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Incremental Mode Superposition Method Using Post Correction Techniques for Nonlinear Dynamic Analysis

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ABSTRACT

This paper describes the application of the incremental mode superposition method using post correction technique for non-linear dynamic response of structure. It is developed on the basis that non-linear terms in equations of motion for non-linear systems may considered as additional external forces, and the fact that by doing so such equations of motions can be interpreted as equation of motion of an equivalent linear system, excited by a ground motion. In the mode superposition method transformation into the modal co-ordinate serves to uncouple the set of simultaneous non-linear differential equation of motion. As a result, one non-linear differential equation representing each mode is obtained. The numerical integration is performed only on onc differential equation at a time in this method, as against the conventional integration technique wherein the numerical integration is performed on the set of simultaneous differential equations at a time. The efficiency of the proposed method is evaluated in terms of timc taken to arrivc at the solution.

INTRODUCTION

It is well known that under the current philosophy of earthquake resistant design, a structure is supposed to resist strong and severe ground motions through inelastic deformations. The capability of a structure to withstand a strong earthquake is measured by its ability to resist the elastic deformations or sustain the ductility demands imposed by such an earthquake. To assess such inelastic deformation dynamic analysis is to be performed taking into consideration the non-linear characteristics of the structures.

The conventional step-by-step integration technique used for non-linear dynamic analysis of structure requires the solution of a set of non-linear differential equations equal in number to the total number of degrees of freedom of the structure. The stiffness matrix needs to be updated at every time step, considering the state of springs during that instant. In these methods small time steps are usually needed to avoid the problem of accuracy and instability of the solution, which in most cases is costly and cumbersome and certainly unacceptable for preliminary designs. Furthermore, such a numerical procedure does not facilitate a direct visualisation of structural behaviour, or sensitivity of response of a structure to design changes and variations in the characteristics of ground motion that excites it.

Few studies have used the mode superposition procedure to solve non-linear problems in structural dynamics. [2,5,6,8,9,10] In mode superposition method transformation into modal co-ordinates serves to uncouple the set of simultaneous non-linear differential equation of motions. As a result, one non-linear differential equation representing each mode is obtained. The post correction approach is based on the assumption that the effect of the non-linear restoring force is equivalent to a series of impulse functions acting on the fundamental linear system. When the post correction approach is combined with the incremental mode superposition method, non-linear dynamic analysis can be performed on each of the single degree of freedom systems representing each mode. The non-linear dynamic analysis can be done by using any of the existing tools of numerical integration. However, the modal velocity and the modal acceleration obtained at the end of each time step have to be modified and these are to be taken as the initial conditions for the succeeding time step. The modal response thus obtained can then be superimposed at the end of time step to give the global response of the system for the time step. The numerical integration is performed only on one differential equation representing each mode at a time in this method, as against the conventional integration technique wherein the numerical integration is performed on the set of simultaneous differential equations at a time. Therefore, the execution time required using the proposed method will be lesser than the execution time required by the conventional methods. Also, for the sake of introducing the basic ideas, it is derived specifically for elastoplastic shear - beam systems with linear damping forces at all response levels. Numerical results have been included in this paper to verify the accuracy of the method.

MATHEMATICAL FORMULATION

Consider a multi degree of freedom system with mass, damping and stiffness properties [M], [C] and [K] respectively. Assuming that mass and damping characteristics remain constant throughout and the springs have the general non-linear characteristics, the equation of the system subjected to a base acceleration $\ddot{x}_g(t)$, can be written as,

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+\{F_s(t)\}= -[M]\{\rho\}\,\ddot{x}_g(t) \tag{1}$$

where, $\ddot{x}(t)$ and $\dot{x}(t)$ are the acceleration & velocity of each mass, respectively at time t.

$\{F_s(t)\}$ is the spring force at time t.

$\{\rho\}$ is influence vector whose elements are unity for unconstrained horizontal degree of freedom and zero for all other.

