after all shipments are completed. This model has the following

parameters expressed in pounds of food: (i) the total supply S ; (ii) the demand D_j in county $j \in J$; and (iii) the receiving capacity C_j of county $j \in J$. The parameter K , *the equity deviation limit*, represents the maximum allowed deviation from equity and takes values between zero and one, allowing us to explore the trade-off between equity and effectiveness. If $K = 0$, we will use the term “perfect equity” to distinguish this case from the case where $K > 0$. The Food Distribution Model can be written as follows:

$$\text{minimize } P \tag{3.1}$$

subject to

$$\left| \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} \right| \leq K \quad j \in J \tag{3.2}$$

$$S - \sum_{l=1}^n X_l - P = 0 \tag{3.3}$$

$$X_j \leq C_j \quad j \in J \tag{3.4}$$

$$X_j, P \geq 0 \quad j \in J. \tag{3.5}$$

Constraint (3.2) ensures that the proportion of all distributed food received by county j deviates from the proportion of total demand represented by county j by no more than the deviation limit K . Constraint (3.3) enforces conservation of flow at the branch level and constrains the total distribution by the amount of available supply. Constraint set (3.4) is the set of receiving capacity constraints for each county j . Constraints (3.5) are nonnegativity constraints. The objective function (3.1) minimizes the total undistributed supply over all branches. The nonlinear constraints (3.2) are mathematically equivalent to the following:

$$-K \leq \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} \leq K \quad j \in J, \quad (3.2.a)$$

which can be linearized by rewriting them as:

$$X_j \sum_{l=1}^n D_l - D_j \sum_{l=1}^n X_l + K \sum_{l=1}^n X_l \sum_{l=1}^n D_l \geq 0 \quad j \in J, \quad (3.6)$$

$$X_j \sum_{l=1}^n D_l - D_j \sum_{l=1}^n X_l - K \sum_{l=1}^n X_l \sum_{l=1}^n D_l \leq 0 \quad j \in J. \quad (3.7)$$

This model addresses the objective of effectiveness by minimizing the amount of undistributed food in the objective function, and that of equity through constraint (3.2). Equity among counties is addressed by limiting the maximum absolute deviation from a perfectly equitable solution through the equity deviation limit K . This is achieved since (3.2) is mathematically equivalent to the following:

$$\max_{j \in J} \left| \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} \right| \leq K \quad (3.2.b)$$

Minimizing the maximum absolute deviation from perfect equity is appropriate for our problem since this improves the condition of the county that receives the worst service in terms of equity, improving overall equity in the entire service region. By modeling equity as a constraint rather than as the objective and linearizing this nonlinear constraint, we can control the level of inequity permitted in the system while preserving the convexity of the feasible region.

It is evident from the formulation that the structure of the optimal solution is defined by the interactions of the total available supply S , the receiving capacities C_j and demands D_j of the counties. To this end we define three important ratios. The first of these is $DF_j = \frac{D_j}{\sum_{l=1}^n D_l}$, the fraction of total demand incurred in that county. The second is the capacity-to-demand (CD) ratio of each county $CD_j = \frac{C_j}{D_j}$. This ratio relates capacity-to-demand, but does not consider equity in any way. In order to incorporate equity, we define the modified capacity-to-demand (MCD) ratio $MCD_j = \frac{C_j}{D_j - K \sum_{l=1}^n D_l}$. Based on these, the set

$$J_0 = \left\{ j \in J \mid \frac{D_j}{\sum_{l=1}^n D_l} = DF_j > K \right\} \quad (3.8)$$

defines the set of counties j that must receive a nonzero allocation of food ($X_j > 0$) to satisfy the equity constraints (3.2). In addition, if

$$R = \min_{j \in J_0} \left\{ \frac{C_j}{D_j - K \sum_{l=1}^n D_l} \right\} = \min_{j \in J_0} \{MCD_j\}, \quad (3.9)$$

we can define the set

$$B = \left\{ j \in J_0 \mid \frac{C_j}{D_j - K \sum_{l=1}^n D_l} = R \right\} \quad (3.10)$$

as the set of bottleneck counties. Recall that the equity constraints require that the fraction of available food sent to county j , $\frac{X_j}{\sum_{l=1}^n X_l}$, deviates from the fraction of total demand required by that county, $\frac{D_j}{\sum_{l=1}^n D_l}$, by no more than the specified equity limit K . Proposition 3.1 shows that

the counties in the set B constrain the amount of food shipped to other counties with higher MCD_j ratios.

Given these preliminaries, Proposition 3.1 defines an optimal solution of the Food Distribution Model under all three possible cases that can arise in a problem instance. For this proposition, we assume that $\sum_{l=1}^n C_l \geq S$. This assumption is not limiting since if $S > \sum_{l=1}^n C_l$, at least $S - \sum_{l=1}^n C_l$ units of supply will go to waste. In that case, we can set $S = \sum_{l=1}^n C_l$ and the proposition will hold.

Proposition 3.1. *An optimal solution to a given instance of the Food Distribution Model is as follows:*

Case 1: *If $\sum_{l=1}^n C_l \geq S$ and $J_0 = \emptyset$, i.e., the instance is partially equity constrained, then the optimal objective function value is:*

$$P^* = 0, \quad (3.11)$$

and the individual distributions to the counties (X_j^) have multiple optimal solutions.*

Case 2: *If $J_0 \neq \emptyset$ and $\sum_{l=1}^n C_l \geq S \geq R \sum_{l=1}^n D_l$, i.e., the instance is capacity and equity constrained, then the optimal objective function value is:*

$$P^* = S - R \sum_{l=1}^n D_l. \quad (3.12)$$

In an optimal solution, the bottleneck counties receive an amount of food equal to their capacity,

$$X_j^* = C_j \text{ for all } j \in B. \quad (3.13)$$

Food shipments to all remaining counties will have multiple optimal solutions.

Case 3: If $J_0 \neq \emptyset$, $\sum_{l=1}^n C_l \geq S$, and $R \sum_{l=1}^n D_l > S$, i.e., the instance is supply and equity constrained, then the optimal objective function value is:

$$P^* = 0. \quad (3.14)$$

The individual distributions to the counties (X_j^*) will have multiple optimal solutions.

Proof of Proposition 3.1. The proof will consider each of the three cases separately. Let $\Delta = \sum_{l=1}^n D_l$.

Case 1: In this case we have $\frac{D_j}{\Delta} \leq K$ for all counties $j \in J$. This implies that $-K + \frac{D_j}{\Delta} \leq 0$ for all j , i.e., $J_0 = \emptyset$. Constraint (3.2) can be written as:

$$-K + \frac{D_j}{\Delta} \leq \frac{X_j}{\sum_{l=1}^n X_l} \leq K + \frac{D_j}{\Delta} \quad j \in J \quad (3.15)$$

which, due to the nonnegativity of X_j , (3.15) can be rewritten as:

$$0 \leq \frac{X_j}{\sum_{l=1}^n X_l} \leq K + \frac{D_j}{\Delta} \quad j \in J. \quad (3.16)$$

Combining (3.16) with the capacity constraint (3.4) yields

$$0 \leq X_j \leq \min \left(C_j, \left(K + \frac{D_j}{\Delta} \right) \sum_{l=1}^n X_l \right) \quad j \in J, \quad (3.17)$$

where constraints (3.3) and (3.5) ensure that

$$\sum_{l=1}^n X_l \leq S. \quad (3.18)$$

We will prove that there always exists a feasible solution set $\{X_j^*, j \in J\}$ such that $\sum_{l=1}^n X_l^* = S$.

For the given value of S , let

$$X_j = \min\left(C_j, S \frac{D_j}{\Delta}\right) \quad j \in J. \quad (3.19)$$

Let

$$X_C \equiv \{j \in J: X_j = C_j\}, \quad (3.20)$$

$$X_S \equiv \left\{j \in J: X_j = S \frac{D_j}{\Delta}\right\}. \quad (3.21)$$

If, $X_C = \emptyset$ and $X_S \neq \emptyset$, i.e., for all j , $X_j = S \frac{D_j}{\Delta}$; then $\sum_{l=1}^n X_l = S$ and an optimal solution has been obtained.

Next, we show that the case when $X_S = \emptyset$ and $X_C \neq \emptyset$ is not possible. This case would imply that for all j , $C_j < S \frac{D_j}{\Delta}$. Summing over all j , we get $\sum_{l=1}^n C_l < S$, which is a contradiction since we assume that $S \leq \sum_{l=1}^n C_l$.

The last case we consider is $X_S \neq \emptyset$ and $X_C \neq \emptyset$. For counties $l \in X_C$, define

$$E_l = S \frac{D_l}{\Delta} - C_l \quad l \in X_C. \quad (3.22)$$

The value E_l represents the extra amount of supply that must be allocated to some county $j \in X_S$ with idle capacity in order to achieve $\sum_{l=1}^n X_l = S$. The total idle capacity available must be greater than or equal to the total extra pounds of food to be shipped in order for this solution to hold, implying that

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} \left(C_l - S \frac{D_l}{\Delta} \right), \quad (3.23)$$

where $C_l - S \frac{D_l}{\Delta}$ represents the idle capacity at county $l \in X_S$.

Since we have $S \leq \sum_{l=1}^n C_l$ by assumption,

$$S \left(\sum_{l \in X_C} \frac{D_l}{\Delta} + \sum_{l \in X_S} \frac{D_l}{\Delta} \right) \leq \sum_{l \in X_C} C_l + \sum_{l \in X_S} C_l, \quad (3.24)$$

$$S \sum_{l \in X_C} \frac{D_l}{\Delta} - \sum_{l \in X_C} C_l \leq \sum_{l \in X_S} C_l - S \sum_{l \in X_S} \frac{D_l}{\Delta}, \quad (3.25)$$

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} C_l - S \sum_{l \in X_S} \frac{D_l}{\Delta}. \quad (3.26)$$

so inequality (3.23) always holds.

Therefore, by assigning this extra $\sum_{l \in X_C} E_l$ pounds of food among the counties in set X_S in an arbitrary manner, we can obtain an optimal solution with $\sum_{l=1}^n X_l^* = S$. Furthermore, since there can be different assignments to the counties with idle capacity, there can be multiple optimal solutions to the Food Distribution Model for Case 1. An algorithm for generating these alternative allocations is given in Figure 3.3.

Case 2: For Case 2 we have at least one county $j \in J$ such that $\frac{D_j}{\Delta} > K$, and $S \geq R\Delta$. Hence, constraint (3.2), in combination with the capacity constraint (3.4), can be written as:

$$\max \left(0, \left(-K + \frac{D_j}{\Delta} \right) \sum_{l=1}^n X_l \right) \leq X_j \leq \min \left(C_j, \left(K + \frac{D_j}{\Delta} \right) \sum_{l=1}^n X_l \right) \quad j \in J_0. \quad (3.27)$$

where constraints (3.3) and (3.5) ensure that

$$\sum_{l=1}^n X_l \leq S. \quad (3.28)$$

For feasibility, we must have

$$\left(-K + \frac{D_j}{\Delta}\right) \sum_{l=1}^n X_l \leq C_j \quad j \in J_0, \quad (3.29)$$

$$\sum_{l=1}^n X_l \leq \frac{C_j \Delta}{D_j - K\Delta} \quad j \in J_0, \quad (3.30)$$

$$\sum_{l=1}^n X_l \leq \min_{j \in J_0} \left\{ \frac{C_j \Delta}{D_j - K\Delta} \right\}, \quad (3.31)$$

$$\sum_{l=1}^n X_l \leq R\Delta. \quad (3.32)$$

Since this case satisfies the condition that $S \geq R\Delta$, we only need $\sum_{l=1}^n X_l \leq R\Delta$.

We will prove that there always exists a feasible solution set $\{X_j^*, j \in J\}$ that satisfies

$$\sum_{l=1}^n X_l^* = R\Delta.$$

For the given value of $R\Delta$, let

$$X_j = \min(C_j, RD_j) \quad j \in J. \quad (3.33)$$

Let

$$X_C \equiv \{j \in J: X_j = C_j\}, \quad (3.34)$$

$$X_S \equiv \{j \in J: X_j = RD_j\}. \quad (3.35)$$

If, $X_C = \emptyset$ and $X_S \neq \emptyset$, i.e., for all j , $X_j = RD_j$, then $\sum_{l=1}^n X_l = R\Delta$ is obviously an optimal and feasible solution.

Next, we show that the case when $X_S = \emptyset$ and $X_C \neq \emptyset$ is not possible. This case would imply that for all j , $C_j < RD_j$. Summing over all j , we get $\sum_{l=1}^n C_l < R\Delta \leq S$ due to the assumption of Case 2. This is a contradiction since we assume that $S \leq \sum_{l=1}^n C_l$.

The last case we consider is $X_S \neq \emptyset$ and $X_C \neq \emptyset$. For counties $l \in X_C$, define

$$E_l = RD_l - C_l \quad l \in X_C. \quad (3.36)$$

The value E_l represents the extra amount of supply to be allocated to any county $j \in X_S$ with idle capacity in order to achieve $\sum_{l=1}^n X_l = R\Delta$. This E_l pounds of food can be allocated. The total idle capacity available should be greater than or equal to the total extra pounds of food to be shipped in order for this solution to hold. So, we must have:

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} (C_l - RD_l), \quad (3.37)$$

where $C_l - RD_l$ represents the idle capacity at county $l \in X_S$.

By using the main assumption of $S \leq \sum_{l=1}^n C_l$ and the condition of this case that $S \geq R\Delta$, we get $\sum_{l=1}^n C_l \geq R\Delta$. It follows that,

$$R \left(\sum_{l \in X_C} D_l + \sum_{l \in X_S} D_l \right) \leq \sum_{l \in X_C} C_l + \sum_{l \in X_S} C_l, \quad (3.38)$$

$$R \sum_{l \in X_C} D_l - \sum_{l \in X_C} C_l \leq \sum_{l \in X_S} C_l - R \sum_{l \in X_S} D_l, \quad (3.39)$$

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} C_l - R \sum_{l \in X_S} D_l. \quad (3.40)$$

So, inequality (3.37) always holds.

Therefore, by assigning this extra $\sum_{l \in X_C} E_l$ pounds of food to the counties in set X_S in an arbitrary manner, we can obtain the optimal solution of $\sum_{l=1}^n X_l^* = R\Delta$. Furthermore, since there can be different assignments to the counties with idle capacity, this shows that there can be multiple optimal solutions to the Food Distribution Model for Case 2.

Proof for distribution to bottleneck counties: According to the definition of a bottleneck county given in Proposition 3.1, bottleneck counties are those with the minimum $\left\{ \frac{C_j}{D_j - K\Delta} \right\}$ ratio among the counties $j \in J_0$. Then, for $j \in B$, since we have shown that $\sum_{l=1}^n X_l^* = R\Delta$, from Constraint (3.2), we have

$$\left(-K + \frac{D_j}{\Delta}\right)R\Delta \leq X_j^* \leq \min\left(C_j, \left(K + \frac{D_j}{\Delta}\right)R\Delta\right) \quad j \in B, \quad (3.41)$$

$$\left(-K + \frac{D_j}{\Delta}\right)\left(\frac{C_j\Delta}{D_j - K\Delta}\right) \leq X_j^* \leq \min\left(C_j, \left(K + \frac{D_j}{\Delta}\right)\left(\frac{C_j\Delta}{D_j - K\Delta}\right)\right) \quad j \in B, \quad (3.42)$$

$$C_j \leq X_j^* \leq C_j \quad j \in B. \quad (3.43)$$

It follows that

$$X_j^* = C_j \text{ for } j \in B. \quad (3.44)$$

Case 3: Case 3 considers the situation where there exists at least one county $j \in J$ such that $\frac{D_j}{\Delta} > K$, and $S < R\Delta$. The proof for this case follows from a combination of the proofs for Cases 1 and 2. We can apply the equations (3.27)-(3.32) exactly to this case. However, due to

the condition of $S < R\Delta$ that this case satisfies, we only need $\sum_{l=1}^n X_l \leq S$. The remaining argument follows along the lines of Case 1 as given in (3.19)-(3.26). The proof is completed.

■

In Case 1 of Proposition 3.1, the equity constraints do not prevent the entire supply from being shipped since every county has $K \geq \frac{D_j}{\sum_{l=1}^n D_l}$. In Cases 2 and 3 the set J_0 is nonempty, implying that the counties $j \in J_0$ will limit the total distribution. In Case 3 there is sufficient capacity for all supply to be distributed, while in Case 2 the equity constraints prevent all supply from being distributed.

The key insight from Proposition 3.1 is the critical role of the MCD ratios in determining the structure of the solution. The MCD ratios combine information on capacity, equity and demand into a quantitative measure of the equity-effectiveness trade-off. Very low MCD ratios may be due to low capacities or a low equity deviation limit. In either case, the optimal food distribution is also low, resulting in high waste. Proposition 3.1 also highlights the fact that all three cases result in multiple optimal solutions for the individual flows X_j to the nonbottleneck counties. We now introduce a Multiple Optima Generation (MOG) algorithm for generating alternative optimal solutions in Figure 3.3. This algorithm uses insights from Proposition 3.1 and its proof to examine alternatives for allocating donated food among the counties in the service region while shipping the maximum possible amount and maintaining the specified level of equity. The correctness of the algorithm follows directly from the proof of Proposition 3.1.

MOG Algorithm first determines which case of Proposition 3.1 applies to the problem instance at hand. If Case 2 applies, all bottleneck counties will receive food equal to their capacity. Once the demand of the bottleneck counties has been met in this fashion, a feasible solution to the Food Distribution Model satisfying $\sum_{j=1}^n X_j^* = R \sum_{l=1}^n D_l$ as shown in Case 2 of Proposition 3.1 is given by $X_j^* = RD_j$. If $RD_j < C_j$, we set $X_j^* = RD_j$ and add county j to the set J_E of counties with excess capacity. After this is done for all $j \in J$, we sort the indices $j \in J_E$ in random order. Then, for all $l \in J \setminus J_E$, we set $X_l^* = C_l$ and calculate the amount of remaining food $E_l = RD_l - C_l$ that must be reallocated from county l in order to achieve the optimal total distribution. We then distribute these extra E_l pounds of food among the counties $j \in J_E$ without violating any capacity constraints. This is done by allocating E_l pounds of food to any county $j \in J_E$ with excess capacity until its capacity is saturated and moving on to the next county with excess capacity, until all E_l pounds of food have been allocated. This allocation scheme results in an optimal solution as discussed in the proof of Proposition 3.1. The second part of the algorithm considers Cases 1 and 3 of Proposition 3.1, the supply constrained instances. In these cases a solution to the Food Distribution Model satisfying $\sum_{j=1}^n X_j^* = S$ is given by $X_j^* = \frac{SD_j}{\sum_{l=1}^n D_l}$ and we use this as a basis to assign food to the counties. Different orderings of the set J_E yield different optimal solutions to the Food Distribution Model. The worst-case time complexity of the MOG Algorithm is $O(n^2)$.

Multiple Optima Generation (MOG) Algorithm

$\Delta = \sum_{l=1}^n D_l, J_0 \leftarrow \{j \in J \mid \frac{D_j}{\Delta} > K\}, R = \min_{j \in J_0} \left\{ \frac{C_j}{D_j - K\Delta} \right\}, B = \left\{ j \in J_0 \mid \frac{C_j}{D_j - K\Delta} = R \right\}$
 $X_j \leftarrow 0, E_j \leftarrow 0, \forall j \in J$
 $J_E \leftarrow \emptyset, B_E \leftarrow \emptyset$
if $R\Delta \leq S$ && $|J_0| > 0$ **then**
 for $j = 1$ **to** n **do**
 if $j \in B$ **then**
 $X_j \leftarrow C_j$
 else if $RD_j \leq C_j$ **then**
 $X_j \leftarrow RD_j, J_E \leftarrow J_E \cup \{j\}$
 end if
 end for
 Sort $j \in J_E$ in a random order
 for $l = 1$ **to** n **do**
 if $l \notin J_E$ **then**
 $X_l \leftarrow C_l, E_l \leftarrow RD_l - C_l$
 $B_E \leftarrow B_E \cup \{l\}$
 for all $j \in J_E$ **do**
 if $X_j + E_l \leq C_j$ **then**
 $X_j \leftarrow X_j + E_l, E_l \leftarrow 0$, Continue to next l
 else
 $E_l \leftarrow E_l - (C_j - X_j), X_j \leftarrow C_j, B_E \leftarrow B_E \cup \{j\}$
 end if
 end for
 end if
 end for
else
 for $j = 1$ **to** n **do**
 if $\frac{SD_j}{\Delta} \leq C_j$ **then**
 $X_j \leftarrow \frac{SD_j}{\Delta}, J_E \leftarrow J_E \cup \{j\}$
 end if
 end for
 Sort $j \in J_E$ in a random order
 for $l = 1$ **to** n **do**
 if $l \notin J_E$ **then**
 $X_l \leftarrow C_l, E_l \leftarrow \frac{SD_l}{\Delta} - C_l, B_E \leftarrow B_E \cup \{l\}$
 for all $j \in J_E$ **do**
 if $X_j + E_l \leq C_j$ **then**
 $X_j \leftarrow X_j + E_l, E_l \leftarrow 0$, Continue to next l
 else
 $E_l \leftarrow E_l - (C_j - X_j), X_j \leftarrow C_j, B_E \leftarrow B_E \cup \{j\}$
 end if
 end for
 end if
 end for
end if

Figure 3.3: Multiple Optima Generation (MOG) Algorithm.

In order to illustrate the operation of the MOG Algorithm, consider the network introduced in Figure 3.2 with $K = 0.01$. This instance is capacity and equity constrained since $R \sum_{l=1}^n D_l = 58.718 < S = 100$, and hence has multiple optimal solutions by Proposition 3.1, with County 3 being the bottleneck county. Table 3.1 shows two alternative optimal solutions to this instance that suggest different shipments to Counties 1 and 2 and maintain the same total distribution.

Table 3.1: Alternative optimal solutions to the simple network in Figure 3.2.

	<i>Alternative Solution 1</i>	<i>Alternative Solution 2</i>
X_1^*	14.2349	14.8221
X_2^*	9.484	8.8968
X_3^*	35	35
$\sum_{l=1}^3 X_l$	58.7189	58.7189

Corollary 3.1 applies Proposition 3.1 to the *perfect equity* case of $K = 0$.

Corollary 3.1. *Let $K = 0$ and assume that $\sum_{l=1}^n C_l \geq S$. In the optimal solution to the Food Distribution Model, if $S \geq R \sum_{l=1}^n D_l$, i.e., the solution is capacity constrained, then $X_j^* = C_j$ for $j \in B$ and $X_j^* < C_j$ for $j \notin B$. On the other hand, if $R \sum_{l=1}^n D_l > S$, i.e., the solution is supply constrained, then $X_j^* < C_j$ for all j . The optimal solution is given by:*

$$X_j^* = \min \left\{ S \frac{D_j}{\sum_{l=1}^n D_l}, RD_j \right\} \quad j \in J \quad (3.45)$$

$$P^* = \max \left\{ 0, S - R \sum_{l=1}^n D_l \right\} \quad (3.46)$$

For the supply constrained perfect equity case, supply is distributed among the counties according to their DF ratios, with each county receiving less food than its capacity. If the instance is capacity constrained, the bottleneck counties $j \in B$ receive food equal to their capacities, while the remaining counties $j \in J \setminus B$ will have excess capacity. Therefore the optimal effectiveness under perfect equity is determined by the minimum CD ratio among all counties. Under perfect equity, Constraint (3.2) becomes $X_j = \frac{D_j}{\sum_{l=1}^n D_l} \sum_{l=1}^n X_l$, yielding a unique optimal solution. If the instance is capacity constrained, then $X_j^* = RD_j \leq C_j$ for all $j \in J$ by definition and $\sum_{l=1}^n X_l^* = R \sum_{l=1}^n D_l$. On the other hand, if the instance is supply constrained, then $X_j^* = S \frac{D_j}{\sum_{l=1}^n D_l} < RD_j \leq C_j$ for all $j \in J$ by assumption and $\sum_{l=1}^n X_l^* = S$.

Since demand is treated as exogenous to the model, Proposition 3.1 and Corollary 3.1 imply that for a capacity constrained instance, the amount of food distributed cannot be increased while maintaining equity unless the capacities of the bottleneck counties are increased. We now examine how to increase the capacities of the bottleneck counties to increase distribution effectiveness while preserving equity.

3.4 Capacity Allocation Problem

A problem of interest to FBCENC is how to allocate a given amount of additional capacity among the counties such that equity is maintained and effectiveness is increased.

This additional capacity can be in the form of storage capacity like extra freezer space, money to increase an agency's purchasing budget, opening a new agency in a county, or increasing distribution capacity by means such as mobile pantries or additional delivery vehicles. FBCENC often solicits grant or donor funding to purchase such additional capacity to enable agencies to distribute more food in their areas. Our results in this section are helpful to the food banks in developing proposals for funding to external sponsors by quantifying the benefit of additional capacity in terms of the additional food that can be distributed.

We now develop a linear programming model to determine the optimal allocation of additional capacity to the counties that will maximize the increase in effectiveness while maintaining the desired level of equity. We do not consider any costs associated with the allocation of additional capacity to counties, as our focus is to identify priorities for the allocation of additional capacity. We only consider counties in set J_0 since only these counties have the potential to make the problem capacity constrained as discussed in Section 3.3.

Without loss of generality, we assume the counties in set J_0 are indexed in increasing order of their unique MCD ratios, i.e., for $j, g \in J_0$ and $j < g$, we have $\frac{C_j}{D_j - K \sum_{l=1}^n D_l} < \frac{C_g}{D_g - K \sum_{l=1}^n D_l}$. The assumption of unique MCD ratios is not restrictive since both capacity and demand are expressed in tons of food, and are hence large, continuous quantities. In the Capacity Allocation Model, the only new parameter we introduce is Y , the amount of extra capacity available to the food bank, in pounds of food. The decision variable ρ_j represents the fraction of the additional capacity Y allocated to county j , and Q the minimum MCD ratio

among the counties after the additional capacity has been allocated. We formulate the Capacity Allocation Model as follows:

$$\text{maximize } Q \quad (3.47)$$

subject to

$$Q \leq \frac{C_j + \rho_j Y}{D_j - K \sum_{l=1}^n D_l} \quad j \in J_0 \quad (3.48)$$

$$\sum_{l \in J_0} \rho_l \leq 1 \quad (3.49)$$

$$Q, \rho_j \geq 0 \quad j \in J_0. \quad (3.50)$$

The objective of this model is to maximize the minimum MCD ratio, i.e., the value of the MCD ratio of all the bottleneck counties, as implemented in the objective function (3.47) and constraints (3.48). Constraint (3.49) ensures that the total capacity distributed cannot exceed the available additional capacity Y . Constraints (3.50) are nonnegativity constraints.

We propose an efficient iterative algorithm to solve the Capacity Allocation Model exactly, and show that this can be used to obtain a closed-form solution. The algorithm is based on the insight from Proposition 3.1: the total amount shipped cannot be increased while preserving equity unless the MCD ratio of the bottleneck counties is increased. In other words, since the problem is to maximize the minimum MCD ratio, the algorithm tries to maximize the number of bottleneck counties with the same MCD ratio. Thus at each iteration the algorithm identifies the MCD ratio of the current bottleneck counties, and the county with the next lowest MCD ratio, which we shall call the target MCD ratio. It then tries to allocate additional capacity to all current bottleneck counties to raise their MCD ratios to the target

MCD ratio. If this is successful, the county associated with the target MCD ratio is added to the set of bottleneck counties and a new iteration begins. If there is insufficient additional capacity available to accomplish this, any as yet unallocated additional capacity is allocated among the current bottleneck counties such that they all have equal MCD ratios, and the algorithm terminates. We now give a more detailed description of the algorithm, which is formally stated in Figure 3.4.

For a given iteration z , let R_z denote the minimum MCD ratio at that iteration, T_z the second-lowest MCD ratio, and $B_z = \{j \in J | \text{MCD}_j = R_z\}$ the current set of bottleneck counties. The variable Y_z denotes the amount of capacity in pounds remaining to be distributed in the beginning of iteration z .

The algorithm is initialized with $z = 1$, $B_1 = \{1\}$ and $Y_1 = Y$. At each iteration z , we first check whether the maximum number of iterations $|J_0|$ has been reached. If not, we determine R_z , B_z , T_z and the associated county u with $\text{MCD}_u = T_z$. We also calculate φ_z , the amount of additional capacity that must be allocated to increase the capacities of the counties $j \in B_z$ such that their MCD ratios become equal to T_z . Note that counties in B_z will not receive equal allocations of additional capacity, since they have different capacities and demands. If $Y_z \geq \varphi_z$, we distribute the φ_z units of capacity among the counties in set B_z such that their revised MCD ratios are equal to T_z . In Figure 3.4, for each iteration z of the algorithm and each county $j \in B_z$, ρ_j represents the fraction of the additional capacity Y allocated to county j prior to iteration z so that $C_j + \rho_j Y$ is the allocated capacity of county j just before iteration z is performed. If we let W_l^z denote the additional capacity to be

allocated to county l at iteration z , this means that the revised MCD ratios for the counties in B_z must satisfy

$$\frac{C_l + \rho_l Y + W_l^z}{D_l - K\Delta} = T_z \quad l \in B_z, \quad (3.51)$$

where $\Delta = \sum_{l=1}^n D_l$, from which it follows immediately that we must have

$$W_l^z = T_z D_l - C_l - \rho_l Y - K T_z \Delta \quad l \in B_z; \quad (3.52)$$

and since by definition

$$\varphi_z = \sum_{l \in B_z} W_l^z, \quad (3.53)$$

the assignment

$$\varphi_z \leftarrow \sum_{l \in B_z} (T_z D_l - C_l - \rho_l Y - K T_z \Delta) \quad (3.54)$$

in Figure 3.4 must hold. Notice that $B_{z+1} = B_z \cup \{u\}$, increasing the number of bottleneck counties by one.

If $Y_z < \varphi_z$, then there is not enough unallocated capacity remaining to move to iteration $z + 1$. In this case, we distribute the remaining capacity on hand among the counties in set B_z such that their MCD ratios (the current R_z) increase but remain equal to each other. We set $Q = R_z$ and terminate the algorithm. If we reach iteration $|J_0|$, we distribute the remaining capacity on hand among all counties such that the MCD ratios remain equal to each other, set $Q = R_z$ and terminate. This case occurs only if Y is sufficiently large to make

every county's MCD ratio equal to the maximum MCD ratio. The worst-case time complexity of the Capacity Allocation Algorithm is $O(n^2)$.

Capacity Allocation Algorithm

$$\Delta = \sum_{l=1}^n D_l, J_0 \leftarrow \left\{ j \in J \mid \frac{D_j}{\Delta} > K \right\}$$

$$\rho_j \leftarrow 0, \forall j \in J_0$$

$$z \leftarrow 1, B_z \leftarrow \{1\}, Y_z \leftarrow Y$$

while $z \leq |J_0|$ **do**

if $z \neq |J_0|$ **then**

$R_z \leftarrow \min_{j \in J_0} \left\{ \frac{C_j + \rho_j Y}{D_j - K\Delta} \right\}, B_z \leftarrow \left\{ j \in J_0 \mid \frac{C_j + \rho_j Y}{D_j - K\Delta} = R_z \right\}$

$T_z \leftarrow \min_{j \in J_0 \setminus B_z} \left\{ \frac{C_j}{D_j - K\Delta} \right\}, B_{z+1} \leftarrow \left\{ j \in J_0 \mid \frac{C_j}{D_j - K\Delta} = T_z \right\}$

$\varphi_z \leftarrow \sum_{l \in B_z} (T_z D_l - C_l - \rho_l Y - K T_z \Delta)$

if $Y_z \geq \varphi_z$ **then**

$\rho_j \leftarrow \frac{T_z D_j - C_j - K T_z \Delta}{Y}, \forall j \in B_z$

$Y_{z+1} \leftarrow Y_z - \varphi_z, z \leftarrow z + 1$

else

$R_z \leftarrow \frac{Y_z + \sum_{l=1}^{|B_z|} (C_l + \rho_l Y)}{\sum_{l=1}^{|B_z|} (D_l - K\Delta)}$

$\rho_j^* \leftarrow \frac{R_z D_j - C_j - K R_z \Delta}{Y}, \forall j \in B_z$

$Q^* \leftarrow R_z$

$\eta \leftarrow z$

Quit

end if

else

$R_z \leftarrow \frac{Y_z + \sum_{l=1}^{|J_0|} (C_l + \rho_l Y)}{\sum_{l=1}^{|J_0|} (D_l - K\Delta)}$

$\rho_j^* \leftarrow \frac{R_z D_j - C_j - K R_z \Delta}{Y}, \forall j \in J_0$

$Q^* \leftarrow R_z$

$\eta \leftarrow z$

Quit

end if

end while

Figure 3.4: Capacity Allocation Algorithm.

In order to prove the optimality of the Capacity Allocation Algorithm, we first show that in the optimal solution to the Capacity Allocation Model, all additional capacity Y will be distributed.

Proposition 3.2. *In the optimal solution to the Capacity Allocation Algorithm, Constraint (3.49) will be satisfied at strict equality.*

Proof of Proposition 3.2. Let $\Delta = \sum_{l=1}^n D_l$. If we associate dual variables π_l with the constraints (3.48) and π_0 with constraint (3.49), then the dual of the Capacity Allocation Model can be written as

$$\min \sum_{l \in J_0} C_l \pi_l + \pi_0 \quad (3.55)$$

$$\sum_{l \in J_0} (D_l - K\Delta) \pi_l \geq 1 \quad (3.56)$$

$$-Y\pi_j + \pi_0 \geq 0 \quad j \in J_0 \quad (3.57)$$

$$\pi_j \geq 0 \quad j \in J_0 \cup \{0\} \quad (3.58)$$

From (3.57), we see that

$$\pi_j \leq \pi_0/Y \quad j \in J_0 \quad (3.59)$$

and thus from (3.56), we have

$$\sum_{l \in J_0} \frac{(D_l - K\Delta)\pi_0}{Y} \geq \sum_{l \in J_0} (D_l - K\Delta)\pi_l \geq 1. \quad (3.60)$$

From (3.60), we have

$$\pi_0 \geq \frac{Y}{\sum_{l \in J_0} (D_l - K\Delta)} > 0 \quad (3.61)$$

and therefore by the complementary slackness theorem (Bertsimas & Tsitsiklis, 1997) constraint (3.49) must be satisfied at strict equality. ■

Proposition 3.3 outlines the structure of the optimal solution for the Capacity Allocation Model.

