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ASSOCIATION-BALANCED ARRAYS WITH
APPLICATIONS TO EXPERIMENTAL DESIGN
(DISSERTATION)

by

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A Dissertation submitted to the faculty of The University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Statistics.

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ABSTRACT

This dissertation considers block designs for the comparison of v treatments where measurements from different blocks are uncorrelated and measurements in the same block have an arbitrary positive definite covariance matrix V, which is the same for all the blocks.

Martin and Eccleston (1991) show that, for any V, a semi-balanced array of strength two, defined in Rao (1961, 1973), is universally optimal for the generalized least squares estimate of treatment effects over binary block designs, and weakly universally optimal for the ordinary least squares estimate over balanced incomplete block designs. The existence of these arrays requires a large number of columns (or blocks). The purpose of this dissertation is to introduce new series of arrays relaxing this constraint and to discuss their performance as block designs.

Based on the concept of association scheme, an s-associate class association-balanced array (or simply ABA) is defined, and some constructions are given. For any V, the variance matrix of the generalized (or the ordinary) least squares estimate of treatment effects for an ABA is shown to be a constant multiple of that under the usual uncorrelated model, and a combinatorial characterization of the latter condition is given.

The performance of ABA's is discussed first in terms of universal optimality, second in terms of the variance balance of elementary treatment contrasts, and third in comparison to randomized block designs. Some two associate class ABA's are shown to be type-1 optimal, in the sense of Cheng (1978), over a subclass of binary block designs.

When residual effects of the treatments are also modeled along with direct effects and the correlation structure, a special class of ABA's is shown to provide partial variance balance of elementary treatment contrasts for residual effects as well as direct effects and for any V.

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CHAPTER I

INTRODUCTION

In the classical theory of experimental design introduced by Fisher (1925), observation units are arranged in b blocks each of size k, a randomization procedure is performed to determine the allocation of v treatments to experimental units within each block, and the analysis is carried out based on the linear model: $Y = X\theta + \epsilon$, $\mathrm{Var}(\epsilon) = \sigma^2 I$.

In field experiments, blocks usually correspond to plots of land arranged side by side; in biological or industrial experiments, they correspond to a sequence of observations made at successive periods of time on the same material (repeated measurement designs). Hence, time or space introduce a neighbourhood system among units. To take this consideration into account, some authors assume a covariance structure on neighbouring units, others assume the presence of fixed residual effects in addition to direct effects of the treatments. These models have been shown to lead to a more precise analysis than in Fisher's theory (see Bardin, 1990).

Consider the linear model $Y = X\theta + \epsilon$, $\mathrm{Var}(\epsilon) = \sigma^2\Omega$, $\theta = [\tau'; \eta']$, where τ is the vector of parameters of interest (treatment effects), η is a vector of some nuisance parameters that are not directly controlled by the experimenter (block effects, plot effects) and X is a design matrix that determines treatment-plot allocations. The theory of optimal designs is concerned with the choice of X that minimizes, in some sense, the covariance matrix of a given estimator $\widehat{\tau}$ of τ (namely $\mathrm{Var}(\widehat{\tau})$) over some class $\mathfrak{S}$. Common optimality criteria are based on the determinant, the trace or the largest eigenvalue of $\mathrm{Var}(\hat{\tau})$. These are respectively known as the D, A and E optimality. Kiefer (1975) introduced the Universal optimality criterion which includes the above D, A and E criteria. A weaker notion known as weak universal optimality is defined in Kiefer and Wynn (1981). Another criterion is the type-1 optimality defined by Cheng (1978).

When $\Omega = I$ (uncorrelated errors) the search for optimal designs has been dealt with by Kiefer (1958, 1975), Cheng (1978), John and Mitchell (1977), Constantine (1983), Eccleston and Hedayat (1974), Cheng and Wu (1981), Kunert (1983) and many others. In particular, balanced incomplete block designs (or BIBD's) and partially balanced incomplete block designs (or PBIBD's) have been shown to provide good designs in terms of many optimality criteria and relative to the variance balance of elementary treatment contrasts.

When $\Omega \neq I$, the problem becomes more difficult. Kiefer and Wynn (1981) suggest the following two-step approach:

(i) Find the class of designs $\mathfrak{S}^*$ in $\mathfrak{S}$ that are optimal using the ordinary least squares estimation and assuming $\Omega = I$.

(ii) Find the class of designs $\mathfrak{S}^{**}$ in $\mathfrak{S}^*$ that are optimal for the ordinary least squares estimation but for some specified correlation Ω.

Examples of this approach are: Cheng (1983), Ipinyomi (1986), Russel and Eccleston (1987), Morgan and Chakravarti (1988).

A more efficient approach is to use generalized least squares (or best linear unbiased) estimation. Examples are Kunert (1985, 1987), Azzalini and Giovagnoli (1987), Gill and Shukla (1985).

Most papers mentioned above assume that observations from different blocks are uncorrelated and observations from the same block have a specified covariance matrix V which is the same for all the blocks. Commonly used covariances are the autoregressive and the moving average models.

Martin and Eccleston (1991) show that, for any V, a combinatorial structure defined by Rao (1961, 1973) as a semi balanced array is universally optimal over binary block designs for the generalized least squares estimate and weakly universally optimal over BIBD's for the ordinary least squares estimate. This generalizes earlier results on the optimality of semi balanced arrays given by Morgan and Chakravarti (1988) and Cheng (1988). However, a severe constraint on the existence of these arrays is that b must be a multiple of $v(v-1)/2$ if v is odd and a multiple of $v(v-1)$ if v is even.

The purpose of this dissertation is to introduce designs closely related to semi balanced arrays but allowing more flexibility for the parameters b, k and v and discuss their statistical performance in terms of optimality theory and variance balance of elementary treatment contrasts, for an arbitrary within block positive definite covariance matrix V.

A description of the standard treatment-block linear model with correlated observations is given in Chapter II, along with a review of needed optimality criteria. An s-associate class association-balanced array (or simply ABA) in b columns, k rows and v symbols is introduced in Section 3.2; it is based on the combinatorial concept of association scheme. Constructions of these arrays for different combinations of the parameters b, k and v and for different types of association schemes are given in Section 3.3. In particular, three rowed ABA's are constructured for $v = 3n$ and $v = n(n-1)/2$; these are respectively based on the group divisible and the triangular association schemes. Some modified versions of existing methods for the construction of balanced arrays (Chakravarti 1956, 1961) and Mukhopadhyay (1978) are shown to provide some series of ABA's. A method of differences analogous to Bose and Bush's method (1952) for the construction of orthogonal arrays, is used to generate some ABA's based on the general cyclic and the group divisible association schemes.

Chapter IV investigates the performance of ABA's as block designs where rows, columns and symbols are respectively identified with plots, blocks and treatments. Section 4.1 reviews some relevant optimality results, essentially on semi balanced arrays. Section 4.2 gives a combinatorial characterization of designs (including ABA's) for which the covariance matrix of $\hat{\tau}$ for an arbitrary V is a scalar multiple of that under the usual uncorrelated model: i.e., $\mathrm{Var}(\hat{\tau}, \mathrm{V}) = \mathrm{g}(\mathrm{V}) \cdot \mathrm{Var}(\hat{\tau}, \mathrm{I})$ for all V where $\hat{\tau}$ is either the generalized or the ordinary least squares estimate of τ. In Section 4.3, using Cheng and Bailey's approach (1991) it is shown that some two-associate class ABA's are type-1 optimal over a restricted subclass of binary designs for all V. In Section 4.4, the efficiency of ABA's relative to universal optimality is assessed by two measures e_A and e_D used by Gill and Shukla (1985a) for nearest neighbour balanced block designs and an autoregressive covariance matrix V; some series of ABA's constructed in Section 3.3 turn out to be highly efficient for an arbitrary V with respect to both measures. Section 4.5 addresses the departure of designs from balancing the variances of elementary treatment contrasts and gives examples of ABA's that are nearly variance balanced for any covariance V. In Section 4.6, an ABA is compared to a randomized BIBD or PBIBD with the same parameters b, k and v and the same number of replications. The corresponding efficiencies for some examples of ABA's are computed in the case of autoregressive and moving average correlation; ABA's turn out to perform better for highly correlated observations.

Chapter V considers the variance balance of elementary treatment contrasts in the context of repeated measurements experiments where residual effects of the treatments are modeled along with direct effects and the correlation structure. It is shown that a special subclass of ABA's, namely ordered ABA's, achieve partial variance balance for direct treatment effects as well as residual treatment effects and for any assumed covariance structure on the periods of observation.

CHAPTER II

TREATMENT-BLOCK LINEAR MODEL WITH CORRELATED OBSERVATIONS

2.1. Model Description.

Suppose v treatments are to be allocated to bk plots arranged in b blocks each of size k, such that no treatment appears more than once in any block. Denote by $D(b,k,v)$ the collection of all such designs. Labeling the blocks $1,2,\ldots,b$ and numbering the plots within a block $1,2,\ldots,k$, block u may be viewed along with the corresponding observations as:

1	Y_{u1}
2	Y_{u2}
$\vdots$	$\vdots$
k	Y_{uk}

where $Y_{u\ell}$ is the observation from the ℓ^{th} plot of the u^{th} block.

Define $d(u,\ell)$ to be the treatment assigned by a design d to the ℓ^{th} plot of the u^{th} block, and consider the fixed effects linear model:

$$Y_{u\ell} = \tau_{d(u,\ell)} + \alpha_\ell + \beta_u + \epsilon_{u\ell} \qquad (2.1.1)$$

$$1 \le u \le b, \quad 1 \le \ell \le k$$

where τ_i is the effect of treatment i, α_ℓ the effect of plot ℓ, β_u the effect of block u and $\epsilon_{u\ell}$'s are random errors with zero means.

Let $Y = (Y_{11},...,Y_{1k}, Y_{21},...,Y_{bk})'$

$\epsilon = (\epsilon_{11},...,\epsilon_{1k}, \epsilon_{21},...,\epsilon_{bk})'$

$\tau = (\tau_1,...,\tau_v)' \qquad \alpha = (\alpha_1,...,\alpha_k)' \qquad \beta = (\beta_1,...,\beta_b)'$

and $T_d = [T'_{d1} \vdots T'_{d2} \vdots \dots \vdots T'_{db}]'$

where T_{du} is the $k \times v$ 0-1 matrix whose $(\ell, i)^{th}$ entry is equal to 1 if and only if $d(u, \ell) = i$ (i.e., T_{du} is the plot-treatment incidence matrix for block u).

Model (2.1.1) may then be written in matrix form as:

$$Y = T_d\tau + (1_b \otimes I_k)\alpha + (I_b \otimes 1_k)\beta + \epsilon \tag{2.1.2}$$

where I_n is the identity matrix of order n, 1_n is the $n \times 1$ vector with all entries equal to 1 and $\otimes$ is the Kronecker product symbol.

It is assumed that all observations have the same variance σ^2 say, observations from different blocks are uncorrelated and observations in the same block have covariance matrix V which is the same for all the blocks,

i.e., $\text{Var}(\epsilon) = (I_b \otimes V)$ (2.1.3)

$V = (v_{\ell\ell'}) = (\sigma^2 \rho_{\ell\ell'}) \qquad 1 \le \ell, \ell' \le k.$

Commonly used correlation structures are:

<u>Moving average of order j (or MAj) model:</u>

$$\text{where } \rho_{\ell\ell'} = \begin{cases} 1 & \text{if } \ell = \ell' \\ \rho_j & \text{if } |\ell - \ell'| = j \\ 0 & \text{otherwise} \end{cases} .$$

First order autoregressive (or AR1) model:

$$\text{where } \rho_{\ell\ell'} = \rho^{|\ell-\ell'|}, \qquad 1 \le \ell, \ell' \le k.$$

2.2. Estimation of Treatment Effects.

To avoid non-identifiability problems, estimation will be based on the set of contrasts $\theta = (I_v - \frac{1}{v}J_v)\tau$, where $J_v = 1_v 1_v'$ is the $v \times v$ matrix with all entries equal to one. It should cause no harm to do so, since $c'\tau = c'\theta$ for any contrast vector c.

Let $\hat{\tau}$ be the best linear unbiased (or, equivalently, the generalized least-squares) estimate of θ, and S a $k \times k$ matrix such that $SVS' = I_k$, then, the transformed model:

$$(I_b \otimes S)Y = (I_b \otimes S)T_d\tau + (1_b \otimes S)\alpha + (I_b \otimes S1_k)\beta + (I_b \otimes S)\epsilon \qquad (2.2.1)$$

has uncorrelated errors.

Let A^- be the Moore-Penrose generalized inverse of an $n \times m$ matrix A, $\omega(A) = A(A'A)^-A'$ the projection operator onto the column space of A and $\omega^\perp(A) = I_n - \omega(A)$.

Write: $\tilde{Y} = (I_b \otimes S)Y \qquad \tilde{T}_d = (I_b \otimes S)T_d$

$U = (I_b \otimes S1_k) \qquad P = (1_b \otimes S)$

The information matrix for $\hat{\tau}$ in model (2.2.1) is:

$$C_d(V) = \tilde{T}_d'\omega^\perp(U \vdots P)\tilde{T}_d \qquad (2.2.2)$$

$$\hat{\tau} = [C_d(V)]^- \tilde{T}_d'\omega^\perp(U \vdots P)\tilde{Y}$$

$$\text{Var}(\hat{\tau}) = [C_d(V)]^- = D_d(V), \text{ say.}$$

A design $d \in D(b,k,v)$ is said to be uniform on the plots if each treatment i occurs equally often, r_i times say, in any plot label.

The following lemma is implicit in Kunert (1985). Because of its importance to the remaining of this dissertation it would be useful to give an explicit version of its proof.

Lemma 2.2.1: Let $d \in D(b,k,v)$; then:

$$C_d(V) = \sum_{u=1}^{b} T'_{du}W(V)T_{du} - b^{-1}\sum_{p=1}^{b}\sum_{q=1}^{b} T'_{dp}W(V)T_{dq} \tag{2.2.3}$$

and if d is uniform on the plots, then:

$$C_d(V) = \sum_{u=1}^{b} T'_{du}W(V)T_{du} \tag{2.2.4}$$

where $W(V) = V^{-1} - (1_k'V^{-1}1_k)^{-1}V^{-1}J_kV^{-1} = W$ say.

Proof: $\omega^{\perp}(U\,|\,P) = \omega^{\perp}(U) - \omega\{\omega^{\perp}(U)P\} = \Omega_1 - \Omega_2$ say. From (2.2.2):

$$C_d(V) = \tilde{T}'_d\Omega_1\tilde{T}_d - \tilde{T}'_d\Omega_2\tilde{T}_d$$

$$\begin{aligned}\omega(U) &= (I_b \otimes S1_k)\{(I_b \otimes 1_k'S')(I_b \otimes S1_k)\}^{-}(I_b \otimes 1_k'S') \\ &= (I_b \otimes S1_k)(I_b \otimes 1_k'S'S1_k)^{-}(I_b \otimes 1_k'S')\end{aligned}$$

$$SVS' = I_k \Rightarrow S'S = V^{-1}.$$

Hence, $\omega(U) = I_b \otimes w^{-1}S1_k1_k'S'$ where $w = 1_k'V^{-1}1_k$ and

$$\begin{aligned}\omega^{\perp}(U) &= I_b \otimes I_k - \omega(U) \\ &= I_b \otimes (I_k - w^{-1}S1_k1_k'S')\end{aligned}$$

$$
\begin{aligned}
\bar{T}_d'\Omega_1\bar{T}_d &= T_d'(I_b \otimes S')\Omega_1(I_b \otimes S)T_d \\
&= T_d'\left\{I_b \otimes (S'S - w^{-1}S'S1_k1_k'S'S)\right\}T_d \\
&= T_d'(I_b \otimes W)T_d \\
&= (T_{d1}' \vdots \ldots \vdots T_{db}')\begin{pmatrix} W & 0 & \ldots & 0 \\ 0 & W & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & W \end{pmatrix}\begin{pmatrix} T_{d1} \\ T_{d2} \\ \vdots \\ T_{db} \end{pmatrix} \\
&= \sum_{u=1}^{b} T_{du}'WT_{du}
\end{aligned}
$$

$$
\begin{aligned}
\omega^{\perp}(U)P &= \left\{I_b \otimes (I_k - w^{-1}S1_k1_k'S')\right\}(1_b \otimes S) \\
&= I_b \otimes (S - w^{-1}S1_k1_k'S'S) \;=\; 1_b \otimes A, \text{ say.}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 &= \omega(1_b \otimes A) \\
&= (1_b \otimes A)\left\{(1_b' \otimes A')(1_b \otimes A)\right\}^{-}(1_b' \otimes A') \\
&= (1_b \otimes A)\left\{b(A'A)\right\}^{-}(1_b' \otimes A') \\
&= b^{-1}\left\{1_b1_b' \otimes A(A'A)^{-}A'\right\} \\
&= b^{-1}J_b \otimes \omega(A).
\end{aligned}
$$

$$
\begin{aligned}
A'A &= (S' - w^{-1}V^{-1}J_kS')(S - w^{-1}J_kV^{-1}) \\
&= V^{-1} - 2w^{-1}V^{-1}J_kV^{-1} + (w^{-1})^2V^{-1}1_k(1_k'V^{-1}1_k)1_k'V^{-1} \\
&= V^{-1} - w^{-1}V^{-1}J_kV^{-1} \;=\; W
\end{aligned}
$$

hence, $\Omega_2 = b^{-1}J_b \otimes AW^{-}A'$.

$$
\begin{aligned}
\bar{T}_d'\Omega_2\bar{T}_d &= b^{-1}T_d'(I_b \otimes S')(J_b \otimes AW^{-}A')(I_b \otimes S)T_d \\
&= b^{-1}T_d'(J_b \otimes S'AW^{-}A'S)T_d
\end{aligned}
$$

$$= b^{-1}T_d'(J_b \otimes WW^-W)T_d$$

$$= b^{-1}T_d'(J_b \otimes W)T_d$$

$$= b^{-1}(T_{d1}' \vdots \ldots \vdots T_{db}')\begin{pmatrix} W & \ldots & W \\ \vdots & & \vdots \\ W & \ldots & W \end{pmatrix}\begin{pmatrix} T_{d1} \\ T_{d2} \\ \vdots \\ T_{db} \end{pmatrix}$$

$$= b^{-1}\sum_{p=1}^{b}\sum_{q=1}^{b} T_{dp}'WT_{dq}.$$

This proves (2.2.3). For (2.2.4), it remains to show that $\bar{T}_d'\Omega_2\bar{T}_d = 0$ if d is uniform on the plots.