The ground motion can be replaced by effective earthquake force,

$$\{P(t)\}= -[M]\{\rho\}\,\ddot{x}_g(t) \tag{2}$$

Therefore Eq.(1) can be written as,

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[F_s(t)]=\{P(t)\} \tag{3}$$

Generally, the spring force, $\{F_s(t)\}$can be expressed by summation of linear and non-linear terms as,

$$\{F_s(t)\}=[K]\{x(t)\}-\{Q(t)\} \tag{4}$$

where, $\{x(t)\}$ is the displacement of each mass.

$\{Q(t)\}$ is the non-linear spring force.

Thus, substituting Eq.(4) into Eq.(3), we get,

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}=\{P(t)\}+\{Q(t)\} \tag{5}$$

Writing equation (5) in the incremental form, we get,

$$[M]\{\Delta\ddot{x}(t)\}+[C]\{\Delta\dot{x}(t)\}+[K]\{\Delta x(t)\}=\{\Delta P(t)\}+\{\Delta Q(t)\} \tag{6}$$

The frequencies $[\omega]$ and modal shape $[\Phi]$ are obtained from the solution of undamped eigen values problem around with Eq. (7)

$$(\,[K]-\omega^2[M]\,)\{\hat{x}(t)\}=0 \tag{7}$$

where, $\hat{x}(t)$ is an eigen vector.

Introducing the normal mode co-ordinate expression,

$$\{x\}=[\Phi]\{X\} \tag{8}$$

where, $\{x\}$ is global displacement,

$[\Phi]$ is mode shape vector,

$\{X\}$ is modal displacement.

Therefore, Eq.(8) in incremental formulation can be written as,

$$\{\Delta x\}=[\Phi]\{\Delta X\} \tag{9}$$

where, $\{\Delta x\}$ is increment in global displacement,

$[\Phi]$ is mode shape vector,

$\{\Delta X\}$ is increment in modal displacement.

Therefore Eq. (6) can be written as,

$$[M]\{\Delta\ddot{X}(t)\}[\Phi]+[C]\{\Delta\dot{X}(t)\}[\Phi]+[K]\{\Delta X(t)\}[\Phi]=\{\Delta P(t)+\{\Delta Q(t)\} \tag{10}$$

Premultiplying Eq.(10) by the transpose of the n^{th} mode shape vector $\{\Phi_n\}^T$ leads to,

$$\begin{aligned}\{\Phi_n\}^T[M][\Phi]\{\Delta\ddot{X}(t)\}+\{\Phi_n\}^T[C][\Phi]\{\Delta\dot{X}(t)\}+\{\Phi_n\}^T[K][\Phi]\{\Delta X(t)\}\\=\{\Phi_n\}^T(\{\Delta P(t)+\{\Delta Q(t)\})\end{aligned} \tag{11}$$

Using orthogonality condition of system mode shape, we get,

$$M_n\Delta\ddot{X}(t)+C_n\Delta\dot{X}(t)+K_n\Delta X(t)=\Delta P_n(t)+\Delta Q_n(t) \tag{12}$$

in which,

$$M_n=\{\Phi_n\}^T[M]\{\Phi_n\} \tag{13a}$$

$$C_n=\{\Phi_n\}^T[C]\{\Phi_n\}=2\xi_n\omega_n M_n \tag{13b}$$

$$K_n=\{\Phi_n\}^T[K]\{\Phi_n\}=\omega_n^2 M_n \tag{13c}$$

$$P_n=\{\Phi_n\}^T[P(t)] \tag{13d}$$

$$Q_n=\{\Phi_n\}^T[Q(t)] \tag{13e}$$

are modal mass, modal damping, modal stiffness, modal load, modal non-linear spring force respectively.

