

**An Improved Iterative Method for Solving a Class of
Coupled Conductive–Radiative Heat-Transfer Problems**

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Abstract

Newton's method of iteration is used along with the the P_N method, also called the spherical-harmonics method, and Hermite cubic splines to develop an iterative solution to a class of nonlinear problems in radiative transfer. Anisotropic scattering and specularly and diffusely reflecting boundaries are allowed for the steady-state, combined-mode, conductive–radiative, heat-transfer problem considered.

1. Introduction

In a recent paper, Siewert and Thomas¹ used a computationally stable version of the P_N method² and Hermite cubic splines to solve a steady-state problem in combined mode, conduction and radiation, heat transfer that was formulated by Özişik.³ Although in Ref. 1, some good numerical results were obtained from a method based on a simple iterative procedure, the authors mentioned that there were cases where that iteration scheme failed (in a very spectacular manner). Here we follow some recent work of Kelley⁴ and use Newton's method to define our method of iteration. Since this work is, except for the method of iteration, very similar to Ref. 1, we assume that Ref. 1 is available, and so our presentation here is brief.

We consider the equation of transfer written³ as

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^L \beta_l P_l(\mu) \int_{-1}^1 P_l(\mu') I(\tau, \mu') d\mu' + (1 - \varpi) \frac{\sigma n^2}{\pi} T^4(\tau), \quad (1)$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$I(0, \mu) = \epsilon_1 \frac{\sigma n^2}{\pi} T_1^4 + \rho_1^s I(0, -\mu) + 2\rho_1^d \int_0^1 I(0, -\mu') \mu' d\mu' \quad (2a)$$

and

$$I(\tau_0, -\mu) = \epsilon_2 \frac{\sigma n^2}{\pi} T_2^4 + \rho_2^s I(\tau_0, \mu) + 2\rho_2^d \int_0^1 I(\tau_0, \mu') \mu' d\mu' \quad (2b)$$

for $\mu \in [0, 1]$. Here $\tau \in [0, \tau_0]$ is the optical variable, μ is the direction cosine measured from the positive τ axis and ϖ is the albedo for single scattering. In addition, we have assumed that the scattering law $p(\Theta)$ can be represented by a finite Legendre expansion in terms of the cosine of the scattering angle Θ , *i.e.*

$$p(\Theta) = \sum_{l=0}^L \beta_l P_l(\cos \Theta) \quad (3)$$

where $\beta_0 = 1$ and $|\beta_l| < 2l + 1$ for $l \geq 1$. In regard to the boundary conditions, we note that ρ_α^s and ρ_α^d , for $\alpha = 1$ and 2, are coefficients for specular and diffuse reflection and that $\epsilon_\alpha = 1 - \rho_\alpha^s - \rho_\alpha^d$, $\alpha = 1$ and 2, are the emissivities for the two surfaces. In addition n is the index of refraction and σ is the Stefan-Boltzmann constant.

The nonlinear aspect of this problem comes from the fact that the temperature distribution $T(\tau)$ appearing in Eq. (1) must satisfy the heat conduction equation³

$$k\beta \frac{d^2}{d\tau^2} T(\tau) = \frac{d}{d\tau} q_r(\tau) \quad (4)$$

subject to the boundary conditions $T(0) = T_1$ and $T(\tau_0) = T_2$, where T_1 and T_2 are the temperatures that also appear in Eqs. (2). In addition, k is the thermal conductivity of the medium, β is the extinction

coefficient and $q_r(\tau)$ is the radiative heat flux, *i.e.*

$$q_r(\tau) = 2\pi \int_{-1}^1 I(\tau, \mu') \mu' d\mu'. \quad (5)$$

2. Basic Development

To follow a tradition in the heat transfer literature,³ we normalize the considered problem by introducing a convenient reference temperature T_r and by using

$$I(\tau, \mu) = \left(\frac{\sigma n^2}{\pi} T_r^4 \right) I^*(\tau, \mu), \quad (6)$$