Proposition 3.3. *In the optimal solution to the Capacity Allocation Model, there will be η bottleneck counties where*

$$\eta = \underset{1 \leq \xi \leq |J_0|}{\operatorname{argmax}} \left\{ \sum_{l=1}^{\xi-1} \left(\frac{C_\xi (D_l - K\Delta)}{D_\xi - K\Delta} - C_l \right) \leq Y \right\} \quad (3.62)$$

and the summation in (3.62) is taken to be zero when $\xi = 1$. The optimal solution is given by:

$$Q^* = \frac{Y + \sum_{g=1}^{\eta} C_g}{\sum_{l=1}^{\eta} (D_l - K\Delta)}, \quad (3.63)$$

$$\rho_j^* = \begin{cases} \frac{Q^*(D_j - K\Delta) - C_j}{Y} & \text{for } 1 \leq j \leq \eta, \\ 0 & \text{for } \eta + 1 \leq j \leq |J_0|. \end{cases} \quad (3.64)$$

Proof of Proposition 3.3. Let $\Delta = \sum_{l=1}^n D_l$. Direct inspection of the Capacity Allocation Algorithm suggests that the solution obtained from the Capacity Allocation Algorithm is optimal. Here, we will use the solution obtained from the algorithm and prove that it is optimal. By the operation of the Capacity Allocation Algorithm, if we terminate the algorithm with η bottleneck counties, the algorithm must terminate at iteration η . All

bottleneck counties have MCD ratios of at least $\frac{C_\eta}{D_\eta - K\Delta}$. Since the algorithm terminated at iteration η , there was not sufficient capacity to perform the next iteration, so the optimal number of bottleneck counties is

$$\eta = \underset{1 \leq \xi \leq |J_0|}{\operatorname{argmax}} \left\{ \sum_{l=1}^{\xi-1} \left(\frac{C_\xi(D_l - K\Delta)}{D_\xi - K\Delta} - C_l \right) \leq Y \right\} \quad (3.62)$$

In order to show how equation (3.62) is obtained, let W_j denote the total additional capacity to be allocated to county j as a result of the Capacity Allocation Algorithm where $j \leq \eta$. Then, to reach iteration $\eta > 1$, we need

$$\frac{C_j + W_j}{D_j - K\Delta} = \frac{C_\eta}{D_\eta - K\Delta} \quad j < \eta \quad (3.65)$$

$$W_j = \frac{C_\eta(D_j - K\Delta)}{D_\eta - K\Delta} - C_j \quad j < \eta \quad (3.66)$$

This implies that the total additional capacity needed to reach iteration η is:

$$\sum_{l=1}^{\eta-1} W_l = \sum_{l=1}^{\eta-1} \left(\frac{C_\eta(D_l - K\Delta)}{D_\eta - K\Delta} - C_l \right) \quad (3.67)$$

and should satisfy

$$\sum_{l=1}^{\eta-1} W_l = \sum_{l=1}^{\eta-1} \left(\frac{C_\eta(D_l - K\Delta)}{D_\eta - K\Delta} - C_l \right) \leq Y. \quad (3.68)$$

The equation (3.62) follows directly. The summation in (3.62) is taken to be zero when $\xi = 1$, from which it follows that we stop at iteration $\eta = 1$.

The solution $[Q^*, (\rho_j^*; j \in J_0)]$, as given in Proposition 3.3, is feasible for the Capacity Allocation Model. Let $[(\pi_j, j \in J_0), \pi_0]$ represent the corresponding dual solution. The vectors $[Q^*, (\rho_j^*; j \in J_0)]$ and $[(\pi_j, j \in J_0), \pi_0]$ are optimal solutions for the two respective problems if and only if, by the Complementary Slackness Theorem (Bertsimas & Tsitsiklis, 1997), $[(\pi_j, j \in J_0), \pi_0]$ is a feasible dual solution and they satisfy the following:

$$\pi_j(Q^*D_j - KQ\Delta - \rho_j^*Y - C_j) = 0 \quad j \in J_0 \quad (3.69)$$

$$\pi_0 \left(\sum_{l \in J_0} \rho_l^* - 1 \right) = 0 \quad (3.70)$$

$$\left(\sum_{l \in J_0} (D_l - K\Delta)\pi_l - 1 \right) Q = 0 \quad (3.71)$$

$$(-Y\pi_j + \pi_0)\rho_j^* = 0 \quad j \in J_0. \quad (3.72)$$

Since by Proposition 3.2, Constraint (3.49) is satisfied at equality in an optimal solution, we obtain no additional information from (3.70).

Assume that the number of bottleneck counties obtained from (3.62) is η . From (3.69), using the proposed optimal solution in Proposition 3.3, for $j \leq \eta$,

$$\begin{aligned} Q^*D_j - KQ^*\Delta - \rho_j^*Y - C_j &= Q^*D_j - KQ^*\Delta - \frac{Q^*(D_j - K\Delta) - C_j}{Y}Y - C_j \\ &= 0. \end{aligned} \quad (3.73)$$

From Equations (3.58) and (3.69),

$$\pi_j \geq 0 \text{ for } 1 \leq j \leq \eta. \quad (3.74)$$

For $\eta + 1 \leq j \leq n$, from Equation (3.69) and using the proposed optimal solution in Proposition 3.3,

$$\begin{aligned} Q^*D_j - KQ^*\Delta - \rho_j^*Y - C_j &= \frac{D_j(Y + \sum_{g=1}^{\eta} C_g)}{\sum_{l=1}^{\eta} (D_l - K\Delta)} - \frac{K\Delta(Y + \sum_{g=1}^{\eta} C_g)}{\sum_{l=1}^{\eta} (D_l - K\Delta)} - C_j \\ &= \frac{D_j(Y + \sum_{g=1}^{\eta} C_g) - K\Delta(Y + \sum_{g=1}^{\eta} C_g) - C_j \sum_{l=1}^{\eta} (D_l - K\Delta)}{\sum_{l=1}^{\eta} (D_l - K\Delta)}. \end{aligned} \quad (3.75)$$

Based on the termination condition of the Capacity Allocation Algorithm, if $\eta < |J_0|$, then we have

$$\frac{C_{|J_0|}}{D_{|J_0|} - K\Delta} > \dots > \frac{C_{\eta+1}}{D_{\eta+1} - K\Delta} > \frac{Y + \sum_{l=1}^{\eta} C_l}{\sum_{l=1}^{\eta} (D_l - K\Delta)} = R_{\eta}; \quad (3.76)$$

and therefore we have,

$$\frac{C_j}{D_j - K\Delta} > \frac{Y + \sum_{l=1}^{\eta} C_l}{\sum_{l=1}^{\eta} (D_l - K\Delta)} \text{ for } \eta + 1 \leq j \leq |J_0|. \quad (3.77)$$

It follows that

$$D_j \left(Y + \sum_{l=1}^{\eta} C_l \right) - K\Delta \left(Y + \sum_{l=1}^{\eta} C_l \right) - C_j \sum_{l=1}^{\eta} (D_l - K\Delta) < 0 \quad (3.78)$$

for $\eta + 1 \leq j \leq |J_0|$. Since this is the numerator of equation (3.75), from (3.69), it follows that

$$\pi_j = 0 \text{ for } \eta + 1 \leq j \leq |J_0|. \quad (3.79)$$

If $\eta = |J_0|$, equations (3.75) – (3.79) are not needed.

We can assume that $Q > 0$ because it represents the minimum MCD ratio after capacity allocation. Then, from (3.71), we have

$$\sum_{l \in J_0} (D_l - K\Delta)\pi_l = 1. \quad (3.80)$$

Using (3.79), we can rewrite (3.80) as

$$\sum_{l=1}^{\eta} (D_l - K\Delta)\pi_l = 1.$$

By using the proposed solution, without loss of generality, we can assume that $\rho_j \geq \varepsilon$ for $j \leq \eta$, where ε is a small positive number. Then, from (3.70), it follows that

$$-Y\pi_j + \pi_0 = 0 \quad \text{for } j \leq \eta \quad (3.81)$$

$$\pi_j = \frac{\pi_0}{Y} \quad \text{for } j \leq \eta \quad (3.82)$$

By inserting (3.83) into (3.81),

$$\sum_{l=1}^{\eta} (D_l - K\Delta) \frac{\pi_0}{Y} = \frac{\pi_0}{Y} \sum_{l=1}^{\eta} (D_l - K\Delta) = 1 \quad (3.83)$$

$$\pi_0 = \frac{Y}{\sum_{l=1}^{\eta} (D_l - K\Delta)}. \quad (3.84)$$

By inserting (3.85) into (3.83), we get

$$\pi_j = \frac{1}{\sum_{l=1}^{\eta} (D_l - K\Delta)} \quad \text{for } j \leq \eta. \quad (3.85)$$

This result is also intuitive; π_j is the marginal benefit of increasing C_j by one unit. If we examine the structure of Q^* in Equation (3.63), we can see that if C_j for $j \leq \eta$ is increased by one unit, Q^* , which is the optimal objective function value, increases by $\left(\frac{1}{\sum_{l=1}^{\eta}(D_l - K\Delta)}\right)$. This solution, $[(\pi_j, j \in J_0), \pi_0]$, as given by equations (3.78), (3.84) and (3.85), is also feasible for the dual problem since it satisfies constraints (3.66)-(3.68).

Finally, calculating the dual objective function value,

$$Z_{dual} = \sum_{l \in J_0} C_l \pi_l + \pi_0 = \frac{\sum_{l=1}^{\eta} C_l}{\sum_{l=1}^{\eta} (D_l - K\Delta)} + \frac{Y}{\sum_{l=1}^{\eta} (D_l - K\Delta)} = Q^* = Z_{primal}. \quad (3.86)$$

Hence, by the Strong Duality Theorem (Bertsimas and Tsitsiklis, 1997), the proposed solution is optimal. ■

Corollary 3.2 follows from Proposition 3.3 for the *perfect equity* case with $K = 0$.

Corollary 3.2. *Let $K = 0$. Then, in the optimal solution to the Capacity Allocation Model, there will be η bottleneck counties where*

$$\eta = \operatorname{argmax}_{1 \leq \xi \leq n} \left\{ \sum_{l=1}^{\xi-1} \left(\frac{C_{\xi} D_l}{D_{\xi}} - C_l \right) \leq Y \right\} \quad (3.87)$$

The optimal solution is given by:

$$Q^* = \frac{Y + \sum_{g=1}^{\eta} C_g}{\sum_{l=1}^{\eta} D_l}, \quad (3.88)$$

$$\rho_j^* = \begin{cases} \frac{Q^* D_j - C_j}{Y} & \text{for } 1 \leq j \leq \eta, \\ 0 & \text{for } \eta + 1 \leq j \leq n. \end{cases} \quad (3.89)$$

Proposition 3.3 and Corollary 3.2 provide closed-form solutions to the Capacity Allocation Model. The fraction of the additional capacity a county receives is an increasing linear function of its demand and a decreasing linear function of its capacity, so that more densely populated areas receive more capacity, as expected. The additional capacity Y on hand is a parameter in this formulation; if the value of K is also known, then the increase in the distribution effectiveness (total food distributed) can be calculated directly using the results from Propositions 3.1 and 3.3. These ideas are illustrated in the next section.

3.5 Computational Results and Discussion

3.5.1 A Case Study

In this section we use data obtained from FBCENC to illustrate the findings from the previous sections. The results are useful for understanding the structure of FBCENC's non-profit supply chain, highlighting problem areas in this system and suggesting policies to be applied under different scenarios.

FBCENC classifies donated goods into four categories: dry goods, produce, refrigerated food and frozen food. This chapter focuses on a single food category (dry goods) which constitutes the largest category in terms of the amount of food donations received by FBCENC. However, our models could be implemented for other food categories as well. The models are solved using IBM ILOG Optimization Programming Language (IBM, 2013).

The actual donations made to FBCENC's five branches during January 2009 constitute the supply data which, since our models assume a single supply node, is

aggregated over all five branches. The donations data are given in pounds of food. The total amount of donated dry goods during January 2009 was 1,277,363 pounds and corresponds to approximately 45% of all donations during this period.

In this problem, it is not possible to predict demand with certainty although it is reasonable to assume it is proportional to the poverty populations in the counties. However, the definition of the poverty population is itself somewhat arbitrary, since families and individuals continually enter and leave this population for reasons such as relocation, job loss or new employment, and the poverty population includes some whose income may exceed the federal poverty line. The poverty populations of the counties for the year 2009 are obtained from Census data (United States Census Bureau, 2009).

FBCENC does not have exact data on the receiving capacities of the counties in its service region. However, it is reasonable to use the historical amount of food sent to the counties to estimate the counties' capacities based on how much each county was able to receive in the past. The 90th percentile of the empirical distribution of the amount of food shipped to that county each month during fiscal year 2009 is used as an estimate of each county's capacity. The 90th percentile statistic was chosen because a value belonging to the upper quartile of the data was desired so it represents what the county is able to receive. However, it should not be as extreme as the maximum since using the maximum could result in overestimating a county's capacity and resulting waste of allocated food.

We first show the results for the perfect equity case and then illustrate how the optimal solution changes as we increase the equity deviation limit K . Setting $K = 0$ in the Food Distribution Model requires that each county receives their exact relative need from the

total food distribution according to the demand residing in that county. In Figure 3.5, the horizontal axis shows the 34 counties in the FBCENC service region in increasing order of their capacity-to-demand (CD) ratios. The dark grey line shows the ratios of the shipments X_j received by the counties to their capacities C_j on the primary vertical axis. The bars represent the CD ratios of the counties with their values on the secondary vertical axis. The shipment to capacity ratio for the bottleneck county, Wilson, is equal to one. All the other counties receive less food than their capacities, and hence are penalized by the requirement of perfectly equitable distribution across counties with different capacities.

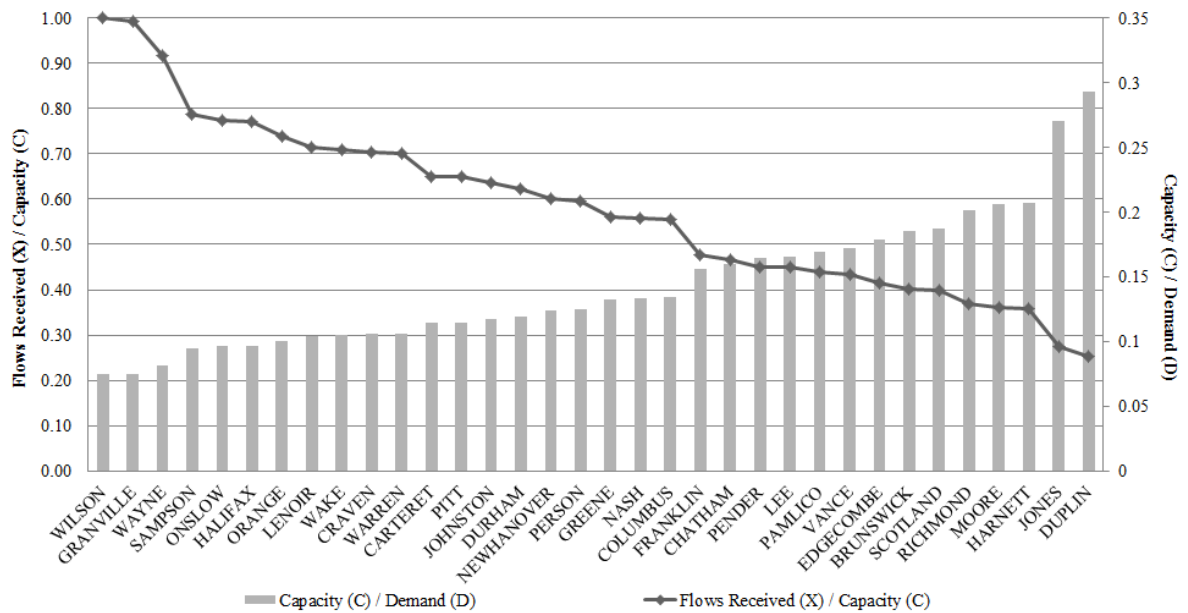


Figure 3.5: The results from the Food Distribution Model for $K = 0$.

To explore the relationship between equity and effectiveness, we use the Food Distribution Model to construct the efficient frontier between the undistributed supply and the permitted deviation from a completely equitable solution. The total undistributed supply decreases as K is increased. Figure 3.6 illustrates this relationship where K is varied between zero and 0.01 with increments of 0.001. We observe that a value of $K = 0.0047$ results in the distribution of all the supply. For $K > 0.0047$ the solution becomes supply constrained as explained in Section 3.3. Thus, permitting a slight deviation from equity of less than 1% results in considerable decrease in waste.

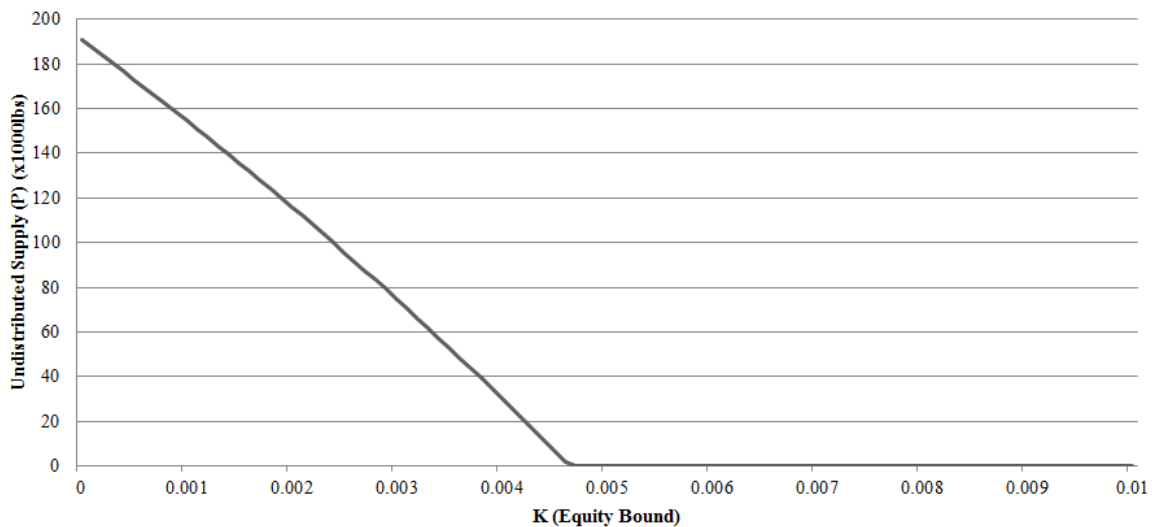


Figure 3.6: P^* Solutions from the Food Distribution Model for increasing K .

We illustrate the results from the Capacity Allocation Model for $K = 0$ in Figures 3.7 and 3.8. The horizontal axes of Figures 3.7 and 3.8 list the counties in increasing order of their CD ratios.

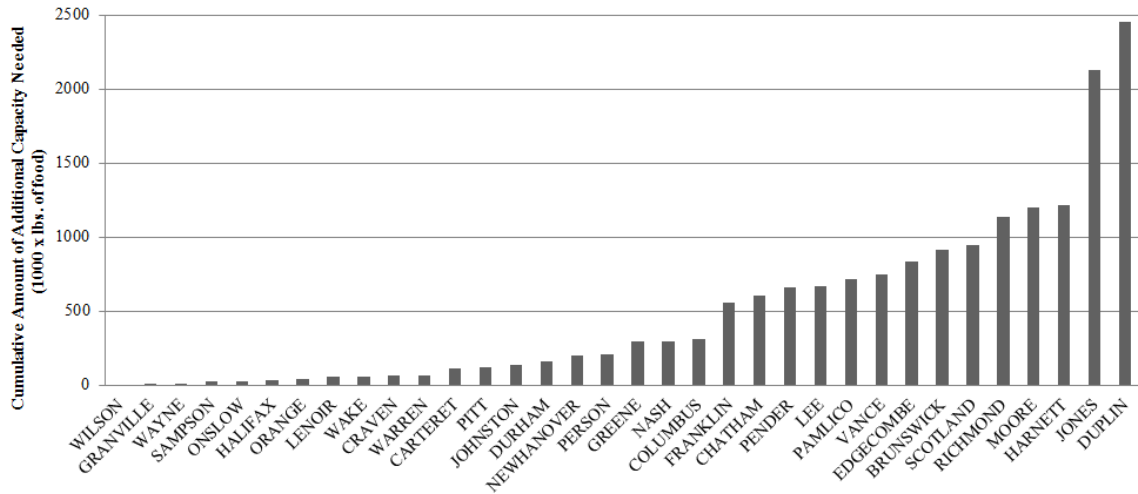


Figure 3.7: Cumulative amounts of additional capacity needed for addition of new bottleneck counties.

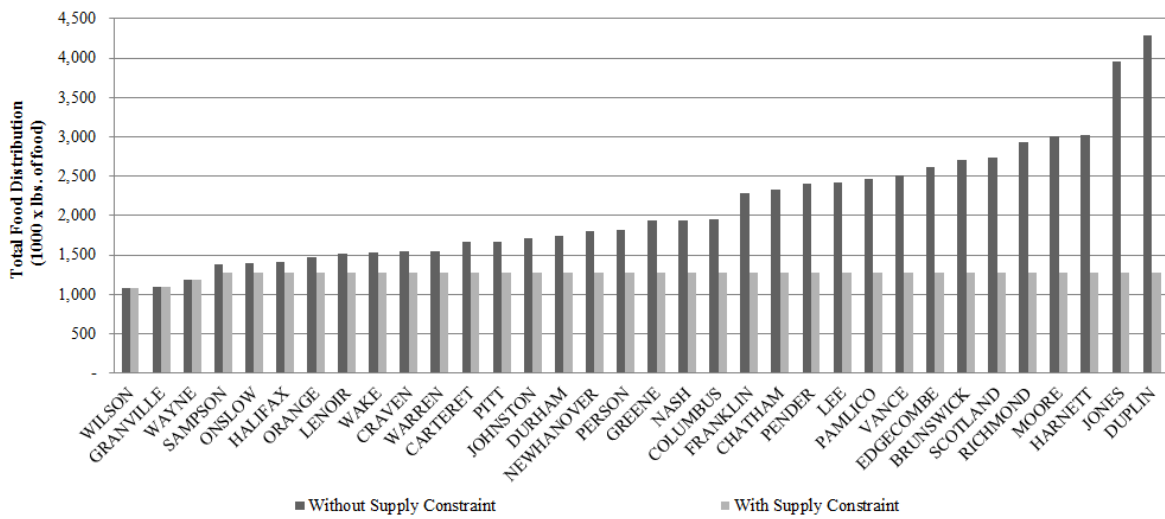


Figure 3.8: Total food distribution as a result of the capacity allocation in Figure 3.7.

Figure 3.7 shows the cumulative additional capacity required for the corresponding county to be added to the set of bottleneck counties. For example, if there are 500,000

pounds of additional capacity to be allocated, we can use this additional capacity to increase the receiving capacities of the counties resulting in an increase of the minimum CD ratio from 0.075, which is Wilson county's CD ratio, to 0.150. From Figure 3.7, we see that 500,000 pounds of additional capacity results in 20 counties (Wilson, Granville, Wayne, Sampson, Onslow, Halifax, Orange, Lenoir, Wake, Craven, Warren, Carteret, Pitt, Johnston, Durham, New Hanover, Person, Greene, Nash and Columbus) becoming bottlenecks that can receive food up to their capacity. Franklin County cannot be included because 559,986 pounds of total additional capacity would be required to include it, as can be seen from the graph. Using (3.63) we can find the resulting minimum CD ratio such that $Q^* = \frac{500,000 + \sum_{g=1}^{20} C_g}{\sum_{l=1}^{20} D_l} = 0.150$.

In Figure 3.8, the dark grey bars show the optimal amounts of food that can be distributed when additional capacity is allocated as in Figure 3.7 and the supply constraint is not considered. As the number of bottleneck counties and their capacities increase, distribution effectiveness increases while equitable distribution is maintained. The light grey bars represent the optimal amount of food that can be distributed when additional capacity is allocated as shown in Figure 3.7 under the supply constraint. As expected, once capacities are increased beyond some point the amount of food distributed starts to be constrained by supply rather than the capacities.

This analysis allows FBCENC to see how they could benefit from increasing their capacities and supply levels and compare different scenarios. For example, consider the case where there are 2,500,000 additional pounds of capacity available for distribution to the

counties. Figure 3.7 shows that this amount of extra capacity is sufficient to achieve equal CD ratios for all counties, resulting in 34 bottleneck counties. Therefore, all counties receive a positive amount of additional capacity. Due to the definition of the optimal fraction of additional capacity ρ_j^* received by county j in (3.89), we would expect densely populated counties with relatively low capacities to obtain the largest proportion of the extra capacity. The data supports this structure such that two counties in the region with highest demands (Wake and Durham) receive the highest proportion of total capacity: Wake county receives 18 percent and Durham county 9 percent of the additional capacity.

3.5.2 Generalizability of the Solutions

In Section 3.3, we showed that the optimal value of the Food Distribution Model is determined by the minimum MCD ratio of the counties. Therefore, any change in the counties' capacities may directly affect the optimal distribution policy. In Section 3.5.1, we used data obtained from FBCENC to calculate nominal estimates of county capacities. However, in reality these capacities are neither deterministic nor constant over time; instead, they vary randomly as individual agencies in the counties add or lose capacity. In this section we explore the sensitivity of our solutions to uncertainty in the capacity values. Some of the questions that we would like to answer are: How well do the nominal solutions obtained in Section 3.5.1 behave under uncertain capacity? Which counties are at greatest risk of becoming bottleneck counties? How does the amount of undistributed supply change when the county capacities follow different probability distributions? We examine these issues by

simulating the behavior of the solutions obtained by our model under different random errors in the nominal values of the county capacities.

3.5.2.1 Design of the Probabilistic Sensitivity Analysis

In order to generate random capacities for each county, we use four different probability distributions which we will refer to as Unif1, Unif2, Beta1 and Beta2. We assume that the counties' capacities are independent of each other. Let C_j^{nom} represent the nominal capacity value for county j as used in Section 3.5.1. We set $C_j \sim \text{Uniform}\left((1 - \theta)C_j^{\text{nom}}, (1 + \theta)C_j^{\text{nom}}\right)$ where $\theta = 0.1$ for the Unif1 distribution and $\theta = 0.2$ for the Unif2 distribution. We also use two Beta distributions to capture the changes in the optimal solutions for a skewed error distribution. We set the Beta distribution ranges such that $C_j \in (a = 0.5C_j^{\text{nom}}, b = 1.15C_j^{\text{nom}})$. These parameters are chosen to obtain a negative skewness based on the 2009 fiscal year FBCENC data; on average the minimum amount of food shipped to a county that year was about half of the 90th percentile, and the maximum approximately 15% greater than the 90th percentile. We set the mode to $m = C_j^{\text{nom}}$. Finally, to be able to compare with the Uniform distributions, we set the variance of the Beta1 distribution equal to that of Unif1 and that of Beta2 to that of Unif2. This allows us to examine the effects of skewness and variance separately. We generate the Beta distributions following Kuhl *et al.* (2010, pp. 96-97) and obtain the following formulas:

$$\text{Beta1: } C_j \sim 0.5C_j^{\text{nom}} + (1.15C_j^{\text{nom}} - 0.5C_j^{\text{nom}}) \text{ BETA}(17.174, 5.852), \quad (3.90)$$

$$\text{Beta2: } C_j \sim 0.5C_j^{\text{nom}} + (1.15C_j^{\text{nom}} - 0.5C_j^{\text{nom}}) \text{BETA}(4.011, 1.903), \quad (3.91)$$

where the expression $a + (b - a)\text{BETA}(\alpha_1, \alpha_2)$ denotes a Beta random variable on the interval $[a, b]$ with first shape parameter α_1 and second shape parameter α_2 as in Kuhl *et al.* (2010, pp. 82-87). We then use each of these four distributions to generate 1000 independent realizations of capacity values for each county for equity deviation limits $K \in \{0, 0.002, 0.004\}$. For each instance generated, we solve the model and record the bottleneck county.

3.5.2.2 Results of the Probabilistic Sensitivity Analysis

We find that Wilson (Wi), Granville (Gr) and Wayne (Wa) counties have the highest probabilities of becoming bottlenecks. In Section 3.5.1, we showed that these counties had the smallest CD ratios for the nominal case as well. Table 3.2 shows the percentages of instances with those counties as bottlenecks. Counties that were bottlenecks in less than 10% of the instances are included in the “Others (O)” row. Table 3.2 also shows the summary statistics (mean, standard deviation, minimum and maximum) of the optimal values (P^*) of the 1000 instances for $K \in \{0, 0.002, 0.004\}$ for Unif1, Unif2, Beta1 and Beta2 distributions.

For each K value, Wilson county always had the highest bottleneck probability. For lower K values, Granville county has the second highest bottleneck probability under all distributions whereas for higher K , Wayne county has the second highest bottleneck probability. The remaining counties have very low bottleneck probabilities in all cases and hence are not shown individually. This analysis is useful to FBCENC by highlighting counties with the highest potential to be bottlenecks and hence the greatest need for capacity

expansion. It shows that in the presence of counties with similar MCD ratios, concentrating exclusively on the county with the lowest nominal MCD ratio is not sufficient; capacity improvements need to be considered for all counties with substantial probability of becoming a bottleneck, in this case Wilson, Granville and Wayne.

Table 3.2: Probabilities of becoming bottleneck county and summary statistics for P^* values (1000 * pounds of food) for different capacity distributions and K limits.

		$K=0$				$K=0.002$				$K=0.004$			
		Unif1	Unif2	Beta1	Beta2	Unif1	Unif2	Beta1	Beta2	Unif1	Unif2	Beta1	Beta2
Prob. of Bottleneck	Wi	50.90%	40.90%	49.60%	37.30%	51.50%	43.20%	52.70%	34.40%	53.40%	42.20%	54.00%	37.30%
	Gr	41.40%	40.60%	43.00%	34.40%	33.00%	33.20%	32.40%	30.70%	21.10%	25.60%	19.20%	23.20%
	Wa	7.70%	15.30%	7.20%	13.80%	15.50%	20.80%	14.70%	22.80%	25.50%	28.10%	26.30%	27.00%
	O	0.00%	3.20%	0.20%	14.50%	0.00%	2.80%	0.20%	12.10%	0.00%	4.10%	0.50%	12.50%
P^* stats	Mean	225.5	280.3	240.4	349.9	148.6	204.7	167.4	294.3	67.1	136.3	86.7	220.9
	S.D.	48.8	87.8	55.3	101.2	48	87.2	59.1	110.2	44.5	83.8	59.5	121.2
	Min	89.1	0	103.7	95.9	0.9	0	11	14.7	0	0	0	0
	Max	299.3	407.8	438.6	639.2	232.1	348.3	379.8	630.2	154.9	278.6	323.3	598.8

For all distributions, as K is increased, average P^* values decrease due to increasing MCD ratios. We also observe that the standard deviations of the P^* values are high, suggesting that variability in capacity has a large effect on system effectiveness. Since the P^* values are driven by the minimum of a number of random variables (MCD ratios), the variability of P^* is higher than the variability in the individual MCD ratios. This is due to the bottleneck structure of the problem; one county with a lower MCD ratio than others will reduce the amount that can be shipped to other counties, even if they have substantial additional capacity. Hence accurate capacity estimation for the counties which are at risk of

becoming bottlenecks, and focused efforts to increase the lowest MCD ratios among counties are crucial.

The equity measure we use in our study, as expressed in (3.2), aims to maintain an overall equity level by improving the condition of the county which receives the worst service in terms of equity. Although this is the measure used by FBCENC, other food banks may evaluate their equity levels differently. For this reason, we also examined how the solutions proposed by our Food Distribution Model perform in terms of four alternative measures of inequity discussed by Marsh and Schilling (1994). We let $E_j = \left| \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} \right|$ and $\bar{E} = \frac{\sum_{l=1}^n E_l}{n}$. As described in Section 3.3, equations (3.2.a) and (3.2.b) imply that the inequity measure used in the Food Distribution Model is equivalent to constraining $\max_j E_j$, the first inequity measure discussed by Marsh and Schilling (1994), to be below a certain limit, K . We compared the results from the 1000 instances described above and calculated the resulting values for the following alternative inequity measures: 1) Variance, $\frac{\sum_{j=1}^n (E_j - \bar{E})^2}{n}$; 2) Average absolute deviation from \bar{E} , $\frac{\sum_{j=1}^n |E_j - \bar{E}|}{n}$; 3) The range, $\max_j E_j - \min_j E_j$; and 4) Maximum absolute deviation from \bar{E} , $\max_j |E_j - \bar{E}|$. We have scaled the first and second measures so that they all take values between zero and one, where zero corresponds to perfect equity and one to the worst level of equity. Our solutions are optimal for each inequity measure considered for $K = 0$ since it requires that $E_j = 0$ for all j . For any $K > 0$, the realizations of the inequity measures were always below K indicating that our solutions

behave well under different commonly used equity measures. Appendix A contains details of this analysis.

3.6 Conclusions and Limitations

We study the food distribution network of a food bank that is an affiliate of Feeding America and develop models to determine the equitable and effective distribution of food donations for such a food bank. Our objective is to maximize the amount of distributed food while limiting the absolute deviation from a perfectly equitable distribution for each county in the food bank's service area. We formulate a linear programming model that maximizes the food bank's effectiveness while satisfying a user-specified constraint on the magnitude of each county's deviation from perfect equity. The structure of the optimal solution depends on the MCD ratios and the total supply, allowing identification of bottleneck counties whose capacity must be increased in order to increase effectiveness. This insight leads us to formulate a second model showing how a food bank should allocate extra capacity to the counties in its service area. We prove that a stepwise capacity allocation algorithm gives the optimal allocation of additional capacity, and allows estimation of the amount of additional food that can be shipped for a given amount of additional capacity. We then use the data obtained from FBCENC to present numerical results for the given models.

The policies and numerical results from this study have been shared with FBCENC's staff. Our results in terms of the bottleneck counties aligned with what they have been observing in their operations and hence, our capacity allocation scheme gave them a useful perspective on how they can allocate their budgets, develop proposals for increasing capacity

as well as generate strategies for recruiting new agencies. Our results provide theoretically optimal solutions and have the potential to considerably improve the level of equity while minimizing waste.

The operations and objectives of FBCENC are representative of many food banks in the United States that are affiliates of Feeding America. The problem of the equitable and effective distribution of food under capacity limitations is faced by many food banks that collect donations from multiple sources and distribute food to their partner agencies. Furthermore, Feeding America faces a similar problem on a larger scale, i.e., the equitable and effective distribution of food among the food banks that have limited capacities throughout the entire nation.

An important assumption in this chapter is that demand and capacity are deterministic. This assumption is appropriate here as our purpose is to understand the general behavior and structural properties of this network-flow problem, especially the interactions between the objectives of equity and effectiveness and the roles of capacity and demand. Our sensitivity analysis shows that capacity variation at the bottleneck counties has an effect on the optimal amount of unshipped food due to the bottleneck nature of the problem. This highlights the importance of accurate capacity estimation and capacity enhancement at the bottleneck counties. Our future work will relax these assumptions to incorporate the stochastic nature of the models to understand how policies more robust to capacity variation can be obtained.

The assumption of no-cost interbranch flows is another direction in which our models can be extended. FBCENC's current operations support this assumption since most donations

are collected at the Raleigh branch and then shipped to the other branches for distribution to the counties according to the fair-share criterion. An interesting question for future work is the following: how does the modeling and solution structure presented in this chapter change if donations are collected at different branches, where each branch serves a set of counties and flows between those branches are incurred at a cost? Our preliminary analysis shows that in such a situation, the structure of the linear programming model and the optimal solutions remain similar; however, in addition to the capacity or supply constrained solutions identified in Proposition 3.1, solutions may also be constrained by the ratio of a particular branch's supply to the total demand it serves, depending on the relative levels of transshipment and waste costs. Moreover in the case that $K > 0$, among the multiple optimal solutions, we would prefer to use the solutions that minimize the interbranch flow costs.

There is a significant relationship between food insecurity and health outcomes of individuals. According to Feeding America's Hunger in America 2014 report, among the households that receive food assistance, 57.8 percent have at least one member with high blood pressure and 33.2 percent have at least one member with diabetes (Feeding America, 2015). However, we have not explicitly modeled the health impact in this study although this is an interesting area for future research.

In this chapter we have limited our analysis to a single food type; the interactions and capacity substitutions between different food groups will be considered in future work. The extension of the models developed here to multiple time periods is also an interesting direction for future research, as is the inclusion of budget constraints on operating costs.

Chapter 4

Single-Period Food Distribution Problem under Stochastic Capacities

4.1 Introduction

Chapter 3 developed deterministic linear models that address the trade-off between equity and effectiveness subject to capacity constraints, yielding valuable insight into the structure of optimal solutions that can inform a food bank's decisions. A major limitation of Chapter 3, which is confirmed through the probabilistic sensitivity analysis, is the assumption of deterministic data.