Alternatively,

$$\Delta\ddot{X}_n(t) + 2\xi_n\omega_n\Delta\dot{X}_n(t) + \omega_n^2\Delta X_n(t) = \frac{\Delta P_n(t) + \Delta Q_n(t)}{M_n} \tag{14}$$

Eq. (14) can be solved by using numerical integration technique to obtain the incremental modal responses. The incremental global responses are then computed as,

$$\Delta x_i(t) = \sum_{n=1}^{N} \Phi_{in}\ \Delta X_n(t) \tag{15}$$

$$\Delta \dot{x}_i(t) = \sum_{n=1}^{N} \Phi_{in}\ \Delta \dot{X}_n(t) \tag{16}$$

The incremental global responses are then added to previous global responses to obtain the displacement and velocities at the end of each time step, as

$$x_i(t_i + \Delta t) = x_i(t) + \Delta x_i(t) \tag{17}$$

$$\dot{x}_i(t_i + \Delta t) = \dot{x}_i(t) + \Delta \dot{x}_i(t) \tag{18}$$

Using the post correction technique the non-linear modal force increment $\{\Delta Q_n(t)\}$is replaced by the post modification of modal velocity and modal acceleration. In this approach the non-linear force increment $\{\Delta Q_n(t)\}$ is expressed by a series of generalized impulses using the impulse momentum principle.

GENERALIZED IMPULSE FUNCTION & POST CORRECTION APPROACH

Consider the fundamental linear SDOF system representing one mode of MDOF system with non-linear modal spring force $Q_n(t)$ as shown in Fig. (1).

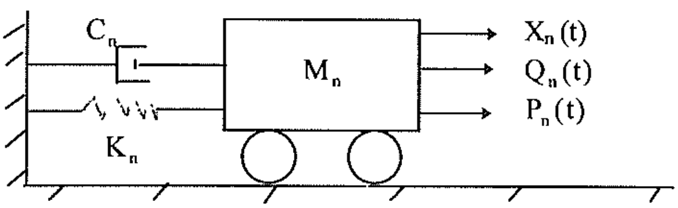

Figure:1 Fundamental linear system and modal spring force.

Assuming that at the beginning of time step 't_i', the initial modal acceleration $\ddot{X}(t_i)$ and modal velocity $\dot{X}(t_i)$ and modal displacement $X(t_i)$ are known, the modal velocity and the modal displacement at the end of time increment, 't_{i+1}' can be assumed as follows.

$$\dot{X}(t_{i+1}) = \dot{X}(t_i) + \Delta\dot{X} + \Delta\dot{Y} \tag{19}$$

$$X(t_{i+1}) = X(t_i) + \Delta X + \Delta Y \tag{20}$$

Where the incremental modal velocity $\Delta\dot{X}$ and incremental modal displacement ΔX are caused by initial modal acceleration, modal velocity and modal displacement at time 't_i' and the modal load increment $\Delta P_n(t)$ at the end of time step. While $\Delta\dot{Y}$ and ΔY are the incremental modal velocity and modal displacement produced by the increment in non-linear part of modal spring force $\Delta Q_n(t)$.

The computation of $\Delta\dot{Y}$ and ΔY is based on the two basic assumptions.

a) The modal non-linear spring force $Q_n(t)$ varies linearly during each time interval Δt.

b) The incremental modal displacement produced by this modal non-linear spring load, $\Delta Q_n(t)$ is negligible for short duration impulsive loading.

For a short time increment, the increment in modal non-linear spring load $\Delta Q_n(t)$, can be considered as short duration impulsive load. Based on the principle of impulse-momentum , change in modal velocity $\Delta\dot{X}$ can be obtained as :

$$\int_{t_i}^{t_{i+1}} M_n d\dot{y} = \int_{t_i}^{t_{i+1}} [Q_n(t) - Q_n(t_i)]dt \tag{21}$$

From the first assumption $\Delta Q_n(t)$ varies linearly in the time interval Δt, Therefore, Eq.(21) can be written as,

$$M_n \Delta\dot{Y} = \frac{\Delta Q_n(t)}{2} \Delta t \tag{22}$$

$$\Delta\dot{Y} = \frac{\Delta t}{2} \frac{\Delta Q_n(t)}{M_n} \tag{23}$$

Thus, it can seen from Eq.(23) that for very small values of Δt, the modal velocity change $\Delta\dot{Y}$ is of the order of Δt, while the modal displacement change, ΔY, developed from short duration impulsive loading is of the order of $(\Delta t)^2$,which is negligibly small.