If $T_{du} = (t_{du\cdot\ell i})$ and $W = (w_{\ell\ell'})$, the $(i,j)^{th}$ entry of $\bar{T}_d'\Omega_2\bar{T}_d$ is:

$$(\bar{T}_d'\Omega_2\bar{T}_d)_{ij} = b^{-1}\sum_{p=1}^{b}\sum_{q=1}^{b}\sum_{\ell=1}^{k}\sum_{\ell'=1}^{k} t_{dp\cdot\ell i}t_{dq\cdot\ell' j}w_{\ell\ell'}$$

$$= b^{-1}\sum_{\ell=1}^{k}\sum_{\ell'=1}^{k}\Big\{\sum_{p=1}^{b}t_{dp\cdot\ell i}\Big\}\Big\{\sum_{q=1}^{b}t_{dq\cdot\ell' j}\Big\}w_{\ell\ell'}$$

$$= r_ir_j b^{-1}\sum_{\ell=1}^{k}\sum_{\ell'=1}^{k}w_{\ell\ell'} = r_ir_j b^{-1}(1_k'W1_k)$$

if treatments i and j respectively occur r_i and r_j times in any plot label, but

$$1_k'W = (W1_k)' = 1_k'V^{-1} - w^{-1}1_kV^{-1}1_k1_k'V^{-1} = 0,$$

i.e., row and column sums of W are equal to 0. □

Clearly, for the reduced model with no plot effects,

$$Y_d = T_d\tau + (I_b \otimes 1_k)\beta + \epsilon,$$

the information matrix for $\hat{\tau}$ is given by (2.2.4) for any $d \in D(b,k,v)$ not necessarily uniform on the plots.

We will be concerned about the search for efficient or optimal designs for the treatment effect estimate $\hat{\tau}$ over a class of designs Δ, assuming an arbitrary positive definite covariance matrix V. The next section reviews some commonly used optimality criteria.

2.3. Optimality Criteria.

We will respectively write C_d and D_d for the matrices $C_d(V)$ and $D_d(V)$ above unless some confusion may arise.

Finding an optimal design for $\hat{\tau}$ over a class of designs Δ is choosing a design d^* so that, in some sense, the matrix $D_d = Var(\hat{\tau})$ is minimized. The usual approach is to minimize some real valued function of D_d or, equivalently, to maximize some real valued function of $C_d = D_d^-$.

Since $\hat{\theta}_i$'s sum to zero $(\theta_i = \tau_i - \sum_{j=1}^{v}\tau_j/v)$, then C_d and D_d have row and column sums zero so that at least one eigenvalue $\mu_{d,v}$ (say) of C_d corresponds to the eigenvector 1_v and is equal to zero. Denote the other eigenvalues by $\mu_{d,1} \geq \mu_{d,2} \geq \cdots \geq \mu_{d,v\text{-}1} \geq 0$. If the estimate $\hat{\tau}$ exists, then $\mu_{d,v\text{-}1} > 0$ and d is said to be connected.

Some well known and appealing criteria are: the A-optimality which minimizes

$$\phi_A(d) = \frac{1}{v-1}\sum_{i=1}^{v-1} \mu_{d,i}^{-1} = \frac{tr(D_d)}{v-1}$$

or the average variance of the $\hat{\tau}_i$'s; the D-optimality which minimizes:

$$\phi_D(d) = \left(\prod_{i=1}^{v-1} \mu_{d,i}^{-1}\right)^{\frac{1}{v-1}}$$

and the E-optimality which minimizes:

$$\phi_E(d) = \frac{1}{\mu_{d,v-1}} = \text{largest eigenvalue of } D_d.$$

A design that optimizes as many criteria as possible is generally preferred. In this regard, Kiefer (1975) gave the following definition.

Definition 2.3.1: A design d^* is said to be universally optimal (or UOP) over a class Δ if it minimizes $\phi(C_d)$ for every ϕ which is convex, invariant under permutations of rows and columns of C_d and has the property that $\phi(hC_d) \leq \phi(C_d)$ for all $h > 1$.

A design which is UOP, in particular, is A, D and E optimal.

Kiefer and Wynn (1981) defined a weaker optimality criteria as follows.

Definition 2.3.2: $d^* \in \Delta$ is said to be weakly universally optimal (WUOP) if it minimizes $\psi(D_d)$ for every ψ which is convex, invariant under permutations of rows and columns of D_d and has the property that $\psi(hD_d) \geq \psi(D_d)$ for all $h > 1$.

As the authors pointed out, a design which is WUOP, in particular, is A and E-optimal but may not be D-optimal.

A $v \times v$ matrix A is said to be completely symmetric if all its diagonal elements are equal and all its off diagonal elements are also equal,

i.e., $A = xI_v + yJ_v$ for some $x \in \mathbf{R}$ and $y \in \mathbf{R}$.

Kiefer (1975) gave a sufficient condition for universal optimality.

Proposition 2.3.1: If $d^* \in \Delta$ has a completely symmetric information matrix C_d and maximizes $tr(C_d)$ over Δ, then d^* is UOP over Δ.

A similar condition is obtained by Kiefer and Wynn (1981) for weak universal optimality.

Proposition 2.3.2: If $d^* \in \Delta$ has a completely symmetric covariance matrix D_d and minimizes $tr(D_d)$ over Δ, then d^* is WUOP over Δ.

Hence, a design satisfying the conditions of Proposition 2.3.1 or at least those of Proposition 2.3.2 is preferred. However, in many cases no such design is known to exist. In these situations, the following criterion introduced by Cheng (1978) is of interest.

Definition 2.3.3: A design d^* is said to be type-1 optimal over Δ if it minimizes $\sum_{i=1}^{v-1} f(\mu_{d,i})$ for all sufficiently differentiable functions f such that $f''(x) > 0$, $f'''(x) < 0$ for $x > 0$ and $\lim_{x \to 0^+} f(x) = \infty$.

As the author pointed out, the A and E-criteria are covered by choosing f(x) $= x^{-1}$ and $-\log x$ respectively; the E-criterion is also covered as a pointwise limit of functions satisfying the conditions of Definition 2.3.3.

CHAPTER III

ASSOCIATION-BALANCED ARRAYS

3.1. Preliminaries.

Definition 3.1.1: A balanced incomplete block design (or BIBD) is an arrangement of v symbols (treatments) into b blocks of k symbols each, such that each symbol occurs r times in the design, no symbol occurs more than once in any block and any unordered pair of symbols occur together in λ blocks. It is denoted by $BIB(b,k,v,r;\lambda)$.

Necessary conditions for the existence of a $BIB(b,k,v,r;\lambda)$ are:

$$bk = vr \qquad r(k-1) = \lambda(v-1) \tag{3.1.1}$$

Hall (1986) provides a comprehensive review on the construction of BIBD's. If no BIBD exists for a given combination of the parameters b, k, v, r and λ, an alternative structure introduced by Bose and Nair (1939) and called partially balanced incomplete block design (or PBIBD) is of interest. To define PBIBD's, one needs the concept of association scheme.

Definition 3.1.2: Given v symbols, a relation satisfying the following conditions is said to be an association scheme with s classes:

(i) Any two symbols are either first, second, ..., or s^{th} associates, the relation of association being symmetrical.

(ii) Each symbol x has n_i i^{th} associates, the numbers $n_1, n_2, ..., n_s$ being independent of x.

(iii) If x and y are i^{th} associates, the number of symbols that are j^{th} associates of x and k^{th} associates of y is p^i_{jk} and is independent of the pair of i^{th} associates x and y.

The numbers s, v, n_i, p^i_{jk} $(i,j,k=1,...,s)$ are called the parameters of the association scheme. P_i $(i=1,...,s)$ denotes the $s \times s$ matrix whose $(j,k)^{th}$ entry is p^i_{jk} and $\mathcal{C}_i$ $(i=1,...,s)$ the i^{th} associate class consisting of all unordered pairs of i^{th} associate symbols.

An immediate consequence of the definition is:

$$\sum_{i=1}^{s} n_i = v-1 \qquad \text{Card } \mathcal{C}_i = \frac{vn_i}{2}, \quad i=1,...,s \tag{3.1.2}$$

The following are well known examples of association schemes needed in subsequent sections.

<u>Trivial association scheme A(v):</u>

There is only one associate class so that, with respect to each symbol, the remaining $v-1$ symbols are first associates. The parameters are:

$$v, s=1, \quad n_1 = v-1, \quad p^1_{11} = v-2.$$

<u>Group divisible association scheme GD(m, n):</u>

There are $v = mn$ symbols divided into m groups of n symbols each. Two symbols in the same group are said to be first associates and two symbols from different groups are said to be second associates. The parameters are:

$$s=2, \quad v=mn, \quad n_1 = n-1, \quad n_2 = n(m-1)$$

$$P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix} \qquad (3.1.3)$$

<u>Triangular association scheme T(n):</u>

There are $v = \binom{n}{2} = \frac{n(n-1)}{2}$ symbols arranged in an array of n rows and n columns with the following conditions:

—The positions in the principal diagonal are left blank.

—The $\binom{n}{2}$ positions above the principal diagonal are filled by the symbol labels.

—The $\binom{n}{2}$ positions below the principal diagonal are filled so that the array is symmetrical.

For a given symbol x, the first associates are those that occur in the same row or in the same column as x, the remaining symbols being second associates. The parameters are:

$$s = 2, \quad v = \binom{n}{2}, \quad n_1 = 2(n-2), \quad n_2 = \frac{(n-2)(n-3)}{2} \qquad (3.1.4)$$

$$P_1 = \begin{bmatrix} n-2 & n-3 \\ n-3 & \frac{(n-3)(n-4)}{2} \end{bmatrix} \quad P_2 = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & \frac{(n-4)(n-5)}{2} \end{bmatrix}$$

An equivalent definition is to identify the $\binom{n}{2}$ treatments with the unordered pairs of distinct symbols (i,j), $1 \le i \ne j \le n$, two symbols being first associates if they have one coordinate in common and second associates otherwise.

<u>Rectangular association scheme R(m, n):</u>

There are $v = mn$ symbols arranged in a rectangle of m rows and n columns. Two symbols are first associates if they are in the same row, second associates if they are in the same column and third associates otherwise. The parameters are:

$$s = 3,\ \ v = mn,\ \ n_1 = n-1,\ \ n_2 = m-1,\ \ n_3 = (n-1)(m-1)$$

$$P_1 = \begin{bmatrix} n-2 & 0 & 0 \\ 0 & 0 & m-1 \\ 0 & m-1 & (m-1)(n-2) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 0 & n-1 \\ 0 & m-2 & 0 \\ n-1 & 0 & (n-1)(m-2) \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 1 & n-2 \\ 1 & 0 & m-2 \\ n-2 & m-2 & (n-2)(m-2) \end{bmatrix} \qquad (3.1.5)$$

L_2-type association scheme $L(n)$:

There are n^2 symbols arranged in an $n \times n$ square array, two symbols are first associates if they occur in the same row or in the same column and second associates otherwise. This is a special case of the rectangular scheme where $m = n$ and the first and second associate classes are combined. The parameters are:

$$s = 2,\ \ v = n^2,\ \ n_1 = 2(n-1),\ \ n_2 = (n-1)^2$$

$$P_1 = \begin{bmatrix} n-2 & n-1 \\ n-1 & (n-1)(n-2) \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 2(n-2) \\ 2(n-2) & (n-2)^2 \end{bmatrix} \qquad (3.1.6)$$

General cyclic association scheme GC(v):

The set of symbols is represented by the set of integers modulo v; two symbols x and y are i^{th} associates if and only if $x-y = \pm i \bmod(v)$. The parameters are:

$$s = \mathrm{int}\left(\frac{v}{2}\right)v, \quad n_1 = n_2 = \ldots = n_{s-1} = 2$$

$$n_s = \begin{cases} 1 & \text{if } v = 2s \\ 2 & \text{if } v = 2s+1 \end{cases}$$

$$p^i_{jk} = \begin{cases} 1 & \begin{array}{l}\text{if } k = \inf\{i+j,\ v-(i+j)\} \\ \text{or } k = \sup\{i,j\} - \inf\{i,j\}\end{array} \\ 0 & \text{otherwise} \end{cases} . \tag{3.1.7}$$

Definition 3.1.3: Given an association scheme with s classes, a partially balanced incomplete block design (or PBIBD) with s associate classes is an arrangement of v symbols into b blocks of size k each such that:

(i) No symbol occurs more than once in a block.

(ii) Any symbol occurs in exactly r blocks.

(iii) If x and y are i^{th} associates, they occur together in λ_i blocks. The numbers λ_i $(i=1,2,\ldots,s)$ being independent of the particular pair of i^{th} associates $\{x,y\}$.

Such a design is denoted by $PBIB(b,k,v,r;\lambda_1,\ldots,\lambda_s)$. The numbers $b,k,v,$ $r;\ \lambda_1,\ldots,\lambda_s$ are called the parameters of the design.

If the design has different block sizes $k_1,k_2,\ldots,k_b$ and satisfies all other conditions of Definition 3.1.3, it is known as a pairwise partially balanced design.

Necessary conditions for the existence of a $PBIB(b,k,v,r;\lambda_1,\ldots,\lambda_s)$ are:

$$bk = vr \qquad r(k-1) = \sum_{i=1}^{s} \lambda_i n_i \tag{3.1.8}$$

Clearly a PBIBD with all λ_i's equal is simply a BIBD.

Further details on association schemes and construction methods of PBIBD's are given in Chapter 8 of Raghavarao (1971). A comprehensive list of two associate class PBIBD's is given by Clatworthy (1973). In the following theorem, a technique of initial blocks is used to generate some series of PBIBD's based on the GC(v) association scheme.

Theorem 3.1.1: Given the general cyclic association scheme GC(v), let $\{B_1, ..., B_t\}$ be a set of t blocks such that:

(i) Each block contains k distinct symbols.

(ii) Among the $t\binom{k}{2}$ unordered pairs arising from the t blocks, there are γ_j pairs from the j^{th} associate class ($j=1,2,...,s$; $s=\text{int}(\frac{v}{2})$). Then, the set of blocks $\{B_i+\theta\text{: } i = 1,...,t,\ \theta = 0,1,...,v-1\}$ provides a general cyclic PBIBD with the parameters $b = vt,\ k, v, r = kt,\ \lambda_1, ..., \lambda_s$; where:

$$\lambda_j = \gamma_j, \quad j = 1,2,...,s-1$$

$$\lambda_s = \begin{cases} \gamma_s & \text{if } v = 2s+1 \\ 2\gamma_s & \text{if } v = 2s \end{cases}$$

$B_i + \theta$ is the block obtained by adding θ to the elements of B_i and reducing modulo v.

Proof: Suppose $v = 2s+1$; let $\{x, y\}$ be any unordered pair of j^{th} associate symbols ($j = 1, ..., s$) occuring in the initial block B_i. Then $\{x, y\}$ generates all possible pairs of j^{th} associate elements in the set of blocks $\{B_i+\theta,\ \theta=0,\ 1, ..., v-1\}$. Since there are γ_j such pairs in the initial blocks $B_1, ..., B_t$, every pair of j^{th}

associate symbols will occur γ_j times in the design.

Similarly, if $v = 2s$, every pair of j^{th} associate symbols will occur γ_j times in the design except for the pairs of s^{th} associate symbols which occur $2\gamma_s$ times since, if $\{x,y\} \in \mathfrak{C}_s$ and $v = 2s$, $(\{x,y\} + \theta;\ \theta = 0,1,...,v-1)$ consists of two replications of $\mathfrak{C}_s$. □

<u>Example 3.1.1</u>: $v = 5,\ s = 2$.

The two initial blocks: $B_1 = (0,1,3)$ $\quad B_2 = (3,1,4)$

are such that: $\gamma_1 = 2,\ \gamma_2 = 4$. The resulting PBIB$(10,3,5,6;2,4)$ is:

0	1	2	3	4	3	4	0	1	2
1	2	3	4	0	1	2	3	4	0
3	4	0	1	2	4	0	1	2	3

with columns corresponding to blocks.

If v is odd, $t = \frac{v-1}{2}$, and $\gamma_1 = \gamma_2 = ... = \gamma_s$, the above theorem yields a series of BIBD's given by Ramanujacharyulu (1966) as illustrated in the following example.

<u>Example 3.1.2</u>: $v = 9,\ s = 4$.

The four initial blocks: $B_1 = (0,1,2,3)$ $\quad B_2 = (0,3,6,8)$

$B_3 = (0,2,4,6)$ $\quad B_4 = (0,4,8,3)$

are such that: $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 6$.