In this chapter, we develop a single-period model for addressing the uncertainty in the receiving capacity of the counties. FBCENC's service region is fairly large and nonhomogeneous in terms of the abilities of different counties to receive, store and distribute food to their beneficiaries. The receiving capacities of the counties vary randomly as individual agencies in the counties add capacity or are disqualified from receiving food from FBCENC.

In Section 4.2, we first develop a single-period, two-stage stochastic optimization model which incorporates the uncertainty in the capacities of the counties. In Section 4.3, we show that this model has a closed-form, newsvendor-type optimal solution under some mild conditions. We show that the closed-form optimal solution depends on the current supply on hand, total demand in the service region and the cumulative distribution function of the

minimum capacity-to-demand (CD) ratio, all of which are assumed to be known. Section 4.4 discusses ways to estimate the probability distribution of the minimum CD ratio when it is not known explicitly. Section 4.5 examines the similarities of the food distribution problem to the newsvendor problem. In Section 4.6, we use a specific probability distribution for the minimum CD ratio and perform sensitivity analysis. Finally, we conclude in Section 4.7.

4.2 Model Formulation

In this section, we develop a single-period stochastic optimization model (SPM) that incorporates uncertainty in the counties' capacities in a period. In a single time period, decisions are made at two stages. The first stage decisions involve determining the amount of food shipped to each county in the service region, while the recourse actions in the second stage determine the shipments from the branch to a county or waste from a county after observing the realized capacities. Thus, in the first stage, X_j pounds of food are shipped to county j for all $j \in J$. In the second stage, the capacities will be observed and either $Y_j \geq 0$ additional pounds of food will be shipped from the branch to each county $j \in J$ to compensate for any food deficiency in county j or any surplus food remaining at a county will be sent to waste, denoted as W_j . Surplus food at a county may arise due either to its realized capacity being less than the food shipped to that county in the first stage or equity limitations, since the total net food remaining in the counties is a function of the minimum CD ratio. Returns from the counties to the branch in the second stage, after capacity has been observed, are not allowed, which reflects the actual operations of the Food Bank. Food transshipments between counties are also not allowed, but will be explored as a future work.

Hence this stochastic model is said to have complete recourse, as an infeasible solution at the second stage is not possible (Birge & Louveaux, 2011).

This model aims to maximize the expected total *benefit* achieved from distributing food to the counties in the service region. There is a benefit associated with shipping food from the branch since keeping food at the branch increases the risk of spoilage and does not satisfy any needs in the current period; thus our model aims to maximize this benefit. The benefit achieved by the first stage shipments is higher than that from the second stage shipments, since keeping food at the branch increases its risk of spoilage. In addition, if the benefit of shipping food in the second stage is higher, a trivial solution would be to postpone any shipment until after capacities are realized and the uncertainty has been removed from the system. There is also a high, positive cost associated with food sent to waste. In the following formulation, we assume perfect equity and identical cost and benefit coefficients for all counties so that shipping food to one location is not preferred over shipping to another. Location specific costs and benefits can be implemented in future work.

We define the following decision variables where $J = \{1, \dots, j, \dots, n\}$ represents the set of counties:

X_j : Amount of shipment in the first stage to county j ,

Y_j : Amount of shipment in the second stage to county j ,

W_j : Amount of waste in the second stage from county j .

We further define the following parameters:

b_X : Value of shipping one pound of food from the branch to county j in the first stage,

b_Y : Value of shipping one pound of food from the branch to county j in the second stage,

$$b_X > b_Y,$$

c_W : Cost of discarding one pound of food as waste in the second stage, $c_W > b_X$,

D_j : Demand of county j ,

$\Delta = \sum_{l=1}^n D_l$: Total demand in the service region,

S : Total supply at the branch.

Finally, C_j represents the capacity of county j and R represents the minimum CD ratio, both of which are random variables in this context. We perform sensitivity analyses on the values of b_X , b_Y and c_W coefficients in Section 4.6. Our sensitivity analysis approach includes fixing b_X and modifying b_Y and c_W coefficients.

The stochastic model can be stated as follows:

$$\max b_X \sum_{l=1}^n X_l + E_C[V(X_j)] \quad (4.1)$$

subject to

$$\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} = 0 \quad j \in J \quad (4.2)$$

$$\sum_{l=1}^n X_l \leq S \quad (4.3)$$

$$X_j \geq 0 \quad j \in J \quad (4.4)$$

The objective function (4.1) maximizes the total benefit from the first stage food shipments plus the expected additional benefit obtained from second stage decisions $E_C[Q(X_j)]$, where the subscript ‘‘C’’ emphasizes that expectation is taken with respect to the realizations of

random capacity. Constraints (4.2) ensure that the first stage shipments are equitable in terms of the fair-share measure defined in Chapter 3. Constraints (4.3) ensure that the total food distributed during the first stage does not exceed the available supply. Constraints (4.4) assert the nonnegativity of X_j , for all j . There is no first stage constraint associated with the capacities of the counties since this information will be revealed in the second stage. The recourse function $V(X_j)$ is defined as follows, where \hat{C}_j represents a specific realization of the random variable C_j for each county j :

$$V(X_j) = \max b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \quad (4.5)$$

subject to

$$\frac{X_j + Y_j - W_j}{\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l} - \frac{D_j}{\Delta} = 0 \quad j \in J \quad (4.6)$$

$$S - \sum_{l=1}^n X_l - \sum_{l=1}^n Y_l \geq 0 \quad (4.7)$$

$$X_j + Y_j - W_j \leq \hat{C}_j \quad j \in J \quad (4.8)$$

$$X_j + Y_j - W_j \geq 0 \quad j \in J \quad (4.9)$$

$$X_j, W_j, Y_j \geq 0 \quad j \in J. \quad (4.10)$$

The objective function (4.5) maximizes the total benefit of second stage food shipments minus the cost of any waste incurred in the second stage, while constraints (4.6) enforce equity. Constraints (4.7) ensure that the amount of food distributed from the branch cannot exceed the total supply. Constraints (4.8) state that the net amount of food remaining in

county j (i.e., the sum of the first stage and second stage shipments into the county minus the waste) cannot exceed the observed capacity of county j . Constraints (4.9) require the net amount of food remaining in county j to be nonnegative. Constraints (4.10) are nonnegativity constraints.

The extended formulation for single-period stochastic model (SPM) can be constructed as follows:

SPM:

$$\max b_X \sum_{l=1}^n X_l + E_C \left[b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \right] \quad (4.11)$$

subject to

$$\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} = 0 \quad j \in J \quad (4.12)$$

$$\frac{X_j + Y_j - W_j}{\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l} - \frac{D_j}{\Delta} = 0 \quad j \in J \quad (4.13)$$

$$S - \sum_{l=1}^n X_l - \sum_{l=1}^n Y_l \geq 0 \quad (4.14)$$

$$X_j + Y_j - W_j \leq C_j \quad j \in J \quad (4.15)$$

$$X_j + Y_j - W_j \geq 0 \quad j \in J \quad (4.16)$$

$$X_j, W_j, Y_j \geq 0 \quad j \in J. \quad (4.17)$$

Constraint (4.3) in the first stage model is not included since it becomes redundant in the extended formulation SPM due to (4.14), since $\sum_{l=1}^n Y_l \geq 0$.

4.3 Optimal Solution Structure for SPM

In Chapter 3, we showed that the deterministic counterpart of this stochastic model has a closed-form optimal solution. Hence once the capacities have been realized, the optimal amount of food to keep in any county will follow the structure of the optimal solution to the deterministic problem. Let Q_j denote the optimal amount of food to keep in county j after \hat{C}_j is observed. At the end of the first stage, when $\sum_{l=1}^n X_l$ pounds of food has been distributed, the counties will either have more food than they need ($X_j > Q_j$), or a shortage ($X_j < Q_j$). Hence in the second stage each county j will either receive additional supply from the branch ($Y_j > 0$) or send surplus food to waste ($W_j > 0$). If $X_j = Q_j$, then $W_j = Y_j = 0$.

We first formulate an aggregate model where we consider the set of all counties as a closed system rather than individual locations. Noting that the first stage decisions are made before capacities are observed, constraint (4.12) can be written as

$$X_j = \frac{D_j}{\Delta} \sum_{l=1}^n X_l \quad j \in J. \quad (4.12a)$$

Using (4.12a), constraint (4.13) can be written as:

$$\frac{Y_j - W_j}{\sum_{l=1}^n Y_l - \sum_{l=1}^n W_l} - \frac{D_j}{\Delta} = 0 \quad j \in J \quad (4.13a)$$

which is equivalent to

$$Y_j - W_j = \frac{D_j(\sum_{l=1}^n Y_l - \sum_{l=1}^n W_l)}{\Delta} \quad j \in J. \quad (4.13b)$$

We then substitute (4.12a) and (4.13b) into (4.15) and (4.16) to develop the following aggregated two-stage, single-period stochastic linear model:

SPM-A:

$$\max b_X \sum_{l=1}^n X_l + E_C \left[b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \right] \quad (4.18)$$

$$S - \sum_{l=1}^n X_l - \sum_{l=1}^n Y_l \geq 0 \quad (4.19)$$

$$\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l \leq \frac{C_j}{D_j} \Delta \quad j \in J \quad (4.20)$$

$$\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l \geq 0 \quad (4.21)$$

$$\sum_{l=1}^n X_l, \sum_{l=1}^n Y_l, \sum_{l=1}^n W_l \geq 0 \quad (4.22)$$

Define the random variable

$$R = \min_{j \in J} \frac{C_j}{D_j} \quad (4.23)$$

that represents the minimum capacity-to-demand (CD) ratio. Since capacity and demand of a county are both large, continuous quantities, R is also assumed to be a continuous random variable. Constraint (4.20) implies that

$$\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l \leq \min_{j \in J} \frac{C_j}{D_j} \Delta = R\Delta. \quad (4.24)$$

Aggregations in this context correspond to the total food shipments and waste over all counties. We define aggregate variables for the first stage shipments, \tilde{X} (total food shipment to all the counties in the first stage); the second stage shipments \tilde{Y} (total food shipment to all the counties in the second stage); and the second stage food waste, \tilde{W} (total food waste from all the counties in the second stage) as follows:

$$\sum_{l=1}^n X_l = \tilde{X}, \quad (4.25)$$

$$\sum_{l=1}^n Y_l = \tilde{Y}, \quad (4.26)$$

$$\sum_{l=1}^n W_l = \tilde{W}. \quad (4.27)$$

Throughout the chapter, we will use \sim over the variables to indicate aggregations over the service region and \wedge to represent realizations of random variables. We can now rewrite the model (4.18 – 4.23), as follows:

SPM-A:

$$\max b_X \tilde{X} + E_R[b_Y \tilde{Y} - c_W \tilde{W}] \quad (4.28)$$

$$S - \tilde{X} - \tilde{Y} \geq 0 \quad (4.29)$$

$$\tilde{X} + \tilde{Y} - \tilde{W} \leq R\Delta \quad (4.30)$$

$$\tilde{X} + \tilde{Y} - \tilde{W} \geq 0 \quad (4.31)$$

$$\tilde{X}, \tilde{Y}, \tilde{W} \geq 0 \quad (4.32)$$

We now prove that with appropriate disaggregation, the LP models SPM-A and SPM are equivalent.

Proposition 4.1: *Assume that the solution $\tilde{\Psi}^* \equiv \{\tilde{X}^*, \tilde{Y}^*, \tilde{W}^*\}$ is optimal for SPM-A and yields the optimal objective function value \tilde{Z}^* . Consider the following disaggregation:*

$$X_j^* = \tilde{X}^* \frac{D_j}{\Delta} \quad (4.33)$$

$$Y_j^* = \tilde{Y}^* \frac{D_j}{\Delta} \quad (4.34)$$

$$W_j^* = \tilde{W}^* \frac{D_j}{\Delta} \quad (4.35)$$

Then, the solution $\Psi^ \equiv \{X_j^*, Y_j^*, W_j^*; j \in J\}$ is optimal for SPM and yields the optimal objective function value $Z^* = \tilde{Z}^*$ if and only if the disaggregation scheme given in (4.33)-(4.35) is used.*

Proof of Proposition 4.1: The optimal objective function value of the aggregated problem provides a lower bound on the optimal objective of the disaggregated (original) problem, i.e., $Z^* \geq \tilde{Z}^*$ (Litvinchev & Tsurkov, 2003). To prove that $Z^* = \tilde{Z}^*$, we will show that the models SPM-A and SPM are equivalent. These two maximization linear programs are equivalent if and only if for each feasible solution $\tilde{\Psi} \equiv \{\tilde{X}^*, \tilde{Y}^*, \tilde{W}^*\}$ to SPM-A with objective \tilde{Z} , there exists a corresponding feasible solution $\Psi \equiv \{X_j^*, Y_j^*, W_j^*; j \in J\}$ to SPM with objective $Z = \tilde{Z}$ and for each feasible solution Ψ to SPM with objective Z , there is a corresponding feasible solution $\tilde{\Psi}$ to SPM with objective $\tilde{Z} = Z$ (Cormen, Leiserson, Rivest & Stein, 2003). Throughout this proof, the “disaggregated solution” will refer to the solution obtained

through (4.33) - (4.35) and the “aggregated solution” to the solution obtained through (4.25) - (4.27).

Consider a feasible solution $\tilde{\Psi}$ to SPM-A with objective \tilde{Z} . Consider the corresponding disaggregated solution Ψ with the objective function Z . We will first show that Ψ is feasible for the model SPM and has the objective $Z = \tilde{Z}$. Since the solution $\tilde{\Psi}$ is feasible for SPM-A, it satisfies constraints (4.29) – (4.32). Furthermore, we have

$$\tilde{Z} = b_X \tilde{X} + E_R [b_Y \tilde{Y} - c_W \tilde{W}]. \quad (4.36)$$

To show that Ψ is feasible for SPM, we substitute (4.33) into constraint (4.12), obtaining

$$\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} = \frac{\tilde{X} \frac{D_j}{\Delta}}{\tilde{X}} - \frac{D_j}{\Delta} = \frac{D_j}{\Delta} - \frac{D_j}{\Delta} = 0 \quad j \in J. \quad (4.37)$$

Hence constraint (4.12) is satisfied.

By substituting the disaggregated solution into (4.13), we obtain

$$\begin{aligned} \frac{X_j + Y_j - W_j}{\sum_{l=1}^n X_l + \sum_{l=1}^n Y_l - \sum_{l=1}^n W_l} - \frac{D_j}{\Delta} &= \frac{\tilde{X} \frac{D_j}{\Delta} + \tilde{Y} \frac{D_j}{\Delta} - \tilde{W} \frac{D_j}{\Delta}}{\tilde{X} + \tilde{Y} - \tilde{W}} - \frac{D_j}{\Delta} = \frac{D_j}{\Delta} - \frac{D_j}{\Delta} \\ &= 0 \quad j \in J. \end{aligned} \quad (4.38)$$

satisfying constraint (4.13).

Substituting the disaggregated solution into (4.14), and by using (4.29), we have

$$S - \sum_{l=1}^n X_l - \sum_{l=1}^n Y_l = S - \tilde{X} - \tilde{Y} \geq 0. \quad (4.39)$$

Hence constraint (4.14) is satisfied.

By inserting the disaggregated solution into (4.15), and using (4.23) and (4.30),

$$\begin{aligned} X_j + Y_j - W_j &= \tilde{X} \frac{D_j}{\Delta} + \tilde{Y} \frac{D_j}{\Delta} - \tilde{W} \frac{D_j}{\Delta} = \frac{D_j}{\Delta} (\tilde{X} + \tilde{Y} - \tilde{W}) \leq R\Delta \frac{D_j}{\Delta} = RD_j \\ &\leq C_j \quad j \in J. \end{aligned} \quad (4.40)$$

Hence constraint (4.15) is satisfied.

By inserting the disaggregated solution into (4.16), and using (4.31), we get

$$X_j + Y_j - W_j = \tilde{X} \frac{D_j}{\Delta} + \tilde{Y} \frac{D_j}{\Delta} - \tilde{W} \frac{D_j}{\Delta} = \frac{D_j}{\Delta} (\tilde{X} + \tilde{Y} - \tilde{W}) \geq 0 \quad j \in J. \quad (4.41)$$

Hence constraint (4.16) is satisfied.

Finally, $X_j = \tilde{X} \frac{D_j}{\Delta} \geq 0$, $Y_j = \tilde{Y} \frac{D_j}{\Delta} \geq 0$, and $W_j = \tilde{W} \frac{D_j}{\Delta} \geq 0$ which follow from (4.32).

Hence, constraints (4.17) are also satisfied. Therefore, the disaggregated solution Ψ defined by (4.33) - (4.35), is feasible for SPM. Finally, Z is calculated as follows:

$$\begin{aligned} Z &= b_X \sum_{l=1}^n X_l + E \left[b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \right] \\ &= b_X \tilde{X} \sum_{l=1}^n \frac{D_l}{\Delta} + E \left[b_Y \tilde{Y} \sum_{l=1}^n \frac{D_l}{\Delta} - c_W \tilde{W} \sum_{l=1}^n \frac{D_l}{\Delta} \right] \\ &= b_X \tilde{X} + E[b_Y \tilde{Y} - c_W \tilde{W}] = \tilde{Z}. \end{aligned} \quad (4.42)$$

Hence, for any feasible solution to SPM-A, $\tilde{\Psi}$ with objective \tilde{Z} , there exists a corresponding disaggregated solution Ψ which is feasible for the model SPM and has the objective $Z = \tilde{Z}$.

Next, we prove the converse case. Consider a feasible solution to SPM, Ψ with objective function value Z and the corresponding aggregated solution $\tilde{\Psi}$ with objective function value \tilde{Z} . We will show that $\tilde{\Psi}$ is feasible for the SPM-A model and has objective function value $\tilde{Z} = Z$. Since the solution Ψ is feasible for SPM, it satisfies constraints (4.12) – (4.17); furthermore, we have

$$Z = b_X \sum_{l=1}^n X_l + E \left[b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \right]. \quad (4.43)$$

By inserting the aggregated solution into (4.29) and using (4.14), we have

$$S - \sum_{l=1}^n X_l - \sum_{l=1}^n Y_l = S - \tilde{X} - \tilde{Y} \geq 0. \quad (4.44)$$

Hence constraint (4.29) is satisfied.

By using (4.12), (4.13) and (4.15); together with the aggregated solution, we get

$$X_j + Y_j - W_j = \tilde{X} \frac{D_j}{\Delta} + \tilde{Y} \frac{D_j}{\Delta} - \tilde{W} \frac{D_j}{\Delta} = \frac{D_j}{\Delta} (\tilde{X} + \tilde{Y} - \tilde{W}) \leq C_j \quad j \in J \quad (4.45)$$

which implies

$$\tilde{X} + \tilde{Y} - \tilde{W} \leq \frac{\Delta C_j}{D_j} \quad j \in J. \quad (4.46)$$

and hence,

$$\tilde{X} + \tilde{Y} - \tilde{W} \leq R\Delta. \quad (4.47)$$

satisfying constraint (4.30).

Constraint (4.16) together with the aggregated solution implies that

$$\tilde{X} + \tilde{Y} - \tilde{W} = \sum_{l \in J} (X_l + Y_l - W_l) \geq 0, \quad (4.48)$$

satisfying constraint (4.31).

Finally, the relations $\tilde{X} = \sum_{l=1}^n X_l \geq 0$, $\tilde{Y} = \sum_{l=1}^n Y_l \geq 0$, and $\tilde{W} = \sum_{l=1}^n W_l \geq 0$ follow from (4.17). Hence, constraints (4.32) are also satisfied. Therefore, the solution $\tilde{\Psi}$, as defined through aggregation equations (4.25) - (4.28), is feasible for the model SPM-A. Finally, \tilde{Z} is calculated as follows:

$$\tilde{Z} = b_X \tilde{X} + E[b_Y \tilde{Y} - c_W \tilde{W}] = b_X \sum_{l=1}^n X_l + E \left[b_Y \sum_{l=1}^n Y_l - c_W \sum_{l=1}^n W_l \right] = Z. \quad (4.49)$$

Hence, for any feasible solution to SPM, Ψ with objective Z , there exists a corresponding aggregated solution $\tilde{\Psi}$ which is feasible for the model SPM-A and has the objective $\tilde{Z} = Z$. Therefore, models SPM and SPM-A are equivalent with aggregation equations (4.25)-(4.28) and disaggregation equations (4.33)-(4.35). Therefore Proposition 4.1 holds. ■

Considering SPM-A as a two-stage model, the first stage model can be formulated as follows:

$$\max b_X \tilde{X} + E_R[V(\tilde{X})] \quad (4.50)$$

$$\tilde{X} \leq S \quad (4.51)$$

$$\tilde{X} \geq 0. \quad (4.52)$$

In the second stage of SPM-A, the objective is to maximize total benefit through second stage shipments \tilde{Y} and \tilde{W} given the first stage decisions \tilde{X} . The aggregated second stage model for a specific realization of the minimum CD ratio \hat{R} can be formulated as follows:

$$\max b_Y \tilde{Y} - c_W \tilde{W} \quad (4.53)$$

$$\tilde{Y} \leq S - \tilde{X} \quad (4.54)$$

$$\tilde{Y} - \tilde{W} \leq \hat{R} \Delta - \tilde{X} \quad (4.55)$$

$$\tilde{Y} - \tilde{W} \geq -\tilde{X} \quad (4.56)$$

$$\tilde{Y}, \tilde{W} \geq 0. \quad (4.57)$$

This model has closed-form optimal solutions characterized by the values \hat{R}, S, Δ and \tilde{X} where \tilde{X} is treated as a constant in the second stage. The observed minimum CD ratio \hat{R} is the source of uncertainty and its value depends on the realized capacity values. In other words, \hat{R} is a specific realization of the random variable R that represents the minimum CD ratio of the counties. The random variable R has support $S_R = (0, \infty)$ and is characterized by the probability density function $f_R(\cdot)$ which is continuous and positive at every $r \in (0, \infty)$ and the cumulative distribution function $F_R(\cdot)$.

The optimal solutions to the aggregated second stage model (4.53 – 4.57) depend on which of the three cases based on the levels of $\hat{R} \Delta$, \tilde{X} , and S given in Table 4.1 the instance satisfies. The aggregation scheme allows us to obtain solutions for each of the three cases that can arise after capacities are realized.

Table 4.1: Optimal Solution Structure for the Aggregated Second Stage Model.

	Case 1 Capacity constrained instance, overshipment in first stage	Case 2 Capacity constrained instance, undershipment in first stage	Case 3 Supply constrained instance
Conditions	$\hat{R}\Delta < \tilde{X}$ $\hat{R}\Delta < S$	$\hat{R}\Delta \geq \tilde{X}$ $\hat{R}\Delta < S$	$\hat{R}\Delta \geq \tilde{X}$ $\hat{R}\Delta \geq S$
	$\hat{R} < \frac{\tilde{X}}{\Delta}$	$\frac{\tilde{X}}{\Delta} \leq \hat{R} < \frac{S}{\Delta}$	$\frac{S}{\Delta} < \hat{R}$
Optimal Solution Structure	$\tilde{W}^* = \tilde{X} - \hat{R}\Delta$ $\tilde{Y}^* = 0$	$\tilde{W}^* = 0$ $\tilde{Y}^* = \hat{R}\Delta - \tilde{X}$	$\tilde{W}^* = 0$ $\tilde{Y}^* = S - \tilde{X}$

Since by definition $\tilde{X} \leq S$, the case where $\hat{R}\Delta < \tilde{X}$ and $\hat{R}\Delta \geq S$ is not possible. Proposition 4.2 gives the optimal solution structure for SPM-A. This result is useful since it gives the optimal first stage shipment amounts based on the information available before any uncertainty is revealed.

Proposition 4.2: *Let R be a random variable with probability density function $f_R(\cdot)$ and cumulative distribution function $F_R(\cdot)$ denote the minimum CD ratio of the counties. Also, assume that $b_Y < b_X < c_W$. Then the optimal solution to SPM-A is as follows:*

$$\tilde{X}^* = \min \left\{ S, \Delta F_R^{-1} \left(\frac{b_X - b_Y}{c_W - b_Y} \right) \right\} \quad (4.58)$$

The values of the decision variables \tilde{W}^ and \tilde{Y}^* are calculated as given in Table 4.1.*

Proof of Proposition 4.2: Using the results from Table 4.1, we can write the expectation of the second stage objective function from (4.50) as:

$$\begin{aligned}
E_R[V(\tilde{X})] &= E_R[b_Y \tilde{Y} - c_W \tilde{W}] \\
&= \int_0^{\frac{\tilde{X}}{\Delta}} [-c_W(\tilde{X} - r\Delta)] f_R(r) dr + \int_{\frac{\tilde{X}}{\Delta}}^{\frac{S}{\Delta}} [b_Y(r\Delta - \tilde{X})] f_R(r) dr \\
&\quad + \int_{\frac{S}{\Delta}}^{\infty} [b_Y(S - \tilde{X})] f_R(r) dr
\end{aligned} \tag{4.59}$$

Since the objective function of the extended aggregated SP is $b_X \tilde{X} + E_R[V(\tilde{X})]$, we seek the \tilde{X} value that maximizes the overall objective subject to the first stage constraint that $\tilde{X} \leq S$. We construct the Lagrangian function $L(\tilde{X}, \lambda) = b_X \tilde{X} + E_R[V(\tilde{X})] + \lambda[S - \tilde{X}]$ where $\lambda \geq 0$ is a Lagrange multiplier. An optimal solution must satisfy the conditions

$$\begin{aligned}
&\frac{\partial [b_X \tilde{X} + E_R[V(\tilde{X})] + \lambda[S - \tilde{X}]]}{\partial \tilde{X}} \\
&= b_X + \int_0^{\frac{\tilde{X}}{\Delta}} -c_W f_R(r) dr + \int_{\frac{\tilde{X}}{\Delta}}^{\frac{S}{\Delta}} -b_Y f_R(r) dr + \int_{\frac{S}{\Delta}}^{\infty} -b_Y f_R(r) dr - \lambda \\
&= b_X - c_W F_R\left(\frac{\tilde{X}}{\Delta}\right) - b_Y \left[F_R\left(\frac{S}{\Delta}\right) - F_R\left(\frac{\tilde{X}}{\Delta}\right) \right] - b_Y \left[1 - F_R\left(\frac{S}{\Delta}\right) \right] \\
&- \lambda = b_X - b_Y + (b_Y - c_W) F_R\left(\frac{\tilde{X}}{\Delta}\right) - \lambda = 0
\end{aligned} \tag{4.60}$$

and

$$\lambda(S - \tilde{X}) = 0. \quad (4.61)$$

We must also check the second derivative of (4.60) to show that the objective function is concave in \tilde{X} .

$$\frac{\partial^2 [b_X \tilde{X} + E_R[V(\tilde{X})] + \lambda[S - \tilde{X}]]}{\partial \tilde{X}^2} = \frac{(b_Y - c_W)}{\Delta} f_R \left(\frac{\tilde{X}}{\Delta} \right). \quad (4.62)$$

Since by definition $f_R \left(\frac{\tilde{X}}{\Delta} \right) \geq 0$, for this expression to be negative, we need $b_Y - c_W < 0$ which is one of the conditions of the proposition. Therefore, the objective is a strictly concave function of \tilde{X} .

The value of the Lagrange multiplier λ represents the dual price of the supply constraint. It reflects the change in the optimal objective function value resulting from a unit change in the total supply in the first stage. Thus,

- **Case 1:** If $\lambda = 0$, then from (4.60),

$$\tilde{X} = \Delta F_R^{-1} \left(\frac{b_X - b_Y}{c_W - b_Y} \right) \quad (4.63)$$

- **Case 2:** If $\lambda > 0$, then from (4.61),

$$\tilde{X} = S \quad (4.64)$$

which, due to (4.60) leads to:

$$\lambda = b_X - b_Y + (b_Y - c_W) F_R \left(\frac{S}{\Delta} \right) \quad (4.65)$$

By assumption of Case 2, $\lambda > 0$. So we obtain,

$$\lambda = b_X - b_Y + (b_Y - c_W)F_R\left(\frac{S}{\Delta}\right) > 0. \quad (4.66)$$

Hence,

$$S < \Delta F_R^{-1}\left(\frac{b_X - b_Y}{c_W - b_Y}\right). \quad (4.67)$$

Collecting these results, we have

$$\tilde{X}^* = \min\left\{S, \Delta F_R^{-1}\left(\frac{b_X - b_Y}{c_W - b_Y}\right)\right\}. \quad (4.68)$$

The proof is completed. ■

We now explore the structure of the term λ as defined in (4.65). This term is the Lagrange multiplier that represents the shadow price of the supply constraint on the first stage of the stochastic model. Since $\lambda > 0$ implies that $\tilde{X}^* = S$, we can calculate the objective function value for SPM-A for this situation as

$$Z^* = b_X S - c_W(S - R\Delta)P\left(R < \frac{S}{\Delta}\right). \quad (4.69)$$

Now assume that the total supply is decreased by 1 unit. In this case, we would expect the objective to decrease by

$$\lambda = b_X - b_Y + (b_Y - c_W)F_R\left(\frac{S}{\Delta}\right) = b_X - b_Y P\left(R \geq \frac{S}{\Delta}\right) - c_W P\left(R < \frac{S}{\Delta}\right). \quad (4.70)$$

The objective function for a supply of $S - 1$ is calculated as

$$\begin{aligned}
Z' &= P\left(R \geq \frac{S}{\Delta}\right) [b_x(S-1) + b_y(1)] \\
&\quad + P\left(R < \frac{S}{\Delta}\right) [b_x(S-1) - c_w(S-1 - R\Delta)] \\
&= b_x(S-1) + P\left(R \geq \frac{S}{\Delta}\right) b_y \\
&\quad - c_w P\left(R < \frac{S}{\Delta}\right) (S-1 - R\Delta).
\end{aligned} \tag{4.71}$$

Finally, we calculate $Z - Z' = b_x - b_y P\left(R \geq \frac{S}{\Delta}\right) - c_w P\left(R < \frac{S}{\Delta}\right) = \lambda$. We notice that λ takes its maximum value if $P\left(R \geq \frac{S}{\Delta}\right) = 1$, i.e., the problem is supply constrained with probability 1; which is reasonable since λ represents the change in the objective function as supply changes.

In order to apply the result from Proposition 4.2, we need to have information regarding the distribution of the minimum CD ratio, $F_R(\cdot)$. If $F_R(\cdot)$ is known, by using Proposition 4.2 together with Table 4.1, we can generate optimal solutions to SPM. In the practical sense, if the distribution of the minimum CD ratio is available, or can be estimated with previous data or expert opinion, then the optimal first stage shipments can be used as conservative estimates of total food shipment prior to having any information regarding capacities. However, if there is no prior information available regarding the distribution of minimum CD ratios, under some conditions, we can use results from order statistics to obtain $F_R(\cdot)$. The next section discusses these ideas.

4.4 Using Order Statistics to Obtain the Distribution of the Minimum CD Ratios

In this section, we discuss two ways that allow us to obtain or estimate $F_R(\cdot)$ when it is not explicitly known. In Subsection 4.4.1, we assume that the CD ratios of the counties are independent and identically distributed with a known cumulative distribution function. We use a result from order statistics to obtain the distribution for the minimum CD ratio. In Subsection 4.4.2, we use an asymptotic result from extreme value statistics that provides an approximation for $F_R(\cdot)$ as the number of counties, n increases.

4.4.1 Exact Result using Order Statistics

Assume that the CD ratios of the counties are independent and identically distributed with the cumulative distribution function $F_C(c)$ and probability density function $f_C(c)$. Let $CD_{(1)}, CD_{(2)}, \dots, CD_{(n)}$ denote the order statistics of a random sample from this distribution. Then the probability density function of the sample minimum, $R = CD_{(1)}$, is given by the following (Casella & Berger, 2002):

$$f_R(r) = f_{CD_{(1)}} = nf_C(c)[1 - F_C(c)]^{n-1}. \quad (4.72)$$

We can use (4.72) for any common distribution of CD ratios that has a finite probability density function and cumulative distribution function and obtain the distribution for R . The assumption of independent CD ratios is not limiting since according to our understanding from the FBCENC, there is no direct correlation between the CD ratios of different counties.

4.4.2 Asymptotic Result by Order Statistics

There may be some situations where the distributions of the CD ratios of the counties in the Food Bank's service region are not known. Then, the classic results by Fisher and Tippett (1928) for the extreme value statistics may be used to obtain a limiting distribution for the minimum CD ratio. The result states the following: We consider n independent random samples, each with size m , from a continuous distribution that is bounded below by a fixed parameter θ . If we assume that the CD ratios of the counties are independent and identically distributed, as in Subsection 4.4.1, then we can consider this as n counties (samples) for a period of m months (sample size). Then, if the cumulative distribution of the CD ratios, $F_R(r)$ behaves like $\left[\frac{(R-\theta)}{\beta}\right]^\alpha + o(|r-\theta|)$ if $r \geq \theta$ and $F_R(r) = 0$ for $r < \theta$ as $R \rightarrow \theta^+$, as the sample size n gets larger, $F_R(r)$ converges to a Weibull distribution with location parameter θ , scale parameter β , and shape parameter α . The form of this cumulative distribution function is given as the following:

$$F_R(r) = \begin{cases} 1 - \exp\left\{-\left(\frac{R-\theta}{\beta}\right)^\alpha\right\} & \text{for } R \geq \theta \\ 0 & \text{for } R < \theta \end{cases} \quad (4.73)$$

In practice, the parameters θ , β , and α are not known with certainty and must be estimated from data (Wilson, King & Wilson, 2004). There are various studies in literature for efficiently estimating these parameters (Giddings, Rardin & Uzsoy, 2014).

For $R \geq \theta$, setting

$$F_R(r) = 1 - \exp\left\{-\left(\frac{R-\theta}{\beta}\right)^\alpha\right\} = \frac{b_X - b_Y}{c_W - b_Y}, \quad (4.74)$$

yields,

$$F_R^{-1}\left(\frac{b_X - b_Y}{c_W - b_Y}\right) = \beta \left[-\ln\left(\frac{c_W - b_X}{c_W - b_Y}\right) \right]^{\frac{1}{\alpha}} + \theta. \quad (4.75)$$

So, using (4.69), we get

$$\tilde{X}^* = \min \left\{ S, \Delta \left(\beta \left[-\ln\left(\frac{c_W - b_X}{c_W - b_Y}\right) \right]^{\frac{1}{\alpha}} + \theta \right) \right\}. \quad (4.76)$$

For this result to hold, we need

$$\ln\left(\frac{c_W - b_X}{c_W - b_Y}\right) \leq 0 \quad (4.77)$$

since $\beta \geq 0$ and $\Delta > 0$ by definition. Inequality (4.77) is equivalent to $\frac{c_W - b_X}{c_W - b_Y} \leq 1$ which is equivalent to $b_X > b_Y$, which is satisfied as a condition of Proposition 4.2. Note that setting $\alpha = 1$ and $\theta = 0$ yields the exponential distribution, which we will be using in our results in Section 4.6.