Thus, the modal displacement at the end of time increment can be expressed approximately by,

$$X(t_{i+1}) = X(t_i) + \Delta X \tag{24}$$

The same concept can be extended for multi degree of freedom system and Eq.(23) and Eq.(24) for modal velocity and modal displacement can be written as,

$$\{\dot{X}(t_{i+1})\} = \{\dot{X}(t_i)\} + \{\Delta\dot{X}\} + \{\Delta\dot{Y}\} \tag{25}$$

$$\{X(t_{i+1})\} = \{X(t_i)\} + \{\Delta X\} \tag{26}$$

RESULTS

The incremental mode superposition method using post correction technique and Post correction integration technique[12] was used to compute the non-linear response of several shear type plane frames (see Appendix I). For incremental mode superposition method using post correction technique a uniform damping ratio for each mode in Eq. (13) was used as proposed by Chopra. A. K[4]. Each frame was subjected to S69E component of Taft and N-S component of Elcentro earthquake[3]. The Wilson θ method[1] was used as a tool of numerical integration. The numerical computation were carried out on IBM 386 machine.

The results of the analysis performed by the this method were compared with the results obtained by using Post correction integration technique proposed by Keming Sun, Jose. A. Pires and J. R. Tao[12]. It has been observed that the results of both the techniques match with each other with a negligible error as can be seen from Fig. 2 and Fig 3. To investigate the efficiency of the incremental mode superposition method using post correction technique as compared to post correction integration technique a summary of CPU time for both the methods for a 2-, 3-, 5- and 8- storey shear type plane frame is given in Table I. The Table shows that the CPU time for the incremental mode superposition method using post correction technique is shorter than that for post correction integration technique.

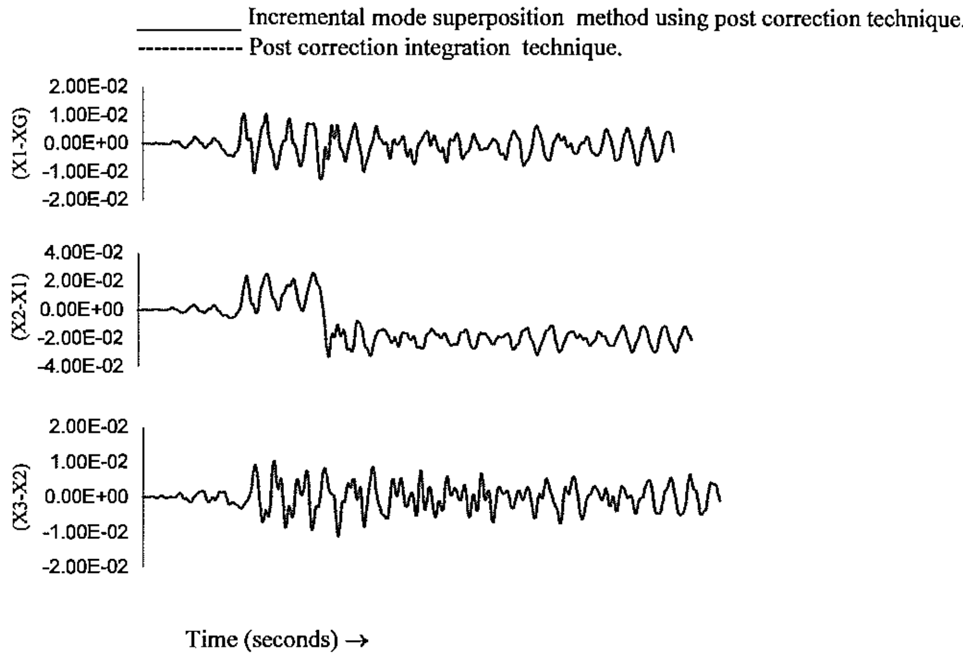

Figure 2: Comparison of relative displacement (meters) time history for three storey shear type plane frame with bilinear elastoplastic column properties and 5% damping subjected to S 69 E component of Taft earthquake.