$$q_r(\tau) = \left(\frac{\sigma n^2}{\pi} T_r^4 \right) q_r^*(\tau) \quad (7)$$

and

$$T(\tau) = T_r \Theta(\tau) \quad (8)$$

to rewrite our problem as

$$\mu \frac{\partial}{\partial \tau} I^*(\tau, \mu) + I^*(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^L \beta_l P_l(\mu) \int_{-1}^1 P_l(\mu') I^*(\tau, \mu') d\mu' + (1 - \varpi) \Theta^4(\tau), \quad (9)$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$,

$$I^*(0, \mu) = \epsilon_1 \Theta_1^4 + \rho_1^s I^*(0, -\mu) + 2\rho_1^d \int_0^1 I^*(0, -\mu') \mu' d\mu' \quad (10a)$$

and

$$I^*(\tau_0, -\mu) = \epsilon_2 \Theta_2^4 + \rho_2^s I^*(\tau_0, \mu) + 2\rho_2^d \int_0^1 I^*(\tau_0, \mu') \mu' d\mu' \quad (10b)$$

for $\mu \in [0, 1]$. In addition

$$\frac{d^2}{d\tau^2} \Theta(\tau) = \frac{1}{4\pi N_c} \frac{d}{d\tau} q_r^*(\tau), \quad (11)$$

with

$$\Theta(0) = \Theta_1 = \frac{T_1}{T_r}, \quad (12a)$$

$$\Theta(\tau_0) = \Theta_2 = \frac{T_2}{T_r} \quad (12b)$$

and

$$q_r^*(\tau) = 2\pi \int_{-1}^1 I^*(\tau, \mu') \mu' d\mu'. \quad (13)$$

Here

$$N_c = \frac{k\beta}{4\sigma n^2 T_r^3} \quad (14)$$

is called the conduction-to-radiation parameter.³

At this point, we choose to express the inhomogeneous source term

$$S(\tau) = (1 - \varpi)\Theta^4(\tau) \quad (15)$$

as

$$S(\tau) = \sum_{k=0}^K a_k \phi_k(\tau/\tau_0) \quad (16)$$

where we consider K to be odd and that the spline functions $\{\phi_k(x)\}$ are as used in Ref. 1. Here, the coefficients $\{a_k\}$ are to be determined. It follows that we can write

$$I^*(\tau, \mu) = I_h^*(\tau, \mu) + \sum_{k=0}^K a_k I_k^*(\tau, \mu) \quad (17)$$

where the basis problems, for $k = 0, 1, 2, \dots, K$, are defined by

$$\mu \frac{\partial}{\partial \tau} I_k^*(\tau, \mu) + I_k^*(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^L \beta_l P_l(\mu) \int_{-1}^1 P_l(\mu') I_k^*(\tau, \mu') d\mu' + \phi_k(\tau/\tau_0), \quad (18)$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$,

$$I_k^*(0, \mu) = \rho_1^s I_k^*(0, -\mu) + 2\rho_1^d \int_0^1 I_k^*(0, -\mu') \mu' d\mu' \quad (19a)$$

and

$$I_k^*(\tau_0, -\mu) = \rho_2^s I_k^*(\tau_0, \mu) + 2\rho_2^d \int_0^1 I_k^*(\tau_0, \mu') \mu' d\mu' \quad (19b)$$

for $\mu \in [0, 1]$. In addition, the homogeneous problem is defined by

$$\mu \frac{\partial}{\partial \tau} I_h^*(\tau, \mu) + I_h^*(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^L \beta_l P_l(\mu) \int_{-1}^1 P_l(\mu') I_h^*(\tau, \mu') d\mu' \quad (20)$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$,