4.5 Comparison of SPM to the Newsvendor Problem

By examining the optimal closed-form solution of SPM given in Proposition 4.2, we see that it has the same structure as the newsvendor model (Arrow, Harris, & Marshak, 1951; Khouja, 1999; Qin, Wang, Vakharia, Chen, & Seref, 2011). If we set

$$\text{Cost of underage: } c_U = b_X - b_Y, \quad (4.78)$$

$$\text{Cost of overage: } c_O = c_W - b_X, \quad (4.79)$$

$$\text{Upper limit (Purchase power): } u = S, \quad (4.80)$$

then the closed-form optimal solution given in Proposition 4.2 becomes

$$\tilde{X}^* = \min \left\{ u, \Delta F_R^{-1} \left(\frac{c_U}{c_O + c_U} \right) \right\}. \quad (4.81)$$

The statement given in (4.81) follows the equivalent structure of the optimal solution for the newsvendor problem. The cost of overage can be considered the opportunity cost of having shipped too little food in the first stage and will be observed when an additional shipment must be made from the branch in the second stage to compensate for undershipment in the first stage. The cost of underage can be thought of as the cost of having shipped too much food in the first stage and will be observed when food needs to be sent to waste due to capacity or equity limitations. In the next section, we will perform sensitivity analysis on these cost parameters for an example case to illustrate this relationship.

4.6 Numerical Results

In order to obtain results for SPM, we assume that the capacity of each county j follows an exponential distribution. In particular, we assume that $C_j \sim \text{Exponential} \left(\frac{1}{Q_j^{90}} \right)$ where Q_j^{90} is estimated to be the 90th percentile of FBCENC's monthly shipments to county j during the 2009 fiscal year for dry goods. For demand and supply, we use the values from the deterministic study in Section 3.5, i.e., supply equals the actual food donations in January 2009 and demand is proportional to the poverty populations in the service region during this time period.

Since the exponential distribution is closed under scaling, we obtain $\frac{c_j}{D_j} \sim \exp\left(\frac{D_j}{Q_j^{90}}\right)$.

Finally, since the minimum of exponential random variables is also exponential, we obtain

$R \sim \exp\left(\sum_l \frac{D_l}{c_l^{90}}\right)$. We then use the closed-form optimal solution given in (4.58) to calculate

the optimal first stage aggregated shipment value \tilde{X}^* . Figure 4.1 and 4.2 illustrate how the optimal total first stage shipment and the optimal objective function value change respectively for varying levels of c_U and c_O .

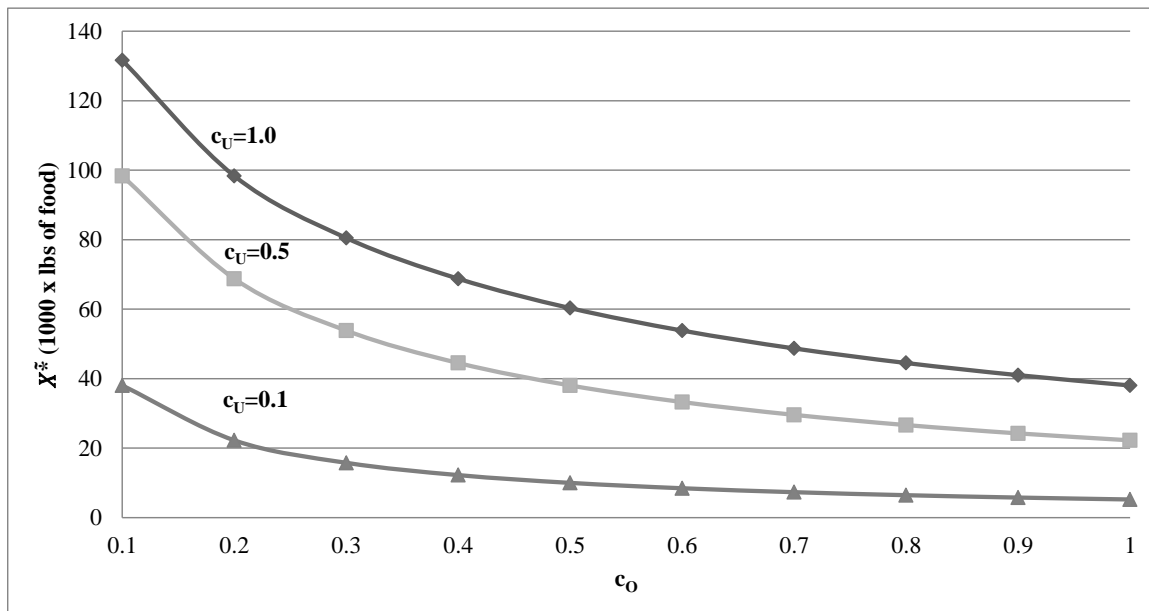


Figure 4.1: First Stage Optimal Total Food Distribution (\tilde{X}^*) for Varying c_U and c_O .

By examining Figure 4.1, we notice that as c_U increases, \tilde{X}^* also increases. This is reasonable because as c_U increases, the penalty associated with sending too little food in the first stage increases, causing the decision maker to take greater risk by shipping more food in

the first stage. Similarly, as c_O increases, \tilde{X}^* decreases. This happens because as c_O increases, the penalty associated with sending too much food in the first stage increases, causing the decision maker to be more risk-averse and ship less food in the first stage.

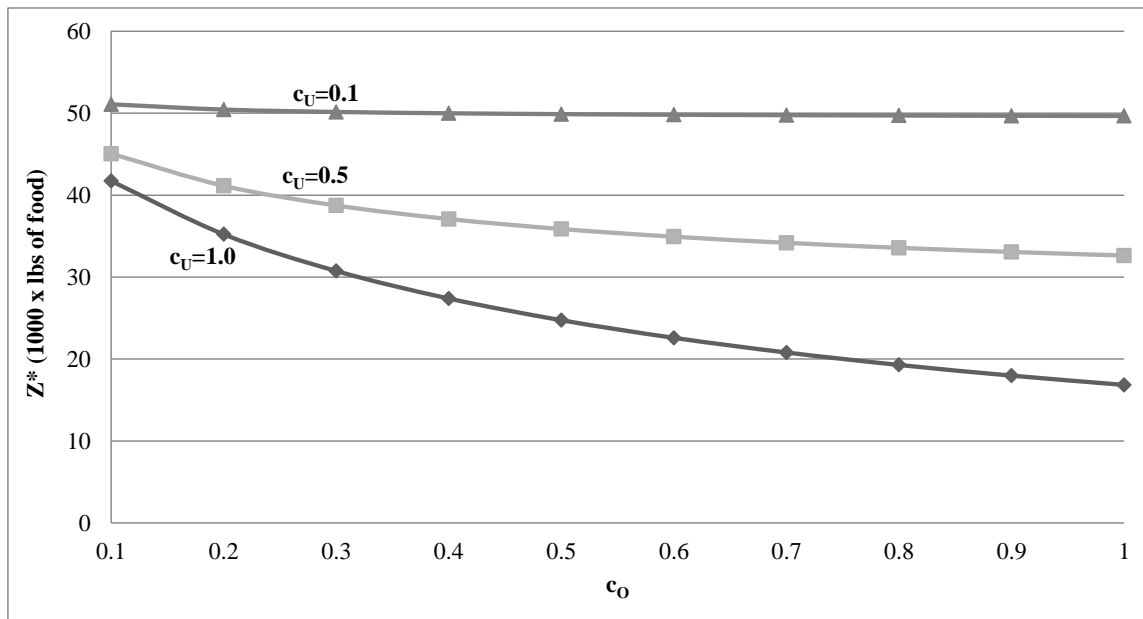


Figure 4.2: Optimal Objective Function Value (Z^*) for Varying c_U and c_O .

Figure 4.2 shows that as c_O or c_U increases, the optimal value of the objective function, Z^* , decreases. This is reasonable since increasing these cost coefficients increases the overall risk associated with the uncertainty in the system decreasing the value of the optimal objective function. In the extreme case, if $c_O = c_U = 0$, the decision maker will wait until the uncertainty is revealed in the second stage and make decisions under perfect information, which would increase the overall benefit of the model.

4.7 Conclusions

We study the problem of obtaining equitable and effective food distribution policies when the capacities of the counties in a food bank's service region are stochastic. In this chapter, we focused on the problem faced by a food bank in a single period. We assumed that in the beginning of the period, food has to be shipped from the food bank to the counties in the service region without knowing the capacities of the counties. Then, capacities are revealed and food shipments can be corrected at an additional cost while preserving perfect equity. We modeled this problem as a two-stage stochastic model and proved its closed-form optimal solution when the minimum CD ratio follows a known probability distribution. We then examined ways to estimate this distribution if it is not known explicitly. We compared our problem to the well-known newsvendor problem and showed its equivalence. Finally, we performed a numerical study on the developed model and explored its sensitivity to various cost parameters.

One limitation of this chapter is that it considers a single time period, whereas a food bank generally has to plan for a longer time horizon. In Chapter 5, we examine the problem of equitable and effective distribution of food donations under capacity uncertainty over multiple time periods. Another limitation of Chapter 4 is the assumption of perfect equity. In our future work, we will study the stochastic single-period food shipment problem under relaxed equity. Moreover, we will explore the structural properties of the single-period problem when supply is also stochastic in addition to the capacities.

Chapter 5

Heuristic Approaches for the Multi-Period Food Distribution

Problem under Stochastic Capacities

5.1 Introduction

In Chapter 4, we considered the problem of equitable and effective food distribution under stochastic capacities in a single period. The decision maker was required to make shipments to counties at the beginning of the period before the capacities are revealed. At the end of the period, capacities were revealed and the decision maker had the opportunity to “correct” shipments through additional shipments with lower benefit or by discarding food as waste at additional cost. We showed that under certain mild conditions this model has a closed-form, newsvendor-type optimal solution.

A natural extension of the single-period model in Chapter 4 is the multi-period problem where food shipments are made and capacities are observed in each period. In this problem, supply is received at the beginning of each period. At the beginning of the planning horizon, the decision maker has no information regarding the county capacities. As capacity is observed, recourse decisions can be made in the form of additional shipments from the branch to the counties or waste that is discarded from the counties. As in the single-period problem, our aim is to maximize the benefit associated with food shipments from the branch less the cost of waste under perfect equity. Waste may arise due to either capacity limitations

or equity restrictions in each period, since we require food distribution to be perfectly equitable.

In this chapter, we develop heuristic approaches for the multi-period food shipment problem under stochastic capacities. Our first heuristic, the Myopic Heuristic, uses the optimal solution structure of SPM from Proposition 4.2 to treat the multi-period problem as a sequence of single-period problems. The One-Period Look-Ahead Heuristic improves the Myopic Heuristic by considering two periods at a time. The third heuristic, the Bounding Heuristic uses the average of lower and upper bounds on the optimal shipment. The Deterministic Approximation Heuristic, in contrast, uses the closed-form optimal solution of the Multi-Period Deterministic Food Distribution Model by using a percentile of the underlying probability distribution of the minimum CD ratio.

As a benchmark for comparing the performance of our heuristics, we develop a Multi-Stage Stochastic Programming Model. The disadvantage of using stochastic programming to model multiple periods is that the model size, as expressed by the size of the extended formulation, grows exponentially with the number of stages and polynomially with the number of possible outcomes considered at each stage, rendering exact solutions impractical for large problem instances. In these situations, heuristic approaches are useful to obtain practical solutions in modest CPU times.

In Section 5.2, we will describe the multi-period food distribution problem under stochastic capacities and define notation. Section 5.3 presents the Myopic Heuristic, and Section 5.4 derives bounds on the value of the optimal solution. We use these structural results in Sections 5.5 and 5.6 to develop the One-Period Look-Ahead Heuristic and the

Bounding Heuristic, respectively. In Section 5.7, we describe the Deterministic Approximation Heuristic. Section 5.8 presents a Multi-Stage Stochastic Programming Model (MSM) that allows corrective actions in the form of additional shipments from the branch or waste from the counties at the end of each period. This model provides a benchmark for evaluating the performance of the heuristic approaches. In Section 5.9, we present computational experiments and results and compare the performance of the stochastic program to that of the heuristics in terms of the objective functions and runtimes. We conclude the chapter with a discussion of findings, limitations and future work.

5.2 Multi-Period Problem Formulation

In this section, we formulate the multi-period food distribution problem and introduce notation that will be used throughout the chapter. The planning horizon consists of multiple decision stages (epochs) $t = 1, \dots, T$, each corresponding to a specific time period. Hence the terms "stage" and "time period" will be used interchangeably. We receive a supply of S_t pounds of food at the beginning of each time period t . In each time period t , food shipments are made prior to observing the capacity at that period. After capacities for that period are observed, food shipments can be corrected at additional cost or lower benefit. Throughout the chapter, we assume perfect equity.

We define the following decision variables:

$X_{j,t}$: Amount of shipment in time period t to county j ,

$Y_{j,t}$: Amount of additional shipment in time period t to county j after the minimum

CD ratio for stage t , \hat{R}_t , has been observed,

$W_{j,t}$: Amount of waste in time period t from county j after the minimum CD ratio for stage t , \hat{R}_t , has been observed.

In Section 4.2, we proved the equivalence of the aggregated and disaggregated formulations over n counties at a given period and derived the optimal disaggregation of an aggregate solution. For this reason, throughout this chapter, the decision variable $\tilde{X}_t = \sum_{l=1}^n X_{l,t} \geq 0$ refers to the total shipment at the beginning of time period t before any information regarding the capacities is revealed. The recourse actions $\tilde{Y}_t = \sum_{l=1}^n Y_{l,t} \geq 0$ and $\tilde{W}_t = \sum_{l=1}^n W_{l,t} \geq 0$ represent the additional shipment and waste at the end of period t , respectively. We will use the hat symbol “^” over a random variable to indicate its observed value. Therefore the value \hat{R}_t denotes the minimum observed capacity-to-demand (CD) ratio in period t and $\hat{C}_{j,t}$ the observed capacity of county j in period t . Given these preliminaries, the sequence of events in each period is as follows and illustrated in Figure 5.1.

- 1) At the beginning of each period the incoming supply of donated food S_t is received.
- 2) This food is combined with the remaining inventory from the previous period to obtain the on-hand inventory $I_t = I_{t-1} - \tilde{X}_{t-1} - \tilde{Y}_{t-1} + S_t$, which represents the amount of food available for distribution in the current period.
- 3) Shipments \tilde{X}_t are made to all counties prior to observing the realized capacities of the counties for that period.
- 4) The realizations of the county capacities for that period are observed.

- 5) Shipment decisions are corrected with additional shipments \tilde{Y}_t or waste from the counties \tilde{W}_t to maintain perfect equity. Any unshipped inventory remaining in the branch at time t is transferred to the next period as beginning inventory for period $t + 1$.

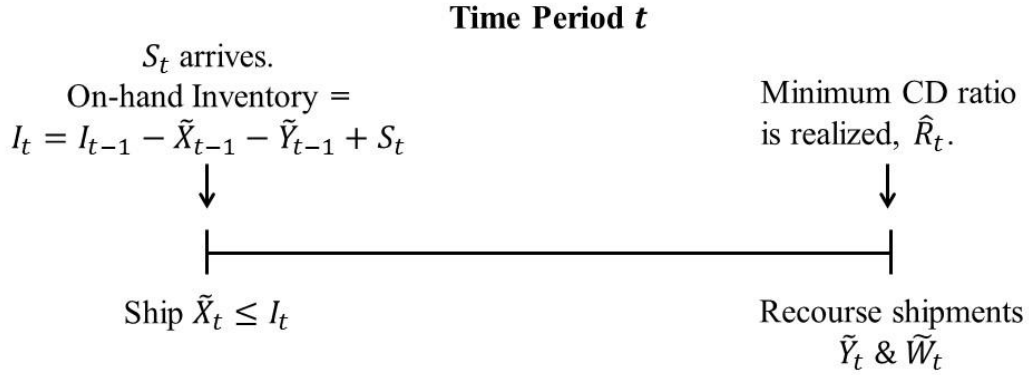


Figure 5.1: Events and Decisions in Period t .

As in Chapter 4, we use the following notation to calculate the objective function:

$b_X(t)$: Benefit of shipping one pound of food from the branch to county j in period t

c_W : Cost of discarding one pound of food as waste

We assume that $b_X(t)$ is decreasing in t , such that $b_X(t) > b_X(t')$ if and only if $t < t'$. We also assume that $b_X(t) < c_W$ for all $t = 1, \dots, T$, as discussed in Section 4.2. In each period $t = 1, \dots, T$, the objective is to maximize the benefit-to-go function given the on-hand inventory, which is defined as

$$B_t(\tilde{X}_t) = E \left[\sum_{\tau=t}^T b_X(\tau) \tilde{X}_\tau + \sum_{\tau=t}^T b_X(\tau+1) \tilde{Y}_\tau - \sum_{\tau=t}^T c_W \tilde{W}_\tau | I_t \right] \quad (5.1)$$

We note that the \tilde{X}_t variables have benefit coefficients $b_X(t)$ but the \tilde{Y}_t variables have benefit coefficients for the following period $b_X(t+1)$. This is because \tilde{Y}_t shipments are made at the end of period t and hence, obtain the benefit for the following period.

At the beginning of period t , since no capacity information is available for the current period, the only constraint imposed upon the shipment decisions \tilde{X}_t is

$$\tilde{X}_t \leq I_t. \quad (5.2)$$

Inequality (5.2) states that the total shipment at the beginning of time t should not exceed the on-hand inventory. After capacities are revealed, recourse actions are taken according to the following constraints:

$$\tilde{Y}_t \leq I_t - \tilde{X}_t \quad (5.3)$$

$$\tilde{Y}_t - \tilde{W}_t \leq \hat{R}_t \Delta - \tilde{X}_t \quad (5.4)$$

$$\tilde{X}_t + \tilde{Y}_t \geq \tilde{W}_t. \quad (5.5)$$

Constraint (5.3) states that the total shipment from the branch at time t cannot exceed the on-hand inventory. Constraints (5.4) and (5.5) state that the total net shipment to the counties cannot exceed $\hat{R}_t \Delta$ and cannot be negative, respectively. Therefore, after a shipment of \tilde{X}_t is made and the minimum CD ratio at time t is revealed, and the recourse shipments \tilde{Y}_t and \tilde{W}_t are made, the actual net amount of food received by the counties \tilde{Q}_t at time t is given by

$$\tilde{Q}_t = \min\{I_t, \hat{R}_t \Delta\}. \quad (5.6)$$

The value \tilde{Q}_t represents the net amount of food that should remain in the counties at time t , i.e., $\tilde{X}_t + \tilde{Y}_t - \tilde{W}_t$. The reason is that due to the assumption of decreasing $b_X(t)$, in each period t after \hat{R}_t is observed, there is no further uncertainty affecting the shipment for period t , making the problem for the current period equivalent to the deterministic food shipment problem studied in Chapter 3. Therefore, the maximum amount of food will be shipped under the realized conditions. If $\tilde{X}_t > \tilde{Q}_t$, there will be waste in the current period to satisfy equity and capacity requirements. Hence, the optimal recourse actions can be calculated as

$$\begin{aligned} \tilde{Y}_t^* &= \max\{0, \tilde{Q}_t - \tilde{X}_t\}, \\ \tilde{W}_t^* &= \max\{0, \tilde{X}_t - \tilde{Q}_t\}. \end{aligned} \quad (5.7)$$

Although, given \tilde{X}_t , the optimal recourse actions \tilde{Y}_t^* and \tilde{W}_t^* are known, the optimal value of \tilde{X}_t is not trivial to determine since it depends on the supply in later periods, as we will see in Section 5.4. We begin by developing a Myopic Heuristic to address this problem.

5.3 The Myopic Heuristic

We use the optimal solution structure of SPM from Chapter 4 to propose a Myopic Heuristic for the multi-period problem. We assume that the cumulative distribution function $F_R(r)$ of the minimum CD ratios is known. We assume without loss of generality that $F_R(r)$ is the same in each period, and use the results from Chapter 4 to generate a Myopic Heuristic for the multi-period problem. We will use the superscript “MH” to indicate a solution obtained by the Myopic Heuristic. Thus the total amount of food shipped in period t by the

Myopic Heuristic is denoted by \tilde{X}_t^{MH} and the shipment to county j in period t by $X_{j,t}^{\text{MH}}$. Similarly, for the recourse actions, the total amount of additional food shipped in period t as a recourse action is denoted by \tilde{Y}_t^{MH} and the individual recourse shipment to county j in period t by $Y_{j,t}^{\text{MH}}$. Any shipment from the branch in period t ; i.e., \tilde{X}_t^{MH} or \tilde{Y}_t^{MH} must be made using the amount of food I_t^{MH} available at the start of period t . The total amount of food wasted by the counties in time period t as a recourse action is denoted by \tilde{W}_t^{MH} and the individual recourse waste from county j in period t by $W_{j,t}^{\text{MH}}$. The remaining parameters have the same definitions as in Chapter 4 and Section 5.2. The Myopic Heuristic (MH) can be stated as follows:

Myopic Heuristic (MH):

1 **Initialization.** Let $t = 1$ and $I_1^{\text{MH}} = S_1$.

2 **For each period** $t = 1, \dots, T$.

2.1 **Calculate shipments.** Calculate the total shipment from the branch $\tilde{X}_t^{\text{MH}} =$

$$\min \left\{ I_t^{\text{MH}}, \Delta F_R^{-1} \left(\frac{b_x(t) - b_x(t+1)}{c_w - b_x(t+1)} \right) \right\} \text{ and set } X_{j,t}^{\text{MH}} = \frac{\tilde{X}_t^{\text{MH}} D_j}{\Delta}.$$

2.2 **Capacity Realization.** After the capacities are observed, calculate $\hat{R}_t = \min_{j \in J} \frac{\hat{c}_{j,t}}{D_j}$.

Calculate the total amount of food that should be shipped at time t as $\tilde{Q}_t^{\text{MH}} = \min \{ I_t^{\text{MH}}, \Delta \hat{R}_t \}$.

2.3 **Calculate the recourse actions:** The food waste in the counties is calculated as

$$\tilde{W}_t^{\text{MH}} = \max \{ 0, \tilde{X}_t^{\text{MH}} - \tilde{Q}_t^{\text{MH}} \} \text{ and } W_{j,t}^{\text{MH}} = \frac{\tilde{W}_t^{\text{MH}} D_j}{\Delta}.$$

The additional food shipment to the counties is calculated as $\tilde{Y}_t^{\text{MH}} = \max \{ 0, \tilde{Q}_t^{\text{MH}} - \tilde{X}_t^{\text{MH}} \}$ and $Y_{j,t}^{\text{MH}} = \frac{\tilde{Y}_t^{\text{MH}} D_j}{\Delta}$.

2.4 Update inventory. If $t \neq T$, compute the beginning inventory at the branch for the next period as $I_{t+1}^{\text{MH}} = I_t^{\text{MH}} - \tilde{X}_t^{\text{MH}} - \tilde{Y}_t^{\text{MH}} + S_{t+1}$. Set $t = t+1$ and go to Step 2.1.

3 Objective Function Calculation. Calculate the objective function value $Z^{\text{MH}} = \sum_{t=1}^T b_X(t) \tilde{X}_t^{\text{MH}} + \sum_{t=1}^T b_X(t+1) \tilde{Y}_t^{\text{MH}} - \sum_{t=1}^T c_W \tilde{W}_t^{\text{MH}}$.

The MH treats the multi-period problem as a sequence of single-period problems that are each solved sequentially independently of other periods. The problem with this approach is that the decision maker does not have any visibility of the changes in supply, demand and benefits of shipment in future periods. In Section 5.4, we develop structural properties and derive bounds on the difference between the optimal solution to the multi-period food shipment problem and the objective function of the Myopic Heuristic.

5.4 Bounds on the Optimal Solution to the Multi-Period Food Distribution Problem

In this section, we use the structural properties of the Myopic Heuristic and the multi-period food distribution problem defined in Section 5.2 to obtain bounds on the value of the optimal solution. First, we show that in this system, the decisions in a given period depend on the past decisions only through the current inventory (Proposition 5.1), and affect future decisions only through the inventory remaining at the end of that period (Corollary 5.2). We emphasize that I_t represents the amount of food available at the start of period t after supply for that period S_t has been received. In other words, I_t is the amount of food available for distribution to the counties in period t .

Proposition 5.1: *For the multi-period food distribution problem, decisions made in the periods t, \dots, T depend on the past decisions $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: \tau = 1, \dots, t - 1\}$ only through $I_t = \sum_{\tau=1}^t S_\tau - \sum_{\tau=1}^{t-1} \tilde{X}_\tau - \sum_{\tau=1}^{t-1} \tilde{Y}_\tau$. That is, given I_t , the decisions in periods $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: \tau = t, \dots, T\}$ are independent of the past decisions $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: \tau = 1, \dots, t - 1\}$.*

Proof of Proposition 5.1: It is directly seen from the formulation of the multi-period problem that the decisions in the future periods $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: j \in J; s \in Y; \tau = t, \dots, T\}$ are independent of the past decisions $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: \tau = 1, \dots, t - 1\}$ as long as $I_t = \sum_{\tau=1}^t S_\tau - \sum_{\tau=1}^{t-1} \tilde{X}_\tau - \sum_{\tau=1}^{t-1} \tilde{Y}_\tau$ is known. In other words, I_t is a sufficient statistic and contains all the information necessary to compute the future evolution of the system. ■

Corollary 5.2 follows directly from Proposition 5.1.

Corollary 5.2: *For the multi-period food distribution problem, given I_{t+1} , decisions made in the future $\{\tilde{X}_\tau, \tilde{Y}_\tau, \tilde{W}_\tau: \tau = t + 1, \dots, T\}$ are independent of the decisions made in period t $\{\tilde{X}_t, \tilde{Y}_t, \tilde{W}_t\}$.*

For the multi-period food distribution problem, for any period t , if $I_t < \hat{R}_t \Delta$, we will call this period “inventory constrained” and if $\hat{R}_t \Delta < I_t$, we will call this period “capacity constrained”. Corollary 5.3 uses the results from Proposition 5.1 to state that the multi-period food distribution process is a regenerative process, i.e., some parts of the multi-period problem can be treated independently from others.

Corollary 5.3: *For the multi-period food distribution problem, any inventory constrained period t is a stochastic regeneration point.*

Corollary 5.3 states that if all of the on-hand inventory is shipped in period t , the multi-period food shipment problem for periods $t + 1, \dots, T$ is statistically independent of the shipment process in the periods $1, \dots, t$. This follows from Proposition 5.1 since if all of the inventory is shipped in period t , there is no information left to carry to the next period $t+1$, hence the process restarts with the supply received in $t+1$. We will now examine the relationship between the values of the myopic solution and the optimal solution.

Let $U_t(\tilde{X}_t)$ represent the objective function of the single-period model derived in Chapter 4 that is optimally solved by the Myopic Heuristic in period t and $V_t(\tilde{X}_t)$ the objective function associated with the multi-period problem for periods $t = t, \dots, T$ in a periodic review environment. Furthermore, let I_t^{MH} denote the on-hand inventory at the start of period t if the Myopic policy is implemented in periods $1, \dots, t - 1$ and I_t^* the on-hand inventory at the start of time t generated by the optimal solution to the multi-period problem. We thus have

$$U_t(\tilde{X}_t) = b_X(t)\tilde{X}_t + E[(b_X(t+1)\tilde{Y}_t - c_W\tilde{W}_t) | \tilde{X}_t] \text{ for } t = 1, \dots, T \quad (5.8)$$

$$V_t(\tilde{X}_t) = \begin{cases} b_X(t)\tilde{X}_t + E[b_X(t+1)\tilde{Y}_t - c_W\tilde{W}_t | \tilde{X}_t] + E[V_{t+1} | I_{t+1}^*] & \text{for } t = 1, \dots, T - 1 \\ b_X(t)\tilde{X}_t + E[b_X(t+1)\tilde{Y}_t - c_W\tilde{W}_t | \tilde{X}_t] & \text{for } t = T \end{cases} \quad (5.9)$$

At any period t , the Myopic Heuristic aims to maximize

$$H_t(\tilde{X}_t, I_t^{\text{MH}}) = E[U_t(\tilde{X}_t) | I_t^{\text{MH}}]. \quad (5.10)$$

Conditioning on I_t^{MH} means conditioning on a sigma algebra that defines the support of the random variable \tilde{X}_t , i.e., $E[U_t(\tilde{X}_t)|I_t^{\text{MH}}] \equiv E[U_t(\tilde{X}_t)|\tilde{X}_t \in [0, I_t^{\text{MH}}]]$.

On the other hand, in any period t , the optimal solution will maximize the following benefit-to-go function for the periods t, \dots, T :

$$B_t(\tilde{X}_t, I_t^*) = E[V_t(\tilde{X}_t)|I_t^*] = E[U_t(\tilde{X}_t)|I_t^*] + E[E[V_{t+1}(\tilde{X}_{t+1})|I_{t+1}^*]|I_t^*] \quad (5.11)$$

where, since given I_{t+1}^* , V_{t+1} is independent of I_t^* by Corollary 5.2, we have $E[E[V_{t+1}(\tilde{X}_{t+1})|I_{t+1}^*]|I_t^*] = E[V_{t+1}(\tilde{X}_{t+1})|I_{t+1}^*]$.

Following this logic, we obtain

$$B_t(\tilde{X}_t, I_t^*) = E[V_t(\tilde{X}_t)|I_t^*] = \sum_{\tau=t}^T E[U_\tau(\tilde{X}_\tau)|I_t^*]. \quad (5.12)$$

Furthermore, let

$$\tilde{X}_t^{\text{MH}} = \left\{ u \in (0, I_t^{\text{MH}}): H_t(u, I_t^{\text{MH}}) = \max_{\tilde{X}_t} H_t(\tilde{X}_t, I_t^{\text{MH}}) \right\} \quad (5.13)$$

$$\tilde{X}_t^* = \left\{ v \in (0, I_t^*): B_t(v, I_t^*) = \max_{\tilde{X}_t} B_t(\tilde{X}_t, I_t^*) \right\} \quad (5.14)$$

denote the arg max of the functions $H_t(\tilde{X}_t, I_t^{\text{MH}})$ and $B_t(\tilde{X}_t, I_t^*)$ subject to all the constraints in the respective models. In other words, \tilde{X}_t^{MH} represents the arg max of the Myopic objective in period t and \tilde{X}_t^* the arg max of the benefit-to-go function for the periods t, \dots, T . Note that $B_t(\tilde{X}_t, I_t^*)$ represents the objective function of the multi-period problem in period t and $B_t(\tilde{X}_t^*, I_t^*)$ the optimal value of the multi-period problem in period t . Similarly,

$H_t(\tilde{X}_t, I_t^{\text{MH}})$ represents the objective function of the Myopic Heuristic in period t and $H_t(\tilde{X}_t^{\text{MH}}, I_t^{\text{MH}})$ the optimal value of the Myopic objective function in period t . We will use Lemma 5.4 to prove various relationships between $H_t(\tilde{X}_t, I_t)$ and $B_t(\tilde{X}_t, I_t)$.

Lemma 5.3: *The functions $H_t(\tilde{X}_t, I_t)$ and $B_t(\tilde{X}_t, I_t)$ are strictly concave in \tilde{X}_t on the interval $[0, I_t]$.*

Proof of Lemma 5.3: In the proof of Proposition 4.2, we showed the strict concavity of the $H_t(\tilde{X}_t)$ function on the interval $[0, I_t]$. Since $B_t(\tilde{X}_t, I_t^*) = \sum_{\tau=t}^T E[U_\tau(\tilde{X}_\tau)|I_t^*]$ and the sum of a strictly concave function and concave functions is strictly concave, to prove the strict concavity of $B_t(\tilde{X}_t)$, it suffices to show that $E[U_\tau(\tilde{X}_\tau)|I_t^*]$ is concave in \tilde{X}_t on the interval $[0, I_t]$ for all $\tau = t + 1, \dots, T$. First, we know that $I_\tau^* = I_t + \sum_{i=t+1}^\tau S_i - \sum_{i=t}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*) = I_t - (\tilde{X}_t^* + \tilde{Y}_t^*) + \sum_{i=t+1}^\tau S_i - \sum_{i=t+1}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*)$. Then

$$\begin{aligned}
U_\tau(\tilde{X}_\tau) &= b_X(\tau)\tilde{X}_\tau + \int_0^{\frac{\tilde{X}_\tau}{\Delta}} [-c_W(\tilde{X}_\tau - r\Delta)]f_R(r)dr \\
&+ \int_{\frac{\tilde{X}_\tau}{\Delta}}^{\frac{I_\tau^*}{\Delta}} [b_X(\tau + 1)(r\Delta - \tilde{X}_\tau)]f_R(r)dr \\
&+ \int_{\frac{I_\tau^*}{\Delta}}^{\infty} [b_X(\tau + 1)(I_\tau^* - \tilde{X}_\tau)]f_R(r)dr.
\end{aligned} \tag{5.15}$$

Taking the derivative of $U_\tau(\tilde{X}_\tau)$ with respect to \tilde{X}_t where $t < \tau \leq T$, we obtain

$$\begin{aligned}
\frac{\partial[U_\tau(\tilde{X}_\tau)]}{\partial\tilde{X}_t} &= \frac{1}{\Delta} \frac{\partial[I_\tau^*(\tilde{X}_t)]}{\partial\tilde{X}_t} \left[b_X(\tau+1) \left(\frac{I_\tau^*}{\Delta} \Delta - \tilde{X}_\tau \right) \right] f_R \left(\frac{I_\tau^*}{\Delta} \right) \\
&\quad + \int_{\frac{I_\tau^*}{\Delta}}^{\infty} \left[b_X(\tau+1) \frac{\partial[I_\tau^*(\tilde{X}_t)]}{\partial\tilde{X}_t} \right] f_R(r) dr \\
&\quad - \frac{1}{\Delta} \frac{\partial[I_\tau^*(\tilde{X}_t)]}{\partial\tilde{X}_t} \left[b_X(\tau+1) \left(\frac{I_\tau^*}{\Delta} \Delta - \tilde{X}_\tau \right) \right] f_R \left(\frac{I_\tau^*}{\Delta} \right) \\
&= -b_X(\tau+1) \left[1 - F_R \left(\frac{I_\tau^*}{\Delta} \right) \right]
\end{aligned} \tag{5.16}$$

where we denote the inventory in a future period $\tau > t$ as $I_\tau^*(\tilde{X}_t)$ to emphasize that the inventory in the future period τ is a function of the shipment in the current period t , \tilde{X}_t . If I_τ^* is independent of \tilde{X}_t , which can happen if any time period between t and τ is inventory constrained, then the process would regenerate at that point and we will have $\frac{\partial[U_\tau(\tilde{X}_\tau)]}{\partial\tilde{X}_t} = 0$, which proves that $B_t(\tilde{X}_t)$ is strictly concave in \tilde{X}_t on the interval $[0, I_t]$. We note that I_τ^* is nonincreasing in \tilde{X}_t ; hence, as \tilde{X}_t is increases, $F_R \left(\frac{I_\tau^*}{\Delta} \right)$ decreases or stays the same. Therefore, as \tilde{X}_t increases from zero to I_t , $\frac{\partial[U_\tau(\tilde{X}_\tau)]}{\partial\tilde{X}_t}$ decreases monotonically. Furthermore,

$$\frac{\partial^2[U_\tau(\tilde{X}_\tau)]}{\partial\tilde{X}_t^2} = -b_X(\tau+1) \frac{\partial \left[F_R \left(\frac{I_\tau^*}{\Delta} \right) \right]}{\partial\tilde{X}_t} \leq 0 \tag{5.17}$$

since

$$\begin{aligned}
F_R\left(\frac{I_\tau^*}{\Delta}\right) &= P\left(R_\tau \leq \frac{I_\tau^*}{\Delta}\right) = P\left(I_t + \sum_{i=t+1}^{\tau} S_i - \sum_{i=t}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*) \geq R_\tau \Delta\right) \\
&= P\left(\tilde{X}_t^* \leq I_t + \sum_{i=t+1}^{\tau} S_i - \tilde{Y}_t^* - \sum_{i=t+1}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*) - R_\tau \Delta\right) \\
&= F_X\left(I_t + \sum_{i=t+1}^{\tau} S_i - \tilde{Y}_t^* - \sum_{i=t+1}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*) - R_\tau \Delta\right)
\end{aligned}$$

and therefore,

$$\frac{\partial \left[F_R\left(\frac{I_\tau^*}{\Delta}\right) \right]}{\partial \tilde{X}_t} = f_X\left(I_t + \sum_{i=t+1}^{\tau} S_i - \tilde{Y}_t^* - \sum_{i=t+1}^{\tau-1} (\tilde{X}_i^* + \tilde{Y}_i^*) - R_\tau \Delta\right) \geq 0$$

by the definition of a probability density function. Hence, we have proved that $E[U_\tau(\tilde{X}_\tau)|I_\tau^*]$ is concave in \tilde{X}_t in the interval $[0, I_t]$ for all $\tau = t + 1, \dots, T$. Consequently, $B_t(\tilde{X}_t, I_t)$ is strictly concave (down) in \tilde{X}_t on the interval $[0, I_t]$. ■

Proposition 5.5 states that the arg max of $H_t(\tilde{X}_t, I_t)$ is never less than the arg max of $B_t(\tilde{X}_t, I_t)$ for any given value of I_t .