The resulting BIB$(36,4,9,16;6)$ is:

0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0	3	4	5	6	7	8	0	1	2
2	3	4	5	6	7	8	0	1	6	7	8	0	1	2	3	4	5
3	4	5	6	7	8	0	1	2	8	0	1	2	3	4	5	6	7

0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	0	1	4	5	6	7	8	0	1	2	3
4	5	6	7	8	0	1	2	3	8	0	1	2	3	4	5	6	7
6	7	8	0	1	2	3	4	5	3	4	5	6	7	8	0	1	2

3.2. Combinatorial Structures of Arrays.

An association-balanced array, explicitly defined later in this section, is a combinatorial arrangement of $k \times b$ arrays of v symbols. It generalizes the concept of semi balanced arrays of strength 2 introduced by Rao (1961, 1973), and yields a particular subclass of PBIBD's. Some construction methods of these arrays given in Section 3.3 involve other well known arrangements of arrays such as: balanced arrays, orthogonal arrays, semi balanced arrays and transitive arrays. Hence, a brief review of these well known structures is of interest.

Rao (1946, 1947) introduced the concept of orthogonal array in connection with the theory of fractional factorial design.

Definition 3.2.1: Orthogonal array.

A $k \times b$ array A with entries from a set of v symbols is called an orthogonal array of strength t if each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = b/v^t$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array.

Example 3.2.1: $OA(8, 4, 2, 3; 1)$.

0	1	1	1	1	0	0	0
1	0	1	1	0	1	0	0
1	1	0	1	0	0	1	0
1	1	1	0	0	0	0	1

Existence and construction of these arrays are investigated, among others, by Bush (1952), Bose and Bush (1952), Seiden (1954), Addelman and Kempthorne (1961), Shrikhande (1964), Seiden and Zemach (1966) and Yamamoto et al. (1984). Chakravarti (1956) generalized the concept of orthogonal array to what he called partially balanced array; this has been renamed by Srivastava and Chopra as balanced array.

Definition 3.2.2: Balanced array.

Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, x_2, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a nonnegative integer $\lambda(x_1, x_2, \ldots, x_t)$ that is invariant under permutations of $x_1, x_2, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$ each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to be a balanced array of strength t. It is denoted $BA(b, k, v, t)$ and the numbers $\lambda(x_1, \ldots, x_t)$ are the index parameters of the array.

Clearly a $BA(b, k, v, t)$ with $\lambda(x_1, \ldots, x_t) = \lambda$ for all t-tuples $(x_1, \ldots, x_t)$ is simply an $OA(b, k, v, t; \lambda)$.

Example 3.2.2: $BA(10, 5, 2, 2)$.

$\lambda(0,0) = \lambda(1,1) = 2 \qquad \lambda(0,1) = 3$

0	1	0	1	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	0
0	0	1	1	1	0	0	0	1	1
1	1	0	0	0	0	1	0	1	1
0	0	0	0	1	1	1	1	1	0

For some constructions of BA's, reference may be made to Chakravarti (1961), Srivastava (1972), Rafter and Seiden (1974), Sinha and Nigam (1983), Saha and Samanta (1985). BA's also play a vital role in the theory of fractional factorial experiments as illustrated by Srivastava (1972), Chopra and Srivastava (1975), Nishii (1981) and many others. An extensive reference list on this regard is given by Srivastava (1990).

Two other arrangements of arrays are defined by Rao (1961) as orthogonal array of type 1 and 2, later renamed transitive array and semi balanced array.

<u>Definition 3.2.3</u>: Semi balanced array.

A $k \times b$ array with entries from a set of v symbols is said to be semi balanced of strength t and index λ, if for any choice of t rows, the b columns contain each of the $\binom{v}{t}$ unordered t-tuples of distinct symbols exactly $\lambda = b/\binom{v}{t}$ times. It is denoted $SB(b, k, v, t; \lambda)$.

In particular, a SBA of strength two, $SBA(b, k, v, 2; \lambda)$, is a $k \times b$ array of v symbols in which each unordered pair of distinct symbols occurs $\lambda = b/\binom{v}{t}$ times as a column of any two-rowed subarray. If $k \geq 3$, this implies that each symbol occurs the same number of times, $r = b/v = \frac{\lambda(v-1)}{2}$ in each row. SBA's with the latter condition for $k = 2$ as well, have been renamed by Martin and Eccleston (1991) as strongly directionally equineighboured (or SDEN) designs. We will adopt their slightly more restrictive definition for a $SBA(b, k, v, 2; \lambda)$.

<u>Example 3.2.3</u>: $SBA(3, 3, 3, 2; 1)$.

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix}$$

Rao (1961) gave a construction method of $SBA(\binom{v}{2}, v, v, 2; 1)$ when v is any odd prime or an odd prime power. Lindner et al. (1987) constructed four and five rowed SBA's of strength 2 and index $\lambda = 1$ for any odd $v \geq 5$ except $v \in \{15, 39\}$.

Mukhopadhyay (1978) constructed SBA's from known SBA's and OA's. Morgan and Chakravarti (1988) constructed three rowed SBA's of strength 2 and index 2 for any even v. SBA's of strength $t \geq 3$ are investigated by Kramer et al. (1989).

Definition 3.2.4: Transitive array.

A transitive array $TA(b, k, v, t; \lambda)$ is a $k \times b$ array of v symbols such that, for any choice of t rows, the $v!/(v-t)!$ ordered t-tuples of distinct symbols each occur λ times as a column.

Example 3.2.4: $TA(12, 4, 4, 2; 1)$.

0	1	2	3	0	1	2	3	0	1	2	3
1	0	3	2	2	3	0	1	3	2	1	0
2	3	0	1	3	2	1	0	1	0	3	2
3	2	1	0	1	0	3	2	2	3	0	1

Clearly a $TA(b, k, v, t; \lambda)$ is also a $SBA(b, k, v, t; t!\lambda)$.

Bose et al. (1960) constructed $TA(v(v-1), k, v, 2; 1)$ from a set of $k-2$ mutually orthogonal latin squares of order v. Suen (1983) constructed $TA(v(v-1), v, v, 2; 1)$ from doubly transitive groups of order v. Application of SBA's and TA's to the theory of block design is reviewed in Section 4.1.

A severe constraint on the existence of SBA's is that the number of columns b must be a multiple of $\binom{v}{2}$; if v is even, Morgan and Chakravarti (1988) showed that b must be a multiple of $2\binom{v}{2} = v(v-1)$. This makes their use as block designs somewhat restrictive. A new combinatorial arrangement relaxing this constraint is introduced in the following definition.

Definition 3.2.5: Association-balanced array.

Given an association scheme with s classes and v symbols a $k \times b$ array with entries from a set of v symbols will be said to be association-balanced if:

(i) Each symbol occurs r times in each row.

(ii) No symbol occurs more than once in any column.

(iii) Two symbols that are i^{th} associates occur λ_i times as a column of any two-rowed subarray. Such an array will be denoted $ABA(b, k, v, r; \lambda_1, ..., \lambda_s)$.

Example 3.2.5: Consider the group divisible association scheme GD(4, 2) with groups $\{0,4\}, \{1,5\}, \{2,6\}$ and $\{3,7\}$. A corresponding $ABA(24, 4, 8, 3; 0, 1)$ is:

0	4	1	5	2	6	3	7	0	4	1	5	2	6	3	7	0	4	1	5	2	6	3	7
1	5	4	0	3	7	6	2	2	6	7	3	4	0	1	5	3	7	2	6	5	1	4	0
2	6	7	3	4	0	1	5	3	7	2	6	5	1	4	0	1	5	4	0	3	7	6	2
3	7	2	6	5	1	4	0	1	5	4	0	3	7	6	2	2	6	7	3	4	0	1	5

The following remarks are immediate consequences of the definitions above.

Remarks 3.2.1:

(i) Any k' rows of an $ABA(b, k, v, r; \lambda_1, ..., \lambda_s)$ form an $ABA(b, k', v, r; \lambda_1, ..., \lambda_s)$.

(ii) The columns of an $ABA(b, k, v, r; \lambda_1, ..., \lambda_s)$ form an s associate class PBIBD with parameters $b, k, v, rk, \lambda_1\binom{k}{2}, ..., \lambda_s\binom{k}{2}$.

(iii) A $SBA(b, k, v, 2; \lambda)$ is an $ABA(b, k, v, r; \lambda, ..., \lambda)$ with respect to the trivial association scheme, where $r = \lambda(v-1)/2$.

Lemma 3.2.1: Necessary conditions for the existence of an $ABA(b, k, v, r; \lambda_1, ..., \lambda_s)$ are:

$$\text{(i)}\ \ b = vr \qquad \text{(ii)}\ \ \sum_{i=1}^{s} \lambda_i n_i = 2r \tag{3.2.1}$$

Proof: (i) is an immediate consequence of the definition. For (ii), a symbol x occurs 2r times in a given two-rowed subarray in which it occurs λ_1 times with its n_1 first associates, λ_2 times with its n_2 second associates and so on. □

In what follows, the abreviations GDABA, TABA, RABA, LABA, and GCABA are used to denote ABA's based respectively on group divisible, triangular, rectangular, L_2-type and general cyclic association scheme.

3.3. Constructions of Association-Balanced Arrays.

Chakravarti (1961) constructed balanced arrays from pairwise partially balanced designs. An analogue of his method can be used to construct some series of ABA's.

Theorem 3.3.1: The existence of a PBIBD with parameters $b, k, v, r, \lambda_1, ..., \lambda_s$ and of a $SBA(\lambda\binom{k}{2}, q, k, 2; \lambda)$ imply the existence of an ABA in v symbols, $\lambda b\binom{k}{2}$ columns, q rows and index parameters $\lambda\lambda_i$ $(i = 1, 2, ..., s)$.

Proof: Let $S = SBA(\lambda\binom{k}{2}, q, k, 2; \lambda)$

and $P = PBIB(b, k, v, r; \lambda_1, ..., \lambda_s)$.

The b blocks of P provide b sets of k symbols each. Using each set once in S, one gets b semi balanced arrays $S_1, S_2, ..., S_b$. If these are put side by side, the resulting array has $\lambda b\binom{k}{2}$ columns, q rows, v symbols and is indeed an ABA with index parameters $\lambda\lambda_i$ $(i = 1, ..., s)$. □

Example 3.3.1: GDABA(36, 4, 6, 6; 4, 2).

Let $G_1 = \{0, 3\}$, $G_2 = \{1, 4\}$ and $G_3 = \{2, 5\}$ be the groups of the group divisible association scheme GD(3, 2). A GDPBIB(3, 4, 6, 2; 2, 1) is:

P:	0	1	2
	3	4	5
	1	2	0
	4	5	3

A $SBA(12,4,4,2;2) = S$, say, is given in Example 3.2.4. Applying the theorem to S and P above, one gets:

S_1:	0	3	1	4	0	3	1	4	0	3	1	4
	3	0	4	1	1	4	0	3	4	1	3	0
	1	4	0	3	4	1	3	0	3	0	4	1
	4	1	3	0	3	0	4	1	1	4	0	3

S_2:	1	4	2	5	1	4	2	5	1	4	2	5
	4	1	5	2	2	5	1	4	5	2	4	1
	2	5	1	4	5	2	4	1	4	1	5	2
	5	2	4	1	4	1	5	2	2	5	1	4

S_3:	2	5	0	3	2	5	0	3	2	5	0	3
	5	2	3	0	0	3	2	5	3	0	5	2
	0	3	2	5	3	0	5	2	5	2	3	0
	3	0	5	2	5	2	3	0	0	3	2	5

The required array is $d = [S_1 \vdots S_2 \vdots S_3]$.

Corollary 3.3.1: If m is an odd prime power and there exists an $OA(n^2, m, n, 2)$, then there exists a GDABA with parameters $b = n^2\binom{m}{2}$, $k = m$, $v = mn$, $r = \frac{n(m-1)}{2}$, $\lambda_1 = 0$ and $\lambda_2 = 1$.

Proof: Bose et al. (1953) established the equivalence between $OA(n^2, m, n, 2)$ and $GDPBIB(n^2, m, mn, n; 0, 1)$ as follows: if the symbols of the OA are labeled $0, 1, \ldots, v-1$, replacing each symbol x in the i^{th} row by $(i-1)n + x$, $i = 1, 2, \ldots, m$, the columns of the resulting array are blocks of the required PBIBD where the symbols of the i^{th} group are numbered $(i-1)n$, $(i-1)n+1$, ..., $(i-1)n+n-1$. It suffices then to apply the theorem above to this PBIBD and to a $SBA(\binom{m}{2}, m, m, 2; 1)$. The assumption of prime power is not necessary if the maximum number of rows, $k = m$, is not required.

Example 3.3.2: $GDABA(27,3,9,3;0,1)$.

The $OA(9,3,3,2;1)$:

0	1	2	0	1	2	0	1	2
0	1	2	1	2	0	2	0	1
0	1	2	2	0	1	1	2	0

yields the $GDPBIB(9,3,9,3;0,1)$.

P:	0	1	2	0	1	2	0	1	2
	3	4	5	4	5	3	5	3	4
	6	7	8	8	6	7	7	8	6

and the required array is:

0	3	6	1	4	7	2	5	8	0	4	8	1	5	6	2	3	7	0	5	7	1	3	8	2	4	6
3	6	0	4	7	1	5	8	2	4	8	0	5	6	1	3	7	2	5	7	0	3	8	1	4	6	2
6	0	3	7	1	4	8	2	5	8	0	4	6	1	5	7	2	3	7	0	5	8	1	3	6	2	4

Corollary 3.3.2: The existence of a $SBA(\lambda\binom{n-1}{2}, k, n-1, 2; \lambda)$ implies the existence of a triangular ABA with parameters $b = \lambda n\binom{n-1}{2}$, k, $v = \binom{n}{2}$, $r = \lambda(n-2)$, $\lambda_1 = \lambda$, and $\lambda_2 = 0$.

Proof: A PBIB with the parameters $b = n$, $k = n-1$, $v = \binom{n}{2}$, $r = 2$, $\lambda_1 = 1$ and $\lambda_2 = 0$ is obtained by writing the n rows of the triangular scheme $T(n)$ as blocks of the PBIB design, and Theorem 3.3.1 applies.

Example 3.3.3: TABA(12, 3, 6, 2; 1, 0).

A triangular scheme T(4) may be represented by:

·	0	1	2
0	·	3	4
1	3	·	5
2	4	5	·

It provides a TPBIB(4, 3, 6, 2; 1, 0):

P:	0	0	1	2
	1	3	3	4
	2	4	5	5

and the required array is:

0	1	2	0	3	4	1	3	5	2	4	5
1	2	0	3	4	0	3	5	1	4	5	2
2	0	1	4	0	3	5	1	3	5	2	4

Corollary 3.3.3: The existence of SBA($\lambda\binom{n}{2}$, k, n, 2; λ) implies the existence of a LABA with parameters $b = \lambda n^2(n-1)$, k, $v = n^2$, $r = \lambda(n-1)$, $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

Proof: The n columns and the n rows of an L_2 scheme L(n) provide 2n blocks of a PBIB($2n, n, n^2, 2; 1, 0$) and Theorem 3.3.1 applies.

Example 3.3.4: LABA(18, 3, 9, 2; 1, 0).

The L_2 scheme L(3) may be represented by:

0	3	6
1	4	7
2	5	8

The resulting PBIB(6, 3, 9, 2; 1, 0) is:

0	0	3	1	2	6
1	3	4	4	5	7
2	6	5	7	8	8

and the required LABA is:

0	1	2	0	3	6	3	4	5	1	4	7	2	5	8	6	7	8
1	2	0	3	6	0	4	5	3	4	7	1	5	8	2	7	8	6
2	0	1	6	0	3	5	3	4	7	1	4	8	2	5	8	6	7

A slightly more general version of Theorem 3.3.1 can be used to construct rectangular ABA's and yields the following corollary.

Corollary 3.3.4: The existence of SBA($\gamma_1\binom{n}{2}$, k_1, n, 2; γ_1) and of SBA($\gamma_2\binom{m}{2}$, k_2, m, 2; γ_2) imply the existence of a rectangular ABA with parameters $b = \gamma_2 n\binom{m}{2} + \gamma_1 m\binom{n}{2}$, $k = \min(k_1, k_2)$, $v = mn$, $r = \frac{b}{v}$, $\lambda_1 = \gamma_1$, $\lambda_2 = \gamma_2$ and $\lambda_3 = 0$.

Proof: Write the m rows of the rectangular scheme R(m, n) as blocks $B_1, B_2, ..., B_m$ each of size n, and the n columns as blocks $C_1, C_2, ..., C_n$ each of size m. Clearly the resulting set of blocks provide a pairwise partially balanced design in $b = m+n$ blocks, $v = mn$ symbols, $r = 2$ replications, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$. Let S_i ($i=1,...,m$) be a SBA($\gamma_1\binom{n}{2}$, k, n, 2) whose symbols are the elements of B_i, and let D_i ($i=1,2,...,n$) be a SBA($\gamma_2\binom{m}{2}$, k, m, 2) whose symbols are the elements of C_i. Then the juxtaposition of the S_i and the D_i arrays provides the required array. □

Example 3.3.5:

A RABA(48, 3, 12, 4; 1, 2, 0) can be constructed from the SBA(3, 3, 3, 2; 1) given in Example 3.2.3 and the SBA(12, 4, 4, 2; 2) given in Example 3.2.4.

Mukhopadhyay (1978) defined a composition operation on semi balanced arrays to obtain other semi balanced arrays. A generalized version of his method can be used to yield some families of association balanced arrays.