$$I_h^*(0, \mu) = \epsilon_1 \Theta_1^4 \quad (21a)$$

and

$$I_h^*(\tau_0, -\mu) = \epsilon_2 \Theta_2^4. \quad (21b)$$

We can solve Eq. (11) to find

$$\Theta(\tau) = \Theta_h(\tau) + \sum_{k=0}^K a_k \Theta_k(\tau) \quad (22)$$

where

$$\Theta_h(\tau) = \Theta_1 + \frac{\tau}{\tau_0}(\Theta_2 - \Theta_1) + \frac{1}{4\pi N_c} \left(\int_0^\tau q_{r,h}^*(x) dx - \frac{\tau}{\tau_0} \int_0^{\tau_0} q_{r,h}^*(x) dx \right) \quad (23a)$$

and, for $k = 0, 1, 2, \dots, K$,

$$\Theta_k(\tau) = \frac{1}{4\pi N_c} \left(\int_0^\tau q_{r,k}^*(x) dx - \frac{\tau}{\tau_0} \int_0^{\tau_0} q_{r,k}^*(x) dx \right) \quad (23b)$$

with

$$q_{r,h}^*(\tau) = 2\pi \int_{-1}^1 I_h^*(\tau, \mu) \mu d\mu \quad (24a)$$

and, for $k = 0, 1, 2, \dots, K$,

$$q_{r,k}^*(\tau) = 2\pi \int_{-1}^1 I_k^*(\tau, \mu) \mu d\mu. \quad (24b)$$

As we can use, for example, the spherical harmonics method (see Ref. 1) to establish the temperature fuctions $\Theta_h(\tau)$ and $\Theta_k(\tau)$, for $k = 0, 1, 2, \dots, K$, we have only to compute the unknown coefficients $\{a_k\}$ to complete our solution. It is clear, from Eqs. (15), (16) and (22), that we can write

$$\sum_{k=0}^K a_k \phi_k(\tau/\tau_0) = (1 - \varpi) \left[\Theta_h(\tau) + \sum_{k=0}^K a_k \Theta_k(\tau) \right]^4. \quad (25)$$

We let $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_M$, where $M = (K - 1)/2$, denote the knots used to define our spline functions and evaluate Eq. (25) and the derivative of Eq. (25) at each of the points $\tau = \zeta_\alpha \tau_0$, for $\alpha = 0, 1, 2, \dots, M$, to obtain the system of nonlinear algebraic equations

$$\sum_{k=0}^K a_k \phi_k(\zeta_\alpha) = (1 - \varpi) \left[\Theta_h(\zeta_\alpha \tau_0) + \sum_{k=0}^K a_k \Theta_k(\zeta_\alpha \tau_0) \right]^4 \quad (26a)$$

and

$$\begin{aligned} \sum_{k=0}^K a_k \frac{d}{d\tau} \phi_k(\tau/\tau_0) \Big|_{\tau=\zeta_\alpha \tau_0} = \\ 4(1 - \varpi) \left[\Theta_h(\zeta_\alpha \tau_0) + \sum_{k=0}^K a_k \Theta_k(\zeta_\alpha \tau_0) \right]^3 \frac{d}{d\tau} \left[\Theta_h(\tau) + \sum_{k=0}^K a_k \Theta_k(\tau) \right] \Big|_{\tau=\zeta_\alpha \tau_0} \end{aligned} \quad (26b)$$

which we consider to define to required constants $\{a_k\}$. If we let the components of a $(K + 1)$ -vector $\mathbf{F}(\mathbf{a})$, where the vector \mathbf{a} has components a_0, a_1, \dots, a_K , be defined, for $\alpha = 0, 1, \dots, M$, by

$$F_\alpha(\mathbf{a}) = \sum_{k=0}^K a_k \phi_k(\zeta_\alpha) - (1 - \varpi) \left[\Theta_h(\zeta_\alpha \tau_0) + \sum_{k=0}^K a_k \Theta_k(\zeta_\alpha \tau_0) \right]^4 \quad (27a)$$