Proposition 5.5: Assume that \tilde{X}_t^{MH} and \tilde{X}_t^* are defined as in (5.13) and (5.14) and assume that $I_t^{\text{MH}} = I_t^* = I_t$ at any time period t for $t = 1, \dots, T$. Then,

- i. The total distribution made by the Myopic Heuristic is an upper bound on the optimal total distribution for the multi-period problem at time t , i.e.,

$$\tilde{X}_t^* \leq \tilde{X}_t^{\text{MH}}. \quad (5.18)$$

ii. *If either of the following conditions is satisfied, then we will have $\tilde{X}_t^* = \tilde{X}_t^{\text{MH}}$, i.e., (5.18) will be satisfied as equality.*

a. *With probability one, all following periods are capacity constrained, i.e.,*

$$P(R\Delta \leq I_\tau) = 1 \text{ for all } \tau = t, \dots, T.$$

b. *With probability one, the current period t is inventory constrained, i.e.,*

$$P(I_t \leq R\Delta) = 1.$$

Proof of Proposition 5.5: We start by noting that the myopic solution \tilde{X}_t^{MH} is always feasible for the multi-period problem for period t since by definition $\tilde{X}_t^{\text{MH}} \leq I_t$.

By Lemma 5.4, we know that both $H_t(\tilde{X}_t, I_t)$ and $B_t(\tilde{X}_t, I_t)$ are strictly concave in \tilde{X}_t on the interval $[0, I_t]$. Therefore, to prove Proposition 5.5, it suffices to show that $\left. \frac{\partial [B_t(\tilde{X}_t, I_t)]}{\partial \tilde{X}_t} \right|_{\tilde{X}_t = \tilde{X}_t^{\text{MH}}} \leq 0$. Figure 5.2 illustrates why this is true.

The quantity of interest is

$$\left. \frac{\partial [B_t(\tilde{X}_t, I_t)]}{\partial \tilde{X}_t} \right|_{\tilde{X}_t = \tilde{X}_t^{\text{MH}}} = \sum_{\tau=t}^T \left. \frac{\partial [E[U_\tau(\tilde{X}_\tau) | I_\tau^*]]}{\partial \tilde{X}_t} \right|_{\tilde{X}_t = \tilde{X}_t^{\text{MH}}}.$$

First, we notice that

$$\left. \frac{\partial [E[U_t(\tilde{X}_t) | I_t]]}{\partial \tilde{X}_t} \right|_{\tilde{X}_t = \tilde{X}_t^{\text{MH}}} = 0$$

by the definition of \tilde{X}_t^{MH} . Then, from (5.16), we see that for $\tau = t + 1, \dots, T$,

$$\left. \frac{\partial \left[E[U_\tau(\tilde{X}_\tau) | I_\tau^*] \right]}{\partial \tilde{X}_t} \right|_{\tilde{X}_t = \tilde{X}_t^{\text{MH}}} \leq 0, \quad (5.19)$$

which completes the proof of part 1 of Proposition 5.5.

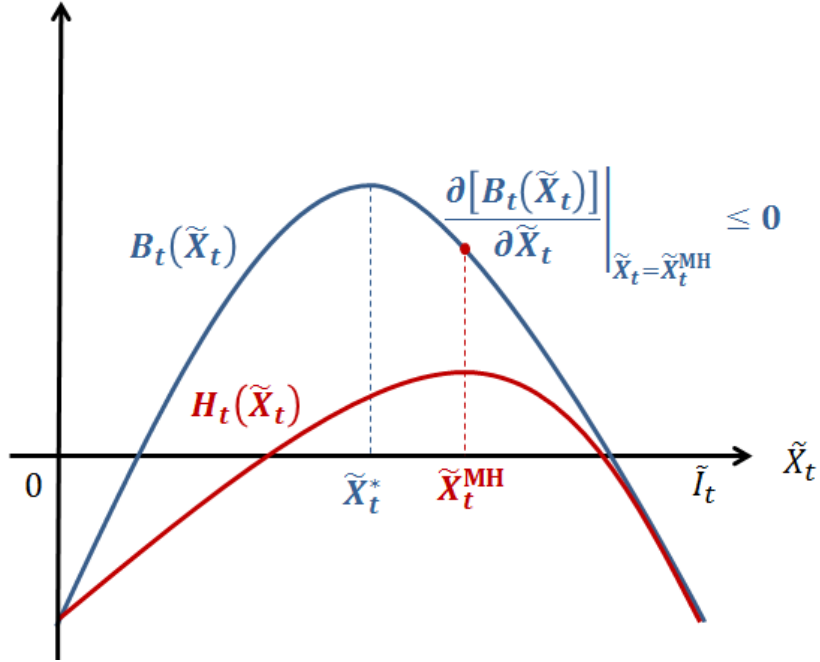


Figure 5.2: Derivative of $B_t(\tilde{X}_t)$ evaluated at \tilde{X}_t^{MH} .

The inequality in (5.19) holds at equality, implying $\tilde{X}_t^* = \tilde{X}_t^{\text{MH}}$, if

$$-b_X(\tau + 1) \left[1 - F_R \left(\frac{I_\tau^*}{\Delta} \right) \right] = 0 \text{ for all } \tau = t + 1, \dots, T. \quad (5.20)$$

which means that for all $\tau = t + 1, \dots, T$,

$$F_R\left(\frac{I_t^*}{\Delta}\right) = P\left(R_\tau \leq \frac{I_t^*}{\Delta}\right) = P(R_\tau \Delta \leq I_t^*) = 1. \quad (5.21)$$

Hence, all periods $\tau = t + 1, \dots, T$ will be capacity constrained with probability one. In this situation we will never need surplus supply in future periods and all food not shipped in the current period will be sent to waste. Hence $\tilde{X}_t^* = \tilde{X}_t^{\text{MH}}$, proving part 2.a of Proposition 5.5.

Finally, consider the case where $P(I_t \leq R\Delta) = 1$, i.e., the solution in period t is inventory constrained with probability 1 resulting in a stochastic regeneration of the inventory process at $t+1$. In this case, regardless of the values of \tilde{X}_t , \tilde{Y}_t and \tilde{W}_t , we will have $I_{t+1}^* = S_{t+1}$ and I_t^* is independent of \tilde{X}_t . Therefore, by Proposition 5.1, the problem solved at time t is equivalent to the single-period model at time t and the myopic solution \tilde{X}_t^{MH} is optimal for period t , yielding $\tilde{X}_t^* = \tilde{X}_t^{\text{MH}}$. This completes the proof. ■

Proposition 5.5 is useful because (i) it provides an upper bound on the optimal shipment at time t by using the myopic solution, and (ii) it characterizes a number of situations where the myopic solution is optimal. The amount of food shipped by the Myopic Heuristic is never less than the optimal food shipment. This is reasonable since the myopic solution lacks knowledge of the future supply and cost coefficients that the multi-period problem has. For example, if supply is only received in the first period, the Myopic Heuristic may ship all of the supply if $S_1 < \Delta F_R^{-1}\left(\frac{b_X(1) - b_X(2)}{c_W - b_X(2)}\right)$. However, an optimal multi-period shipment policy will try to save some supply for future periods, if necessary, which will cause it to ship less than the Myopic Heuristic. Therefore, the Myopic Heuristic will perform badly if the future periods are supply constrained with high probability and the current period

is capacity constrained with a low realized \hat{R}_t value since the myopic approach cannot plan ahead for low supply and creates high waste in the current period. The following corollary derives the relationship between the inventories of the Myopic and optimal solutions.

Corollary 5.6: *In any time period $t = 1, \dots, T$, we will have $I_t^{\text{MH}} \leq I_t^*$.*

Corollary 5.6 states that in any period in the planning horizon, the on-hand inventory at time t corresponding to the optimal solution will always be greater than or equal to the on-hand inventory at time t corresponding to the solution obtained by the Myopic Heuristic. This result follows directly from Proposition 5.5.

We have shown that the Myopic solution provides an upper bound on the optimal shipments. Since the Myopic policy provides a feasible solution to the multi-period problem and ships greater than or equal to the optimal shipments in each stage, the Myopic objective gives a lower bound on the optimal value of the objective function for the multi-period problem. The objective value of the Myopic solution is a lower bound on the optimal value, since the Myopic solution ignores the effect of the current actions on future benefits and aims only to maximize the benefit of the current period. Hence, we will have

$$\sum_{\tau=t}^T E[U_{\tau}(\tilde{X}_{\tau}^*)|I_{\tau}^*] \geq \sum_{\tau=t}^T E[U_{\tau}(\tilde{X}_{\tau}^{\text{MH}})|I_{\tau}^{\text{MH}}]. \quad (5.22)$$

Proposition 5.5 implies that in order to have $\tilde{X}_t^* < \tilde{X}_t^{\text{MH}}$, the complements of conditions 2.a and 2.b of Proposition 5.5 must be satisfied simultaneously. In order to obtain an understanding of the situations where the Myopic Heuristic will perform badly, assume that in period t , $I_t^{\text{MH}} = I_t^* = I_t$ and $\tilde{X}_t^{\text{MH}} - \tilde{X}_t^* = \delta$, i.e., the optimal solution ships δ pounds less

food than the Myopic solution. Also, assume that the following conditions, which are complements of conditions 2.a and 2.b in Proposition 5.5, are satisfied: (i) $P(\hat{R}_t \Delta < I_t) = 1$, i.e., the current period is capacity constrained with probability one and (ii) $P(S_{t+1} + \delta < \hat{R}_{t+1} \Delta) = 1$, i.e., period $t + 1$ is supply constrained with probability one. For example, assume that $\tilde{X}_t^{MH} = I_t$ and $\tilde{X}_{t+1}^{MH} = S_{t+1}$, i.e., the Myopic policy ships all of the on-hand inventory in periods t and $t+1$. We will calculate the additional benefit gained by the optimal shipment. We have

$$U_t(\tilde{X}_t^{MH}) = b_X(t)I_t - c_W(I_t - \hat{R}_t \Delta) \quad (5.23)$$

and

$$U_t(\tilde{X}_t^*) = b_X(t)(I_t - \delta) - c_W(I_t - \delta - \hat{R}_t \Delta). \quad (5.24)$$

Also, since $I_{t+1}^* = S_{t+1} + \delta$, we will have

$$U_{t+1}(\tilde{X}_{t+1}^{MH}) = b_X(t+1)S_{t+1} \quad (5.25)$$

and

$$U_{t+1}(\tilde{X}_{t+1}^*) = b_X(t+1)(S_{t+1} + \delta). \quad (5.26)$$

Collecting these terms, we obtain

$$\begin{aligned}
& U_t(\tilde{X}_t^* | I_t) + U_{t+1}(\tilde{X}_{t+1}^* | I_t^*) - \left(U_t(\tilde{X}_t^{MH} | I_t) + U_{t+1}(\tilde{X}_{t+1}^{MH} | I_t^{MH}) \right) \\
& = \delta \left(c_W - (b_X(t) - b_X(\tau)) \right).
\end{aligned} \tag{5.27}$$

We notice that the bound given in (5.27) will decrease, and the Myopic heuristic will perform better, if

- i. the periods t and τ are further apart from each other. In that case, the value $b_X(t) - b_X(\tau)$ will increase and the gap will decrease. This means that the Myopic Heuristic will perform better if the supply available in period t is needed in a period that is further away in the future. This observation motivates the One-Period Look-Ahead Heuristic discussed in the next section which simultaneously plans shipments for the current and next period.
- ii. the difference between the shipments made by the Myopic and optimal policies δ decreases, in other words if the supply needed in the next period decreases. This is also reasonable since the Myopic solution cannot anticipate the need for future supply. If there is a need for a large amount of supply in a future period that the Myopic Heuristic cannot observe, then the performance of the Myopic solution, in comparison to the optimal solution, will be worse. We will use both these ideas in our numerical experiments to evaluate the performance of the Myopic policy under different situations.

Taken together, these results suggest that the Myopic Heuristic should perform well if all time periods are capacity constrained or all time periods are inventory constrained. It will perform poorly if the periods alternate between being capacity and inventory constrained. In

that situation, the Myopic Heuristic cannot anticipate a future need for supply and may end up shipping too much food in the current period, creating waste. We now develop a lower bound on the optimal shipment at time t , yielding an upper bound on the optimal value.

Proposition 5.7: *Assume that the minimum CD ratio at time t , R_t , follows a known distribution $R_t \sim F_{R_t}(\cdot)$ with a known, finite support $R_t \in [R_t^L, R_t^U]$. Consider the benefit-to-go function defined in (5.11). Assume that optimal decisions have been taken prior to time t and the current inventory $I_t^U = I_t^*$. If we replace the coefficients of \tilde{Y}_t with $b_X(t)$ such that*

$$B_t^U(\tilde{X}_t) = \sum_{\tau=t}^T E[b_X(t)\tilde{X}_t + E[(b_X(t)\tilde{Y}_t - c_W\tilde{W}_t) | \tilde{X}_t] | I_t^U] \quad (5.28)$$

Then, we have the following results:

1. The arg max of $B_t^U(\tilde{X}_t)$ is

$$\tilde{X}_t^L = \min\{I_t^U, R_t^L \Delta\}. \quad (5.29)$$

2. We have $\tilde{X}_t^L \leq \tilde{X}_t^*$, i.e., the arg max of $B_t^U(\tilde{X}_t)$ provides a lower bound on the total shipment at time t .
3. Finally, the benefit-to-go function defined in (5.42) provides an upper bound on the optimal benefit-to-go function, i.e., $B_t^U(\tilde{X}_t) \geq B_t^*(\tilde{X}_t)$.

Proof of Proposition 5.7:

Part 1: In this system, the recourse shipments \tilde{Y}_t have the same benefit as \tilde{X}_t , which allows the decision maker to wait until \hat{R}_t is observed and then make the shipments for time t . Alternatively, the decision maker can ship in the beginning of the period the minimum

amount of shipment that will definitely be shipped in the current period. We would like to emphasize that, in this system, there will never be waste, since the decision maker would never overship at the beginning of a stage. Because there will never be waste, the system solved at time t becomes equivalent to a single-stage problem considered in Chapter 4 with the optimal solution:

$$\tilde{X}_t^L = \min \left\{ I_t^U, \Delta F_R^{-1} \left(\frac{b_x(t) - b_x(t)}{c_w - b_x(t)} \right) \right\} = \min \{ I_t^U, \Delta F_R^{-1}(0) \} = \min \{ I_t^U, R_t^L \Delta \}.$$

This value provides a lower bound on the amount of food that will be shipped at time t without any risk of the food going to waste. Therefore any value of $\tilde{X}_t \in [0, \tilde{X}_t^L]$ provides the optimum solution for the benefit-to-go function $B_t^U(\tilde{X}_t)$.

Part 2: We will prove the result by contradiction. Assume that $\tilde{X}_t^C = \tilde{X}_t^* = \min \{ I_t^*, R_t^L \Delta \} - \delta$ where δ is a small positive number, and let $\tilde{X}_t^A = \min \{ I_t^*, R_t^L \Delta \}$. We will prove that shipping \tilde{X}_t^A will always result in a higher objective function value than \tilde{X}_t^C . We also define \tilde{Y}_t^C (\tilde{Y}_t^A) to be the additional shipment made at time t after the minimum CD ratio is observed, given we have shipped \tilde{X}_t^C (\tilde{X}_t^A), respectively. We note that the optimal total amount of food shipped in period t after the minimum CD ratios are observed is equal to $\min \{ I_t^*, \hat{R}_t \Delta \}$ where, by definition, $\hat{R}_t \geq R_t^L$.

Case 1: $I_t^* < R_t^L \Delta$

Since $I_t^* < \hat{R}_t \Delta$ we have $\tilde{X}_t^A = I_t^*$ and $\tilde{X}_t^C = I_t^* - \delta$. Furthermore, since the shipment benefit coefficients $b_x(t)$ are decreasing in time, the maximum benefit that can be gained from shipping δ pounds of food at time t is $\delta b_x(t)$. Therefore, we have $\tilde{Y}_t^C = \delta$ whereas $\tilde{Y}_t^A = 0$.

We see that \tilde{X}_t^A is a feasible solution that provides a higher benefit function, i.e., $B_t(\tilde{X}_t^A) > B_t(\tilde{X}_t^C)$, contradicting the initial hypothesis that $\tilde{X}_t^C = \tilde{X}_t^*$.

Case 2a: $R_t^L \Delta < I_t^*$ and $R_t^L \Delta < \hat{R}_t \Delta < I_t^*$

In this case, we have $\tilde{X}_t^A = R_t^L \Delta$ and $\tilde{X}_t^C = R_t^L \Delta - \delta$. The optimal amount of food to be shipped at time t is $\hat{R}_t \Delta$. Therefore after the minimum CD ratio is observed, additional shipments in the amount of $\tilde{Y}_t^A = \hat{R}_t \Delta - R_t^L \Delta$ and $\tilde{Y}_t^C = \hat{R}_t \Delta - (R_t^L \Delta - \delta)$ will be made. We see that \tilde{X}_t^A is a feasible solution that provides a higher benefit, i.e., $B_t(\tilde{X}_t^A) > B_t(\tilde{X}_t^C)$, contradicting the initial hypothesis that $\tilde{X}_t^C = \tilde{X}_t^*$.

Case 2b: $R_t^L \Delta < I_t^*$ and $R_t^L \Delta < I_t^* < \hat{R}_t \Delta$

In this case, we have $\tilde{X}_t^A = R_t^L \Delta$ and $\tilde{X}_t^C = R_t^L \Delta - \delta$. The optimal amount of food that should be shipped at time t is I_t^* . Therefore after the minimum CD ratio is observed, additional shipments in the amount of $\tilde{Y}_t^A = I_t^* - R_t^L \Delta$ and $\tilde{Y}_t^C = I_t^* - (R_t^L \Delta - \delta)$ will be made. We see that \tilde{X}_t^A is a feasible solution and that $B_t(\tilde{X}_t^A) > B_t(\tilde{X}_t^C)$, contradicting the initial hypothesis that $\tilde{X}_t^C = \tilde{X}_t^*$.

Therefore, for all realizations of \hat{R}_t , the initial shipment \tilde{X}_t^A always provides a higher benefit value. Therefore we have a contradiction.

Part 3: Since the $b_X(t)$ are decreasing in time, and since we will always have $I_t^U \geq I_t^*$, it is straightforward to see that

$$\begin{aligned}
& \sum_{\tau=t}^T E[b_X(t)\tilde{X}_t + E[(b_X(t)\tilde{Y}_t - c_W\tilde{W}_t) | \tilde{X}_t] | I_\tau^U] \\
& \geq \sum_{\tau=t}^T E[b_X(t)\tilde{X}_t + E[(b_X(t+1)\tilde{Y}_t - c_W\tilde{W}_t) | \tilde{X}_t] | I_\tau^*].
\end{aligned}$$

Proof is completed. ■

In this section, we have derived upper and lower bounds on the optimal food shipment in each time period t and the optimal value of the objective function for the multi-period problem. To summarize our findings, let I_t^* represent the on-hand inventory in any time period t assuming that optimal shipment decisions have been made in periods $1, \dots, t - 1$. We then have

$$\tilde{X}_t^L \leq \tilde{X}_t^* \leq \tilde{X}_t^U, \quad (5.30)$$

where

$$\tilde{X}_t^U = \min \left\{ I_t^*, \Delta F_R^{-1} \left(\frac{b_X(t) - b_X(t+1)}{c_W - b_X(t+1)} \right) \right\}, \quad (5.31)$$

and

$$\tilde{X}_t^L = \min\{I_t^*, R_t^L \Delta\}. \quad (5.32)$$

We have also shown that

$$B_t(\tilde{X}_t^U) \leq B_t(\tilde{X}_t^*) \leq B_t^U(\tilde{X}_t^L). \quad (5.33)$$

We will generate two heuristics using these ideas. First, in Section 5.5, we will develop a heuristic that uses information from both the current and the next period to compute shipments. Then, in Section 5.6, we will introduce a simple heuristic that ships the average of \tilde{X}_t^L and \tilde{X}_t^U in each period.

5.5 The One-Period Look-Ahead Heuristic

In this section, we will use the ideas from Section 5.4 to present a slight modification to the Myopic Heuristic. We will assume that the supply and benefit coefficient values for both the current period t and the next period $t + 1$ are available at time t . All definitions made in Section 5.3 hold here. We will use the superscript LAH to denote a solution obtained by the One-Period Look-Ahead Heuristic, which we will refer to as the Look-Ahead solution. We state the heuristic as follows:

One-Period Look-Ahead Heuristic (LAH):

1 Initialization. Let $t = 1$ and $I_1^{\text{LAH}} = S_1$.

2 For each period $t = 1, \dots, T$.

2.1 Calculate myopic shipments. Calculate the total myopic shipment from the

$$\text{branch } \tilde{X}_t^{\text{LAH}} = \min \left\{ I_t^{\text{LAH}}, \Delta F_R^{-1} \left(\frac{b_x(t) - b_x(t+1)}{c_w - b_x(t+1)} \right) \right\}.$$

2.2 Check remaining supply for the next period. If $I_t^{\text{LAH}} + S_{t+1} - \tilde{X}_t^{\text{LAH}} <$

$$\Delta F_R^{-1} \left(\frac{b_x(t+1) - b_x(t+2)}{c_w - b_x(t+2)} \right), \text{ set}$$

$$\tilde{X}_t^{\text{LAH}} = \max \left\{ \min\{I_t^{\text{LAH}}, R_t^L \Delta\}, I_t^{\text{LAH}} + S_{t+1} - \Delta F_R^{-1} \left(\frac{b_x(t+1) - b_x(t+2)}{c_w - b_x(t+2)} \right) \right\}.$$

2.3 Set $X_{j,t}^{\text{LAH}} = \frac{\tilde{X}_t^{\text{LAH}} D_j}{\Delta}$.

2.4 *Capacity Realization.* After the capacities are observed, calculate $\hat{R}_t = \min_{j \in J} \frac{\hat{c}_{j,t}}{D_j}$.

Calculate the total amount of food that should be shipped at time t as $\tilde{Q}_t^{\text{LAH}} = \min\{I_t^{\text{LAH}}, \Delta \hat{R}_t\}$.

2.5 *Calculate the recourse actions:* The remaining food waste in the counties is

calculated as $\tilde{W}_t^{\text{LAH}} = \max\{0, \tilde{X}_t^{\text{LAH}} - \tilde{Q}_t^{\text{LAH}}\}$ and let $W_{j,t}^{\text{LAH}} = \frac{\tilde{W}_t^{\text{LAH}} D_j}{\Delta}$. The

additional food shipment to the counties is calculated as $\tilde{Y}_t^{\text{LAH}} = \max\{0, \tilde{Q}_t^{\text{LAH}} -$

$\tilde{X}_t^{\text{LAH}}\}$ and let $Y_{j,t}^{\text{LAH}} = \frac{\tilde{Y}_t^{\text{LAH}} D_j}{\Delta}$.

2.6 *Update inventory.* If $t \neq T$, the beginning inventory at the branch for the next

period is calculated as $I_{t+1}^{\text{LAH}} = I_t^{\text{LAH}} - \tilde{X}_t^{\text{LAH}} - \tilde{Y}_t^{\text{LAH}} + S_{t+1}$. Increment t by 1 and go to Step 2.1.

3 *Objective Function Calculation.* Calculate the objective function value $Z^{\text{LAH}} =$

$$\sum_{t=1}^T b_x(t) \tilde{X}_t^{\text{LAH}} + \sum_{t=1}^T b_x(t+1) \tilde{Y}_t^{\text{LAH}} - \sum_{t=1}^T c_w \tilde{W}_t^{\text{LAH}}.$$

The improvement of the One-Period Look-Ahead Heuristic over the Myopic Heuristic is embedded in Step 2.2. In this step, we first check to see if the planned on-hand inventory at the beginning of next period if the Myopic policy is followed will cause the next period to be inventory constrained. If so, we decrease our shipment in the current period, but use the lower bound derived in Proposition 5.7 to ensure that we do not undership in the

current period. Thus the decision maker has the advantage of anticipating the amount of supply that will be needed in the next period, and therefore can choose to save some supply for the following period. Hence, the difference between the performances of the LAH and the MH can be viewed as the expected value of information associated with knowing the supply and benefit values for the next period $t+1$. Hence, the performance of the LAH is expected to be higher in situations where periods with high supply are followed by periods with low supply. We will explore this idea through the numerical experiments.

5.6 Bounding Heuristic

In this section, we use the bounds generated in Section 5.4 and introduce a heuristic that uses the average of the upper and lower bounds for each period. This approach is motivated by Shang and Song (2003) who consider a multi-echelon production-inventory system and develop a simple heuristic that takes the average of the lower and upper bounds on the optimal order quantities in each period to use as the base stock level. The Bounding Heuristic does not require any knowledge from the future periods. We use the superscript BH to denote a solution obtained by the Bounding Heuristic, and will refer to this as the Bounding solution. We state the heuristic as follows:

Bounding Heuristic (BH):

1 Initialization. Let $t = 1$ and $I_1^{\text{BH}} = S_1$.

2 For each period $t = 1, \dots, T$.

2.1 Calculate bounds. Calculate the upper and lower bounds as $\tilde{X}_t^{\text{U}} =$

$$\min \left\{ I_t^{\text{BH}}, \Delta F_R^{-1} \left(\frac{b_X(t) - b_X(t+1)}{c_W - b_X(t+1)} \right) \right\}, \text{ and } \tilde{X}_t^{\text{L}} = \min \{ I_t^{\text{BH}}, R_t^{\text{L}} \Delta \}, \text{ respectively.}$$

2.2 Calculate shipments. Take the average of the bounds to calculate the shipment

$$\tilde{X}_t^{\text{BH}} = \frac{\tilde{X}_t^{\text{U}} + \tilde{X}_t^{\text{L}}}{2} \text{ and set } X_{j,t}^{\text{BH}} = \frac{\tilde{X}_t^{\text{BH}} D_j}{\Delta}.$$

2.3 Capacity Realization. After the capacities are observed, calculate $\hat{R}_t = \min_{j \in J} \frac{c_{j,t}}{D_j}$.

Calculate the total amount of food that should be shipped at time t as $\tilde{Q}_t^{\text{BH}} = \min\{I_t^{\text{BH}}, \Delta \hat{R}_t\}$.

2.4 Calculate the recourse actions: The remaining food waste in the counties is

calculated as $\tilde{W}_t^{\text{BH}} = \max\{0, \tilde{X}_t^{\text{BH}} - \tilde{Q}_t^{\text{BH}}\}$ and let $W_{j,t}^{\text{BH}} = \frac{\tilde{W}_t^{\text{BH}} D_j}{\Delta}$. The additional

food shipment to the counties is calculated as $\tilde{Y}_t^{\text{BH}} = \max\{0, \tilde{Q}_t^{\text{BH}} - \tilde{X}_t^{\text{BH}}\}$ and let

$$Y_{j,t}^{\text{BH}} = \frac{\tilde{Y}_t^{\text{BH}} D_j}{\Delta}.$$

2.5 Update inventory. If $t \neq T$, the beginning inventory at the branch for the next

period is calculated as $I_{t+1}^{\text{BH}} = I_t^{\text{BH}} - \tilde{X}_t^{\text{BH}} - \tilde{Y}_t^{\text{BH}} + S_{t+1}$. Set $t = t+1$ and go to

Step 2.1.

3 Objective Function Calculation. Calculate the objective function value $Z^{\text{BH}} =$

$$\sum_{t=1}^T b_X(t) \tilde{X}_t^{\text{BH}} + \sum_{t=1}^T b_X(t+1) \tilde{Y}_t^{\text{BH}} - \sum_{t=1}^T c_W \tilde{W}_t^{\text{BH}}.$$

The Bounding Heuristic uses the average of the lower and upper bounds of the total optimal food shipment, and hence provides a more conservative policy than the Myopic policy, i.e., $\tilde{X}_t^{\text{BH}} \leq \tilde{X}_t^{\text{MH}}$. Consequently, we would expect the BH to perform better in situations where $\tilde{X}_t^{\text{MH}} - \tilde{X}_t^{\text{BH}}$ is large and the MH is suboptimal. In that situation, if the MH is performing badly and giving solutions with high waste, we would expect the BH to perform better. We will evaluate the performance of the BH in Section 5.9. In the next

section, we introduce a new heuristic that uses the deterministic food distribution model and a percentile of the minimum CD ratio distribution to make food shipments at each stage.

5.7 Deterministic Approximation Heuristic

In this section, we introduce another heuristic based on the deterministic multi-period food distribution model. This idea is motivated by the production planning literature where a common heuristic for solving continuous-review inventory systems is to replace the stochastic demand by its mean and determine the order quantity Q according to the EOQ formula (Zheng, 1992; Axsäter, 1996). Then, the optimal reorder point r is calculated in the second step given Q . The Deterministic Approximation Heuristic uses the closed-form optimal solution for the deterministic multi-period food shipment model where we replace the minimum CD ratio for each period R_t by a percentile ρ_t of the distribution of R_t . We introduce the deterministic multi-period food shipment model (D-MPM) where the R_t ratio is replaced by ρ_t as follows:

D-MPM:

$$\max \sum_{t=1}^T b_X(t) \tilde{X}_t \quad (5.34)$$

subject to

$$I_1 = S_1 \quad (5.35)$$

$$I_t = I_{t-1} + S_t - \tilde{X}_{t-1} \quad t = 2, \dots, T \quad (5.36)$$

$$\tilde{X}_t \leq I_t \quad t = 1, \dots, T \quad (5.37)$$

$$\tilde{X}_t \leq \rho_t \Delta \quad t = 1, \dots, T \quad (5.38)$$

$$\tilde{X}_t \geq 0 \quad t = 1, \dots, T \quad (5.39)$$

The interpretation of the constraints follows directly from the Food Distribution Model in Chapter 3 under perfect equity. Proposition 5.8 gives the closed-form optimal solution to D-MPM.

Proposition 5.8: *The closed-form optimal solution to D-MPM is as follows:*

$$\tilde{X}_t^* = \min\{I_t, \rho_t \Delta\} \quad t = 1, \dots, T \quad (5.40)$$

$$X_{j,t}^* = \tilde{X}_t^* \frac{D_j}{\Delta} \quad j \in J; t = 1, \dots, T \quad (5.41)$$

Proof of Proposition 5.8: Due to (5.37) and (5.38), we have $\tilde{X}_t \leq Q_t = \min\{I_t, \rho_t \Delta\}$. We will prove that this inequality is always satisfied at equality. Assume that at time t , $\tilde{X}_t^A = Q_t - \delta$ where $\delta > 0$. Then, the benefit gained from the current period equals $b_X(t)(Q_t - \delta)$. If we are able to ship this additional δ pounds of food in a future period $\tau > t$, we will obtain a benefit of $b_X(\tau)(\delta)$. Hence, the total benefit obtained from Q_t pounds of food equals $b_X(t)Q_t + \delta(b_X(\tau) - b_X(t)) < b_X(t)Q_t$, since $b_X(t)$ is assumed to be decreasing in time. Therefore, shipping $\tilde{X}_t = Q_t$ provides a higher benefit. Hence, the decision maker will aim to ship the maximum amount of food in each period. For this reason, the problem separates into multiple single-period models solved one after another given the current on-hand inventory. The result (5.40) follows directly from this observation. Finally, we obtain (5.41) due to the perfect equity constraint. The proof is completed. ■

Using the result from Proposition 5.8, we develop the Deterministic Approximation Heuristic as follows. We use the superscript “DAH” to emphasize that the solution belongs to

the Deterministic Approximation Heuristic, which we will refer to as the Deterministic Approximation solution.

Deterministic Approximation Heuristic (DAH):

1 Initialization. Let $t = 1$ and $I_1^{\text{DAH}} = S_1$.

2 For each period $t = 1, \dots, T$.

2.1 Calculate shipments. Calculate the total deterministic shipment from the branch

$\tilde{X}_t^{\text{DAH}} = \min\{I_t, \rho_t \Delta\}$ by using a percentile ρ_t of the minimum CD ratios and set

$$X_{j,t}^{\text{DAH}} = \frac{\tilde{X}_t^{\text{DAH}} D_j}{\Delta}.$$

2.2 Capacity Realization. After the capacities are observed, calculate $\hat{R}_t = \min_{j \in J} \frac{\hat{c}_{j,t}}{D_j}$.

Calculate the optimal amount of total food shipment at time t as $\tilde{Q}_t^{\text{DAH}} = \min\{I_t^{\text{DAH}}, \Delta \hat{R}_t\}$.

2.3 Calculate the recourse actions: The remaining food waste in the counties is

calculated as $\tilde{W}_t^{\text{DAH}} = \max\{0, \tilde{X}_t^{\text{DAH}} - \tilde{Q}_t^{\text{DAH}}\}$ and let $W_{j,t}^{\text{DAH}} = \frac{\tilde{W}_t^{\text{DAH}} D_j}{\Delta}$. The

additional food shipment to the counties is calculated as $\tilde{Y}_t^{\text{DAH}} = \max\{0, \tilde{Q}_t^{\text{DAH}} -$

$\tilde{X}_t^{\text{DAH}}\}$ and let $Y_{j,t}^{\text{DAH}} = \frac{\tilde{Y}_t^{\text{DAH}} D_j}{\Delta}$.

2.4 Update inventory. If $t \neq T$, the beginning inventory at the branch for the next

period is calculated as $I_{t+1}^{\text{DAH}} = I_t^{\text{DAH}} - \tilde{X}_t^{\text{DAH}} - \tilde{Y}_t^{\text{DAH}} + S_{t+1}$. Set $t = t+1$ and go to Step 2.1.

3 Objective Function Calculation. Calculate the objective function value $Z^{\text{DAH}} =$

$$\sum_{t=1}^T b_X(t) \tilde{X}_t^{\text{DAH}} + \sum_{t=1}^T b_X(t+1) \tilde{Y}_t^{\text{DAH}} - \sum_{t=1}^T c_W \tilde{W}_t^{\text{DAH}}.$$

The DAH uses a percentile of the underlying probability distribution of the R_t ratios to make shipments. The performance of this heuristic will highly depend on the percentile value chosen. Using the mean of the distribution has been a common approach in literature (Zheng, 1992; Axsäter, 1996). Although using the mean may provide good solutions for symmetrical distributions, it may not perform very well for asymmetric distributions where estimating the tails of the distribution becomes important.

In the next section, we will describe the Multi-Stage Stochastic Programming Model that we will use as a benchmark to evaluate the performances of the heuristics generated.