<u>Theorem 3.3.2</u>: The existence of SBA($\gamma_i\binom{s_i}{2}, p_i, s_i, 2; \gamma_i$), $i = 1,2$; and of OA(δs_1^2, q, s_1, 2) imply the existence of a GDABA with parameters $b = \gamma_1 s_2\binom{s_1}{2} + \delta\gamma_2 s_1^2\binom{s_2}{2}$, $k = \min(p_1, p_2, q)$, $v = s_1 s_2$, $r = \frac{b}{v}$, $\lambda_1 = \gamma_1$ and $\lambda_2 = \delta\gamma_2$.

<u>Proof:</u>

Let $$A = \begin{bmatrix} a_{11} & \cdots & a_{1c} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kc} \end{bmatrix} \quad c = \delta s_1^2 \quad a_{ij} \in \{0, 1, ..., s_1 - 1\}$$

represent OA($\delta s_1^2, k, s_1, 2$)

and $$B_i = \begin{bmatrix} b_{11}^{(i)} & \cdots & b_{1d_i}^{(i)} \\ \vdots & & \vdots \\ b_{k1}^{(i)} & \cdots & b_{kd_i}^{(i)} \end{bmatrix} \quad d_i = \gamma_i\binom{s_i}{2} \quad b_{pq}^{(i)} \in \{0, 1, ..., s_i - 1\}$$

represents SBA($\gamma_i\binom{s_i}{2}, k, s_i, 2; \gamma_i$), $i = 1, 2$.

Define $$A(j) = \begin{bmatrix} (a_{11}, b_{1j}^{(2)}) & \cdots & (a_{1c}, b_{1j}^{(2)}) \\ \vdots & & \vdots \\ (a_{k1}, b_{kj}^{(2)}) & \cdots & (a_{kc}, b_{kj}^{(2)}) \end{bmatrix}, \quad j = 1, 2, ..., d_2.$$

Write $C = [A(1) \vdots A(2) \vdots ... \vdots A(d_2)]$.

$$\text{Define} \qquad B(j) = \begin{bmatrix} (b_{11}^{(1)},j) & \cdots & (b_{1d_1}^{(1)},j) \\ \vdots & & \vdots \\ (b_{k1}^{(1)},j) & \cdots & (b_{kd_1}^{(1)},j) \end{bmatrix}, \quad j = 0,1,\ldots,s_2-1$$

and write $B = [B(0) \vdots B(1) \vdots \ldots \vdots B(s_2-1)]$.

Let $D = [C \vdots B]$ be the juxtaposition of C and B. D is then an array with k rows, $b = \gamma_1 s_2 \binom{s_1}{2} + \delta\gamma_2 s_1^2 \binom{s_2}{2}$ columns whose entries are from the set $\Gamma = \{(i,j)$: $i = 0,1,\ldots,s_1-1; j = 0,1,\ldots,s_2-1\}$.

Two distinct pairs (i,j) and (i',j') will be called first associates if they have the same second coordinate $(j = j')$ and second associates otherwise. This is clearly an equivalent definition of the group divisible association scheme $GC(s_2, s_1)$.

Consider a two-rowed subarray of D. In the part obtained from C, each unordered pair of second associate elements of Γ occurs $\delta\gamma_2$ times as a column; in the part obtained from B, each unordered pair of first associate element of Γ occurs γ_1 times as a column. Hence D is the required ABA. □

Corollary 3.3.5: The existence of SBA $(\gamma\binom{m}{2}, p, m, 2; \gamma)$ and of $OA(n^2, q, n, 2)$ imply the existence of a GDABA with parameters $b = \gamma n^2 \binom{m}{2}$, $k = \min(p,q)$, $v = mn$, $r = \gamma n(m-1)/2$, $\lambda_1 = 0$ and $\lambda_2 = \gamma$.

Proof: Apply the construction of Theorem 3.3.2 with $s_1 = n$, $s_2 = m$, $\gamma_1 = 0$, $\gamma_2 = \gamma$, $\delta = 1$ and take $D = C$ omitting the part coming from B.

Example 3.3.6: GDABA(40, 3, 10, 4; 0, 1).

An OA(4, 3, 2, 2) is:

A:				
	0	1	0	1
	0	1	1	0
	0	0	1	1

and a SBA(10, 3, 5, 2; 1) is:

B_1:										
	0	1	2	3	4	0	1	2	3	4
	1	2	3	4	0	2	3	4	0	1 .
	2	3	4	0	1	4	0	1	2	3

Writing xy for the ordered pair (x, y), the required ABA is:

00	10	00	10	01	11	01	11	02	12	02	12	03	13	03	13	04	14	04	14
01	11	11	01	02	12	12	02	03	13	13	03	04	14	14	04	00	10	10	00
02	02	12	12	03	03	13	13	04	04	14	14	00	00	10	10	01	01	11	11
00	10	00	10	01	11	01	11	02	12	02	12	03	13	03	13	04	14	04	14
02	12	12	02	03	13	13	03	04	14	14	04	00	10	10	00	01	11	11	01
04	04	14	14	00	00	10	10	01	01	11	11	02	02	12	12	03	03	13	13

The next two theorems are direct methods for the construction of some three rowed group divisible and triangular ABA's.

Theorem 3.3.3: A GDABA with parameters $b = 3n^2$, $k = 3$, $v = 3n$, $r = n$, $\lambda_1 = 0$, and $\lambda_2 = 1$ can always be constructed.

Proof: Consider the group divisible association scheme GD(3, n). Let the three groups of symbols be:

$$G_i = \{t_{i1}, t_{i2}, ..., t_{in}\} \quad i = 1, 2, 3.$$

Write $\vec{t}_{ij} = \underbrace{(t_{ij}, t_{ij}, \dots, t_{ij})}_{n \text{ times}} \quad i = 1,2,3.$

$$t^{u}_{i\cdot} = (t_{iu+1}, \dots, t_{in}, t_{i1}, \dots, t_{iu}) \quad u = 1,\dots,n-1, \quad t^{0}_{i\cdot} = t_{i\cdot} = (t_{i1},\cdots,t_{in})$$

i.e., $t^{u}_{i\cdot}$ is a shift of length u of the entries of $t_{i\cdot}$ to the right. The required array is given by the following juxtaposition:

$$\begin{bmatrix} \vec{t}_{11} \dots \vec{t}_{1n} & \vec{t}_{21} \dots \vec{t}_{2n} & \vec{t}_{31} \dots \vec{t}_{3n} \\ t_{2\cdot} \dots t_{2\cdot} & t_{3\cdot} \dots t_{3\cdot} & t_{1\cdot} \dots t_{1\cdot} \\ t^{0}_{3\cdot} \dots t^{n-1}_{3\cdot} & t^{0}_{1\cdot} \dots t^{n-1}_{1\cdot} & t^{0}_{2\cdot} \dots t^{n-1}_{2\cdot} \end{bmatrix}.$$

Example 3.3.7: GDABA(12, 3, 6, 2; 0, 1).

$$G_1 = \{0,3\} \quad G_2 = \{1,4\} \quad G_3 = \{2,5\}$$

$$t^{1}_{1\cdot} = (3,0) \quad t'_{2\cdot} = (4,1) \quad t^{1}_{3\cdot} = (5,2).$$

The required ABA is:

0	0	3	3	1	1	4	4	2	2	5	5
1	4	1	4	2	5	2	5	0	3	0	3
2	5	5	2	0	3	3	0	1	4	4	1

Theorem 3.3.4: A triangular ABA with parameters $b = (n-2)\binom{n}{2}$, $k = 3$, $v = \binom{n}{2}$, $r = n-2$, $\lambda_1 = 1$ and $\lambda_2 = 0$ can always be constructed.

Proof: The proof is illustrated in the particular case $n = 5$, so that $v = 10$. Let the symbol labels $0, 1, \dots, 9$ be arranged in a triangular scheme as follows:

T:	·	0	1	2	3
	0	·	4	5	6
	1	4	·	7	8
	2	5	7	·	9
	3	6	8	9	·

First, form a 2 rowed array whose columns $(a_i, b_i)'$ are all possible unordered pairs arising from the columns of T, with the condition that: $a_i > b_i$ if a_i and b_i are in opposite sides to the principal diagonal of T, and $a_i < b_i$ if they are in the same side. This ensures that each symbol is replicated $n-2$ times in each row of the array. For any pair of first associates a_i and b_i, there is a unique pair of rows in T in which a_i and b_i appear in the same column of T; further, these two rows have one and only one symbol in common, c_i say. The required array consists of all column vectors $(a_i, b_i, c_i)'$ so defined.

In our example, the initial two-rowed array is:

0 0 0 1 1 2 4 4 5 7 1 2 2 5 3 3 3 6 6 8 6 5 4 8 8 7 7 9 9 9
1 2 3 2 3 3 5 6 6 8 4 5 7 7 6 8 9 8 9 9 0 0 0 4 1 4 1 5 7 2

and the third row corresponding to the c_i's is:

4 5 6 7 8 9 7 8 9 9 0 0 1 4 0 1 2 4 5 7 3 2 1 6 3 5 2 6 8 3

Bose and Bush (1952) constructed orthogonal arrays of strength 2 by a method of difference arrays. An analogue of their method can be used to generate some general cyclic association balanced arrays.

<u>Theorem 3.3.5</u>: Let it be possible to find a $k \times b_0$ array D with entries from a group $M = \{0, 1, ..., v-1\}$ $(b_0 = \ell_1 + \ell_2 + ... + \ell_s,\ s = \mathrm{int}(v/2))$ such that: among the b_0 differences arising from any two rows of D, each difference $\pm j$ modulo v $(j = 1, 2, ..., s)$ occurs ℓ_j times. Then, by adding the elements of M to the elements

of D and reducing modulo v, one generates $b = vb_0$ columns resulting to GCABA$(b,k,v,r;\ell_1,...,\ell_s)$ or GCABA$(b,k,v,r;\ell_1,...,\ell_{s-1},2\ell_s)$ according as v is odd or even.

Proof: Any two elements, in a given column of the initial array D whose difference is j or $-j$ $(j = 1,2,...,s-1)$, generate one replication of the j^{th} associate class $\mathfrak{C}_j$ of the GC(v) association scheme. If the difference is s or $-s$ mod v, it generates one replication of $\mathfrak{C}_s$ if $v = 2s+1$ and two replications of $\mathfrak{C}_s$ if $v = 2s$. □

For a GCABA$(b,k,v,r;\lambda_1,...,\lambda_s)$, (3.1.7) and (3.2.1) imply:

$$r = \begin{cases} \sum_{i=1}^{s}\lambda_i & \text{if } v = 2s+1 \\ \sum_{i=1}^{s-1}\lambda_i + \frac{\lambda_s}{2} & \text{if } v = 2s \end{cases}.$$

Hence λ_s must be even if v is even in accordance with Theorem 3.3.5.

Lemma 3.3.1: A GCABA$(b,k,v,r;\lambda_1,...,\lambda_s)$ with $v = mn$ and

$$\lambda_i = \begin{cases} \gamma_1 & \text{if i is a multiple of m} \\ \gamma_2 & \text{otherwise} \end{cases}$$

is equivalent to a GDABA$(b,k,v,r;\gamma_1,\gamma_2)$ in m groups of v symbols each.

Proof: Let the symbols $0,1,...,v-1$ be arranged in m groups:

$$G_j = \{j, m+j, 2m+j, ..., (n-1)m+j\} \quad j = 0,1,...,m-1.$$

Clearly, each pair of elements in G_j occurs γ_1 times in any two rowed subarrays of the GCABA, and each pair of elements from different groups occurs γ_2 times. □

The following are examples of initial arrays for GDABA's.

Example 3.3.8:

(i) GDABA(8, 2, 4, 2; 2, 1):

$$\begin{matrix} 0 & 0 \\ 1 & 2 \end{matrix}$$

(ii) GDABA(18, 3, 6, 3; 2, 1):

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 3 & 5 \end{matrix}$$

(iii) GDABA(32, 3, 8, 4; 2, 1):

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 6 & 4 & 7 & 5 \end{matrix}$$

However GCABA's that are not group divisible do exist as illustrated by the following example.

Example 3.3.9:

(i) GCABA(24, 4, 8, 3; 1, 0, 1, 2):

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 4 & 3 \\ 5 & 7 & 4 \\ 4 & 3 & 7 \end{matrix}$$

(ii) GCABA(15, 3, 5, 3; 1, 2):

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{matrix}$$

(iii) GCABA(27, 3, 9, 3; 0, 1, 1, 1):

$$\begin{matrix} 0 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 7 & 6 \end{matrix}$$

(iii) GCABA(44, 3, 11, 4; 0, 1, 1, 1, 1):

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 6 & 9 & 7 & 3 \end{matrix}$$

CHAPTER IV

ASSOCIATION-BALANCED ARRAYS AS TREATMENT-BLOCK DESIGNS

4.1. A Review on Optimality of Semi Balanced Arrays.

In what follows all combinatorial arrays such as SBA'a and ABA's are considered as block designs whose blocks, plots and treatments are respectively identified with columns, rows and symbols. Both terminologies are used for the sake of convenience. Hence ABA's form a particular subclass of $D(b, k, v)$.

Early contributions to the theory of optimal designs assume uncorrelated errors ($V = I$). Under this assumption BIBD's and a generalized version of them known as regular graph designs turn out to have very desirable qualities with respect to many optimality criteria and over different classes of designs. Examples are Kiefer (1975), John and Mitchell (1977) and Cheng (1979).

Theorem 4.1.1: Kiefer 1975. Assuming uncorrelated errors and no plot effects (i.e., $V = I$ and $\alpha = 0$ in Model 2.1.2), if a BIBD d^* with parameters b, k and v exists, then it is universally optimal over (b, k, v)-designs.

In the presence of plot effects, Kiefer (1975) also showed that some Latin squares and generalized Yonden designs are efficient with respect to universal optimality.

As pointed out earlier, for many combinations of the parameters b, k and v there exists no BIBD; this motivates the search for optimal designs among

PBIBD's. Examples of such contributions are Cheng (1978), Constantine (1983) and Cheng and Bailey (1991).

Given a design $d \in D(b, k, v)$, its incidence matrix is defined as the $v \times b$ matrix N_d whose $(i, u)^{th}$ entry is equal to 1 if treatment i occurs in block u and 0 otherwise, the concurrence matrix of d in the $v \times v$ matrix $B_d = N_d N_d$.

Theorem 4.1.2: Cheng and Bailey (1991). Assuming uncorrelated errors and no plot effects, if there exists a two-associate class $PBIB(b, k, v, r; \lambda_1, \lambda_2)$ with $|\lambda_1 - \lambda_2| = 1$, which is connected and has a singular concurrence matrix, then it is type-1 optimal over all equireplicate designs in $D(b, k, v)$.

In practice, the assumption of uncorrelated errors seems unrealistic. Recent contributions have considered models with a specified within block covariance matrix V; commonly used correlations are MA1, MA2 and AR1 recalled earlier. Kiefer and Wynn (1981) used ordinary least-squares in a two-step approach and obtained some weak universal optimality results within the class of Latin squares; they assumed a moving average covariance model. Other examples to this approach are Cheng (1983), Ipinyomi (1986), Russel and Eccleston (1987), Morgan and Chakravarti (1988).

Theorem 4.1.3: Morgan and Chakravarti (1988). Assuming an MA2 covariance structure, the existence of $SBA(b, k, v, 2; \lambda)$ implies the existence of a weakly universally optimal design over all BIBD's with parameters b, k and v.

Theorem 4.1.4: Morgan and Chakravarti (1988). Assuming an MA1 or MA2 covariance structure, a weakly universally optimal BIBD with block size 3 is equivalent to a semi balanced array.

A more efficient approach is to use generalized least squares (or best linear unbiased) estimation procedure instead of ordinary least squares. Examples are Kunert (1985, 1987), Azzalini and Giovagnoli (1987), Gill and Shukla (1985) and many others. In most of these papers, a first order autoregressive correlation is assumed.

Cheng (1988) was the first to consider an arbitrary covariance matrix V and obtained the following theorem.

<u>Theorem 4.1.5</u>: Cheng (1988). If a $SBA(b,k,v,2;\lambda)$ exists, then it is universally optimal over $D(b,k,v)$ for the generalized least-square estimation of treatment effects and for any within block correlation matrix assuming equivarible errors.

Martin and Eccleston (1991) investigated in more detail the optimality properties of SBA's and gave the following theorem.

<u>Theorem 4.1.6</u>: Martin and Eccleston (1991). If a $SBA(b,k,v,2;\lambda)$ exists, then:

(i) It is universally optimal under generalized least squares for any covariance matrix V over $D(b,k,v)$.

(ii) It is universally optimal under generalized least squares over all block designs for any V such that $w_{ij} \leq 0$ for $i \neq j$, where w_{ij}'s are the entries of $W(V) = V^{-1} - (1_k' V^{-1} 1_k)^{-1} V^{-1} J_k V^{-1}$.

(iii) It is weakly universally optimal under ordinary least squares over all BIBD's with parameters b, k and v and for any covariance V.

The above theorem, clearly, generalizes all preexisting results on the optimality of SBA's.

The authors obtained similar results with slightly less restrictive combinatorial conditions than those of a SBA by limiting the class of covariances to centrosymmetric V's (i.e., $v_{ij} = v_{k+1-i,\,k+1-j}$).

However, in many combinations of the parameters b, k and v, either there exists no SBA or the number of columns (blocks) required for its existence is very large. In these situations, the use of association-balanced arrays is suggested as an efficient alternative, as illustrated in the following sections.