and

$$\begin{aligned} F_{M+1+\alpha}(\mathbf{a}) = \sum_{k=0}^K a_k \frac{d}{d\tau} \phi_k(\tau/\tau_0) \Big|_{\tau=\zeta_\alpha \tau_0} \\ - 4(1 - \varpi) \left[\Theta_h(\zeta_\alpha \tau_0) + \sum_{k=0}^K a_k \Theta_k(\zeta_\alpha \tau_0) \right]^3 \frac{d}{d\tau} \left[\Theta_h(\tau) + \sum_{k=0}^K a_k \Theta_k(\tau) \right] \Big|_{\tau=\zeta_\alpha \tau_0} \end{aligned} \quad (27b)$$

then we can express the Newton iteration scheme as

$$\mathbf{a}_{j+1} = \mathbf{a}_j + \mathbf{J}^{-1}(\mathbf{a}_j)\mathbf{F}(\mathbf{a}_j) \quad (28)$$

for $j = 0, 1, \dots$. Here the Jacobian matrix is given by

$$\mathbf{J}(\mathbf{a}) = \left(\frac{\partial}{\partial a_0}\mathbf{F}(\mathbf{a}), \frac{\partial}{\partial a_1}\mathbf{F}(\mathbf{a}), \dots, \frac{\partial}{\partial a_K}\mathbf{F}(\mathbf{a}) \right). \quad (29)$$

For our initial approximation we take \mathbf{a}_0 as obtained by using a linear temperature distribution to define a first approximation of the source term of $S(\tau)$.

Assuming that we have found a converged result from Eq. (28), we can use Eqs. (15) and (16) to find the required normalized temperature distribution, viz.

$$\Theta(\tau) = \left[\left(\frac{1}{1-\varpi} \right) \sum_{k=0}^K a_k \phi_k(\tau/\tau_0) \right]^{1/4}. \quad (30)$$

We can then differentiate Eq. (30) and compute the normalized conductive heat flux from

$$Q_c(\tau) = -\frac{d}{d\tau}\Theta(\tau) \quad (31)$$

The normalized radiative heat flux can be evaluated from

$$Q_r(\tau) = \frac{1}{2N_c} \left[q_{r,h}^*(\tau) + \sum_{k=0}^K a_k q_{r,k}^*(\tau) \right], \quad (32)$$

and finally the normalized total heat flux is given by

$$Q(\tau) = Q_c(\tau) + Q_r(\tau). \quad (33)$$

3. Numerical Results

To have a specific scattering law for testing our solution technique, and to avoid having to provide a table of the scattering law coefficients $\{\beta_l\}$, we use here the binomial scattering law⁵

$$p(\cos \Theta) = \frac{L+1}{2^L} (1 + \cos \Theta)^L \quad (34)$$

which can be represented exactly with $L+1$ Legendre coefficients that can be computed with $\beta_0 = 1$ and⁶

$$\beta_l = \left(\frac{2l+1}{2l-1} \right) \left(\frac{L+1-l}{L+1+l} \right) \beta_{l-1}. \quad (35)$$

To make available some numerical results that have been obtained with the Newton method of iteration discussed here, we consider the four test problems defined in Table 1. As we wish to solve problems here that we were unable to solve with the iteration procedure used in Ref. 1, we elect to use here $N_c = 0.00001$ and to keep all of the other defining parameters the same as for the related problems solved in Ref. 1 for the case $N_c = 0.05$.

Our converged results for the normalized temperature distribution and the normalized heat fluxes are given in Tables 2–5. Having varied the order of the P_N approximation, the number of Hermite splines used and the number of Gauss points used to evaluate some integrals over the spline functions, we have some confidence that the results given in Tables 2–5 are correct to within one unit in the last digit given.