5.8 Benchmark Approach - Multi-Stage Stochastic Programming Model (MSM)

In the previous sections, we developed heuristics to address the multi-period food shipment problem. To find an exact optimal solution to the multi-period problem would require integrating over multiple random variables for multiple time periods, which makes the computation challenging. In order to evaluate the performance of the heuristics, in this section, we develop a Multi-Stage Stochastic Programming Model that models the evolution of the minimum CD ratios over the planning horizon using a discrete set of scenarios. Stochastic linear programs arise when one or more of the data elements in a linear programming model can be represented as random variables (Sen & Hige, 1999).

Scenario Generation. A scenario in this formulation represents a set of realizations of the R_t ratios in all time periods. The scenario tree is organized in stages which represent the evolution of R_t ratios over the time horizon, for $t = 1, \dots, T$. Each node m of this scenario tree represents a realization of the R_t ratio for the time period $t(m)$ and has probability p^m .

There are a total of M nodes. Each node except the root node $m=1$ has an ancestor node denoted by $a(m)$. A path $P(m)$ from the root node to node m represents a scenario, a possible realization of R_t ratios for the time periods $t = 1$ to $t = t(m)$. Figure 5.3 illustrates a scenario tree for this formulation where the number of branches at each period $Q_R = 2$ and $T = 2$. The time horizon T represents the number of periods where we observe a realization of the minimum CD ratio. The scenario tree will have $T + 1$ layers since no realization is observed at the root node. From Figure 5.3, we see that the size of the model (size of the scenario tree) grows exponentially with the length of the time horizon T and polynomially with the number of branches at each node. More specifically, for a tree with T periods and Q_R branches from each node, we will have $(Q_R)^T$ scenarios and the total number of nodes will be $M = \sum_{t=0}^T (Q_R)^t$.

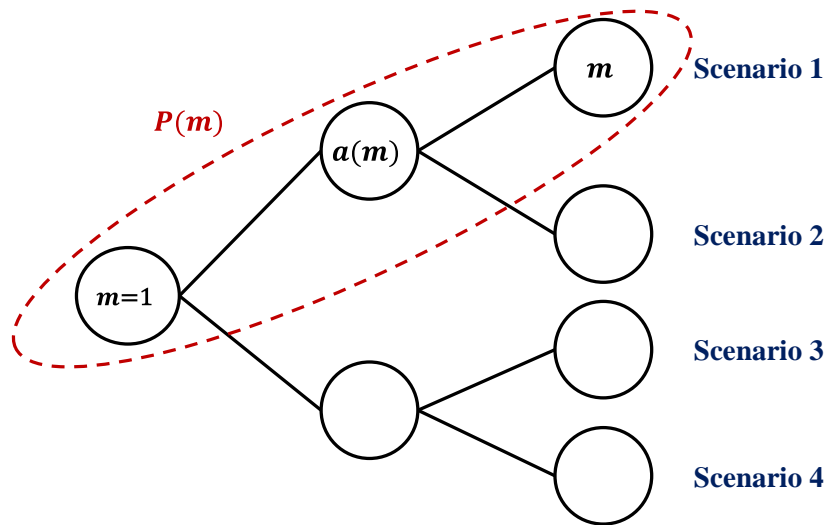


Figure 5.3: Scenario Tree for the MSM for $Q_R=2$ and $T=2$.

Formulation of the MSM. We now present the mathematical formulation of the MSM. The decision variable X_j^m represents the amount of food shipped to county j at node m before the minimum CD ratio for time $t(m)$ is observed. The recourse variable Y_j^m represents the additional shipment to county j and W_j^m represents the waste from county j as a consequence of shipment decisions after all capacities in the path $P(m)$ are observed. As before, we use the aggregated decision variables to develop the formulation. Since we enforce perfect equity, the disaggregation scheme given in Section 4.2 is known to be optimal. We define the aggregate decision variables as $\tilde{X}^m = \sum_{l=1}^n X_l^m$, $\tilde{Y}^m = \sum_{l=1}^n Y_l^m$, and $\tilde{W}^m = \sum_{l=1}^n W_l^m$. The parameter $S_{t(m)}$ represents the additional supply received in period $t(m)$ after recourse decisions Y_j^m and W_j^m have been made, but before any new shipment X_j^m has occurred. The inventory level I^m is the total supply remaining on hand after all R_t ratios in the path $P(m)$ are observed and after $S_{t(m)}$ has been received. Hence $I^m = I^{a(m)} + S_{t(m)} - \tilde{X}^{a(m)} - \tilde{Y}^m$.

The formulation is given as follows:

MSM:

$$\max \sum_{i=1}^M p^i [b_X(t(i))\tilde{X}^i + b_Y(t(i))\tilde{Y}^i - c_W\tilde{W}^i] \quad (5.42)$$

subject to

$$\tilde{Y}^1 = 0 \quad (5.43)$$

$$\tilde{W}^1 = 0 \quad (5.44)$$

$$\tilde{X}^m = 0 \quad m = M - (Q_R)^T + 1, \dots, M \quad (5.45)$$

$$I^1 = S_1 \quad (5.46)$$

$$I^m = I^{a(m)} + S_{t(m)} - \tilde{X}^{a(m)} - \tilde{Y}^m \quad m = 2, \dots, M \quad (5.47)$$

$$I^{a(m)} - \tilde{X}^{a(m)} - \tilde{Y}^m \geq 0 \quad m = 2, \dots, M \quad (5.48)$$

$$\tilde{X}^{a(m)} + \tilde{Y}^m - \tilde{W}^m \leq R^m \Delta \quad m = 2, \dots, M \quad (5.49)$$

$$\tilde{X}^{a(m)} + \tilde{Y}^m - \tilde{W}^m \geq 0 \quad m = 2, \dots, M \quad (5.50)$$

$$\tilde{X}^m, \tilde{Y}^m, \tilde{W}^m, I^m \geq 0 \quad m = 1, \dots, M \quad (5.51)$$

In this model, the objective (5.42) is to maximize the expected benefit over the entire planning horizon. In each node, shipment \tilde{X}^m is made without observing the capacities for that period. At the end of the period, the R_t ratio for that period is observed and recourse decisions in the form of shipments from the branch \tilde{Y}^m and waste \tilde{W}^m are made. Constraints (5.43) and (5.44) state that no recourse actions are taken at the root node. Constraints (5.45) ensure that only recourse actions are taken at the leaf nodes. Constraints (5.46) and (5.47) are material balance equations for the branch. Constraint (5.48) limits the food shipment from the branch in a given period by the on-hand inventory. Constraint (5.49) states that the total net food distribution cannot exceed $R^m \Delta$. Constraint (5.50) states that the total net food distribution should be nonnegative. Finally, constraints (5.51) are nonnegativity constraints.

Implementation. To implement the MSM, we consider a periodic review environment with discrete time periods (Snyder & Shen, 2011). A disadvantage of the MSM is that due to the size of the scenario tree and the model, it becomes impractical to solve the model for the entire planning horizon. For this reason, we use a rolling horizon approach. Figure 5.4 depicts the rolling horizon simulation approach that we will take in order to calculate the performance of the MSM.

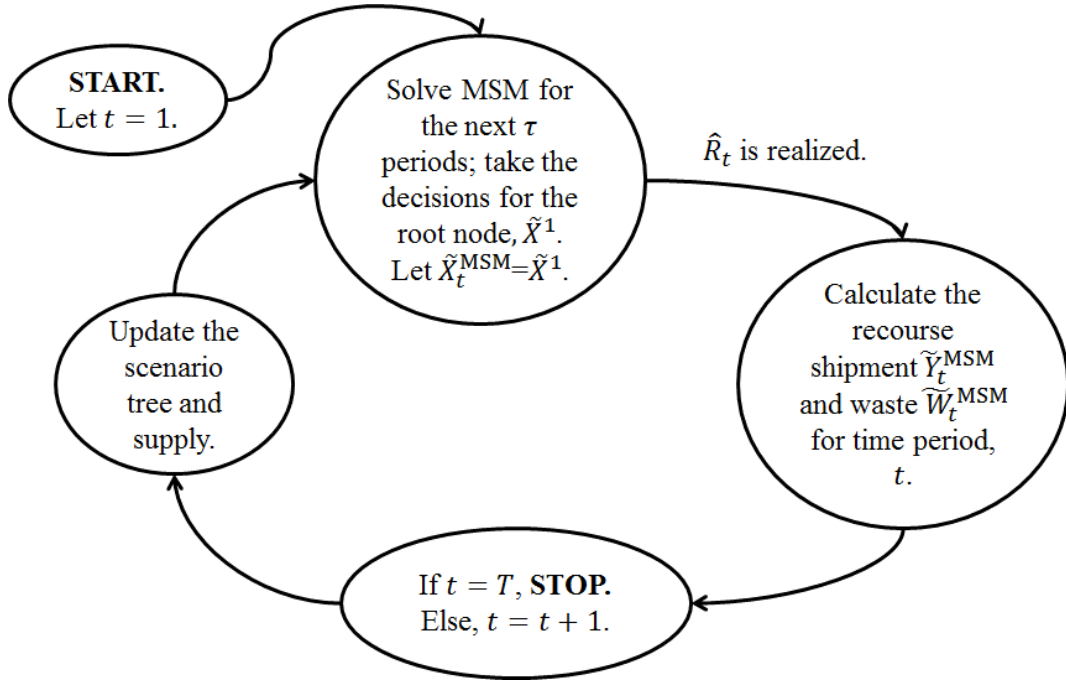


Figure 5.4: Rolling horizon simulation for the MSM.

After the simulation is finished, the final value of the objective function is calculated as:

$$Z^{\text{MSM}} = \sum_{t=1}^T b_X(t) \tilde{X}_t^{\text{MSM}} + \sum_{t=1}^T b_X(t+1) \tilde{Y}_t^{\text{MSM}} - \sum_{t=1}^T c_W \tilde{W}_t^{\text{MSM}} \quad (5.52)$$

similar to the previous sections. As a limitation of the rolling horizon approach, the performance of the MSM will depend on the depth of the scenario tree τ . Therefore, as τ and Q_R increase, the solution from the MSM will approach the true value of the optimal solution, but the model size increases as a function of τ and Q_R as well, making it impractical to solve for large τ and Q_R values.

In order to solve the MSM, we build the scenario tree based on the possible realizations of the minimum CD ratios rather than individual capacities of the counties as

described in Appendix B. In Section 5.9, we will discuss numerical results using this model as a benchmark for evaluating the performances of the heuristics.

5.9 Computational Experiments

In this section, we present the numerical results from our analysis.

5.9.1 Experimental Design

In order to explore the performance of the heuristics under different conditions, we perform an extensive experimental study varying

- the length of time horizon, T ,
- the discretization scheme used, i.e., the number of possible outcomes of R_t at each period, Q_R ,
- the number of periods for which the rolling horizon simulation is carried out for MSM, τ , and
- the supply at time t , S_t .

We perform each experiment with 100 simulated capacity realizations. Table 5.1 summarizes our experimental design. We use an exponential function to represent the change of benefit coefficients over time. We also assume that the total demand is normally distributed with a mean of 100 units and a standard deviation of 25 units where we perform experiments for 100 independent demand realizations. The experiments are conducted using MATLAB (The MathWorks Inc., 2000) and IBM CPLEX Solver (IBM, 2013).

Table 5.1: Experimental Design Settings.

Parameter	Settings
T	10, 30, 50
Q_R	2,3,4,5
τ	3,4,5
S_t	Supply of 50 units received on every ω periods where $\omega = 1,2,3$
	Supply of 100 units received on every ω periods where $\omega = 1,2,3$
$b_X(t)$	$b_X(t) = e^{\frac{1-t}{1+t}}$
Δ	Normal(100,25)

We use an exponential distribution with a mean of one to represent the distribution of the minimum capacity-to-demand (CD) ratio R^t in each period. We use the discrete approximation of Miller & Rice (1983), together with the Gaussian Quadrature values tabulated in Stroud & Secrest (1966) to represent the probability distribution with a discrete number of scenarios as summarized in Table 5.2. We will now present the numerical results.

5.9.2 Numerical Results

In this section, we first examine the relationship between the size of the scenario tree and the performance of the MSM. We then compare the performance of the heuristics to the performance of the MSM under different supply functions and explore the relationship between the relative performances of the heuristics and the length of the time horizon T . We conclude the section with a discussion of the value of information in this system.

Table 5.2: Discretization Schemes for Exponential(1) Distribution (Miller & Rice, 1983; Stroud & Secrest, 1966).

Q_R	Values	Probabilities
2	0.585786	0.853553
	3.414214	0.146447
3	0.415775	0.711093
	2.294280	0.278518
	6.289945	0.010389
4	0.322548	0.603154
	1.745761	0.357419
	4.536620	0.038888
	9.395071	0.000539
5	0.263560	0.521756
	1.413403	0.398667
	3.596426	0.075942
	7.085810	0.003612
	12.640801	0.000023

5.9.2.1 The Effect of Scenario Tree Size on the Performance of the MSM

The number of branches from each node of the scenario tree, which we denote as Q_R , is an important parameter in multi-stage stochastic programs, since it determines the quality of the representation of the underlying probability distribution. Although higher Q_R values should improve the performance of MSM, they also increase the computational requirements of the model, making it impractical to solve. Figure 5.5 illustrates the change in the average objective function value and the average CPU time for the MSM for varying Q_R when $T = 30$, $\tau = 4$ and $S = 50$ is received every period.

The x axis of Figure 5.5 represents different Q_R values. The primary y axis represents the value of the objective function of the MSM and the secondary y axis represents the total CPU times for solving the model. As Q_R increases, the performance of the MSM improves. This is expected since as the number of outcomes at each node of the scenario tree increases, we obtain a more accurate representation of the underlying probability distribution. However, the CPU times increase polynomially as Q_R increases. For the rest of our analysis, we set $Q_R = 4$.

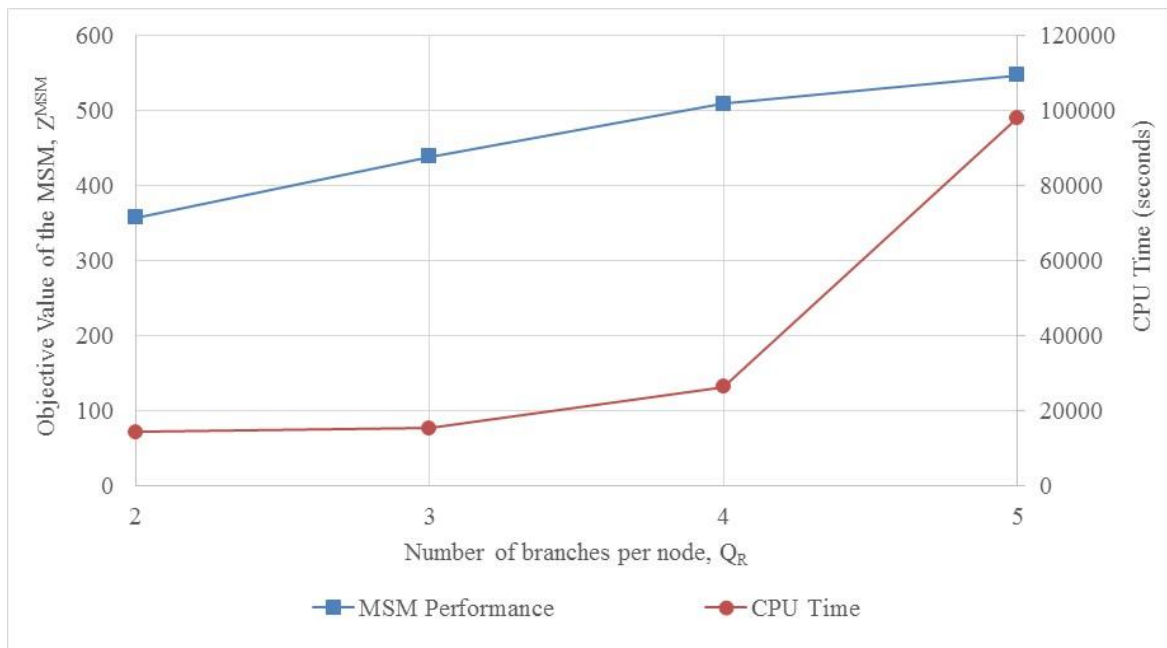


Figure 5.5: The Impact of Changing Number of Branches per Node, Q_R .

Another parameter that affects the performance of the MSM as well as the solution time is the rolling horizon length τ for solving the MSM. Since solving the MSM for the

entire planning horizon T is not possible due to the computation time, we use the rolling horizon simulation and solve the MSM for τ periods iteratively, as explained in Figure 5.4. Figure 5.6 illustrates the change in the average objective function value and the average CPU time for the MSM for varying τ when $T = 30$, $Q_R = 4$ and $S = 50$ is received every period.

The x axis of Figure 5.6 represents different τ values. The y axes have the same definitions as in Figure 5.5. As τ increases, the performance of the MSM improves, as expected. However, the CPU times increase exponentially as τ increases. For this reason, for the rest of our analysis, we set $\tau = 4$ for solving the MSM. Hence, for solving the MSM, the scenario tree has 4^4 scenarios and $M = \sum_{t=0}^4 (4)^t = 341$ nodes at each iteration of the rolling horizon simulation.

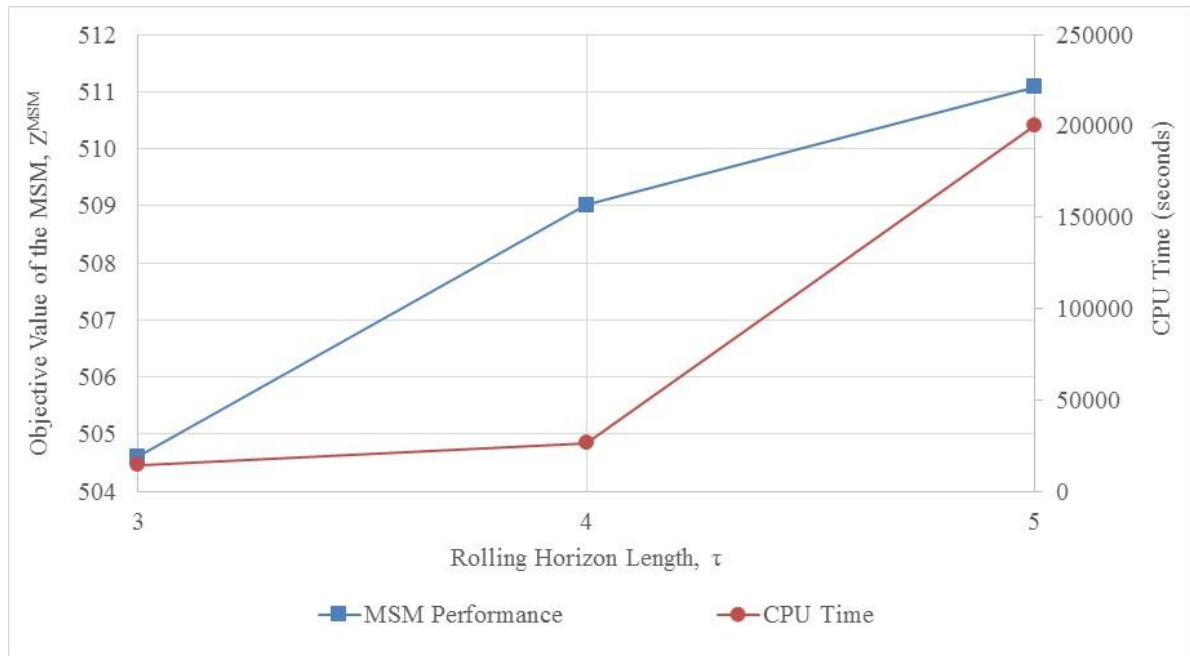


Figure 5.6: The Impact of Changing the Rolling Horizon Length, τ .

5.9.2.2 Comparison of the Heuristic Performances

In this section, we compare the performance of the MH, LAH, BH and DAH heuristics when supply is varied. We use the results from the MSM as a benchmark to evaluate the performances of these heuristics. For the DAH, we use the mean of the underlying distribution to approximate the R_t values, i.e., we set $\rho_t = 1$ for an exponential distribution with mean 1 in (5.38). In this section, we set $T=30$.

We define the following measure to evaluate the performance of the heuristics:

$$\text{Error Ratio} = \frac{Z^{\text{MSM}} - Z^{\text{H}}}{Z^{\text{MSM}}} (100\%)$$

where Z^{H} represents the value of the objective function obtained from the heuristics (MH, LAH, BH or DAH) and Z^{MSM} represents the objective function value of MSM. We examine the Average Error Ratio and the Worst Case Error Ratio for each of the heuristics. The Average Error Ratio is the Error Ratio obtained from the average values of the objectives, i.e., $\frac{\bar{Z}^{\text{MSM}} - \bar{Z}^{\text{H}}}{\bar{Z}^{\text{MSM}}} (100\%)$ whereas the Worst Case Error Ratio represents the Maximum Error Ratio over all replications. We consider the cases when $S=100$ units and $S=50$ units of supply is received every $w = 1, 2$ or 3 periods. As w increases, supply is received less frequently, increasing the need to plan for shipments in future periods and the probability that a period will be inventory constrained. The results are tabulated in Table 5.3 where columns represent the results for different heuristics and the rows represent different experiments. The last two rows show the Average and the Worst Case performance of each heuristic over all experiments. The “Overall” column represents the Average and the Worst Case performance

of all the heuristics for a given experiment. The “Except DAH” column is similar to the “Overall” column, except the results for the DAH are excluded from calculations.

Table 5.3: Error Ratios for $S=100$ and $S=50$ Received Every $w = 1, 2$ or 3 Periods (Grey-highlighted cells indicate the worst performances).

			MH	LAH	BH	DAH	Overall	Except DAH
$S=100$ received every w periods	$w=1$	Average	0.62%	0.62%	0.04%	69.89%	17.79%	0.42%
		Worst Case	1.20%	1.20%	0.33%	86.56%	86.56%	1.20%
	$w=2$	Average	2.17%	0.76%	0.49%	69.29%	18.18%	1.14%
		Worst Case	4.78%	2.99%	1.44%	85.60%	85.60%	4.78%
	$w=3$	Average	2.95%	1.01%	0.62%	65.95%	17.63%	1.53%
		Worst Case	6.73%	5.37%	2.14%	80.41%	80.41%	6.73%
$S=50$ received every w periods	$w=1$	Average	0.18%	0.18%	0.02%	36.14%	9.13%	0.12%
		Worst Case	0.97%	0.97%	0.29%	80.02%	80.02%	0.97%
	$w=2$	Average	0.46%	0.01%	0.03%	33.99%	8.62%	0.17%
		Worst Case	2.97%	1.39%	0.93%	80.19%	80.19%	2.97%
	$w=3$	Average	0.67%	0.01%	0.05%	31.82%	8.14%	0.25%
		Worst Case	3.28%	0.54%	0.81%	70.18%	70.18%	3.28%
Overall	Average	1.17%	0.43%	0.21%	51.18%	13.25%	0.60%	
	Worst Case	6.73%	5.37%	2.14%	86.56%	86.56%	6.73%	

Examining Table 5.3, we see that for every case considered, the LAH and the BH outperforms the MH except when $w=1$, in which case the MH and the LAH perform the same. This is expected since when $w=1$, supply is received every period and hence there is no benefit gained from saving supply for the future periods. The grey highlighted cases show the worst performances for the heuristics and the last two rows summarize the overall performance of the heuristics. The average error for the MH is 1.17% and the worst case

error is 6.73%. For the LAH, the average error is 0.43% and the worst case error is 5.37%. The BH performs the best with an average error is 0.21% and the worst case error is 2.14%.

The DAH performs much worse than any other heuristic with an average error of 51.18% and worst case error of 86.56%. The performance of the DAH improves as supply becomes scarcer and this shows that the DAH overships food since it does not take into consideration the changes in the benefit function. The low error ratios for the LAH and the BH illustrate that these heuristics perform very close to the optimal solution. Furthermore, for the MH and the BH, as w increases, the Error Ratio increases since these heuristics do not have any information regarding future periods. For the LAH, when $S=100$, as w increases, the Error Ratio increases because saving supply for future periods results in too much waste due to the abundance of supply in the system. However, when $S=50$, the Error Ratio for the LAH decreases as w increases, because in that case it becomes beneficial to save supply for future periods, due to the scarcity of supply in the future. Hence, in situations where supply is scarce, using the LAH is more beneficial whereas if there is plenty of supply BH provides better results.

We would also like to evaluate the performance of the heuristics in terms of their total waste. In Figures 5.7 and 5.8, we consider the cases when supply is received every $w=1, 2,$ and 3 periods in the amount of $S=50$ and $S=100$, respectively. We show the results for the MSM, the MH, the LAH and the BH. We omit the results for the DAH since it performs much worse than the others. The three points for both graphs from left to right correspond to $w=1, w=2$ and $w=3$.

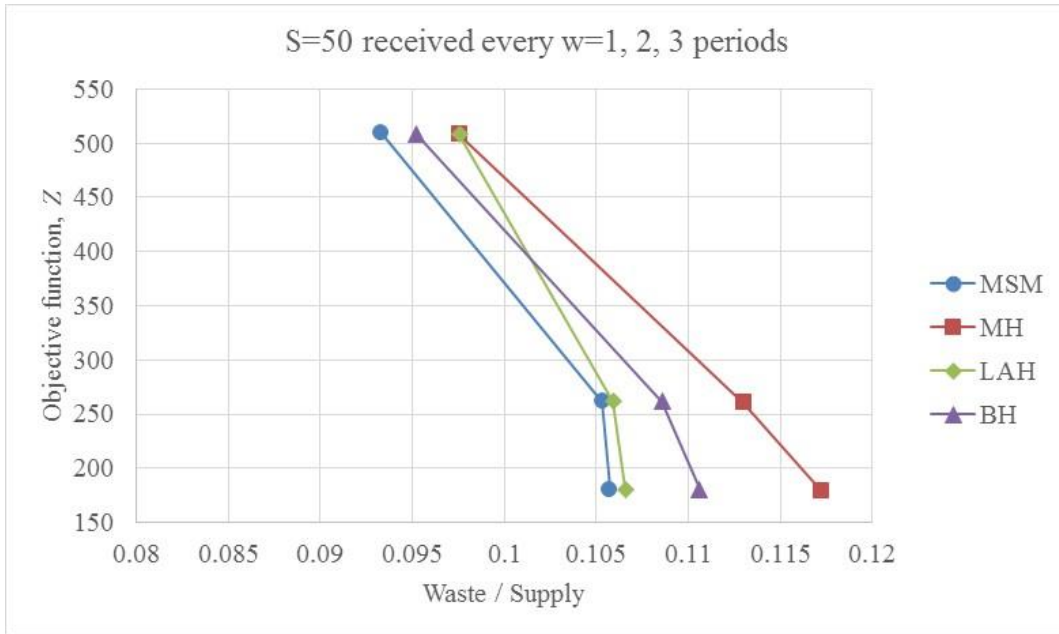


Figure 5.7: Objective versus Waste / Supply for $S=50$ units.

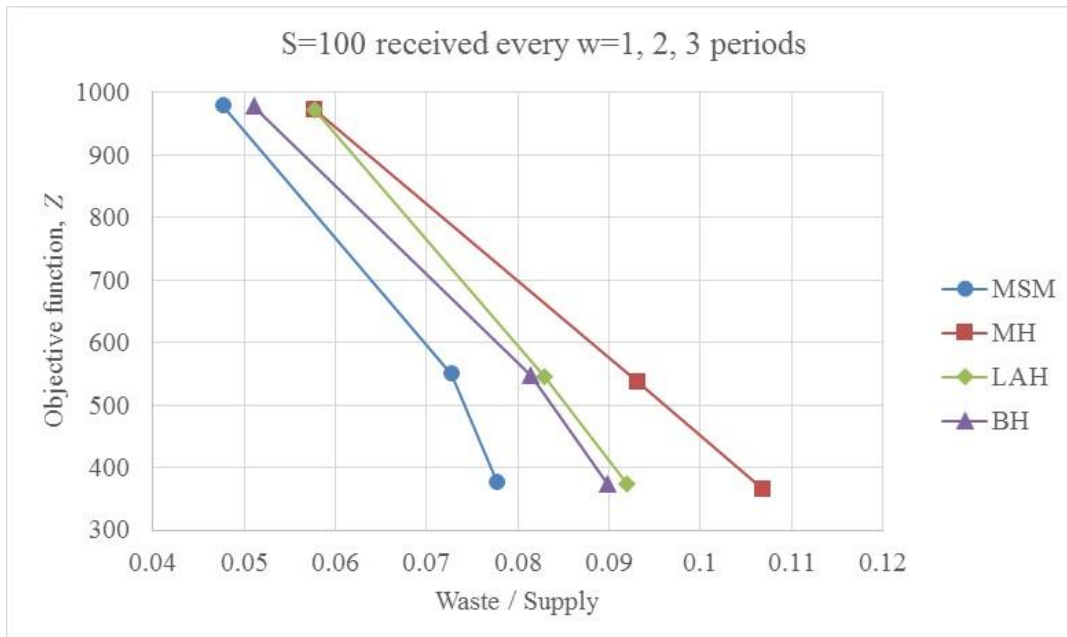


Figure 5.8: Objective versus Waste / Supply for $S=100$ units.

From Figures 5.7 and 5.8, we see that the MSM provides an efficient frontier for the heuristics, as expected. The MH results in the most waste since it does not have the capability to plan ahead. As w increases, the waste from the LAH decreases. This happens since as w increases, the effectiveness of the LAH increases. The BH performs well under all cases. We also see that higher waste results in a lower objective function value in all cases.

5.9.2.3 Effect of Changing the Length of the Time Horizon, T

In this section, we explore the relationship between the time horizon T and the performance of the heuristics. We focus on the MH, the LAH and the BH. Figure 5.9 shows the change in the Error Ratios as a function of T with $S=100$ and $w=3$.



Figure 5.9: Error Ratios (%) of the Heuristics when T is varied.

From Figure 5.9, we observe that the error ratios for all heuristics decrease as T increases. This may be for two reasons. The first reason is that as T increases, the total supply entering

the system increases, making the instances more likely to be capacity constrained over time. As we saw in Section 5.4, this causes the MH to perform better and since the LAH and the BH are upper bounds on the MH, they also perform better. The second reason may be that since we use the rolling horizon simulation to calculate the objective of the MSM, as T increases, the effectiveness of the rolling horizon simulation decreases which increases the difference between the objective of the MSM and the true optimal value for the problem.

5.9.2.4 Value of Perfect Information

In this section, we discuss the expected value of perfect information (EVPI) in this system as the number of branches per node in the scenario tree Q_R increases. EVPI is defined as the difference between the average objective value under perfect information, i.e., when all the \hat{R}_t values are known from the beginning, and the average objective value of the Multi-Stage Stochastic Program, MSM (Birge & Louveaux, 2011). We also calculate the EVPI Percentage Benefit which represents the percentage increase in the objective function by obtaining the perfect information (Maggioni, Bertocchi, Allevi, Potra, & Wallace, 2012) as $\frac{z^{PI} - z^{MSM}}{z^{PI}} (100\%)$. We consider the case when $T = 30$ and supply in the amount of 50 units is received every period. In Table 5.4, the first two rows present the average objective values of the MSM and perfect information (PI) with standard deviations given in parentheses. The last two rows show EVPI and EVPI Percentage Benefit.

We observe that the EVPI and EVPI Percentage Benefit decrease as Q_R increases, since the scenario tree becomes a better representation of perfect information and therefore

the relative benefit of perfect information decreases. However, as shown in Section 5.9.2.1, the MSM becomes impractical to solve for large Q_R values.

Table 5.4: The Expected Value of Perfect Information.

Average (SD)	Q_R			
	2	3	4	5
MSM	357.75 (39.48)	438.94 (27.30)	509.61 (20.68)	549.78 (11.96)
PI	667.94 (15.49)	667.94 (15.49)	667.90 (15.45)	667.90 (15.45)
EVPI	310.18	229.00	158.89	118.12
EVPI Percentage Benefit	46.44%	34.28%	23.70%	17.69%

5.10 Conclusions

In this chapter, we examined the problem of food shipment over multiple periods when the capacities of the counties are not known deterministically. In this problem, the decision maker has to ship food to the counties in each period without knowing their capacities. At the end of the period, capacities are observed and shipments made to the counties can be corrected by either additional shipment or waste, incurring extra cost. Any unshipped food in the branch is carried over to the following period as starting inventory. First, we used the results from Chapter 4 to develop a Myopic Heuristic that optimizes shipment decisions in individual periods without considering later periods. We showed that this heuristic provides an upper bound on optimal shipments and a lower bound on the optimal objective value. We then explored the structural properties of the multi-period food

shipment problem to develop a lower bound on the optimal shipments that gives an upper bound on the optimal objective value. We used these structural results to improve the Myopic Heuristic and obtain a One-Period Look-Ahead Heuristic and a Bounding Heuristic. We then developed a final heuristic, the Deterministic Approximation Heuristic that uses a percentile of the underlying probability distribution to make food shipment decisions. Lastly, we developed a Multi-Stage Stochastic Programming Model to serve as a benchmark for comparing the performances of the heuristics.

We performed an extensive numerical study to explore the performances of the heuristics. Our results showed that the Bounding Heuristic and the One-Period Look-Ahead Heuristic provide the best performances, both having average relative errors of less than 0.5%. We also saw that using the LAH is preferred in environments where supply is scarce whereas when supply is abundant, the BH performs better. We observed that DAH does not perform well when the R_t ratios are replaced by the mean of the underlying probability distribution. As a part of our future work, we will extend our experiments for DAH using other percentiles of the underlying probability distributions.

Future research directions for this chapter include exploring the structural properties of the multi-period food shipment problem under relaxed equity and the addition of stochastic supply. We would also like to understand how the results in this chapter extend for the infinite horizon food shipment problem where the objective would be to maximize the long-run average benefit of food shipments.

Chapter 6

Robust Optimization Models

6.1 Introduction

A major limitation of the deterministic models presented in Chapter 3 is the assumption of deterministic capacity data. In Chapters 4 and 5, we addressed capacity uncertainty in the single- and multi-period food shipment problems by developing stochastic optimization models based on the probability distribution of the minimum CD ratios. However, a food bank may be faced with situations where the entire probability distribution of the underlying uncertain parameter is not available but estimates of the ranges of the parameters are known. In this chapter, we first pursue this idea to develop a robust optimization model that protects the solution against deviations in capacities. We then develop a nontraditional robust model that considers deviations in the equity parameter K while limiting the total deviation from perfect equity over all counties.

The problem with using point estimates of stochastic parameters to formulate a nominal optimization model is that small deviations from the nominal estimates of data can render the nominal solution suboptimal or even infeasible (Ben-Tal, Ghaoui, & Nemirovski, 2009). Robust optimization has been proposed as a method for obtaining solutions that are flexible under uncertain problem parameters, i.e., less vulnerable to uncertainty in parameters (Marla, 2007). Given pre-specified ranges over which the model parameters are assumed to vary randomly, robust models seek solutions that remain feasible and near-optimal after small perturbations in the data (Bertsimas and Sim, 2004). Bertsimas and Sim (2004) also

introduced the idea of limiting the number of parameters that are allowed to achieve their worst-case behavior, avoiding overly conservative solutions.