4.2. A Characterization Theorem.

In this section, it is assumed that $\alpha = 0$ in Model (2.1.2) so that:

$$Y = T_d\tau + (I_b \otimes 1_k)\beta + \epsilon \qquad Var(\epsilon) = I_b \otimes V.$$

The information matrix for a design $d \in D(b,k,v)$, under generalized least squares estimation, is given in (2.2.4) as:

$$C_d(V) = \sum_{u=1}^{b} T'_{du} W T_{du}.$$

The optimality properties of SBA's recalled above essentially follow from the fact that:

$$C_d(V) = h(V)\, C_d(I) \quad \text{for all } V$$

whenever d is a SBA; i.e., the information matrix or, equivalently, the variance matrix for $\hat{\tau}$, differs from that under the usual uncorrelated model only by a constant. The next theorem characterizes the class of designs satisfying this condition for all positive definite V.

The following lemma given by Chakravarti (1975) will be needed.

Lemma 4.2.1: Let B be an $n \times n$ matrix with non-negative entries, s_i the sum of the i^{th} row of B and $D(s) = diag(s_1, ..., s_n)$. Then, B is irreducible if and only if

$Q = B - D(s)$ has rank $n-1$.

Definition 4.2.1: A $k \times b$ array d with v symbols will be called strongly uniform on the rows if:

(i) d is uniform on the rows (i.e., each symbol i occurs r_i times, say, in every row of d);

(ii) each unordered pair of symbols $\{i,j\}$ occurs λ_{ij} times as a column of any two-rowed subarray of d, λ_{ij}'s being independent of the particular choice of two rows.

It follows from the definition above that

$$b = \sum_{i=1}^{v} r_i \qquad \sum_{\substack{j=1 \\ j \neq i}}^{v} \lambda_{ij} = 2r_i \qquad i = 1, \ldots, v.$$

Denote by $\Delta(k, v, r_1, \ldots, r_v)$ the collection of all arrays with k rows and $b = \sum_{i=1}^{v} r_i$ columns that are strongly uniform on the rows and $\Delta(k, v, r, \ldots, r) = \Delta(k, v, r)$.

Clearly SBA's and ABA's are particular elements of $\Delta(k, v, r)$.

Theorem 4.2.1: Let $d \in D(b, k, v)$ and $k \geq 3$. Then, there exists a scalar $h(V)$ such that:

$$C_d(V) = h(V)\, C_d(I) \quad \text{for all positive definite } V \tag{4.2.1}$$

if and only if d is strongly uniform on the rows.

Proof: Let $r_{i\ell}$ be the number of occurrences of treatment i in row ℓ, R_i the number of occurrences of treatment i in the whole design d, $\lambda_{ij}^{\ell\ell'}$ the number of occurrences of the unordered pair of treatments $\{i,j\}$ as a column of the two-rowed subarray consisting of rows ℓ and ℓ' of d, so that:

$$\sum_{i=1}^{v} r_{i\ell} = b \qquad \sum_{\ell=1}^{k} r_{i\ell} = R_i \qquad \sum_{i=1}^{v} R_i = bk$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{v} \lambda_{ij}^{\ell\ell'} = r_{i\ell} + r_{i\ell'} \qquad 1 \leq \ell \neq \ell' \leq k.$$

Let $d \in D(b,k,v)$ be strongly uniform on the rows and write $r_i = R_i/k$ for the number of occurrences of treatment i in any row of d. Let $r^\delta = \text{diag}(r_1, ..., r_v)$ and $R^\delta = \text{diag}(R_1, ..., R_v) = kr^\delta$. The $(i,j)^{\text{th}}$ entry of $C_d(V)$ is:

$$C_{d \cdot ij} = \sum_{u=1}^{b} \sum_{\ell=1}^{k} \sum_{\ell'=1}^{k} t_{du \cdot \ell i} t_{du \cdot \ell' j} w_{\ell\ell'}$$

where $t_{du \cdot \ell i}$'s and $w_{\ell\ell'}$'s are, respectively, the entries of T_{du} and W.

Since no treatment is replicated more than once in any column, $t_{du \cdot \ell i} t_{du \cdot \ell' i} = 0$ if $\ell \neq \ell'$; hence,

$$C_{d \cdot ii} = \sum_{u=1}^{b} \sum_{\ell=1}^{k} t_{du \cdot \ell i} w_{\ell\ell} = \sum_{\ell=1}^{k} \Big\{ \sum_{u=1}^{b} t_{du \cdot \ell i} \Big\} w_{\ell\ell}$$

$$= r_i \sum_{\ell=1}^{k} w_{\ell\ell} = r_i \, \text{tr}(W); \qquad (4.2.2)$$

if $i \neq j$:

$$C_{d \cdot ij} = \sum_{\ell=1}^{k} \sum_{\ell'=1}^{k} \sum_{u=1}^{b} t_{du \cdot \ell i} t_{du \cdot \ell' j} w_{\ell\ell'}$$

$$= \sum_{1 \leq \ell < \ell' \leq k} \Big\{ \sum_{u=1}^{b} (t_{du \cdot \ell i} t_{du \cdot \ell' j} + t_{du \cdot \ell' i} t_{du \cdot \ell j}) \Big\} w_{\ell\ell'}$$

because W is symmetric and $t_{du \cdot \ell i} t_{du \cdot \ell j} = 0$.

The sum between brackets is equal to λ_{ij} by Definition 4.2.1(ii). Hence

$$C_{d \cdot ij} = \lambda_{ij} \sum_{\ell < \ell'} w_{\ell\ell'} = -\frac{\lambda_{ij}}{2} \operatorname{tr}(W) \tag{4.2.3}$$

because W is symmetric with row and column sums equal to zero.

A matrix expression of (4.2.2) and (4.2.3) is:

$$C_d(V) = \operatorname{tr}(W)\,(r^{\delta} - \tfrac{1}{2}\Lambda) \tag{4.2.4}$$

where $\Lambda_{ii} = 0$, $\Lambda_{ij} = \lambda_{ij}$ if $i \neq j$. For $V = I$, $W = I_k - \frac{1}{k} J_k$ and $\operatorname{tr}(W) = k-1$, so that $C_d(I) = (k-1)(r^{\delta} - \frac{1}{2}\Lambda)$; hence, $C_d(V) = h(V)\, C_d(I)$ where $h(V) = \frac{\operatorname{tr}(W)}{k-1}$.

Conversely, let $d \in D(b,k,v)$ satisfy (4.2.1). $C_d(V)$ has entries:

$$C_{d \cdot ii}(V) = \sum_{\ell=1}^{k} r_{i\ell} w_{\ell\ell} \qquad i = 1, \ldots, v$$

$$C_{d \cdot ij}(V) = \sum_{1 \le \ell < \ell' \le k} \lambda_{ij}^{\ell\ell'} w_{\ell\ell'} \qquad 1 \le i \neq j \le v,$$

so that

$$C_{d \cdot ii}(I) = \frac{k-1}{k} R_i$$

$$C_{d \cdot ij}(I) = -\frac{1}{k} \sum_{\ell < \ell'} \lambda_{ij}^{\ell\ell'}.$$

(4.2.1) implies:

$$\sum_{\ell=1}^{k} r_{i\ell} w_{\ell\ell} = h(V) \frac{k-1}{k} R_i \qquad i = 1, \ldots, v \tag{4.2.5}$$

$$\sum_{\ell<\ell'} \lambda_{ij}^{\ell\ell'} w_{\ell\ell'} = -\frac{h(V)}{k} \sum_{\ell<\ell'} \lambda_{ij}^{\ell\ell'} \qquad i \neq j \qquad (4.2.6);$$

summing equations (4.2.5) over i:

$$b \operatorname{tr}(W) = h(V) \frac{k-1}{k} \sum_{i=1}^{v} R_i = h(V)\,(k-1)b \;\Rightarrow\; h(V) = \frac{\operatorname{tr}(W)}{k-1}.$$

Let $V^{(p,q)}$ $(1 \le p < q \le k)$ be the covariance matrix corresponding to the model where only observations within the same column in row labels p and q are correlated, all other observations being uncorrelated.

Write $$\mathcal{V} = \left\{V^{(1,2)}, \ldots, V^{(1,k)}, V^{(2,3)}, \ldots, V^{(k-1,k)}\right\}.$$

In particular, d satisfies (4.2.1) for each element of $\mathcal{V}$.

Without loss of generality, assume $\sigma^2 = 1$. Hence

$$V^{(1,2)} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and I is the identity matrix of order $k-2$

$$[V^{(1,2)}]^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix} \qquad A^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

The corresponding $W[V^{(1,2)}] = W^{(1,2)}$, say, has entries:

$$w_{11}^{(1,2)} = w_{22}^{(1,2)} = \frac{1}{1-\rho^2} - \frac{2}{w(1+\rho)^2} = g, \text{ say,}$$

$$w_{\ell\ell}^{(1,2)} = 1 - \tfrac{1}{w} = h, \text{ say,} \qquad \ell \notin \{1,2\}$$

$$w_{12}^{(1,2)} = -\frac{\rho}{1-\rho^2} - \frac{2}{w(1+\rho)^2} = a_0', \text{ say,}$$

$$w_{1\ell}^{(1,2)} = w_{2\ell}^{(1,2)} = -\frac{2}{w(1+\rho)} = a_1', \text{ say,} \qquad \ell \notin \{1,2\}$$

$$w_{\ell\ell'}^{(1,2)} = -\tfrac{1}{w} = a_2', \text{ say} \qquad \{\ell, \ell'\} \cap \{1,2\} = \phi$$

where $$w = 1_k'[V^{(1,2)}]^{-1}1_k = (k-2) + \tfrac{2}{1+\rho}$$

and $$\text{tr}\,[W^{(1,2)}] = 2g + (k-2)h = t, \text{ say.}$$

For any $V^{(p,q)} \in \mathcal{V}$:

$$V^{(p,g)} = P\,V^{(1,2)}P'$$

where P is a permutation matrix permuting rows 1 and 2 of $V^{(1,2)}$ with rows p and q. Let $W^{(p,q)} = W(V^{(p,q)})$. It is easily seen that:

$$W^{(p,q)} = PW^{(1,2)}P'$$

so that:

$$w_{pp}^{(p,q)} = w_{qq}^{(p,q)} = g$$

$$w_{\ell\ell}^{(p,q)} = h \quad \text{if } \ell \notin \{p,q\}$$

$$w_{p\ell}^{(p,q)} = w_{q\ell}^{(p,q)} = a_1' \quad \ell \notin \{p,q\}$$

$$w_{pq}^{(p,q)} = a_0'$$

$$w_{\ell\ell'}^{(p,q)} = a_2' \quad \text{if } \{\ell,\ell'\} \cap \{p,q\} = \phi$$

and tr[W] = t is invariant over $\mathcal{V}$.

Writing (4.2.5) respectively for $V = V^{(p,q)}$ and $V = V^{(p,q')}$:

$$(r_{ip}+r_{iq})g + h \sum_{\substack{\ell=1 \\ \ell \notin \{p,q\}}}^{k} r_{i\ell} = \tfrac{t}{k} R_i$$

$$(r_{ip}+r_{iq'})g + h \sum_{\substack{\ell=1 \\ \ell \notin \{p,q'\}}}^{k} r_{i\ell} = \tfrac{t}{k} R_i$$

$$\Rightarrow (r_{iq}-r_{iq'})(g-h) = 0.$$

For $\rho = \frac{1}{2}$ $\quad g-h = \frac{1}{3}(1 + \frac{5}{3k-2}) \neq 0 \Rightarrow r_{iq} = r_{iq'}$

hence d must be uniform on the rows.

Similarly, writing (4.2.6) for each $V \in \mathscr{V}$ yields a system of $\binom{k}{2}$ equations:

$$a_0' \lambda_{ij}^{pq} + a_1' \sum_{\substack{\ell=1 \\ \ell \notin \{p,q\}}}^{k} \left(\lambda_{ij}^{p\ell} + \lambda_{ij}^{q\ell}\right) + a_2' \sum_{\substack{1 \leq \ell < \ell' \leq k \\ \{\ell,\ell'\} \cap \{p,q\} = \phi}}^{k} \lambda_{ij}^{\ell\ell'}$$

$$= -\frac{tr(W)}{k(k-1)} \sum_{1 \leq \ell < \ell' \leq k} \lambda_{ij}^{\ell\ell'}, \quad 1 \leq p < q \leq k,$$

or equivalently:

$$a_1 \Big\{ \sum_{\substack{\ell=1 \\ \ell \notin \{p,q\}}}^{k} (\lambda_{ij}^{p\ell} + \lambda_{ij}^{q\ell}) \Big\} + a_2 \Big\{ \sum_{\substack{1 \leq \ell < \ell' \leq k \\ \{\ell,\ell'\} \cap \{p,q\} = \phi}} \lambda_{ij}^{\ell\ell'} \Big\} - a_0 \lambda_{ij}^{pq} = 0$$

$$1 \leq p < q \leq k.$$

where

$$a_0 = -a_0' - \frac{tr(W)}{k(k-1)}$$

$$a_1 = a_1' + \frac{tr(W)}{k(k-1)}$$

$$a_2 = a_2' + \frac{tr(W)}{k(k-1)}.$$

Let $\lambda = (\lambda_{ij}^{12}, \lambda_{ij}^{13}, \ldots, \lambda_{ij}^{1k}, \lambda_{ij}^{23}, \ldots, \lambda_{ij}^{k-1,k})'$. The above equations can be written in matrix form as:

$$Q\lambda = 0 \tag{4.2.7}$$

where Q is the square matrix of order $\binom{k}{2}$ given by:

$$Q = a_1 B_1 + a_2 B_2 - a_0 I$$

where B_1 and B_2 are 0–1 matrices of order $\binom{k}{2}$ such that:

$$B_1 + B_2 = J - I.$$

From the first part of the proof, it follows that a vector λ with all entries equal is a solution to (4.2.7). To complete the proof, it remains to show that the null space of Q has dimension 1 or, equivalently, $\mathrm{rank}(Q) = \binom{k}{2} - 1$ for some value of ρ. Each row of Q sums to:

$$s(Q) = 2(k-2)a_1 + \frac{(k-2)(k-3)}{2} a_2 - a_0$$

$$= 2(k-2)\, a_1' + \frac{(k-2)(k-3)}{2} a_2' + a_0' + \frac{tr(W)}{2}$$

$$= \sum_{\ell < \ell'} w_{\ell\ell'} + \frac{tr(W)}{2} = 0\,.$$

Hence, each row of $B = a_1 B_1 + a_2 B_2$ sums to $s(B) = a_0$.

For $\rho = \frac{1}{2}$ and $k > 2$, simple computations yield:

$$a_0 = \frac{(k-2)(2k^2-k+1)}{k(k-1)(3k-2)} > 0$$

$$a_1 = \frac{k^2-k+2}{k(k-1)(3k-2)} > 0$$

$$a_2 = \frac{2}{k(k-1)(3k-2)} > 0\,.$$

B is irreducible and lemma 4.2.1 implies $Q = B - a_0 I$ is of rank $\dim(B) - 1$.

A similar characterization theorem can be obtained for the ordinary least squares estimator $\tilde{\tau}$ of $\theta = (I_v - \frac{1}{v} J_v)\tau$.

Let $D_d(\tilde{\tau}, V)$ be the variance matrix of $\tilde{\tau}$ for a design d and a covariance matrix V.

Theorem 4.2.2: Let $d \in D(b,k,v)$ and $k \geq 3$. Then:

$$D_d(\bar{\tau}, V) = g(V) D_d(\bar{\tau}, I) \text{ for all positive definite } V \qquad (4.2.8)$$

if and only if d is strongly uniform on the rows.

Proof: The expression of $D_d(\bar{\tau}, V)$ is given, for example, in Martin and Eccleston (1991) by:

$$D_d(\bar{\tau}, V) = [A_d(I)]^- [A_d(V)] [A_d(I)]^-$$

where $$A_d(V) = T_d'(I_b \otimes W_0) T_d = \sum_{u=1}^{b} T_{du}' W_0 T_{du}$$

and $$W_0 = E_k V E_k \qquad E_k = I_k - \tfrac{1}{k} J_k.$$

(4.2.8) is equivalent to:

$$A_d(V) = g(V) A_d(I).$$

The rest of the proof goes along the lines of the proof of Theorem 4.2.1 and yields:

$$g(V) = \frac{tr(W_0)}{k-1}. \qquad \square$$

For $k = 2$:

$$V = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

$$W[V] = \frac{1}{1-\rho}(I - \tfrac{1}{2}J) = tr[W_0(V)]\, W_0(I)$$

$$W_0(V) = (1-\rho)(I - \tfrac{1}{2}J) = tr[W_0(V)]\, W_0(I).$$

Hence, (4.2.1) and (4.2.8) hold for any $d \in D(b,2,v)$.

It is to be noted that strong uniformity on the rows is a sufficient condition for (4.2.1) to hold even in the presence of row effects ($\alpha \neq 0$), since $C_d(V)$ for the general model (2.1.2) is the same as for the reduced model if d is uniform on the rows.

4.3. Type-1 Optimal Association-Balanced Arrays.

Recall that $\Delta(k,v,r)$ denotes the class of $k \times b$ arrays that are strongly uniform on the rows with all r_i's equal to r. Using Cheng and Bailey's approach (Theorem 4.1.2 above), the objective is to show that some ABA's constructed in Section 3.3 are type-1 optimal over $\Delta(k,v,r)$.