To conclude this work we would like to record a few remarks. In regard to the existence and uniqueness issues mentioned in Ref. 1, we note that Kelley⁴ has provided existence and uniqueness theorems that apply to our problem for the case of isotropic scattering and non-reflecting boundaries; we note also that work aimed at extending those theorems is continuing.⁴ Also, as we have no proof that the Newton iteration scheme we use actually converges, we can only conjecture that the results given in Tables 2–7 are correct. Finally, we must conclude that Newton’s method of iteration is, in general, vastly superior to the simple iteration scheme used in Ref. 1, and so it is clear that Newton’s method can also be used to extend the class of problems solved for spheres and cylinders in Refs. 7 and 8.

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References

1. **C. E. Siewert and J. R. Thomas, Jr.**, *JQSRT* **45**, 273 (1991).
2. **M. Benassi, R. D. M. Garcia, A. H. Karp and C. E. Siewert**, *Astrophys. J.* **280**, 853 (1984).
3. **M. N. Özışık**, *Radiative Transfer and Interaction with Conduction and Convection*, John Wiley & Sons, New York (1973).
4. **C. T. Kelley**, *private communication and preprint* (1994).
5. **H. G. Kaper, J. K. Shultis and J. G. Veninga**, *J. Comp. Phys.* **6**, 288 (1970).
6. **N. J. McCormick and R. Sanchez**, *J. Math. Phys.* **22**, 199 (1981).
7. **C. E. Siewert and J. R. Thomas, Jr.**, *JQSRT* **46**, 63 (1991).
8. **C. E. Siewert and J. R. Thomas, Jr.**, *JQSRT* **48**, 227 (1991).

Table 1. Physical Data for Different Problems

Problem	ϵ_1	ϵ_2	ρ_1^s	ρ_2^s	ρ_1^d	ρ_2^d	Θ_1	Θ_2	ϖ	τ_0	N_c	L
1	1.0	1.0	0.0	0.0	0.0	0.0	1.0	0.5	0.9	1.0	0.00001	0
2	0.7	0.6	0.1	0.3	0.2	0.1	1.0	0.5	0.9	3.0	0.00001	0
3	1.0	1.0	0.0	0.0	0.0	0.0	1.0	0.5	0.95	1.0	0.00001	299
4	0.8	0.8	0.1	0.1	0.1	0.1	1.0	0.5	0.99	3.0	0.00001	299

centerlineTable 2. Normalized Temperature Distribution and Heat Fluxes for Problem 1

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	1.192(1)	1.29680(4)	1.29799(4)
0.01	9.45293(-1)	2.024	1.29779(4)	1.29799(4)
0.05	9.27635(-1)	1.787(-1)	1.29797(4)	1.29799(4)
0.10	9.19142(-1)	1.64429(-1)	1.29797(4)	1.29799(4)
0.20	9.03101(-1)	1.58610(-1)	1.29797(4)	1.29799(4)
0.30	8.87169(-1)	1.60829(-1)	1.29797(4)	1.29799(4)
0.40	8.70808(-1)	1.66932(-1)	1.29797(4)	1.29799(4)
0.50	8.53678(-1)	1.76206(-1)	1.29797(4)	1.29799(4)
0.60	8.35449(-1)	1.89059(-1)	1.29797(4)	1.29799(4)
0.70	8.15700(-1)	2.06973(-1)	1.29797(4)	1.29799(4)
0.80	7.93766(-1)	2.33741(-1)	1.29797(4)	1.29799(4)
0.90	7.68273(-1)	2.82207(-1)	1.29796(4)	1.29799(4)
0.95	7.52599(-1)	3.916(-1)	1.29795(4)	1.29799(4)
0.99	6.64184(-1)	9.358	1.29706(4)	1.29799(4)
1.00	5.0(-1)	2.559(1)	1.29543(4)	1.29799(4)