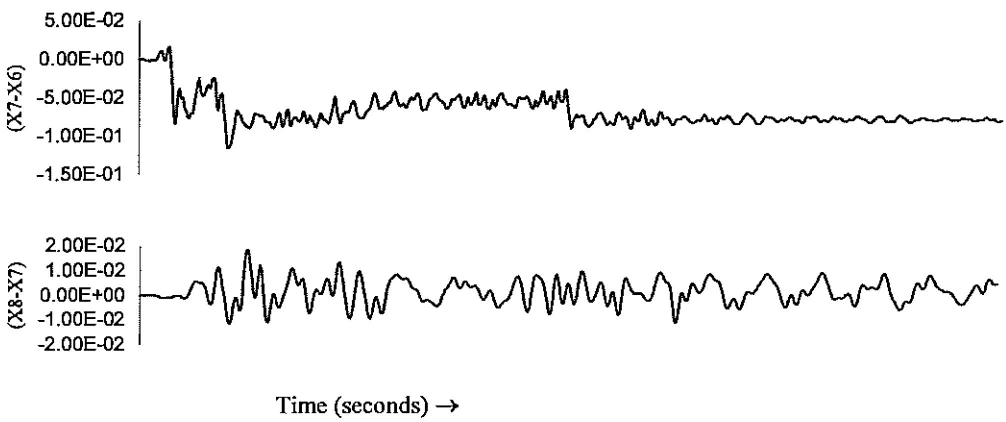

Figure 2 : Comparison of relative displacement (meters) time history for top two stories of eight storey shear type plane frame with bilinear elastoplastic column properties and 5% damping subjected to N-S component of Elcentro earthquake.

Table : I COMPARISON OF CPU TIME FOR INCREMENTAL MODE SUPERPOSITION METHOD USING POST CORRECTION TECHNIQUE AND POST CORRECTION INTEGRATION TECHNIQUE.

Sr. No	Earthquake	Degrees of freedom	CPU Time in seconds for (a)	(b)
1	TAFT S69E	2	7.17	7.45
		3	8.82	9.86
		5	14.64	15.14
		8	24.75	26.18
2	ELCENTRO N- S	2	14.92	15.90
		3	23.43	24.64
		5	41.88	46.00
		8	62.43	69.68

(a) Incremental mode superposition method using post correction technique

(b) Post correction integration technique

CONCLUSION

Analysis of shear type plane frame with elastoplastic bilinear stiffness properties indicates that incremental mode superposition method using post correction technique requires less computational time than Post correction integration technique. When all modes are considered the method gives results identical to those from direct integration. Fewer modes may be used to compute the response with reasonable accuracy by carrying out the modal truncation for each time increment. Thus, the method herein proposed offers a convenient alternative over a direct integration method for the time analysis of large non-linear structures under dynamic loads.

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APPENDIX I

TABLE : II Properties of 3-, 5- and 8 storey shear type plane frames used in numerical examples.

1.Three storey.

Level	Mass (Kg)	Stiffness (N/m)	Yield deformation (m)
1 (Bottom)	25000	10500000.00	0.012
2	25000	6076388.890	0.012
3 (Top)	25000	6076388.890	0.012

2. Five storey.

Level	Mass (Kg)	Stiffness (N/m)	Yield deformation (m)
1 (Bottom)	25000	24888888.90	0.012
2	25000	24888888.90	0.012
3	25000	10500000.00	0.012
4	25000	10500000.00	0.012
5 (Top)	25000	6076388.890	0.012

3.Eight storey.

Level	Mass (Kg)	Stiffness (N/m)	Yield deformation (m)
1 (Bottom)	25000	48611111.10	0.012
2	25000	24888888.90	0.012
3	25000	24888888.90	0.012
4	25000	10500000.00	0.012
5	25000	10500000.00	0.012
6	25000	10500000.00	0.012
7	25000	6076388.890	0.012
8 (Top)	25000	6076388.890	0.012

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