Bose and Mesner (1959) found the eigenvalues of the concurrence matrix B_d of a two associate class PBIB$(b,k,v,r;\lambda_1,\lambda_2)$ to be:

$$\begin{cases} \theta_0 = rk \\ \theta_1 = r-\frac{1}{2}\{(\lambda_1-\lambda_2)(-\gamma-\sqrt{\delta})+(\lambda_1+\lambda_2)\} \\ \theta_2 = r-\frac{1}{2}\{(\lambda_1-\lambda_2)(-\gamma+\sqrt{\delta})+(\lambda_1+\lambda_2)\} \end{cases} \tag{4.3.1}$$

with respective multiplicities:

$$m_0 = 1$$

$$m_1 = \frac{n_1+n_2}{2} - \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\delta}}$$

$$m_2 = \frac{n_1+n_2}{2} + \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\delta}}$$

where $\gamma = p^2_{12} + p^1_{12}$, $\beta = p^2_{12} - p^1_{12}$, $\delta = \gamma^2+2\beta+1$ and n_i, p^i_{jk} $(i,j,k=1,2)$ are the parameters of the underlying association scheme.

(4.3.1) and Remark 3.2.1 (ii) above give the eigenvalues of the concurrence matrix of a two associate class ABA$(b,k,v,r;\lambda_1,\lambda_2)$ as:

$$
\begin{cases}
\nu_0 = rk^2 \\
\nu_1 = k[r - \frac{k-1}{4}\{(\lambda_1-\lambda_2)(-\gamma-\sqrt{\delta}) + (\lambda_1+\lambda_2)\}] \\
\nu_2 = k[r - \frac{k-1}{4}\{(\lambda_1-\lambda_2)(-\gamma+\sqrt{\delta}) + (\lambda_1+\lambda_2)\}]
\end{cases}
\tag{4.3.2}
$$

with respective multiplicities m_0, m_1 and m_2 above.

The next lemma is stated as Theorem 2.1 in Cheng and Bailey (1991) and proved in Cheng (1981).

Lemma 4.3.1: Let $\mathfrak{C} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n;\ x_i \geq 0,\ i = 1, ..., n;\ \sum_{i=1}^n x_i = c\}$ for some constant c. If $\mathfrak{C}$ contains an element x^* such that:

(i) $x_i^* > 0 \quad i = 1, 2, ..., n$;

(ii) there are only two distinct values among $x_1^*, ..., x_n^*$;

(iii) x^* minimizes $\sum x_i^2$ over $\mathfrak{C}$;

(iv) x^* maximizes $\max_i\{x_i\}$ over $\mathfrak{C}$.

Then, x^* minimizes $\sum f(x_i)$ over $\mathfrak{C}$ for all sufficiently differentiable real valued functions f such that $f''(x) > 0$, $f'''(x) < 0$ for $x > 0$ and $\lim_{x \to 0^+} f(x) = \infty$.

The following lemma, stated as Lemma 3.1 in Cheng (1978), is also needed.

Lemma 4.3.2: For given positive integers p and q, the maximum of $\sum_{i=1}^p e_i^2$ subject to $\sum_{i=1}^p e_i = q$, where the e_i's are non-negative integers, is attained when $q - p \cdot \text{int}(q/p)$ of the e_i's are equal to $\text{int}(q/p) + 1$ and the remaining e_i's are equal to $\text{int}(q/p)$.

Theorem 4.3.1: If there exists a connected two associate class $ABA(b, k, v, r; \lambda_1, \lambda_2)$ with $|\lambda_1 - \lambda_2| = 1$, whose concurrence matrix is singular, then it is type-1 optimal over $\Delta(k, v, r)$ for any within block covariance matrix V.

Proof: Let $\mu_{d1} \geq \mu_{d2} \geq \cdots \geq \mu_{dv} = 0$ be the eigenvalues of C_d. Write

$$\mu_d = (\mu_{d1}, \ldots, \mu_{dv-1}) \quad \text{and} \quad \mathfrak{C} = \{\mu_d\colon\ d \in \Delta(k,v,r)\}.$$

Let d^* be an ABA satisfying the conditions of Theorem 4.3.1. To prove the theorem it suffices to show that $\mathfrak{C}$ and μ_{d^*} satisfy the condition of Lemma 4.3.1.

For any $d \in \Delta(k,v,r)$, (4.2.4) implies:

$$\mathrm{tr}(C_d) = \sum_{i=1}^{V-1} \mu_{di} = b\,\mathrm{tr}(W) = c, \text{ say;}$$

d^* is connected $\Rightarrow$ $\mu_{d^*i} > 0$ for $i = 1, 2, \ldots, v-1$. The concurrence matrix of d is:

$$B_d = rk\,I + \binom{k}{2}\Lambda$$

where Λ is defined in (4.2.4). Hence

$$C_{d^*} = \mathrm{tr}(W)\,[rI - \tfrac{1}{2}\Lambda] = \mathrm{tr}(W)\,[\tfrac{rk}{k-1}I - \tfrac{1}{k(k-1)}B_d]. \tag{4.3.3}$$

(4.3.2) and (4.3.3) yield the nonzero eigenvalues of C_{d^*} as:

$$\psi_1 = \mathrm{tr}(W)\,[r - \tfrac{1}{4}\{(\lambda_1 - \lambda_2)(-\gamma - \sqrt{\delta}) + (\lambda_1 + \lambda_2)\}$$

$$\psi_2 = \mathrm{tr}(W)\,[r - \tfrac{1}{4}\{(\lambda_1 - \lambda_2)(-\gamma + \sqrt{\delta}) + (\lambda_1 + \lambda_2)\} \tag{4.3.4}$$

with multiplicities m_1 and m_2 such that $m_1 + m_2 = v-1$. Hence, the entries of μ_{d^*} take only 2 distinct values ψ_1 and ψ_2. This proves conditions (i) and (ii) of Lemma 4.3.1.

Since B_d is non-negative definite, (4.3.3) implies:

$$\mu_{di} \leq \tfrac{rk}{k-1}\mathrm{tr}(W) \quad \text{for any } d \in \Delta(k,v,r) \text{ and } i = 1, 2, \ldots, v-1$$

hence $$\max_i \{\mu_{di}\} \leq \tfrac{rk}{k-1}\mathrm{tr}(W) \quad \text{for all } d \in \Delta(k,v,r).$$

Since B_{d^*} is singular, (4.3.3) implies that C_{d^*} has at least one eigenvalue equal to $\frac{rk}{k-1}$ tr(W). Hence μ_{d^*} maximizes $\max_i\{\mu_{di}\}$ over $\mathfrak{C}$. This proves condition (iv) of Lemma 4.3.1.

It remains to verify condition (iii). Let $A = \frac{2}{tr(W)} C_d = 2rI - \Lambda$.

$$\Sigma_{i=1}^{v-1} \mu_{di}^2 = tr(\mathfrak{C}_d^2) = [\tfrac{tr(W)}{2}]^2 \, tr(A^2).$$

Simple computations yield:

$$tr(A^2) = 4vr^2 + 2 \sum_{1 \le i < j \le v} \lambda_{ij}^2 .$$

Apply Lemma 4.3.2 to the $p = \binom{v}{2}$ non-negative integers λ_{ij}'s $(1 \le i < j \le v)$ whose sum is:

$$q = \sum_{i<j} \lambda_{ij} = b.$$

Suppose first $\lambda_1 - \lambda_2 = 1$; put $\lambda_1 = \lambda+1$ and $\lambda_2 = \lambda$ if λ_{ij}^*'s are the index parameters of d^*; then:

$$|\{\lambda_{ij}^*;\, i<j,\ \text{i and j first associates}\}| = \frac{vn_1}{2}$$

$$|\{\lambda_{ij}^*;\, i<j,\ \text{i and j second associates}\}| = \frac{vn_2}{2} .$$

It remains to show that:

$$q - p \cdot int(q/p) = \frac{vn_1}{2} .$$

but $$q/p = b/\binom{v}{2} = \frac{2r}{v-1} = \frac{n_1\lambda_1 + n_2\lambda_2}{v-1} = \lambda + \frac{n_1}{v-1}$$

$$n_1 < v-1 \Rightarrow int(q/p) = \lambda$$

$$q - p\cdot \mathrm{int}(q/p) = vr - \frac{v(v-1)}{2}\lambda$$

$$= v\left\{\frac{n_1(\lambda+1)+n_2\lambda}{2} - \frac{v-1}{2}\right\}$$

$$= \frac{v}{2}\{n_1(\lambda+1) + n_2\lambda - (n_1+n_2)\lambda\}$$

$$= \frac{vn_1}{2}.$$

The case $\lambda_2 - \lambda_1 = 1$ is proved similarly. □

The following corollaries give some series of ABA's constructed in Section 3.3 and satisfying the conditions of Theorem 4.3.1.

Corollary 4.3.1: The GDABA's with parameters $b = n^2\binom{m}{2}$, $k = m$, $v = mn$, $r = n(m-1)/2$, $\lambda_1 = 0$ and $\lambda_2 = 1$ constructed in Corollary 3.3.1 are type-1 optimal over $\Delta(k, v, r)$.

Proof: Let d^* be a GDABA with the above parameters. The eigenvalues of its concurrence matrix B_{d^*} computed from (4.3.2) and (3.1.3) are:

$$\nu_0 = rk^2 \quad \text{with multiplicity } m_0 = 1;$$

$$\nu_1 = rk \quad \text{with multiplicity } m_1 = m(n-1);$$

$$\nu_2 = 0 \quad \text{with multiplicity } m_2 = m-1;$$

and those of $\mathcal{C}_{d^*}$ computed from (4.3.4) are:

$$\psi_1 = 0 \qquad \psi_1 = r\,\mathrm{tr}(W) \qquad \psi_2 = \tfrac{v}{2}\,\mathrm{tr}(W) \tag{4.3.5}$$

with the same multiplicities m_0, m_1 and m_2. Hence, B_{d^*} is singular, d^* is connected and Theorem 4.3.1 applies. □

Example 4.3.1: Let $d_1 \in \Delta(3,6,2)$ be the array:

0	0	1	1	2	2	3	3	4	4	5	5
2	2	3	3	4	4	5	5	0	0	1	1
4	4	5	5	0	0	1	1	2	2	3	3

and let d_2 be the GDABA(12,3,6,2;0,1) given in Example 3.3.7. Corollary 4.3.1 implies that d_2 is type-1 better than d_1 for any within block covariance matrix V; in particular, it is A, D and E better over all association-balanced arrays.

Corollary 4.3.2: The TABA's with parameters $b = (n-2)\binom{n}{2}$, $k = n-1$, $v = \binom{n}{2}$, $r = n-2$, $\lambda_1 = 1$ and $\lambda_2 = 0$ constructed in Corollary 3.3.2 are type-1 optimal over $\Delta(k,v,r)$.

Proof: Let d^* be a TABA with the above parameters; the eigenvalues of its concurrence matrix B_{d^*} computed from (4.3.2) and (3.1.4) are:

$$\nu_0 = r(n-1) \quad \text{with multiplicity } m_0 = 1;$$

$$\nu_1 = m + \tfrac{n(n-1)(n-4)}{2} \quad \text{with multiplicity } m_1 = n-1;$$

$$\nu_2 = 0 \quad \text{with multiplicity } m_2 = \tfrac{n(n-3)}{2};$$

and those of C_{d^*} computed from (4.3.4) are:

$$\psi_0 = 0 \qquad \psi_1 = \tfrac{n}{2}\,\mathrm{tr}(W) \qquad \psi_2 = (n-1)\,\mathrm{tr}(W) \tag{4.3.6}$$

with the same multiplicites m_0, m_1 and m_2. Hence, B_{d^*} is singular, d^* is connected and Theorem 4.3.1 applies. □

4.4. Efficiency Relative to Universal Optimality.

Gill and Shukla (1985) used two efficiency criteria to measure the performance of designs for a first order autoregressive covariance structure. The same criteria are adopted here to measure the performance of association-balanced arrays with an arbitrary covariance matrix V.

If d is a connected design, the A and D optimalities call for the maximization of the functionals:

$$\varphi_A(d) = (v-1)\Big(\sum_{i=1}^{v-1} \mu_{d,i}^{-1}\Big)^{-1}$$
$$\varphi_D(d) = \Big(\prod_{i=1}^{v-1} \mu_{d,i}\Big)^{\frac{1}{v-1}} \tag{4.4.1}$$

where $\mu_{d,i}$'s are the nonzero eigenvalues of $C_d(V)$.

Since $tr(C_d) = b\,tr(W)$ is constant over $D(b,k,v)$, a design d in $D(b,k,v)$ would be universally optimal if its information matrix is completely symmetric (see Proposition 2.3.1) or, equivalently, if all $\mu_{d,i}$'s are equal. Let d^* be a hypothetical universally optimal design whose information matrix has a single nonzero eigenvalue $\mu = (\mu_{d,1} + \dots + \mu_{d,v-1})/(v-1)$ with multiplicity $v-1$. Then $\varphi_A(d)$ and $\varphi_D(d)$ are maximized for this design; their maximum value is:

$$\varphi_A(d^*) = \varphi_D(d^*) = \mu.$$

The A and D efficiencies relative to the hypothetical universally optimal design are then defined by:

$$e_A(d) = \frac{\varphi_A(d)}{\varphi_A(d^*)}$$
$$e_D(d) = \frac{\varphi_D(d)}{\varphi_D(d^*)}. \tag{4.4.2}$$

If d is a SBA, both efficiencies attain their maximum value 1 in agreement with the universal optimality of these arrays (Theorem 4.1.6). In the remainder of this section it is shown that the ABA's constructed in Section 3.3 are highly efficient with respect to both e_A and e_D.

<u>GDABA's:</u>

Let d be a GDABA with parameters $b = n^2\binom{m}{2}$, k, $v = mn$, $r = n(m-1)/2$, $\lambda_1 = 0$ and $\lambda_2 = 1$; the nonzero eigenvalues of its information matrix C_d are ψ_1 and ψ_2 with respective multiplicities m_1 and m_2 given in (4.3.5). Hence

$$\varphi_A(d) = (v-1)\left(\frac{m_1}{\psi_1} + \frac{m_2}{\psi_2}\right)^{-1} = \frac{v(v-1)(m-1)}{2m(v-2)+2}\,tr(W)$$

$$\varphi_D(d) = \left(\psi_1^{m_1} \times \psi_2^{m_2}\right)^{\frac{1}{v-1}} = \tfrac{n}{2}\,tr(W)\left((m-1)^{m_1} \times m^{m_2}\right)^{\frac{1}{v-1}}$$

$$\mu = \tfrac{1}{v-1}\,tr(C_d) = \tfrac{b}{v-1}\,tr(W) = \tfrac{vr}{v-1}\,tr(W).$$

Replacing $m_1 = m(n-1)$, $m_2 = m-1$ in the expressions above, the efficiencies of the design computed from (4.4.2) are:

$$e_A(d) = \frac{(v-1)^2}{v(v-2)+n} > \frac{v-1}{v}$$

$$e_D(d) = \frac{v-1}{v}\left(\frac{m}{m-1}\right)^{\frac{m-1}{v-1}} > \frac{v-1}{v}.$$

In general, the eigenvalues of the information matrix of a GDABA with arbitrary index parameters λ_1 and λ_2 can be computed from (4.3.4) and (3.1.3).

Table 4.4.1: e_A and e_D efficiencies of GDABA's.

m	n	v	b	k	λ_1	λ_2	e_A	e_D	Construction
2	2	4	4	2	0	1	0.900	0.944	Trivial
			8	2	2	1	0.964	0.982	Ex.3.3.8
	4	8	16	2	0	1	0.942	0.966	Trivial
3	2	6	12	3	0	1	0.961	0.980	Ex.3.3.7
			18	3	2	1	0.980	0.990	Ex.3.3.8
	3	9	27	3	0	1	0.969	0.983	Ex.3.3.2
	4	12	48	3	0	1	0.975	0.986	Theor.3.3.3
4	2	8	24	4	0	1	0.980	0.989	Ex.3.2.5
			32	3	2	1	0.987	0.993	Ex.3.3.8
5	2	10	40	3	0	1	0.987	0.993	Ex.3.3.6
	3	15	90	4	0	1	0.989	0.994	Coroll.3.3.5

TABA's:

Let d be a TABA with parameters $b = (n-2)\binom{n}{2}$, $k, v = \binom{n}{2}$, $r = n-2$, $\lambda_1 = 1$ and $\lambda_2 = 0$. The nonzero eigenvalues of its information matrix C_d are ψ_1 and ψ_2 with respective multiplicities m_1 and m_2 given in (4.3.6).

$$\varphi_A(d) = (v-1)\left(\frac{m_1}{\psi_1} + \frac{m_2}{\psi_2}\right)^{-1} = (v-1)\left\{\frac{(n-2)(n^2+3n-2)}{2n(n-1)}\right\}^{-1} tr(W)$$

$$\varphi_D(d) = \left(\psi_1^{m_1} \times \psi_2^{m_2}\right)^{\frac{1}{v-1}} = \tfrac{1}{2}\left[n^{m_1} \times \{2(n-1)\}^{m_2}\right]^{\frac{1}{v-1}} tr(W)$$

$$\mu = \tfrac{1}{v-1}\, tr(C_d) = \tfrac{v}{v-1}(n-2)\, tr(W).$$

Hence:
$$e_A^{-1}(d) = \frac{\mu}{\varphi_A(d)} = 1 + \frac{n-2}{(n+1)^2}$$

$$\varphi_D(d) = \frac{\varphi_D(d)}{\mu} = \frac{n+1}{n}\left\{\frac{n}{2(n-1)}\right\}^{\frac{n-1}{v-1}}.$$

Table 4.4.2: e_A and e_D efficiencies of TABA's.

n	v	b	k	λ_1	λ_1	e_A	e_D	Construction
4	6	12	3	1	0	0.961	0.980	Ex.3.3.3
5	10	30	3	1	0	0.947	0.973	Thm.3.3.4
6	15	60	5	1	0	0.942	0.972	Coroll.3.3.2

LABA's:

If d is an LABA with parameters $b = n^2(n-1)$, $k, v = n^2$, $r = n-1$, $\lambda_1 = 1$ and $\lambda_2 = 0$, the eigenvalues of its information matrix and their respective multiplicities can be computed from (4.3.4) and (3.1.6). Simular computations to the above yield:

$$e_A(d) = 1-\frac{n-1}{n(n+3)} \qquad e_D(d) = \frac{n+1}{n} 2^{-2/(n+1)}.$$

The LABA(18, 3, 9, 2; 1, 0) given in Example 3.3.4 has efficiencies:

$$e_A = 0.888 \qquad e_D = 0.942.$$

GCABA's:

Let d be a GCABA$(b, k, v, r; \lambda_1, ..., \lambda_s)$, $s = \text{int}(v/2)$.