Robust optimization models are worst-case oriented since they seek solutions that remain feasible under the parameter deviations that cause the most adverse effect on the problem. Some underlying assumptions of robust optimization are the following: 1) the decision model has to be solved before observing the uncertain outcomes; 2) the decision has to be protected against uncertainty in the pre-specified ranges; 3) the constraints cannot be violated, in contrast to chance-constrained formulations where constraints can be violated with specified probabilities. (Ben-Tal *et al.*, 2009; Birge & Louveaux, 2011)

Section 6.2 discusses several robust optimization models from the literature. In Section 6.3, we formulate a robust optimization model that incorporates uncertainty in the capacity parameters. Assuming that the capacity of each county can vary within a pre-specified range, the model aims to achieve solutions that are robust to data uncertainty while managing the trade-off between the robustness and conservativeness of the solution. We prove several structural properties of this model and develop an efficient algorithm to solve this model optimally. In Section 6.4, we extend the Deterministic Food Distribution Model from Chapter 3. We use the methodology of robust optimization in a nontraditional manner by treating the equity constant K as an uncertain parameter and limiting the total deviation from a perfectly equitable distribution over all counties. We also prove the optimal solution structure of this model and develop an algorithm to solve it optimally. For both of these models, we present numerical results. We finish the chapter with conclusions and directions for future work.

6.2 Models in Robust Optimization

Soyster (1973), in the initial study of robust optimization, considers a classic linear program (LP) and focuses on uncertainty observed over multiple constraints, which we will refer to as column-wise uncertainty. He considers the following LP:

$$\begin{aligned} & \max \mathbf{c}'\mathbf{x} \\ & \text{subject to} \\ & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{6.1}$$

where the columns of the \mathbf{A} matrix are subject to uncertainty such that each column vector $\mathbf{A}_j \in \mathbf{K}_j$ where \mathbf{K}_j is a convex set. He shows that this model is equivalent to

$$\begin{aligned} & \max \mathbf{c}'\mathbf{x} \\ & \text{subject to} \\ & \bar{\mathbf{A}}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\bar{a}_{ij} = \sup_{\mathbf{A}_j \in \mathbf{K}_j} a_{ij}$. Hence, in order to be robust to any deviation of \mathbf{A}_j within the given convex set, the solution is protected against the worst case. If we consider our Deterministic Food Distribution Model and focus on the uncertainty in capacity values, this becomes equivalent to modeling the capacity constraint as

$$X_j \leq \hat{C}_j - \varepsilon_j \quad \forall j \in J$$

where X_j denotes the food shipment to county j , \hat{C}_j the nominal capacity estimates and $\varepsilon_j > 0$ the maximum permitted deviations from these nominal estimates. It is straightforward to see that in this case, the optimal amount of food distributed by the Deterministic Food Distribution Model is equal to

$$\sum_{l=1}^n X_l^* = \min \left\{ S, \min_{j \in J} \left(\frac{\hat{C}_j - \varepsilon_j}{D_j} \right) \sum_{l=1}^n D_l \right\}$$

where S denotes the total supply and D_j the demand at county j , as before. This solution becomes overly conservative since it focuses on the worst-case deviations of the counties. In this context, a solution is more conservative if it results in a lower objective function value as compared to the nominal solution for a maximization problem, i.e., “we give up (too much) optimality for the nominal problem in order to ensure robustness” (Bertsimas and Sim, 2004). Hence, for our problem, a conservative solution results in lower total shipment than the nominal solution.

Several studies have proposed robust optimization models that are less conservative than Soyster’s method. Ben-Tal and Nemirovski (2000) aim to achieve less conservative models by considering ellipsoidal uncertainty sets. Although their models are less conservative, the resulting models are nonlinear which increases their computational complexity.

Bertsimas and Sim (2004) introduce a novel approach to control the conservatism of the solution. Consider the i^{th} constraint of the LP given in (6.1), i.e., $\mathbf{a}'_i \mathbf{x} \leq b_i$ where \mathbf{a}'_i represents the i^{th} row of the \mathbf{A} matrix and b_i the i^{th} element of the right-hand side vector.

They define the set J_i as the index set of a_{ij} coefficients subject to uncertainty in row i . Assuming that each element of the A matrix $\{a_{ij} | j \in J_i\}$ can vary within a symmetric range such that $a_{ij} \in [\hat{a}_{ij} - \varepsilon_{ij}, \hat{a}_{ij} + \varepsilon_{ij}]$, they introduce a parameter $\Gamma_i \in [0, |J_i|]$ that controls the level of robustness of the model against the level of conservatism of the obtained robust solution. For example, if $\Gamma_i = 0$, then none of the coefficients of constraint i are allowed to vary and hence all take their nominal values. The robustness of the obtained solution will increase as Γ_i increases. The approach of Bertsimas and Sim (2004) is useful because (i) it can control the trade-off between robustness and conservatism, (ii) the resulting model is an LP and has almost the same computational complexity as the original LP, and (iii) the resulting solution remains near-optimal for the nominal values of data. However, there are also some drawbacks with their method that make it unsuitable for our problem. The method of Bertsimas and Sim (2004) works best if uncertainty is focused on a small number of constraints. The parameter Γ_i controls the level of robustness for any constraint i ; hence we can say that it controls row-wise robustness. In our case, uncertainty is present over n constraints of the form

$$X_j \leq C_j \quad \forall j \in J.$$

Therefore, we would need a separate Γ_j value for each of these constraints and hence, the method by Bertsimas and Sim (2004) does not allow us to control column-wise robustness.

Marla (2007) argues that the approach of Bertsimas and Sim (2004) is not efficient for large-scale problems due to the need to estimate a Γ_i parameter for each constraint. To address this, she develops the “Extended Bertsimas-Sim (Delta) Formulation” that eliminates

this drawback. She considers binary integer programs with binary decision variables x_j and defines a variable Δ_i to be the maximum number of binary decision variables x in the solution whose coefficients must take their nominal values for a constraint i to remain feasible. Then, she introduces another decision variable $\vartheta = \max_i \Delta_i$ to be the maximum number of uncertain coefficients over all constraints that should take their nominal values for a feasible solution. The objective is to minimize ϑ and hence increase the robustness of the solution. In order to avoid an overly conservative solution, she introduces a constraint limiting the difference between the optimal objective function values of the nominal and robust models. Our models use some of the ideas by Marla (2007) although we focus on linear programs and, instead of using a constraint to control the level of conservativeness of the solution, we use an objective function that simultaneously considers the optimal amount of food shipment and the degree of robustness of the solution.

6.3 Robust Optimization Model for Capacity Uncertainty (C-RM)

In this section, we formulate the Robust Optimization Model for Capacity Uncertainty and derive some structural properties.

6.3.1 The formulation

The purpose of the C-RM model is to obtain food distribution policies that are robust to changes in capacity values. In order to avoid an overly conservative solution, we introduce a coefficient that represents the value of robustness. We will show that as the value of robustness increases, we obtain more robust solutions where the solution is protected against larger deviations in capacity. As the value of robustness decreases, more counties have to

take their nominal values for a feasible solution. Before introducing C-RM, we define the following notation:

D_j : Demand of county j ,

$\Delta = \sum_{i=1}^n D_i$, Total demand for service area,

\hat{C}_j : The nominal capacity value for county j ,

ε_j : Maximum deviation of the capacity of county j from its nominal value \hat{C}_j ,

S : Total supply,

ρ : Value of robustness.

The decision variables are the following:

X_j : Amount of the food shipment to county j ,

φ_j : The fraction of error allowed in the capacity of county j from its nominal value,

$0 \leq \varphi_j \leq 1$,

ϑ : Level of robustness of the solution.

In our model, we focus on negative deviations of the capacity parameters from their nominal values. Specifically, we allow each county's capacity to vary between $C_j \in [\hat{C}_j - \varepsilon_j, \hat{C}_j]$. We focus on the negative deviation for two reasons: (i) Robust optimization is a worst-case oriented optimization, so the solution is protected against the worst case scenario. In our case since the capacities determine the minimum CD ratio, which in turn determines the optimal amount of food distribution, the robust model must consider the lowest capacity values depending upon the given value of robustness. (ii) Practically, overestimating the capacity for

a county is worse than underestimating. Assume that county j is the bottleneck county whose capacity we estimate as \hat{C}_j . After shipping the food, we realize that the actual capacity of county j is lower than \hat{C}_j . In that case, a positive amount of food is sent to waste and equity is violated. However, if the actual capacity of county j is higher than \hat{C}_j , we do not have any waste and any unshipped food remaining at the branch may be shipped in a later period.

In this section, we assume perfect equity as in Chapters 4 and 5. The food distribution problem under capacity uncertainty assuming imperfect equity will be considered in future work. We formulate the Robust Optimization Model for Capacity Uncertainty (C-RM) as follows:

C-RM

$$\max \sum_{l=1}^n X_l + \rho\vartheta \quad (6.2)$$

subject to

$$\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} = 0 \quad \forall j \in J \quad (6.3)$$

$$X_j \leq \hat{C}_j - \varepsilon_j \varphi_j \quad \forall j \in J \quad (6.4)$$

$$\vartheta \leq \sum_{l=1}^n \varphi_l \quad (6.5)$$

$$\sum_{l=1}^n X_l \leq S \quad (6.6)$$

$$0 \leq \varphi_j \leq 1 \quad \forall j \in J \quad (6.7)$$

$$X_j, \vartheta \geq 0 \quad \forall j \in J \quad (6.8)$$

Constraint (6.3) enforces perfect equity. Constraint (6.4) states that the amount of food sent to a county j should be less than its nominal capacity minus the allowed deviation. Constraint (6.5) defines the variable ϑ . Constraint (6.6) is the supply constraint. Constraint (6.7) ensures that the variables φ_j take values between zero and one. Finally, (6.8) are nonnegativity constraints. The objective function maximizes the sum of total food distribution and the robustness of the system. The parameter ρ represents the value of robustness given in pounds of food. This parameter can be interpreted as the maximum pounds of food that the decision maker is willing to unship in order to achieve complete deviation from nominal capacity at any county j . For example, if $\rho = 0$, then the value of robustness is zero, i.e., we are not willing to sacrifice any food to obtain a robust solution, hence, the nominal solution is optimal with $\varphi_j^* = 0$ for all j . Alternatively, if $\rho = \infty$, then we are willing to sacrifice all of our food to achieve a solution that is robust to any capacity deviation and the optimal solution becomes $\varphi_j^* = 1$ for all j . Thus, the objective function considers the trade-off between the objectives of robustness and effectiveness. As the value of robustness, ρ , increases, the solution will become more robust to changes in the counties' capacities, causing the total food distribution to decrease due to (6.4).

6.3.2 Structural Properties of C-RM

By examining C-RM, we notice that as $\sum_{l=1}^n \varphi_l$ increases, we obtain a more robust solution. However, this results in lower $\sum_{l=1}^n X_l$. Therefore, to find a closed-form optimal solution for C-RM, we focus on the benefit of increasing any one of the φ_j . We first explore

the structural properties of this model and then introduce an efficient algorithm for optimally solving C-RM.

First, we define the initial bottleneck county B as in Chapter 3:

$$B = \left\{ j \in J \mid \frac{\hat{C}_j}{D_j} = \min_{i \in J} \left\{ \frac{\hat{C}_i}{D_i} \right\} \right\} \quad (6.9)$$

and set J_E to be the set of counties that can achieve maximum deviation from their nominal values without having an effect on the optimal solution. The elements in this set depend on whether the problem instance is supply or capacity constrained.

Case 1: Capacity constrained instance: $S > \frac{\Delta \hat{C}_B}{D_B}$

In this case, some counties can deviate from their nominal values without having any effect on the optimal solution. Define the set

$$J_E = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j}{D_j} > \frac{\hat{C}_B}{D_B} \right\}. \quad (6.10)$$

The total food shipment will not change as long as the bottleneck county does not deviate from its nominal value, so we set $\varphi_B^{\min} = 0$ for now. The subscript “min” is used to emphasize that this deviation in the bottleneck county’s capacity does not have any effect on the optimal solution, so it is the “minimum” deviation. We emphasize that this value φ_B^{\min} may not be the optimal value of this decision variable. We find its optimal value φ_B^* in the next steps. Furthermore, we set

$$\varphi_j^{\min} = \min \left\{ \frac{\hat{C}_j D_B - \hat{C}_B D_j}{\varepsilon_j D_B}, 1 \right\} \quad (6.11)$$

for all $j \in J$ which is obtained by setting $\frac{\hat{C}_j - \varepsilon_j \varphi_j^{\min}}{D_j} = \frac{\hat{C}_B}{D_B}$ where $\varphi_j^{\min} \leq 1$.

Case 2: Supply constrained instance: $S < \frac{\Delta \hat{C}_B}{D_B}$

If the solution is supply constrained, we know that we can decrease all the counties' capacities until we achieve $S = \Delta \min_{j \in J} \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j}$ without affecting the total food distribution. Hence, we set

$$\varphi_j^{\min} = \min \left\{ \frac{\hat{C}_j \Delta - S D_j}{\varepsilon_j \Delta}, 1 \right\} \quad (6.12)$$

for all $j \in J$.

The new bottleneck, after the capacities are decreased, becomes

$$B = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j \varphi_j^{\min}}{D_j} = \min_{i \in J} \left\{ \frac{\hat{C}_i - \varepsilon_i \varphi_i^{\min}}{D_i} \right\} \right\}. \quad (6.13)$$

Hence, the bottleneck B is the county i with the lowest $\frac{\hat{C}_i - \varepsilon_i \varphi_i^{\min}}{D_i}$ ratio. The total food distribution after the capacity deviations remains equal to S due to (6.12). Finally, we define the set J_E as the counties that can achieve maximal deviation without impacting the optimal food distribution according to the following expression:

$$J_E = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j}{D_j} > \frac{\hat{C}_B - \varepsilon_B \varphi_B^{\min}}{D_B} \right\}. \quad (6.14)$$

After set J_E is determined according to whether the instance is capacity or supply constrained as just shown, the rest of the proof is valid for both cases. It is trivial to see that in the optimal solution $\varphi_j^* = 1$ for $j \in J_E$.

We define the set J_I as the complement of J_E , i.e., $J_I = J \setminus J_E$. For the counties in J_I , we have $0 \leq \varphi_j \leq 1$. Without loss of generality, assume that the counties in set J_I are indexed in decreasing order of their $\frac{\hat{C}_j - \varepsilon_j}{D_j}$ ratios. Furthermore, we observe that at optimality, constraint (6.5) is satisfied at equality. Combining constraints (6.3) and (6.4), and setting $\tilde{X} = \sum_{l=1}^n X_l$, we obtain

$$\tilde{X}^*(\varphi_j) = \min \left\{ S, \Delta \min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} \right\} \quad (6.15)$$

where we write $\tilde{X}^*(\varphi_j)$ to emphasize that \tilde{X} is a function of φ_j . Furthermore, we will always have

$$\Delta \min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} \leq S \quad (6.16)$$

since the values φ_j are at least equal to their calculated values in (6.12), i.e., $\varphi_j \geq \varphi_j^{\min}$.

Then, the objective function becomes

$$Z^* = \tilde{X}^*(\varphi_j) - \rho \sum_{l=1}^n (1 - \varphi_l) = \Delta \min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} + \rho \sum_{l=1}^n \varphi_l - \rho n. \quad (6.17)$$

Since we know that for $j \in J_E$, $\varphi_j^* = 1$,

$$\begin{aligned}
Z^* &= \Delta \min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} + \rho \sum_{j \in J_E} 1 + \rho \sum_{j \in J_I} \varphi_j - \rho n \\
&= \Delta \min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} + \rho \sum_{j \in J_I} \varphi_j - \rho(n - |J_E|).
\end{aligned} \tag{6.18}$$

We know from (6.9) and (6.13) that we can set

$$\min_{j \in J} \left\{ \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} \right\} = \frac{\hat{C}_B - \varepsilon_B \varphi_B^{\min}}{D_B} \tag{6.19}$$

without causing a decrease in the total distribution. However, any increase in φ_B^{\min} will result in an increase in φ_j for $j \in J_I$ and consequently, a decrease in \tilde{X}^* .

Assume that we increase φ_B^{\min} by a very small amount δ such that $\varphi'_B = \varphi_B^{\min} + \delta <$

1. Then, for $j \in J_I$, we have

$$\frac{\hat{C}_j - \varepsilon_j \varphi'_j}{D_j} = \frac{\hat{C}_B - \varepsilon_B \varphi'_B}{D_B} \tag{6.20}$$

which is

$$\varphi'_j = \min \left\{ 1, \frac{\hat{C}_j D_B - \hat{C}_B D_j + \varepsilon_B \varphi'_B D_j}{\varepsilon_j D_B} \right\}. \tag{6.21}$$

We assume that the value δ is selected such that

$$\max_{j \in J_I} \left\{ \frac{\hat{C}_j D_B - \hat{C}_B D_j + \varepsilon_B D_j (\varphi_B^{\min} + \delta)}{\varepsilon_j D_B} \right\} = 1. \tag{6.22}$$

Hence, the calculated value of φ'_j in (6.21) is equal to one for one county in set J_I . Let us denote this county by the index L since it represents the county with the largest $\frac{\hat{c}_j - \varepsilon_j \varphi'_j}{D_j}$ ratio.

Therefore, in this new solution, we have

$$\varphi'_L = \frac{\hat{c}_L D_B - \hat{c}_B D_L + \varepsilon_B D_L (\varphi_B^{\min} + \delta)}{\varepsilon_L D_B} = 1 \quad (6.23)$$

and

$$\varphi'_j = \frac{\hat{c}_j D_B - \hat{c}_B D_j + \varepsilon_B D_j (\varphi_B^{\min} + \delta)}{\varepsilon_j D_B} < 1 \quad j \in J_I \setminus \{L\}. \quad (6.24)$$

By combining (6.18) with (6.22) and (6.23), the net benefit from this increase is calculated as

$$Z(\varphi'_B) - Z(\varphi_B^{\min}) = -\Delta \frac{\varepsilon_B \delta}{D_G} + \rho \frac{\varepsilon_B \delta}{D_B} \sum_{l \in J_I} \frac{D_l}{\varepsilon_l} \quad (6.25)$$

Therefore, we increase φ_B^{\min} as long as the value in (6.20) is positive and δ satisfies (6.22) since $Z(\varphi'_G) - Z(\varphi_G)$ is a linear function of δ . Hence, if

$$\rho > \frac{\Delta}{\sum_{l \in J_I} \frac{D_l}{\varepsilon_l}} \quad (6.26)$$

we increase φ_B^{\min} to the point that $\varphi'_L = 1$. Then, we set φ_j according to (6.21) and remove county L from set J_I and add it to set J_E . We repeat the process for the remaining counties in J_I . The following Robust Optimization Algorithm for Capacity Uncertainty (C-RA) in Figure 6.1 is built upon these ideas and provides the optimal solution for a given ρ .

Robust Food Distribution Algorithm

if $\Delta \min_{i \in J} \left\{ \frac{\hat{C}_i}{D_i} \right\} < S$ **then**

$$B = \left\{ j \in J \mid \frac{\hat{C}_j}{D_j} = \min_{i \in J} \left\{ \frac{\hat{C}_i}{D_i} \right\} \right\}$$

$$J_E = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j}{D_j} > \frac{\hat{C}_B}{D_B} \right\}$$

else

$$\varphi_j = \min \left\{ \frac{\hat{C}_j \Delta - S D_j}{\varepsilon_j \Delta}, 1 \right\}$$

$$B = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j \varphi_j}{D_j} = \min_{i \in J} \left\{ \frac{\hat{C}_i - \varepsilon_i \varphi_i}{D_i} \right\} \right\}$$

$$J_E = \left\{ j \in J \mid \frac{\hat{C}_j - \varepsilon_j}{D_j} > \frac{\hat{C}_B - \varepsilon_B \varphi_B}{D_B} \right\}$$

end if

$$\varphi_j \leftarrow 1, \forall j \in J_E$$

$$J_I = (J_E)^C$$

Sort J_I in a decreasing order of $\frac{\hat{C}_j - \varepsilon_j}{D_j}$

while $i \leq |J_I|$ **do**

if $\rho \geq \frac{\Delta}{\sum_{l \in J_I} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}} - \sum_{l=1}^i \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}}}$ **then**

for $1 \leq k \leq i$ **do**

$$\varphi_{J_I(k)} \leftarrow 1$$

end for

for $i + 1 \leq k \leq |J_I|$ **do**

$$\varphi_{J_I(k)} \leftarrow \frac{\hat{C}_{J_I(k)} D_{J_I(i)} - \hat{C}_G D_{J_I(i)} + \varepsilon_G D_{J_I(i)}}{\varepsilon_{J_I(k)} D_{J_I(i)}}$$

end for

else

Quit

end if

end while

Figure 6.1: Robust Optimization Algorithm for Capacity Uncertainty (C-RA).

The Robust Food Distribution Algorithm for Capacity Uncertainty starts by determining the elements of set J_E according to the instance (without any capacity deviations) being supply or capacity constrained. This set contains the counties that can achieve maximal deviation from their nominal capacities without any effect on the total food distribution. We set $\varphi_j^* = 1$ for $j \in J_E$. The complement of this set, J_I contains the counties whose maximum deviations from their nominal capacities have an effect on the total food distribution. We sort J_I in a decreasing order of $\frac{\hat{c}_j - \varepsilon_j}{D_j}$. This means that if we set $\varphi_{J_I(1)} = 1$ and set $\frac{\hat{c}_j - \varepsilon_j \varphi_j}{D_j} = \frac{\hat{c}_{J_I(1)} - \varepsilon_{J_I(1)}}{D_{J_I(1)}}$ for $j \in J_I$, we have $\varphi_j < 1$. Then, we find the county i that satisfies

$$\frac{\Delta}{\sum_{l \in J_I} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}} - \sum_{l \in J_I}^i \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}}} \leq \rho < \frac{\Delta}{\sum_{l \in J_I} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}} - \sum_{l \in J_I}^{i+1} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}}}. \quad (6.27)$$

We then set $\varphi_{J_I(l)} = 1$ for $l = 1, \dots, i$ and calculate the remaining counties φ_j by setting

$$\frac{\hat{c}_j - \varepsilon_j \varphi_j}{D_j} = \frac{\hat{c}_i - \varepsilon_i \varphi_i}{D_i}.$$

The worst-case time complexity of the Robust Food Distribution Algorithm is $O(n^2)$.

We now explain the intuition behind the inequality (6.27). The value of $\frac{\varepsilon_j}{D_j}$ represents the maximum deviation of county j 's CD ratio. In this sense, if ρ decreases, this causes the algorithm to stop at a lower i index. Thus, if the value of robustness ρ is lower, fewer counties achieve their maximal deviation and the overall robustness of the model to changes in capacity values decreases. On the other hand, assume that ρ is a very large value. In that

case, all of the counties achieve their maximal deviation and hence, the solution is protected against any uncertainty in the counties' capacities within the given ranges.

6.3.3 Numerical Results

In this subsection, we use the data obtained from FBCENC to solve C-RM. As discussed in Chapter 3, the “capacity” of a county also incorporates a number of aspects beyond simple storage space, including its ability to receive and purchase food, its agency suspensions, its transportation resources and so on. For this reason, instead of using some measure of physical storage space, we use the amount of food shipped into the counties to estimate the amount of food each county can reliably accept each month.

We assume that the capacity of county j can vary in the range defined by $C_j \in [\hat{C}_j - \varepsilon_j, \hat{C}_j]$ where \hat{C}_j represents the nominal capacity value and is equal to the nominal estimates used in Chapter 3, i.e., the 90th percentile of the empirical distribution of the amount of food shipped to each county during fiscal year 2009. We set $\varepsilon_j = \theta \hat{C}_j$ where we perform experiments for $\theta = \{0.1, 0.5, 0.9\}$. For the supply, we consider three cases: the actual food distribution amount of donated dry goods during January 2009 (1,277,363 pounds of food), $S = 1,000,000$ pounds of food and $S = 800,000$ pounds of food. For demand, the poverty populations of the counties for the year 2009 are used as in Chapter 3 (United States Census Bureau, 2009). There are $n = 34$ counties in FBCENC's service region.

In Figure 6.2, we consider the case where total supply is 1,277,363 pounds of food and the maximum capacity deviations are varied. The maximum deviation of the capacity of a county, which is a parameter, is set to $\varepsilon_j = \theta \hat{C}_j$ where the different θ values correspond to

different lines in Figure 6.2. In this graph, the value of robustness ρ is increased from zero pounds of food to 1,000,000 pounds of food. As stated before, this value represents the maximum pounds of food that the decision maker is willing to unship, or send to waste, in order to achieve complete deviation from nominal capacity at any county j . The x axis gives the value of robustness ρ . On the y -axis, we show the change in total food distribution \tilde{X}^* as ρ is varied.

Figure 6.2 illustrates the effectiveness – robustness trade-off. We see that as ρ increases, the total food distribution \tilde{X}^* decreases for all θ values. This results in a more robust solution which causes a lower food shipment in order to be protected against larger capacity deviations. Also, we notice that as ρ is increased, \tilde{X}^* decreases in a piecewise linear manner. This is expected since we proved in Section 6.3.2 that when ρ is within some ranges

such as $\frac{\Delta}{\sum_{l \in J_I} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}} - \sum_{l \in J_I}^i \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}}} \leq \rho < \frac{\Delta}{\sum_{l \in J_I} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}} - \sum_{l \in J_I}^{i+1} \frac{D_{J_I(l)}}{\varepsilon_{J_I(l)}}}$, the solution remains the same. We also

observe that higher θ means higher values of robustness are required to cause a decrease in \tilde{X}^* . This is reasonable since setting $\varphi_j = 1$ causes a higher decrease in \tilde{X}^* for larger θ values. For example, for $\theta = 0.1$, we achieve protection against maximum deviation for all counties, i.e., $\vartheta^* = n = 34$ at $\rho = 109$. The minimum value of robustness for maximum deviation is $\rho = 544$ for $\theta = 0.5$ and $\rho = 981$ for $\theta = 0.9$. The reason is that when θ is larger, capacity deviations have greater effect on the optimal shipment. In that case, we have to sacrifice a greater amount of total shipment in order to achieve a solution that is robust against all capacity deviations.

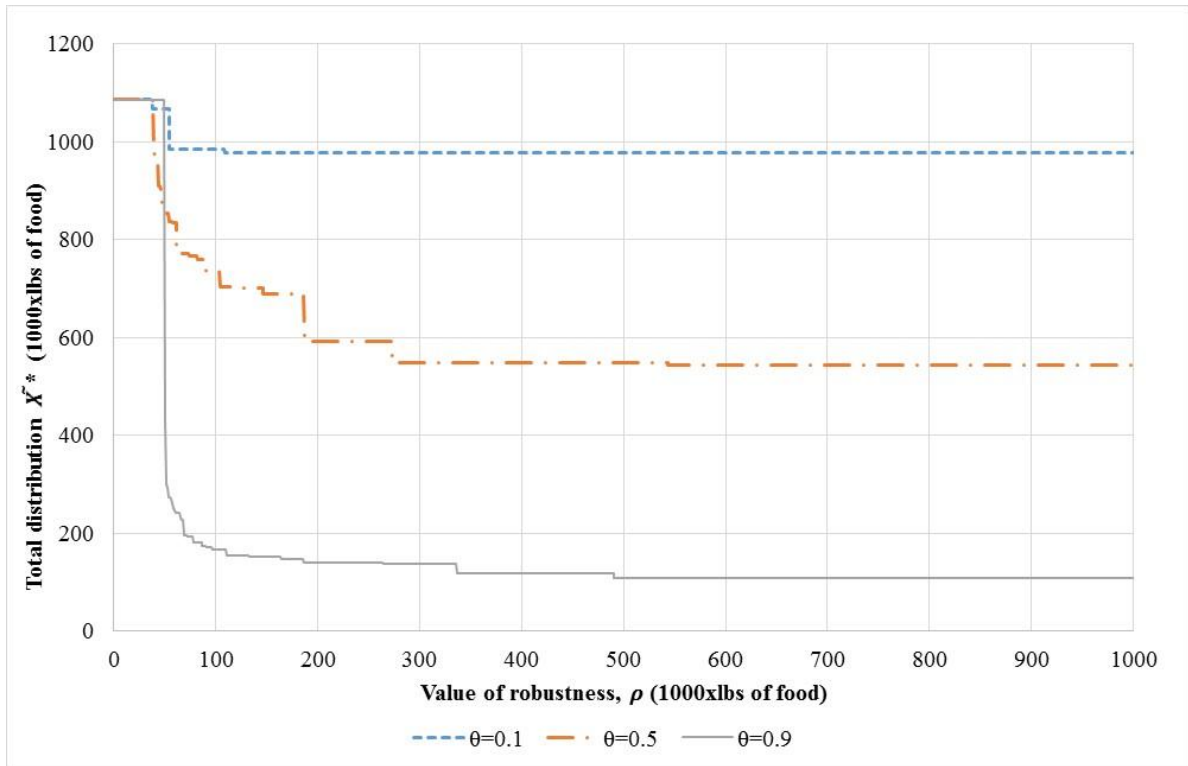


Figure 6.2: Total Optimal Food Distribution \tilde{X}^* versus ρ for Changing θ .

Figure 6.3 considers the situation where $\varepsilon_j = 0.5\hat{C}_j$ for all counties and illustrates the effect of changing the total supply. In this graph, the value of robustness ρ is increased from 0 to 100,000 pounds. On y-axis, we show the change in total food distribution \tilde{X}^* as ρ is varied.

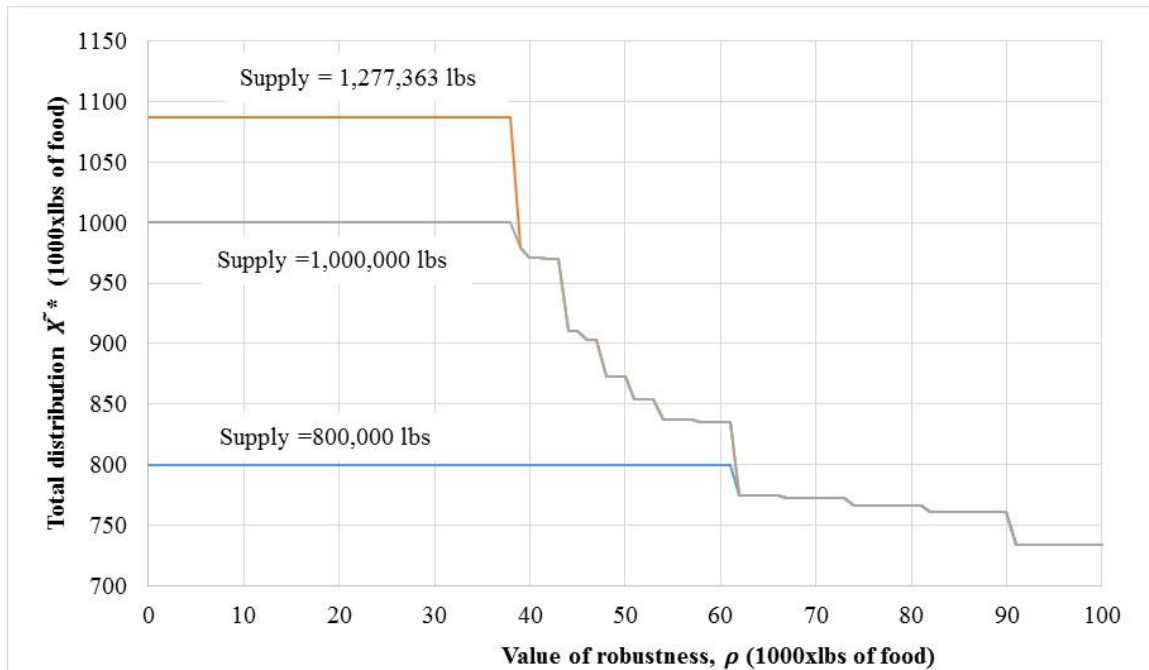


Figure 6.3: Total Optimal Food Distribution \tilde{X}^* versus ρ for Changing Supply.

We make two observations from Figure 6.3. First, we notice that as ρ is increased, \tilde{X}^* decreases in a piecewise linear manner similar to Figure 6.2. This is due to the solution remaining the same for ranges of ρ values as explained above. The second observation we make is that for $S = 1,000,000$ and $S = 800,000$, the initial solution when $\rho = 0$ is supply constrained. However, as ρ increases, we have to obtain a solution that is protected against higher deviations in capacities, which cause the solutions to become capacity constrained when ρ increases beyond a certain value. For example, for $S = 1,000,000$, the solution switches from supply constrained to capacity constrained when $\rho = 39,000$. For $\rho > 39,000$, the solution remains capacity constrained. For $S = 1,277,363$, solution is capacity constrained for any $\rho > 0$ since the nominal solution for this supply setting is also capacity

constrained. Another observation we make is associated with the convergence of the solutions. We observe that the lines overlap after certain values of ρ . These overlaps occur when the solution is capacity constrained with the bottleneck county having a positive deviation from its nominal capacity. After that point, the solution is constrained by the counties' deviations from capacities and the original supply has no effect on the optimal solution.

We would like to finish this section with a discussion of how this model can be used in practice by a food bank. Although estimating the value of robustness ρ is a challenging task, a food bank can determine the minimum amount of food that should be shipped in a time period considering the risks of spoilage and limitations of their storage space. Then, according to that minimum shipment value, we can use Figure 6.3 to find the corresponding value of robustness ρ . Finally, by using the Robust Optimization Algorithm for Capacity Uncertainty, we can determine the φ_j values for each of the counties. If for a county j , $\varphi_j = 1$, this means that the solution is feasible for any deviation of that county's capacity within the given range. If $\varphi_j < 1$, then it becomes important to get additional information about that county's capacity since the solution will be infeasible if that county's capacity is varied by more than $\varphi_j \varepsilon_j$. If $\varphi_j = 0$, then that county's capacity should assume its nominal value for the solution to remain feasible.

In the next section, we consider the Deterministic Food Distribution Model with imperfect equity and develop a robust optimization model based on the equity deviation constant, K .

6.4 Robust Optimization Model for Equity Deviation (E-RM)

In this section, we build upon the Deterministic Food Distribution Model and allow a subset of the counties to deviate from perfectly equitable distribution by a pre-specified value K . In Chapter 3, through a series of models and propositions, we showed that a subset of the counties, denoted the *bottleneck counties*, have a greater effect on the optimal objective function value than other counties, and that the objective function can be constrained by a proportion of the total demand depending on the (modified) CD ratios of the bottleneck counties and therefore, the equity constant, K .

In the formulation of the Food Distribution Model, we do not distinguish between the counties in terms of deviation from equity and hence, the allowed deviation is the same for all counties. In other words, all counties can deviate from equity within a specified limit, K , in order to obtain higher effectiveness, while it might be possible to obtain the same level of effectiveness by allowing only a subset of counties to deviate but with a high deviation. In other words, in the Food Distribution Model, in order to maximize effectiveness, all counties deviate from perfect equity by the maximum allowed amount since there is no control over the total equity deviation over all counties. Hence, this approach results in worst-case oriented solutions due to each county achieving maximum deviation from equity. The question we address in this section is: is it possible to limit the total deviation from equity over all the counties such that deviation is restricted to a certain subset of counties while still increasing distribution effectiveness? In this way, we can limit the total deviation from equity while requiring some counties to maintain perfect equity and increase distribution effectiveness. The subset of counties that are allowed to deviate are selected by the model to

maximize food distribution. As a result of this, if a county has been overserved or underserved in a given period, the Food Bank may compensate for this by adjusting shipments in the following periods. In Section 6.4.1, we formulate the Robust Optimization Model for Equity (E-RM). In Section 6.4.2, we prove its structural properties and introduce an algorithm that solves E-RM optimally. Finally, Section 6.4.3 presents numerical results.

6.4.1 The formulation

We use ideas from Bertsimas and Sim (2004) to model an individual county's deviation from equity while constraining the total deviation over all counties. We obtain a solution that is feasible (though suboptimal) for the nominal solution, i.e., the optimal solution to the Deterministic Food Distribution Model under perfect equity.