Table 3. Normalized Temperature Distribution and Heat Fluxes for Problem 2

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	9.36	5.29641(3)	5.30577(3)
0.01	9.48894(-1)	1.10(-1)	5.30566(3)	5.30577(3)
0.05	9.40977(-1)	5.91125(-2)	5.30571(3)	5.30577(3)
0.10	9.32452(-1)	5.53844(-2)	5.30571(3)	5.30577(3)
0.20	9.16047(-1)	5.46797(-2)	5.30571(3)	5.30577(3)
0.30	8.99417(-1)	5.64179(-2)	5.30571(3)	5.30577(3)
0.40	8.82090(-1)	5.92490(-2)	5.30571(3)	5.30577(3)
0.50	8.63780(-1)	6.29676(-2)	5.30571(3)	5.30577(3)
0.60	8.44207(-1)	6.77151(-2)	5.30570(3)	5.30577(3)
0.70	8.23002(-1)	7.39731(-2)	5.30569(3)	5.30577(3)
0.80	7.99552(-1)	8.30248(-2)	5.30568(3)	5.30577(3)
0.90	7.72487(-1)	9.96011(-2)	5.30567(3)	5.30577(3)
0.95	7.56331(-1)	1.18386(-1)	5.30565(3)	5.30577(3)
0.99	7.337(-1)	9.54(-1)	5.30481(3)	5.30577(3)
1.00	5.0(-1)	2.54(1)	5.28040(3)	5.30577(3)

Table 4. Normalized Temperature Distribution and Heat Fluxes for Problem 3

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	1.527(1)	2.20768(4)	2.20921(4)
0.01	9.13550(-1)	4.405	2.20877(4)	2.20921(4)
0.05	8.74381(-1)	1.059(-1)	2.20920(4)	2.20921(4)
0.10	8.71112(-1)	5.34518(-2)	2.20920(4)	2.20921(4)
0.20	8.66268(-1)	4.51222(-2)	2.20920(4)	2.20921(4)
0.30	8.61925(-1)	4.21786(-2)	2.20920(4)	2.20921(4)
0.40	8.57770(-1)	4.11385(-2)	2.20920(4)	2.20921(4)
0.50	8.53659(-1)	4.12457(-2)	2.20920(4)	2.20921(4)
0.60	8.49488(-1)	4.23702(-2)	2.20920(4)	2.20921(4)
0.70	8.45143(-1)	4.47814(-2)	2.20920(4)	2.20921(4)
0.80	8.40457(-1)	4.94963(-2)	2.20920(4)	2.20921(4)
0.90	8.35032(-1)	6.17824(-2)	2.20920(4)	2.20921(4)
0.95	8.29743(-1)	2.826(-1)	2.20918(4)	2.20921(4)
0.99	6.99929(-1)	1.275(1)	2.20793(4)	2.20921(4)
1.00	5.0(-1)	2.872(1)	2.20633(4)	2.20921(4)

Table 5. Normalized Temperature Distribution and Heat Fluxes for Problem 4

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	6.993	1.49489(4)	1.49559(4)
0.01	8.98702(-1)	1.354	1.49546(4)	1.49559(4)
0.05	8.70402(-1)	2.298(-2)	1.49559(4)	1.49559(4)
0.10	8.67735(-1)	1.54505(-2)	1.49559(4)	1.49559(4)
0.20	8.63661(-1)	1.23167(-2)	1.49559(4)	1.49559(4)
0.30	8.60163(-1)	1.11720(-2)	1.49559(4)	1.49559(4)
0.40	8.56891(-1)	1.07183(-2)	1.49559(4)	1.49559(4)
0.50	8.53692(-1)	1.06676(-2)	1.49559(4)	1.49559(4)
0.60	8.50456(-1)	1.09659(-2)	1.49559(4)	1.49559(4)
0.70	8.47069(-1)	1.17040(-2)	1.49559(4)	1.49559(4)
0.80	8.43356(-1)	1.32407(-2)	1.49559(4)	1.49559(4)
0.90	8.38908(-1)	1.71475(-2)	1.49559(4)	1.49559(4)
0.95	8.35678(-1)	3.741(-2)	1.49559(4)	1.49559(4)
0.99	7.40963(-1)	4.198	1.49517(4)	1.49559(4)
1.00	5.0(-1)	1.308(1)	1.49429(4)	1.49559(4)