A square matrix $A = (a_{ij})$ of order v is said to be regular circulant if $a_{ij} = a_{pq}$ whenever $j-i = q-p$ modulo v; such a matrix is then completely determined by its first row. A good review of these matrices and their algebraic properties can be found in Graybill (1981). In particular, the eigenvalues of a regular circulant matrix with first row $(a_0, a_1, ..., a_{v-1})$ are:

$$\psi_j = a_0 + a_1\omega_j^1 + a_2\omega_j^2 + ... + a_{v-1}\omega_j^{v-1} \quad j = 1, ..., v \tag{4.4.3}$$

where $\omega_j = \exp\left(\frac{2ij\pi}{v}\right)$ is the j^{th} root of unity. The entries of $C_d(V)$ are:

$$C_{d \cdot i,j} = r\,\mathrm{tr}(W) = c_0, \text{ say,}$$

$$C_{d \cdot i,i+j} = C_{d \cdot i+j,i} = -\frac{\lambda_j}{2}\,\mathrm{tr}(W) = c_j, \text{ say,} \quad j = 1, \ldots, s \tag{4.4.4}$$

Clearly, $C_d(V)$ is regular circulant with first row:

$$(c_0, c_1, \ldots, c_s, c_s, \ldots, c_1) \qquad \text{if } v = 2s+1$$

$$(c_0, c_1, \ldots, c_{s-1}, c_s, c_{s-1}, \ldots, c_1) \qquad \text{if } v = 2s.$$

Suppose $v = 2s+1$. From (4.4.3) the eigenvalues of C_d are:

$$\mu_{dj} = c_0 + \sum_{i=1}^{s} c_i\left(\omega_j^i + \omega_j^{2s+1-i}\right)$$

$$= c_0 + 2\sum_{i=1}^{s} c_i \cos\left(\frac{2ij\pi}{v}\right) \qquad j = 1, \ldots, v$$

but $\qquad c_0 = -2\sum_{i=1}^{s} c_i.$

The eigenvalues of C_d can then be expressed as:

$$\psi_0 = 0 \quad \text{with multiplicity 1,}$$

$$\psi_j = \sum_{i=1}^{s} \lambda_i\left\{1 - \cos\left(\frac{2ij\pi}{v}\right)\right\}\mathrm{tr}(W) \qquad j = 1, \ldots, s$$

each with multiplicity 2.

Hence $\qquad \varphi_A(d) = s\left(\sum_{j=1}^{s} \frac{1}{\psi_j}\right)^{-1}$

$$\varphi_D(d) = \prod_{j=1}^{s} \psi_j^{\frac{1}{s}}$$

and $\qquad \mu = \frac{1}{v-1}\,\mathrm{tr}(C_d) = \mathrm{tr}(W)\,\frac{v}{v-1}\sum_{i=1}^{s} \lambda_i$

yield the e_A and e_D efficiencies of d.

If v = 2s, similar computations to the above yield the eigenvalues of $C_d(V)$ as:

$$\psi_0 = 0 \quad \text{with multiplicity 1,}$$

$$\psi_s = \sum_{i=1}^{s} \gamma_i\left\{1-(-1)^i\right\} \text{tr(W)} \quad \text{with multiplicity 1}$$

$$\psi_j = \sum_{i=1}^{s} \gamma_i\left\{1-\cos\left(\tfrac{2ij\pi}{v}\right)\right\} \text{tr(W)} \qquad j = 1,\dots,s-1$$

each with multiplicity 2

where $\gamma_j = \lambda_j$ for $j = 1,\dots,s-1$ and $\gamma_s = \frac{\lambda_s}{2}$. Hence:

$$\varphi_A(d) = (2s-1)\left\{\sum_{j=1}^{s-1} \frac{2}{\psi_j} + \frac{1}{\psi_s}\right\}^{-1}$$

$$\varphi_D(d) = \left(\psi_s \prod_{j=1}^{s-1} \psi_j^2\right)^{\frac{1}{2s-1}}$$

and $$\mu = \tfrac{1}{v-1}\,\text{tr}(C_d) = \tfrac{2s}{2s-1}\,\text{tr(W)} \sum_{i=1}^{s} \gamma_j.$$

The efficiencies are then computed from 4.4.2.

Table 4.4.3: e_A and e_D efficiencies of GCABA's.

	e_A	e_D
GCABA(15, 3, 5, 3; 1, 2)	0.977	0.988
GCABA(24, 4, 8, 4; 1, 0, 1, 2)	0.907	0.957
GCABA(27, 3, 9, 3; 0, 1, 1, 1)	0.963	0.981
GCABA(44, 3, 11, 4; 0, 1, 1, 1, 1)	0.977	0.988

The initial arrays for the construction of the arrays above are given in Example 3.3.9.

4.5. Variances of Elementary Treatment Contrasts.

If all observations are uncorrelated and a BIBD is used in Model (2.1.2), it is well known that all elementary treatment contrasts $\tau_i - \tau_j$, $1 \le i \ne j \le v$, are estimated with the same variance. Such a desirable property is known as variance balance. It is also known that PBIBD's achieve partial variance balance, in the sense that any two elementary treatment contrasts are estimated with the same variance if the corresponding pair of treatments belong to the same associate class. Hence, (4.2.1) implies that any association-balanced array achieves partial variance balance for any within block positive definite covariance matrix V.

Explicit expressions for the variances of elementary treatment contrasts for an association-balanced array can be derived, using some algebraic properties of association schemes given, for example, in Delsarte (1973) and Bailey (1985).

An association scheme in v symbols and s associate classes is also characterized by s matrices $A_1, A_2, \ldots, A_s$ where A_i is the $v \times v$ $0-1$ matrix whose $(p,q)^{th}$ entry is equal to 1 if and only if p and q are i^{th} associates.

Writing $A_0 = I_v$, the $s+1$ matrices $A_0, A_1, \ldots, A_v$ satisfy the following relations:

$$\begin{cases} A_i \text{ is symmetric} \quad i = 0, 1, \ldots, v \\ \sum_{i=0}^{s} A_i = J_v \\ A_i A_j = \sum_{k=0}^{s} p_{ij}^k A_k \end{cases} \qquad (4.5.1)$$

For any association matrix A_j, there exist real numbers e_{ij} $(i = 0, 1, \ldots, s)$ such that:

$$A_j = \sum_{i=0}^{s} e_{ij} S_i \qquad j = 0, 1, \ldots, s \qquad (4.5.2)$$

where $S_0, S_1, ..., S_s$ are symmetric idempotent and mutually orthogonal matrices. The $(s+1)\times(s+1)$ matrix $E = (e_{ij})$ is called the character table of the scheme. E is invertible with inverse $F = (f_{ij})$, say, so that:

$$S_i = \sum_{j=0}^{s} f_{ji} A_j \qquad i = 0, 1, ..., s. \tag{4.5.3}$$

If d is an $ABA(b, k, v, r; \lambda_1, ..., \lambda_s)$, its information matrix $C_d(V)$ can be expressed as:

$$C_d(V) = tr(W)\,[rI - \tfrac{1}{2}(\lambda_1 A_1 + ... + \lambda_s A_s)]$$

$$= tr(W) \sum_{i=0}^{s} \theta_i A_i, \text{ say,} \tag{4.5.4}$$

where A_i's are the association matrices of the underlying association scheme, $\theta_o = r$, $\theta_i = -\lambda_i/2$, $i = 1, 2, ..., s$.

$$(4.5.2) \Rightarrow \quad C_d(V) = tr(W)\left\{\sum_{i=0}^{s} \theta_i \sum_{j=0}^{s} e_{ji} S_j\right\}$$

$$= tr(W) \sum_{i=0}^{s}\left\{\sum_{j=0}^{s} \theta_i e_{ji}\right\} S_j$$

$$= tr(W) \sum_{i=0}^{s} e_j S_j \text{ , say,} \tag{4.5.5}$$

where $e_j = \sum_{i=0}^{s} \theta_i e_{ji}$;

hence $\qquad C_d^- = Var(\hat{\tau}) = \frac{1}{tr(W)} \sum_{\{j=e_j \neq 0\}} e_j^{-1} S_j.,$

S_j's being idempotent and mutually orthogonal.

Replacing S_j by its expression in (4.5.3):

$$Var(\hat{\tau}) = \frac{1}{tr(W)} \sum_{i=0}^{s} \varphi_i A_i \qquad \varphi_i = \sum_{\{j=e_j \neq 0\}} e_j^{-1} f_{ij}. \tag{4.5.6}$$

Hence $\text{var}(\hat{\tau}_p) = \varphi_0/\text{tr}(W)$, $p = 1, ..., v$, and if p and q are i^{th} associates:

$$\text{Cov}(\hat{\tau}_p, \hat{\tau}_q) = \frac{\varphi_i}{\text{tr}(W)}$$

so that:

$$\text{Var}(\hat{\tau}_p - \hat{\tau}_q) = \frac{2}{\text{tr}(W)}(\varphi_0 - \varphi_i) = v_i, \text{ say,} \tag{4.5.7}$$

GDABAS:

The character table of the group divisible association scheme GD(m, n) is given in Bailey (1985) as:

$$E = \begin{bmatrix} 1 & n-1 & n(m-1) \\ 1 & n-1 & -n \\ 1 & -1 & 0 \end{bmatrix};$$

its inverse is:

$$F = \frac{1}{nm}\begin{bmatrix} 1 & m-1 & m(n-1) \\ 1 & m-1 & -m \\ 1 & -1 & 0 \end{bmatrix}.$$

Let d be a GDABA with parameters $b = n^2\binom{m}{2}$, $k, v = nm$, $r = n(m-1)/2$, $\lambda_1 = 0$ and $\lambda_2 = 1$.

$$(4.5.4) \Rightarrow \theta_0 = \frac{n(m-1)}{2} \quad \theta_1 = 0 \quad \theta_2 = -\frac{1}{2}$$

$$(4.5.5) \Rightarrow e_0 = 0 \quad e_1 = \frac{nm}{2} \quad e_2 = \frac{n(m-1)}{2}$$

$$(4.5.6) \Rightarrow \varphi_0 = \frac{2}{nm}\left(\frac{n-1}{nm} + \frac{m(n-1)}{n(m-1)}\right)$$

$$\varphi_1 = \frac{2}{nm}\left[\frac{m-1}{nm} - \frac{m}{n(m-1)}\right]$$

$$\varphi_2 = \frac{2}{(nm)^2}$$

$$(4.5.7) \Rightarrow \begin{cases} v_1 = \frac{4}{n(m-1)\, tr(W)} \\ v_2 = \frac{v-1}{v} v_1 \end{cases} \tag{4.5.8}$$

TABA'S:

The inverse of the character table of the triangular association scheme T(n) is given in Ogawa and Ishii (1965) as:

$$F = \frac{2}{n(n-1)} \begin{bmatrix} 1 & n-1 & n(n-3)/2 \\ 1 & (n-1)(n-4)/2(n-2) & -n(n-3)/2(n-2) \\ 1 & -2(n-1)/(n-2) & n/(n-2) \end{bmatrix};$$

its inverse is:

$$E = \begin{bmatrix} 1 & 2(n-2) & (n-2)(n-3)/2 \\ 1 & n-4 & -(n-3) \\ 1 & -2 & 1 \end{bmatrix}.$$

Similar computations to the above with:

$$\theta_0 = (n-2) \quad \theta_1 = -\tfrac{1}{2} \quad \theta_2 = 0$$

yield: $$e_0 = 0 \quad e_1 = \tfrac{n}{2} \quad e_2 = n-1$$

$$\varphi_0 = \frac{4}{n^2} + \frac{n-3}{(n-1)^2}$$

$$\varphi_1 = \frac{1}{n-2}\left\{\frac{2(n-4)}{n^2} - \frac{(n-3)}{(n-1)^2}\right\}$$

$$\varphi_2 = \frac{1}{n-2}\left\{\frac{2}{(n-1)^2} - \frac{8}{n^2}\right\}$$

and

$$\begin{cases} v_1 = \frac{n+1}{v\, tr(W)} \\ v_2 = \frac{n+2}{v\, tr(W)} = v_1 + \frac{1}{v\, tr(W)} \end{cases} \tag{4.5.9}$$

RABA's:

Let d be a RABA with parameters $b = n\binom{m}{2} + m\binom{n}{2}$, $k, v = mn$, $r = (m+n)/2$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$.

The character table of the rectangular association scheme R(m,n) is also given in Bailey (1985) as:

$$E = \begin{bmatrix} 1 & n-1 & m-1 & (n-1)(m-1) \\ 1 & -1 & m-1 & -(m-1) \\ 1 & n-1 & -1 & -(n-1) \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Computing $F = E^{-1}$ and proceeding as before, the three distinct values of elementary treatment contrast variances corresponding to the three associate classes of the scheme turn out to be:

$$\begin{cases} v_1 = \frac{4}{v\,tr(W)}\left\{1 + \frac{n(m-1)}{n+m}\right\} \\ v_2 = \frac{4}{v\,tr(W)}\left\{1 + \frac{m(n-1)}{m+n}\right\} \\ v_3 = \frac{4}{v\,tr(W)}\left\{1 + \frac{mn}{m+n}\right\} \end{cases} \qquad (4.5.10)$$

Taking m = n in the expressions above, one obtains the variances corresponding to an L_2-type ABA with $v = n^2$, $\lambda_1 = 1$ and $\lambda_2 = 0$. these are:

$$\begin{cases} v_1 = \frac{4}{n^2\,tr(W)}\left\{1 + \frac{n-1}{2}\right\} \\ v_2 = \frac{4}{n^2\,tr(W)}\left\{1 + \frac{n}{2}\right\} \end{cases} \qquad (4.5.11)$$

Expressions (4.5.8) through (4.5.11) suggest that the departure from exact variance balance is not very large and tends to 0 as the number of treatment increases.

GCABA's:

Clearly, the association matrices of the GC(v) association scheme are regular circulant, and their eigenvalues are easily obtained from (4.4.3).

Let d be a GCABA$(b,k,v,r;\lambda_1,\dots,\lambda_s)$, suppose $v=2s+1$. Simple but lengthy algebraic computations give the entries e_{ij}'s of the character table of the scheme as:

$$e_{i0} = 1 \quad i=0,1,\dots,s$$

$$e_{ij} = 2\cos\left(\tfrac{2ij\pi}{v}\right) \quad j=1,\dots,s \quad i=0,\dots,s.$$

The idempotents of the scheme are:

$$S_0 = \tfrac{1}{v}J_v \quad S_j = \tfrac{2}{v}\Omega_j \quad j=1,\dots,s$$

where Ω_j is the $v\times v$ matrix whose $(p,q)^{th}$ entry is $\cos(2ij\pi(p-q)/v)$. Hence the variances of elementary treatment contrasts are:

$$v_i = \frac{4}{v\,tr(W)} \sum_{\{j,\,e_j \neq 0\}} e_j^{-1}\left\{1-\cos\left(\tfrac{2ij\pi}{v}\right)\right\} \quad i=1,\dots,s$$

$$\text{where } e_j = \sum_{i=1}^{s}\lambda_i\left\{1-\cos\left(\tfrac{2ij\pi}{v}\right)\right\} \quad j=0,1,\dots,s.$$

Similar but slightly different expressions can be obtained for the case $v=2s$.

4.6. Efficiency Relative to Randomization.

In this section an ABA$(b,k,v,r;\lambda_1,\lambda_2)$, d say, is compared to a competing design, g say, in terms of the average variance of elementary treatment contrasts. A competing design considered here is either a BIB$(b,k,v,rk;\gamma)$ if it exists, or a PBIB$(b,k,v,rk;\gamma_1,\gamma_2)$ with $|\gamma_1-\gamma_2|=1$.

For the competing design, there are b sets of k treatments each, the individual treatments of a set are assigned randomly to the plots of a block.

For a design d, consider the model:

$$Y = T_d\tau + (I_b \otimes 1_k)\beta + \epsilon$$

$$E(\epsilon) = 0 \qquad \text{Var}(\epsilon) = I_b \otimes V = \sigma^2(I_b \otimes R)$$

where R is a correlation matrix within each block.

The information matrix of the generalized least squares estimate $\hat{\theta}$ of $\theta = (I_V - \frac{1}{V}J_V)\tau$ is given by:

$$C_d(R) = \sigma^{-2}T'_d[I_b \otimes (E_kRE_k)^-]T_d = \sigma^{-2}A_d(R) \text{ say,}$$

where $E_k = I_k - \frac{1}{k}J_k$.