Consider the equity constraint from the Food Distribution Model:

$$-K \leq \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} \leq K \quad j \in J, \quad (6.28)$$

$$-K + \frac{D_j}{\Delta} \leq \frac{X_j}{\sum_{l=1}^n X_l} \leq K + \frac{D_j}{\Delta} \quad j \in J. \quad (6.29)$$

We can rewrite this inequality as

$$\frac{X_j}{\sum_{l=1}^n X_l} \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right] \quad j \in J. \quad (6.30)$$

Now define the parameter a_j such that

$$a_j \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right] \quad j \in J. \quad (6.31)$$

Following Bertsimas and Sim (2004), we can write the Food Distribution Model as follows:

$$\min P \quad (6.32)$$

subject to

$$X_j - a_j \sum_{l=1}^n X_l = 0 \quad j \in J \quad (6.33)$$

$$S - \sum_{l=1}^n X_l - P = 0 \quad (6.34)$$

$$X_j \leq C_j \quad j \in J \quad (6.35)$$

$$X_j, P \geq 0 \quad j \in J \quad (6.36)$$

$$a_j \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right] \quad j \in J \quad (6.37)$$

If in the optimal solution $\frac{X_j^*}{\sum_{l=1}^n X_l^*} = \frac{D_j}{\Delta} - K$, this means that county j is being underserved, i.e.,

it is receiving less than its fair-share. If, on the other hand, in the optimal solution we have

$\frac{X_j^*}{\sum_{l=1}^n X_l^*} = \frac{D_j}{\Delta} + K$, this means that county j is being overserved, i.e., it is receiving more than

its fair-share. In this section, we see that if we sacrifice equity slightly on the bottleneck

counties by allowing them to be underserved to a limited extent, we can distribute a

considerable amount of additional food to the remaining counties with idle capacity. This

approach is also reasonable practically since, if the Food Bank is faced with a situation where

they have some food that has to be shipped, they ship the food to whichever agency has the

capacity, sacrificing equity in order to avoid sending food to waste. We emphasize that we

are currently assuming the demand D_j and capacity C_j of county j are known with certainty.

The dual of this model is

$$\max S\gamma - \sum_{l=1}^n C_l \delta_l \quad (6.38)$$

subject to

$$\alpha_j - \sum_{l=1}^n a_l \alpha_l + \gamma - \delta_j \leq 0 \quad j \in J \quad (6.39)$$

$$\gamma \leq 1 \quad (6.40)$$

$$\gamma, \alpha_j \text{ free} \quad j \in J \quad (6.41)$$

$$\delta_j \geq 0 \quad j \in J \quad (6.42)$$

$$a_j \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right] \quad j \in J \quad (6.43)$$

We formulate the robust model based on the dual formulation given in (6.38)-(6.43) because we would like to achieve a solution that relaxes equity and increases the total shipment. Since robust optimization aims to achieve conservative solutions that are protected against uncertainty, if we build the model based on the primal, the perfect equity solution will always be optimal since it is the most conservative solution in terms of the equity constant ($K=0$). In robust optimization, “optimizing under the worst case scenario in the primal is the same as optimizing under the best case scenario in the dual, i.e., “primal worst equals dual best” (Beck & Ben-Tal, 2009). Hence, a conservative solution for the primal problem is decreasing food shipments whereas for the dual problem, a conservative solution would increase shipments while relaxing equity. The E-RM can be considered as an extension of the Deterministic Food Distribution Model introduced in Chapter 3. The difference between the robust model and the Food Distribution Model is that the former can control the overall level

of equity in the system whereas the Food Distribution Model controls equity at individual counties.

In order to derive the robust optimization model, we follow Bertsimas and Sim (2004), introducing the following definitions.

J_K : The set of indices for which the parameters (a_j) are allowed to vary

$\equiv J$ if we allow for any a_j to change

$$\Gamma_K \in [0, |J_K|]$$

The set J_K indicates the parameters that are *allowed* to vary. In our model, we make no distinctions between the counties, so we take this set to be the set of all counties, J . Notice that this is *not* the set of counties for which the parameter a_j is required to change. The set of counties for which the parameter a_j changes will be determined by the model such that the total variation is limited by the given threshold value Γ_K . Define the variation space of a_j , i.e., the space that contains the set of values that a_j can take, as

$$\Sigma = \left\{ A \in \mathbb{R}^n: a_j \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right] \forall j \in J, \sum_{j \in J} \frac{|a_j - \frac{D_j}{\Delta}|}{K} \leq \Gamma_K \right\}. \quad (6.44)$$

The constraint (6.39) for county j can be written as

$$\max_{A \in \Sigma} - \sum_{l=1}^n (a_l \alpha_l) \leq \delta_j - \gamma - \alpha_j. \quad (6.45)$$

Therefore, for row j , we have to solve the linear program

$$\max \sum_{l=1}^n -\alpha_l \left(\frac{D_l}{\Delta} + z_l K \right) \quad (6.46)$$

subject to

$$\sum_{l \in J} |z_l| \leq \Gamma_K \quad (6.47)$$

$$0 \leq |z_j| \leq 1 \quad j \in J. \quad (6.48)$$

The dual of this linear program is:

$$\min u \Gamma_K + \sum_{l \in J} r_l \quad (6.49)$$

subject to

$$u + r_j \geq K |\alpha_j| \quad j \in J \quad (6.50)$$

$$u, r_j \geq 0 \quad j \in J. \quad (6.51)$$

Inserting (6.49)-(6.51) into (6.38)-(6.43), we obtain the Robust Optimization Model for Equity Deviation (E-RM) as follows:

E-RM:

$$\max S\gamma - \sum_{l=1}^n C_l \delta_l \quad (6.52)$$

subject to

$$\alpha_j - \sum_{l=1}^n \left(\alpha_l \frac{D_l}{\Delta} \right) + u \Gamma_K + \sum_{l \in J} r_l + \gamma - \delta_j \leq 0 \quad j \in J \quad (6.53)$$

$$u + r_j \geq K q_j \quad j \in J \quad (6.54)$$

$$-q_j \leq \alpha_j \leq q_j \quad j \in J \quad (6.55)$$

$$\gamma \leq 1 \quad (6.56)$$

$$\gamma, \alpha_j \text{ free} \quad j \in J \quad (6.57)$$

$$u, \delta_j, r_j, q_j \geq 0 \quad j \in J \quad (6.58)$$

In the next subsection, we explore structural properties of this model.

6.4.2 Structural Properties of E-RM

We start by proving structural properties of E-RM and then, we use these to develop an efficient algorithm that optimally solves E-RM. The equity constraint is stated in (6.33) as

$$\frac{X_j}{\sum_{l=1}^n X_l} = a_j \quad j \in J$$

where $a_j \in \left[\frac{D_j}{\Delta} - K, \frac{D_j}{\Delta} + K \right]$ for all $j \in J$. Therefore, a_j is treated as an uncertain parameter that is allowed to vary within these ranges. Then, we can write

$$\sum_{l=1}^n X_l = \frac{X_j}{a_j} \quad j \in J \quad (6.59)$$

Furthermore, since $X_j \leq C_j$, we obtain

$$\sum_{l=1}^n X_l \leq \min_{j \in J} \left\{ \frac{C_j}{a_j} \right\} \quad (6.60)$$

Therefore, in order to increase the total distribution, we must set the value of a_j to its minimum possible value. We can write

$$a_j = \frac{D_j - K\Delta\varphi_j}{\Delta}$$

where $-1 \leq \varphi_j \leq 1$, yielding

$$\sum_{l=1}^n X_l \leq \min_{j \in J} \left\{ \frac{\Delta C_j}{D_j - K\varphi_j \Delta} \right\} \quad (6.61)$$

Therefore, as the right-hand side of (6.61) increases, the total distribution also increases. This idea is similar to the Capacity Allocation Problem we considered in Chapter 3. In Chapter 3, the purpose was to increase the modified CD ratios of the counties, $\frac{C_j}{D_j - K\Delta}$ by allocating additional capacity starting with the counties that have high risk of becoming bottlenecks, i.e., low modified CD ratios. Here, we follow a similar approach. We notice that if the parameter φ_j assumes a higher value for the counties with low CD ratios, then the right-hand side of inequality (6.61) increases, resulting in a higher total distribution. However, as in Chapter 3, where the solution was constrained by the amount of additional capacity, in this case, the total net deviation is constrained by the given threshold. More specifically, we have

$$\sum_{j \in J} |\varphi_j| \leq \Gamma_K. \quad (6.62)$$

The following Robust Optimization Algorithm for Equity Deviation (E-RA) provides an efficient way to solve E-RM optimally.

Robust Optimization Algorithm for Equity Deviation

Sort J in an increasing order of $\frac{C_j}{D_j}$ ratios

$\varphi_j \leftarrow 0, \forall j \in J$

while $i \leq |J|$ **do**

for $1 \leq l \leq i$ **do**

$$\varphi_l \leftarrow \frac{C_{J(i+1)}D_{J(l)} - C_{J(l)}D_{J(i+1)}}{K\Delta C_{J(i+1)}}$$

end for

if $\sum_{l \in J} \varphi_l > \frac{\Gamma_K}{2}$

$$R \leftarrow \frac{\sum_{l=1}^i C_{J(l)}}{\sum_{l=1}^i D_{J(l)} - \frac{\Gamma_K}{2} K\Delta}$$

if $R > \min_{l \in J} \left\{ \frac{C_l}{D_l - K\Delta} \right\}$

$$R \leftarrow \min_{l \in J} \left\{ \frac{C_l}{D_l - K\Delta} \right\}$$

end if

$$\sum_{l=1}^n X_l^* \leftarrow \min\{S, R\Delta\}$$

Quit

end if

end while

Figure 6.4: Robust Optimization Algorithm for Equity Deviation (E-RA).

The algorithm starts by sorting the set of all counties, J , in increasing order of their CD ratios. Initially, we set all $\varphi_j = 0$. For i^{th} county in set J , we start by calculating the values of φ_j , where $1 \leq j \leq i$, i.e., all counties j have lower CD ratios than $\frac{C_{i+1}}{D_{i+1}}$. County $i + 1$ represents the target county. We calculate φ_j by setting

$$\frac{C_j}{D_j - K\varphi_j\Delta} = \frac{C_{i+1}}{D_{i+1}} \quad (6.63)$$

and obtain

$$\varphi_j = \frac{C_{i+1}D_j - C_jD_{i+1}}{K\Delta C_{i+1}} \geq 0. \quad (6.64)$$

Then, we check if the sum of the φ_j 's exceed $\Gamma_K/2$. The reason we use $\Gamma_K/2$ instead of Γ_K is as follows: if we were to allocate Γ_K completely towards increasing the bound in (6.61), we would not be able to distribute this additional food as the bottleneck counties with positive φ_j are receiving less food than they would receive under perfect equity since

$$\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} = \frac{-K\Delta\varphi_j}{\Delta} < 0.$$

If we allocate all Γ_K towards increasing the bound in (6.61), the non-bottleneck counties would be forced to achieve perfect equity ($\varphi_j = 0$). For this reason, we allocate half of the given threshold to increasing the bound in (6.61) and the other half to distributing this additional food to nonbottleneck counties. Consider the situation where the threshold is Γ_K and as a result of the E-RA, the optimal food distribution after equity deviations equals $\sum_{l=1}^n X_l^* = R_{\text{new}}\Delta$ where $R_{\text{new}} > \min_{j \in J} \left\{ \frac{C_j}{D_j} \right\}$. In order to achieve this increase in food distribution, since by (6.33) and (6.37) we have $X_j = \left(\frac{D_j - K\Delta\varphi_j}{\Delta} \right) \sum_{l=1}^n X_l$, some counties are underserved, i.e., $0 < \varphi_j^* < 1$ and some counties are overserved $-1 < \varphi_j^* < 0$ in the optimal solution. Let us denote the set of counties that are underserved as J_U . The total distribution to the counties that are underserved equals

$$\sum_{l \in J_U} X_l = \sum_{l \in J_U} R_{\text{new}}\Delta \left(\frac{D_l - K\Delta\varphi_l}{\Delta} \right) = R_{\text{new}}\Delta \left(\frac{\sum_{l \in J_U} D_l}{\Delta} - K \sum_{l \in J_U} \varphi_l \right) \quad (6.65)$$

Then, in order to distribute $R_{\text{new}}\Delta$ pounds of food, the overserved counties in set $J \setminus J_U$ should receive

$$\begin{aligned} \sum_{l \in J \setminus J_U} X_l &= R_{\text{new}}\Delta - R_{\text{new}}\Delta \left(\frac{\sum_{l \in J_U} D_l}{\Delta} - K \sum_{l \in J_U} \varphi_l \right) \\ &= R_{\text{new}}\Delta \left(1 - \frac{\sum_{l \in J_U} D_l}{\Delta} + K \sum_{l \in J_U} \varphi_l \right) \end{aligned} \quad (6.66)$$

pounds of food. However, the maximum amount of food that they can receive equals

$$\begin{aligned} \sum_{l \in J \setminus J_U} X_l &\leq \sum_{l \in J \setminus J_U} R_{\text{new}}\Delta \left(\frac{D_l + K\Delta|\varphi_l|}{\Delta} \right) \\ &= R_{\text{new}}\Delta \left(\frac{\sum_{l \in J \setminus J_U} D_l}{\Delta} + K \sum_{l \in J \setminus J_U} |\varphi_l| \right). \end{aligned} \quad (6.67)$$

Equating the terms in (6.66) and (6.67), we obtain

$$1 - \frac{\sum_{l \in J_U} D_l}{\Delta} + K \sum_{l \in J_U} \varphi_l = \frac{\sum_{l \in J \setminus J_U} D_l}{\Delta} + K \sum_{l \in J \setminus J_U} |\varphi_l| \quad (6.68)$$

which yields

$$\sum_{l \in J_U} \varphi_l = \sum_{l \in J \setminus J_U} |\varphi_l|. \quad (6.69)$$

Since by definition $\sum_{l \in J} |\varphi_l| \leq \Gamma_K$, we get $\sum_{l \in J_U} \varphi_l = \sum_{l \in J \setminus J_U} |\varphi_l| = \Gamma_K/2$.

If $\sum_{l \in J} \varphi_l \leq \frac{\Gamma_K}{2}$, we increment i by one and continue. If $\sum_{l \in J} \varphi_l > \frac{\Gamma_K}{2}$, then the total equity deviation violates the threshold hence we cannot increase all of MCD_j ratios to the level of CD_{i+1} . In that case, we calculate the maximum MCD value that we can achieve as:

$$\frac{C_j}{D_j - K\varphi_j\Delta} = R \quad (6.70)$$

which means

$$\varphi_j = \frac{RD_j - C_j}{K\Delta R} \quad (6.71)$$

and we set

$$\sum_{l=1}^i \varphi_l = \sum_{l=1}^i \frac{RD_l - C_l}{K\Delta R} = \frac{R \sum_{l=1}^i D_l - \sum_{l=1}^i C_l}{K\Delta R} = \frac{\Gamma_K}{2}. \quad (6.72)$$

We obtain

$$R = \frac{\sum_{l=1}^i C_l}{\sum_{l=1}^i D_l - K\Delta \frac{\Gamma_K}{2}}. \quad (6.73)$$

However, we have to check that the value obtained in (6.66) is less than or equal to one. So, we get

$$\varphi_j = \frac{RD_j - C_j}{K\Delta R} \leq 1$$

which yields

$$R \leq \min_{j \in J} \left\{ \frac{C_j}{D_j - K\Delta} \right\}. \quad (6.74)$$

Hence, we set the value of R according to the minimum of (6.73) and (6.74). Finally, we set

$$\sum_{l=1}^n X_l^* = R \min\{S, R\Delta\} \quad (6.75)$$

and find the optimal solution. The worst-case time complexity of E-RA is $O(n^2)$. In the next section, we perform a numerical study and illustrate the results.

6.4.3 Numerical Results

In this subsection, we solve E-RM for K varying between 0 and 0.1 in increments of 0.001 and Γ_K between 0 and 34 in increments of 0.5. We use the capacity, demand and supply parameters the same as Chapter 3. The solutions have an efficient frontier structure. The total undistributed amount of food supply (P^*) for the range $K \in [0, 0.02]$ and $\Gamma \in [1, 4]$ for each combination of parameters is shown in Table 6.1.

As expected, when $K = 0$, implying perfect equity, we obtain the same optimal objective function value for all values of Γ_K . Furthermore, this solution is the same, capacity constrained solution that is obtained by the Food Distribution Model under perfect equity.

If we keep $K > 0$ constant and increase Γ_K , the total food distribution increases and eventually becomes constant. Furthermore, these threshold values are the same as those we would obtain if we solve the Food Distribution Model under relaxed equity with the corresponding K values. This happens because the $\min_{l \in J} \left\{ \frac{c_l}{D_l - K\Delta} \right\}$ ratio starts to constrain the total distribution and hence, increasing the threshold does not affect the solution.

On the other hand, when Γ_K is kept constant and K is increased, we see that P decreases and becomes zero after a point. The areas in Table 6.1 that are shaded grey

represent supply constrained solutions. This happens because after that point, we are able to distribute all of the food supply due to increased MCD ratios.

Table 6.1: P^* Values for E-RM for $K \in [0, 0.02]$ and $\Gamma_K \in [1, 4]$ in thousands of pounds

(Grey-highlighted cells indicate supply constrained solutions).

K/Γ_K	0	0.5	1	1.5	2	2.5	3	3.5	4
0	190.8	190.8	190.8	190.8	190.8	190.8	190.8	190.8	190.8
0.001	190.8	182.1	177.2	172.2	167.2	162.1	157.0	154.7	154.7
0.002	190.8	177.2	167.2	157.0	146.6	136.0	125.3	116.0	116.0
0.003	190.8	172.2	157.0	141.3	125.3	108.7	91.9	83.0	74.7
0.004	190.8	167.2	146.6	125.3	103.1	86.0	74.0	61.8	49.3
0.005	190.8	162.1	136.0	108.7	86.0	71.0	55.6	39.8	23.6
0.006	190.8	157.0	125.3	91.9	74.0	55.6	36.6	17.0	0.0
0.007	190.8	151.8	114.3	83.0	61.8	39.8	17.0	0.0	0.0
0.008	190.8	146.6	103.1	74.0	49.3	23.6	0.0	0.0	0.0
0.009	190.8	141.3	91.9	64.9	36.6	6.9	0.0	0.0	0.0
0.01	190.8	136.0	86.0	55.6	23.6	0.0	0.0	0.0	0.0
0.011	190.8	130.7	80.0	46.1	10.3	0.0	0.0	0.0	0.0
0.012	190.8	125.3	74.0	36.6	0.0	0.0	0.0	0.0	0.0
0.013	190.8	119.8	67.9	26.8	0.0	0.0	0.0	0.0	0.0
0.014	190.8	114.3	61.8	17.0	0.0	0.0	0.0	0.0	0.0
0.015	190.8	108.7	55.6	6.9	0.0	0.0	0.0	0.0	0.0
0.016	190.8	103.1	49.3	0.0	0.0	0.0	0.0	0.0	0.0
0.017	190.8	97.4	43.0	0.0	0.0	0.0	0.0	0.0	0.0
0.018	190.8	91.9	36.6	0.0	0.0	0.0	0.0	0.0	0.0
0.019	190.8	88.9	30.1	0.0	0.0	0.0	0.0	0.0	0.0
0.02	190.8	86.0	23.6	0.0	0.0	0.0	0.0	0.0	0.0

The results from E-RM show that this model can be used to restrict the total deviation from equity and still increase the total food distribution. For example, from the results of the Food Distribution Model, we know that if we increase the level of K to 0.005, we obtain

complete distribution ($P^* = 0$). We calculate the corresponding total absolute deviation from equity for that solution as follows:

$$\sum_{j=1}^n \frac{\left| \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} \right|}{K} = 32.284. \quad (6.76)$$

However, from Table 6.1, we see that we can obtain complete distribution of supply by setting the total deviation threshold $\Gamma = 1$ and increasing K to 0.024. In that case, the sum calculated in (6.76) for the robust solution equals 1. Therefore, the level of equity constant (K) is higher but the total deviation is limited to a lower threshold value that can be more desirable. If we examine the individual shipments, we see that counties with low CD ratios, such as Wilson, Wayne and Granville, are underserved whereas counties with idle capacities are overserved. The worst case equity deviation for the robust solution equals

$$\frac{X_{Wilson}}{\sum_{l=1}^n X_l} - \frac{D_{Wilson}}{\Delta} = -0.00492.$$

Therefore, the Robust Model for Equity Deviation allowed us to obtain maximum effectiveness with lower total deviation from equity. However, in this case, some counties experience a high level of inequity while others maintain equity. We observed that for 25 counties out of 34, the achieved solution was perfectly equitable, i.e., $\frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\Delta} = 0$. Hence, we were able to increase total distribution while satisfying perfect equity at many counties, whereas we had to sacrifice equity in some counties through either undershipment or overshipment.

6.5 Conclusions

In this chapter, we used Robust Optimization methodology to address uncertainty in capacity estimates and the equity deviations.

First, we first considered the case where the capacities of the counties in FBCENC's service region are allowed to vary within given ranges. More specifically, we examined the case where the capacity of a county, C_j , was allowed to vary between $[\hat{C}_j - \varepsilon_j, \hat{C}_j]$ where \hat{C}_j are the nominal capacity estimates and ε_j are the maximum deviations from these nominal estimates. We developed the Robust Optimization Model for Capacity Uncertainty aimed at controlling the trade-off between the total amount of food distributed and the robustness of the solution towards these capacity estimates. We then proved structural properties of this model and developed an efficient algorithm that solves this model optimally. We showed that the optimal solution to this model preserves the bottleneck structure. In addition, we showed that being protected against deviations in the bottleneck's capacity implies being protected against other counties' deviations as well. We then used data from FBCENC to illustrate the trade-off between effectiveness and robustness for varying levels of the robustness coefficient.

Our second model used the robust optimization methodology in a non-traditional way to obtain a deterministic model that controls the overall equity in the system while aiming to maximize total food shipment. We developed the Robust Optimization Model for Equity Deviation that limits the sum of absolute equity deviations to be below a certain threshold. We proved the optimal solution structure of this model and developed an algorithm that identifies the optimal solution. By sacrificing equity at locations that have a

high risk of becoming bottlenecks (low CD ratios), we were able to achieve a considerable increase in the total distribution while most counties continue to receive food in a perfectly equitable manner.

In our future work, we will consider the problem of capacity uncertainty under relaxed equity, extending C-RM to consider capacity deviations when a certain level of inequity is allowed in the system. We would also like to address the problem of data uncertainty in more than one set of parameters. For example, we would like to consider the case where both the county capacity and total supply are allowed to vary within pre-specified ranges and examine the structure of the resulting model and optimal solution.

Chapter 7

Conclusions and Future Directions

Food insecurity has been an increasing problem both in the United States and in the world (FAO, 2013; United Nations, 2010). Statistics show that there is a direct correlation between food insecurity and people's health status and quality of life (Vailas *et al.*, 1998; Stuff *et al.*, 2004; Vozoris and Tarasuk, 2003; Nelson *et al.*, 2001; Olson, 1999). In the United States, Feeding America is the leading nationwide domestic hunger relief organization serving the food-insecure population through 200 food banks (Feeding America, 2015). These food banks serve the local food-insecure population through the strategic distribution of food donations. Feeding America requires its food banks to distribute food in an equitable manner such that ideally, each person in the service region receives the same amount of food. On the other hand, timely distribution of food donations is also important due to the risk of spoilage. This dissertation addresses the problem of food insecurity by developing easily implementable policies for the equitable and effective distribution of food donations by a large food bank.

The Food Bank of Central and Eastern North Carolina (FBCENC) is an affiliate of Feeding America and distributes donations from many sources such as Feeding America, companies, individuals and the government to the charitable agencies in its 34-county service region (FBCENC, 2013). The partner agencies of FBCENC consist of food pantries, soup kitchens, churches, etc., that have limited capacity to serve their demand. In our models, we aggregate the agencies in terms of the counties to which they belong, and consider counties

to be the smallest distribution locations. The capacity of a county, i.e., its ability to receive, store and distribute food, depends on the storage and transportation capabilities, work force and budget of the agencies located in that county. In addition, when new agencies are recruited to FBCENC's network or existing agencies are suspended due to the failure to satisfy their reporting requirements, this affects the overall capacity of a county.

In this dissertation, we develop mathematical models which aim to maximize the total and timely distribution of food donations while maintaining equity in the service region under capacity constraints. This is a challenging problem since (i) the objectives of equity and effectiveness are in direct conflict with each other, and (ii) the capacity constraints, which are highly dynamic, add another level of complexity to the system.

Chapter 3 addresses this problem by developing mathematical network-flow models that address the equity-effectiveness trade-off under deterministic capacity constraints. We show that these models have closed-form optimal solutions that can be used as easily-implementable policies by food banks. More specifically, we find that the county that has the smallest capacity-to-demand ratio, *the bottleneck county*, constrains the entire food distribution network because of the requirement to distribute food equitably. We also address the problem of optimally allocating additional capacity to the counties. We illustrate our results by using actual data from FBCENC and perform probabilistic sensitivity analysis to explore the effect of the counties' capacities on the optimal solution. The main limitation of Chapter 3 is the assumption of deterministic capacities, which is relaxed in Chapters 4, 5 and 6.

Chapter 4 formulates a two-stage, single-period stochastic model that addresses the objectives of equity and effectiveness under capacity uncertainty. In this model, the decision maker is required to make food shipments in the beginning of the period without knowing the realized capacities. After the capacities are observed, food shipments are corrected at an additional cost. We prove the closed-form optimal solution of this model and show its equivalency to the newsvendor problem. We present numerical results by using data from FBCENC and also perform sensitivity analysis on the cost coefficients.

In Chapter 5, we extend the single-period food shipment problem studied in Chapter 4 to multiple periods. In this problem, supply is received at the beginning of each period and the decision maker is required to make shipment decisions for the current period prior to knowing the realized capacities for that period. After capacities are realized, food shipments made at the beginning can be corrected at additional cost. Any unshipped food remaining in the food bank after all shipments are finalized is considered to be the starting inventory in the next period. We examine the structural properties of the multi-period food shipment problem and develop several heuristics that provide easily-implementable near-optimal solutions. We test the performance of these heuristics by using a Multi-Stage Stochastic Programming Model as a benchmark. The main limitation of Chapters 4 and 5 is the assumption of perfect equity, i.e., each person in the service region receives the same amount of food. In our future work, we will relax this assumption and examine the single-period and multi-period food shipment problems under relaxed equity.

Chapter 6 develops two robust optimization models. The first robust model addresses capacity uncertainty within a single period. The difference between this model and the

single-period stochastic model considered in Chapter 4 is that instead of using the underlying probability distributions for capacity, Chapter 6 uses ranges to represent the uncertainty in a county's capacity. More specifically, the robust solution is feasible when the capacities of counties deviate within pre-specified ranges. This model also assumes perfect equity. The second robust model extends the deterministic food shipment problem by limiting the overall deviation from equity instead of the individual counties' deviations from equity. We find that if equity is slightly violated at the bottleneck location, total distribution can be increased significantly while maintaining equity for the remaining counties. We illustrate these results using data from FBCENC.

In addition to examining the stochastic food shipment problem under relaxed equity, our future work also includes considering multiple food groups simultaneously. The models in this dissertation are based on a single food group and do not consider correlations and interactions between different food groups. In our future work, we aim to formulate models that consider equity and effectiveness over multiple food groups. In these models, the nutritional requirements of the food-insecure population as well as the correlations between the capacities of multiple food groups will be considered for satisfying equitable and effective food distribution. Another possible extension of our work is to consider stochastic supply for the single- and multi-period food shipment problems. Also, in our models, we used the min-max equity measure of FBCENC. As future work, we will study stochastic formulations with different equity measures such as the range of the absolute deviations from perfect equity. Another extension that is relevant for the stochastic models in Chapters 4 and 5 is the inclusion of transshipments between counties after the capacities of the counties are

revealed at the end of each period. Our current work assumes that the minimum CD ratio, R , is a continuous random variable. A possible extension is to consider a discrete capacity distribution resulting in a discrete distribution of R ratio and study the results for that case. Furthermore, numerical methods and simulation techniques may also be explored for estimating the cumulative distribution function of the R ratio.

The results of this dissertation are applicable to any food bank that aims to achieve the equitable and effective distribution of food donations under capacity limitations. The results can also be applied to the larger scale problem of Feeding America distributing food to its food banks. Additionally, any system which aims to achieve the equitable and effective distribution of scarce resources under capacity constraints can benefit from our results. Some examples of these systems are the assignment of teachers to schools and districts for K-12 education, the location of ambulance dispatch centers within counties and the distribution of donated blood to blood banks.

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Appendices

Appendix A: Analysis of Different Equity Measures

We examine how the solutions proposed by the Food Distribution Model perform in terms of four alternative measures of inequity discussed by Marsh and Schilling (1994). First, we make the following definitions in accordance with Marsh and Schilling (1994):

$$E_j = \left| \frac{X_j}{\sum_{l=1}^n X_l} - \frac{D_j}{\sum_{l=1}^n D_l} \right| \quad (\text{A.1})$$

$$\bar{E} = \frac{\sum_{l=1}^n E_l}{n} \quad (\text{A.2})$$

As described in Section 3.3, through equations 3.2.a and 3.2.b, the inequity measure used in the Food Distribution Model is equivalent to constraining $\max_j E_j$, which is the first inequity measure discussed by Marsh and Schilling (1994), to be below a certain limit. By doing this, we enforce that this inequity measure remains below an equity deviation limit, K . We will compare our results from the Food Distribution Model to four alternative inequity measures.

The measures we will use are: 1) Variance, $\frac{\sum_{j=1}^n (E_j - \bar{E})^2}{n}$; 2) Average absolute deviation from \bar{E} , $\frac{\sum_{j=1}^n |E_j - \bar{E}|}{n}$; 3) The range, $\max_j E_j - \min_j E_j$; and 4) Maximum absolute deviation from \bar{E} , $\max_j |E_j - \bar{E}|$. We have scaled some of the measures from Marsh and Schilling (1994) to normalize the inequity measures so that they all take values between zero and one. Since these are all measures of inequity, smaller values indicate a better equity level.

In terms of the experimental design, we use the same approach as explained in Section 3.5.2. We will again use the equity deviation limits, $K = 0, 0.002$, and 0.004 since we would like to select K values corresponding to capacity and equity constrained instances.

We then use the obtained optimal solutions for each instance and calculate the inequity levels for each of the four measures considered. The average inequity levels from the 1000 instances for Beta2 distribution are summarized in Table A.1 where the values in parentheses show the corresponding standard deviations. The remaining distributions are not shown here since this distribution has the highest level of variance and skewness among the considered distributions and hence exhibits the widest variability.

Table A.1: Analysis of different inequity measures from Marsh and Schilling (1994) for Beta2 distribution.

Mean (S.D.)		<i>Equity Deviation Limit, K</i>		
		0	0.002	0.004
Inequity Measures	$\frac{\sum_{j=1}^n (E_j - \bar{E})^2}{n}$	0.00000 (0.00000)	0.00000 (0.00000)	0.00000 (0.00000)
	$\frac{\sum_{j=1}^n E_j - \bar{E} }{n}$	0.00000 (0.00000)	0.00008 (0.00009)	0.00031 (0.00019)
	$\max_j E_j - \min_j E_j$	0.00000 (0.00000)	0.00073 (0.00075)	0.00242 (0.00133)
	$\max_j E_j - \bar{E} $	0.00000 (0.00000)	0.00068 (0.00070)	0.00225 (0.00122)

The inequity measure used in the Food Distribution Model limits all E_j values to stay below a certain limit and hence, forces a certain equity level on each county. The perfect equity case, $K = 0$ requires that $E_j = 0$ for all j . Hence, all the other measures are also equal to zero indicating that our solutions are optimal for each inequity measure considered for $K = 0$. When $K > 0$, each E_j is required to be below K , and hence \bar{E} is also required to be less than

K. This causes all the measures containing the $(E_j - \bar{E})$ term to achieve low levels. The measure that behaves the worst is the range, $\max_j E_j - \min_j E_j$, but that measure is also constrained to be lower than the original measure since $\min_j E_j > 0$. The results show that the measure we use in our study is a very strong equity measure and enforces a certain level of equity at each of the locations considered. This causes the resulting policies to behave well under different commonly used equity measures.

Appendix B: Capacity-Based versus Bottleneck-Based Scenario Trees

In order to generate the scenario trees for the Multi-Stage Stochastic Program, we discretize the underlying probability distributions. In this context, if the scenario trees are built based upon the capacities of individual counties such that each county's capacity is represented by a different probability distribution, the size of the scenario tree associated with the MSM grows exponentially in the number of counties, making the models grow very fast. In order to illustrate this, assume that there are n counties in the food bank's service region and that each county's capacity, C_j follows a discrete probability distribution taking the value ϑ_i with probability π_i , $i = 1, \dots, Q_C$. This means that each county's capacity can take one of the Q_C capacity values with the given probabilities. In general, if there are n counties and each county's capacity can take Q_C values, there are Q_C^n possible outcomes in each period. If T time periods are modeled, the number of nodes in the scenario tree will be $\sum_{t=0}^T (Q_C^n)^t$ which grows very fast even for small values of T . We will call the scenario tree generated in this manner the "Capacity-Based Scenario Tree".

From Chapter 3, we know that the optimal amount of food distributed under perfect equity depends on the minimum capacity-to-demand (CD) ratio, i.e., the CD ratio of the bottleneck county. We use this property to generate our scenario trees based on the minimum capacity demand ratio, R , rather than the capacity of each county in the service region. Let us assume that the minimum capacity-to-demand ratio, R , follows a discrete empirical probability distribution such that R equals γ_i with probability ρ_i , $i = 1, \dots, Q_R$. Without loss of generality, we assume that R has the same distribution at each period t . Therefore, the

minimum capacity-to-demand ratio R can take on one of the Q_R values with given probabilities. For this situation, there are Q_R possible outcomes at each period. Furthermore, if T time periods are modeled, the number of nodes in the scenario tree will be $\sum_{t=0}^T (Q_R^n)^t$ which is independent of the number of counties n . We call the scenario tree generated in this manner the “Bottleneck-Based Scenario Tree”. Let us consider the following example to compare these two approaches.

Example. We assume that $n = 3$ and $Q_R = Q_C = Q$. Table B.1 illustrates how the sizes of the scenario tree grow for both approaches when $Q = 4$ and T is increased from one to seven. Table B.2 illustrates how the sizes of the scenario tree grow for both approaches when $T = 3$ and Q is increased from two to ten.

Table B.1: Comparison of the Scenario Tree Size for Capacity-Based Scenario Tree and Bottleneck-Based Scenario Tree for increasing T .

T	Capacity-Based Scenario Tree Size	Bottleneck-Based Scenario Tree Size
1	65	5
2	4,161	21
3	266,305	85
4	17,043,521	341
5	1,090,785,345	1,365
6	69,810,262,081	5,461
7	4,467,856,773,185	21,845

Table B.2: Comparison of the Scenario Tree Size for Capacity-Based Scenario Tree and Bottleneck-Based Scenario Tree for increasing Q .

Q	Capacity-Based Scenario Tree Size	Bottleneck-Based Scenario Tree Size
2	585	15
3	20,440	40
4	266,305	85
5	1,968,876	156
6	10,124,569	259
7	40,471,600	400
8	134,480,385	585
9	387,952,660	820
10	1,001,001,001	1,111

Tables B.1 and B.2 both show that by using the Bottleneck-Based Scenario Tree, we obtain a significant reduction in the size of the scenario trees. Since the minimum CD ratios provide sufficient information to obtain the optimal solution, we build the Multi-Stage Stochastic Program based on the Bottleneck-Based Scenario Tree.