The average variance of elementary treatment contrasts, estimated from a design d is given by Kempthorne (1956) as:

$$V_d = \frac{2\sigma^2}{v-1}\sum_{i=1}^{v-1}\gamma_{d,i}^{-1} \qquad (4.6.1)$$

where γ_{di}'s are the nonzero eigenvalues of $A_d(R)$. This average for the proposed systematic designs will be compared to the average variance obtained by using randomization with ordinary least squares for a BIB or a PBIB (cf. Gill and Shukla, 1985b).

The average coreelation matrix over the randomization is:

$$\bar{R} = \frac{1}{k!}\sum_{P \in \mathcal{P}} PRP'$$

where $\mathcal{P}$ is the set of all $k \times k$ permutation matrices.

Simple computations give:

$$\bar{R} = \alpha_1 I_k + \alpha_2 J_k \quad \text{with } \alpha_1 = 1 - \frac{s(R)}{k} \quad \alpha_2 = \frac{s(R)}{k}$$

where s(R) is the sum of the upper (or lower) off diagonal entries of R.

Since $E_k J_k = 0$, the corresponding average information matrix for the competing design g can be written as:

$$C_g(\bar{R}) = (\alpha_1 \sigma^2)^{-1} C_g(I).$$

Hence, the average variance of elementary treatment contrasts estimated from g can be obtained by using the ordinary least squares analysis where observations are uncorrelated each with variance $\alpha_1 \sigma^2$.

If g is a BIB or a PBIB design, the expression of this average is:

$$V_g = \frac{2\alpha_1 \sigma^2}{pE} \tag{4.6.2}$$

where p is the number of replications and E the overall efficiency factor of g.

Define the efficiency of a design d relative to a competing design g by:

$$e = \frac{V_g}{V_d}$$

Table 4.6.1 below, gives the efficiencies of four examples of ABA's d_1, d_2, d_3 and d_4 , where:

$$\begin{aligned} d_1 &= \text{GDABA}(12,3,6,2;0,1) \\ d_2 &= \text{GDABA}(24,3,8,3;0,1) \\ d_3 &= \text{TABA}(30,3,10,3;1,0) \\ d_4 &= \text{LABA}(18,3,9,2;1,0) \end{aligned}$$

The respective competing designs are:

$$g_1 = \text{GDPBIB}(12,3,6,6;3,2)$$

$$g_2 = \text{GDPBIB}(24,3,8,9;2,3)$$

$$g_3 = \text{BIB}(30,3,10,9;2).$$

$$g_4 = \text{LPBIB}(18,3,9,6;1,2).$$

The PBIBD's g_1, g_2 and g_4 and their efficiency factors are given in

Clatworthy(1973) where they are respectively labeled R43, R58 and LS12.

The efficiency of d_i relative to g_i, $i = 1.2.3.4$, is evaluated for both autoregressive (AR1) and moving average (MA1) correlations.

For the MA1 model with k = 3, the correlation matrix

$$R = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$$

is positive definite if and only if $|\rho| < \frac{1}{\sqrt{2}}$. $\alpha_1 = 1 - \frac{2\rho}{3}$.

For the AR1 model with k = 3, the correlation matrix

$$R = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}$$

is positive definite for $|\rho| < 1$. $\alpha_1 = 1 - \frac{2\rho + \rho^2}{3}$.

For the autoregressive case, the designs d_i $(i = 1, 2, 3, 4)$ turn out to perform better for highly correlated observations. For the moving average, they perform better for highly and positively correlated observations.

Table 4.6.1: e efficiencies of association-balanced arrays.

Autoregressive

ρ	d_1	d_2	d_3	d_4
–0.9000	3.6701	3.7031	3.5744	3.3957
–0.8000	2.0670	2.0856	2.0131	1.9124
–0.7000	1.5468	1.5607	1.5065	1.4312
–0.6000	1.2983	1.3099	1.2644	1.2012
–0.5000	1.1592	1.1696	1.1289	1.0725
–0.4000	1.0755	1.0852	1.0475	0.9951
–0.3000	1.0243	1.0335	0.9976	0.9477
–0.2000	0.9940	1.0029	0.9681	0.9197
–0.1000	0.9784	0.9872	0.9528	0.9052
0	0.9737	0.9825	0.9483	0.9009
0.1000	0.9778	0.9866	0.9523	0.9047
0.2000	0.9892	0.9981	0.9634	0.9152
0.3000	1.0070	1.0160	0.9807	0.9317
0.4000	1.0308	1.0400	1.0039	0.9537
0.5000	1.0603	1.0698	1.0326	0.9810
0.6000	1.0954	1.1053	1.0669	1.0135
0.7000	1.1364	1.1466	1.1068	1.0514
0.8000	1.1835	1.1942	1.1527	1.0950
0.9000	1.2373	1.2484	1.2050	1.1448
	Moving Average			
–0.7000	1.0834	1.0931	1.0551	1.0024
–0.6000	1.0603	1.0698	1.0326	0.9810
–0.5000	1.0386	1.0480	1.0115	0.9610
–0.4000	1.0189	1.0280	0.9923	0.9427
–0.3000	1.0015	1.0105	0.9754	0.9266
–0.2000	0.9874	0.9962	0.9616	0.9135
–0.1000	0.9775	0.9863	0.9520	0.9044
0	0.9737	0.9825	0.9483	0.9009
0.1000	0.9087	0.9875	0.9532	0.9055
0.2000	0.9973	1.0063	0.9713	0.9227
0.3000	1.0386	1.0480	1.0115	0.9610
0.4000	1.1221	1.1322	1.0928	1.0382
0.5000	1.2983	1.3099	1.2644	1.2012
0.6000	1.7527	1.7684	1.7070	1.6216
0.7000	4.1545	4.1918	4.0462	3.8438

CHAPTER V

PARTIALLY BALANCED REPEATED MEASUREMENTS DESIGNS

5.1. Introduction.

An experiment in which each of b experimental units is exposed to a sequence of treatments during k periods of time, there being v treatments in all, is called a repeated measurements experiment. A repeated measurements design is one that determines which treatment is applied to which unit at which period, and the collection of all such designs is denoted by RMD(b, k, v). A design in RMD(b, k, v) may then be considered as a $k \times b$ array of v symbols whose columns and rows correspond, respectively, to units and periods.

In such an experiment it is usually assumed that each treatment manifests its effect not only during the period of its application (direct effect), but also during the subsequent period in which some other treatment is applied (residual effect).

A comprehensive review on the theory of optimal repeated measurement designs is given by Matthews (1988). In the case of uncorrelated observations, Constantine and Hedayat (1982) constructed some series of RMD's that are variance balanced for residual effects; Blaisdell and Raghavarao (1980) constructed some RMD's that are partially variance balanced for residual effects as well as for direct effects.

In the following, a special class of ABA's is defined, namely, ordered association-balanced arrays (or OABA's).. They are shown to achieve partial variance balanced for both direct and residual effects, assuming an arbitrary within period covariance structure but no fixed period effects. In particular, transitive arrays provide variance-balanced RMD's for both effects.

Definition 5.1.1: Given an association scheme with v symbols and s classes a $k \times b$ array of v symbols will be said ordered association-balanced if

(i) Each symbol occurs r times in each row.

(ii) No symbol occurs more than once in any column.

(iii) Any ordered pair of symbols that are i^{th} associates occurs λ_i times as a column of any two rowed subarray, $i = 1, ..., s$.

Denote such an array by $OABA(b, k, v, r; \lambda_1, ..., \lambda_s)$.

Clearly an $OABA(b, k, v, r; \lambda_1, ..., \lambda_s)$ is also an $ABA(b, k, v, r; 2\lambda_1, ..., 2\lambda_s)$, and if all λ_i's are equal to λ the array is simply a $TA(b, k, v, r; \lambda)$.

Hence, necessary conditions for the existence of these arrays are:

$$b = vr \qquad r = \sum_{i=1}^{s} n_i \lambda_i$$

where n_i's are parameters of the underlying association scheme.

For $d \in RMD(b, k, v)$, assume the linear model:

$$y_{u\ell} = \tau_{d(u,\ell)} + \beta_u + \rho_{d(u,\ell-1)} + \epsilon_{u\ell} \qquad (5.1.1)$$

$$u = 1, 2, ..., b \qquad \ell = 1, 2, ..., k$$

where $d(u, \ell)$ is the treatment assigned by d to the u^{th} unit at the ℓ^{th} period, $d(u, 0) = 0$, τ_i the direct effect of treatment i, ρ_i the residual effect of treatment i, β_u the effect of unit u and $\epsilon_{u\ell}$'s are random errors with zero means such that:

$$\mathrm{cov}(\epsilon_{u\ell}, \epsilon_{u'\ell'}) = \begin{cases} v_{\ell\ell'} & \text{if } u = u' \\ 0 & \text{if } u \neq u' \end{cases} .$$

Model (5.1.1) is clearly the same as Model (2.1.1) except for the addition of residual effects ρ_i's and the absence of period (or plot) effects α_ℓ's. Hence, adopting similar notations to those of Section 2.1, the matrix form of (5.1.1) is:

$$Y = T_d\tau + (I_b \otimes 1_k)\beta + F_d\rho + \epsilon \tag{5.1.2}$$

$$\mathrm{Var}(\epsilon) = I_b \otimes V \qquad V = [v_{\ell\ell'}]$$

where $F_d = [F'_{d1} \mid \ldots \mid F'_{db}]'$, and $F_{du}(u=1,\ldots,b)$ is the $k \times v$ 0–1 matrix whose $(\ell, i)^{th}$ entry is equal to 1 if and only if treatment i is applied to period $\ell - 1$ of unit u.

The variance stabilizing transform of (5.1.2) is:

$$\begin{aligned} (I_b \otimes S)Y &= (I_b \otimes S)T_d\tau + (I_b \otimes S1_k)\beta + (I_b \otimes S)F_d\rho + (I_b \otimes S)\epsilon \\ &= \tilde{T}_d\tau + U\beta + \tilde{F}_d\rho + \tilde{\epsilon}, \text{ say,} \end{aligned} \tag{5.1.3}$$

where S is a $k \times k$ matrix such that $SVS' = I_k$.

5.2. Partial Variance Balance.

The information matrix for direct effects can be derived as:

$$C_d = C_{d\cdot 11} - C_{d\cdot 12} C^-_{d\cdot 22} C_{d\cdot 21} \tag{5.2.1}$$

and for residual effects as:

$$\tilde{C}_d = C_{d\cdot 22} - C_{d\cdot 21} C^-_{d\cdot 11} C_{d\cdot 12} . \tag{5.2.2}$$

where $\quad C_{d\cdot 11} = \tilde{T}'_d \omega^\perp(U) \tilde{T}_d$

$$C_{d\cdot 12} = \tilde{T}'_d \omega^{\perp}(U)\tilde{F}_d$$

$$C_{d\cdot 21} = \tilde{F}'_d \omega^{\perp}(U)\tilde{T}_d$$

$$C_{d\cdot 22} = \tilde{F}'_d \omega^{\perp}(U)\tilde{F}_d \tag{5.2.3}$$

and $\omega^{\perp}(U)$ is the projection operator onto the orthogonal complement of the column space of $U = I_b \otimes S1_k$.

The design matrix for residual effects F_d can be related to T_d by the relations: $F_{du} = RT_{du}$ $(u = 1, ..., b)$ where R is the $k \times k$ matrix whose entry r_{ij} is equal to 1 if $i = j+1$ and $j < k$ and 0 otherwise.

$$R = \begin{bmatrix} 0 & & \cdots & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$$

If d is such that no treatment occurs more than once in any period, similar computations to those in the proof of Lemma 2.2.1 yield:

$$\begin{cases} C_{d\cdot 11} = \sum_{u=1}^{b} T'_{du} W T_{du} \\ C_{d\cdot 12} = C'_{d\cdot 21} = \sum_{u=1}^{b} T'_{du} W R T_{du} \\ C_{d\cdot 22} = \sum_{u=1}^{b} T'_{du} R' W R T_{du} \end{cases} \tag{5.2.4}$$

where $W = V^{-1} - (1'_k W 1_k)^{-1} V^{-1} J_k V^{-1} = [w_{\ell\ell'}]$.

<u>Lemma 5.2.1</u>: If d is $OABA(b, k, v, \tau; \lambda_1, ..., \lambda_s)$, then

(i) $C_{d\cdot 11} = tr(W)(rI - \lambda_1 A_1 - ... - \lambda_s A_s)$,

(ii) $C_{d\cdot 12} = C'_{d\cdot 21} = tr(WR)(rI - \lambda_1 A_1 - ... - \lambda_s A_s)$,

(iii) $C_{d\cdot 22} = r\{tr(W) - w_{11}\} I - \{tr(W) - 2w_{11}\}(\lambda_1 A_1 + ... + \lambda_s A_s)$, where $A_1, A_2, \ldots, A_s$ are the association matrices of the underlying association scheme.

Proof: Write $C_A = \sum_{u=1}^{b} T'_{du} A T_{du}$ where $A = [a_{ij}]$ is an arbitrary $k \times k$ matrix.

The entries of C_A can be expressed as:

$$\begin{cases} [C_A]_{ii} = r\,\mathrm{tr}(A) & i = 1, \ldots, v \\ [C_A]_{ij} = \lambda_\ell \sum_{p \neq q} a_{pq} & \text{if i and j are } \ell^{th} \text{ associates.} \end{cases} \tag{5.2.5}$$

Proof of (i): $C_{d \cdot 11} = C_W$.

Recall that all entries of W sum to 0, so that $\sum_{p \neq q} w_{pq} = -\mathrm{tr}(W)$; hence (i) is a matrix form of (5.2.5).

Proof of (ii): $C_{d \cdot 12} = C_{WR}$.

Let $w_1, w_2, \ldots, w_k$ be the columns of W, then $A = WR = [w_2 \,|\, w_3 \,|\, \ldots \,|\, w_k \,|\, 0]$, since each column of W also sums to 0, $\sum_{p \neq q} a_{pq} = -\mathrm{tr}(WR)$; hence (ii) is also a matrix form of (5.2.5).

Proof of (iii): $C_{d \cdot 22} = C_{R'WR}$.

Let W be partitioned as:

$$W = \left[\begin{array}{c|ccc} w_{11} & w_{12} & \cdots & w_{1k} \\ \hline w_{12} & & & \\ \vdots & & W_{22} & \\ w_{1k} & & & \end{array}\right];$$

then:

$$A = R'WR = \left[\begin{array}{c|c} W_{22} & 0 \\ \hline 0 & 0 \end{array}\right].$$

$$\mathrm{tr}(A) = \mathrm{tr}(W_{22}) = \mathrm{tr}(W) - w_{11}$$

$$\sum_{p,q} a_{pq} = -\sum_{j=2}^{k} w_{1j} = w_{11}$$

and
$$\sum_{p \neq q} a_{pq} = w_{11} - \mathrm{tr}(A) = 2w_{11} - \mathrm{tr}(W).$$

Hence (iii) is a matrix form of (5.2.5) with $A = R'WR$. □

Theorem 5.2.1: the existence of an ordered association-balanced array implies the existence of a partially variance balanced repeated measurements design with respect to both residual and direct treatment effects and for any within period covariance matrix V.

Proof: Lemma 5.2.1 above shows that the matrices $C_{d \cdot ij}$, $i,j = 1,2$, belong to the association algebra $\sigma(A_0, A_1, ..., A_s)$ generated by the association matrices $A_0 = I$, $A_1, ..., A_s$ which is closed under the g-inverse operation. Hence, (5.2.1) and (5.2.2) imply that C_d and $\tilde{C}_d$ also belong to the algebra and so does their g-inverse, i.e.,

$$\mathrm{Var}(\widehat{\tau}) = \sum_{i=0}^{s} \varphi_i A_i \qquad \mathrm{Var}(\widehat{\rho}) = \sum_{i=0}^{s} \psi_i A_i$$

for some real numbers φ_i and ψ_i $(i = 0, 1, ..., s)$ so that, if p and q are i^{th} associates:

$$\begin{cases} \mathrm{var}(\widehat{\tau}_p - \widehat{\tau}_q) = 2(\varphi_0 - \varphi_i) \\ \mathrm{var}(\widehat{\rho}_p - \widehat{\rho}_q) = 2(\psi_0 - \psi_i) \end{cases} \qquad i = 1, 2, ..., s.$$

5.3. Construction of Ordered Association-Balanced Arrays.

Two period RMD's are of great importance in clinical trials (Grizzle 1965, O'Neill 1977, Hill and Armitage 1979, Armitage and Hill 1982, Willan and Pater 1986). The construction of the corresponding 2 rowed OABA's is straightforward.

Example 5.3.1: Group divisible OABA$(8,2,4,2;0,1)$ with groups $G_1 = \{0,2\}$ and $G_2 = \{1,3\}$.

$$\begin{matrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 & 3 & 0 & 1 & 2 \end{matrix}.$$

Some construction methods of ABA's given in Section 3.3 can be easily adjusted to the construction of OABA's.

Theorem 5.3.1: The existence of a PBIB$(b,k,v,r;\lambda_1,...,\lambda_s)$ and of a TA$(\lambda k(k-1), q,k,r;\lambda)$ imply the existence of an OABA in $\lambda bk(k-1)$ columns, q rows, v symbols and index parameters $\lambda\lambda_i$ $(i = 1,2,...,s)$.

Proof: Apply the construction given in the proof of Theorem 3.3.1 to the transitive array $S = TA(\lambda k(k-1),q,k,r;\lambda)$ and the PBIBD $P = PBIB(b,k,v,r;\lambda_1,...,\lambda_s)$.

Hence the GDABA$(36,4,6,6;2,4)$ in Example 3.3.1 is also a GDOABA$(36,4,6,6;1,2)$.

Theorem 5.3.2: The existence of a TA$(\gamma m(m-1),p,m,2;\gamma)$ and of OA$(n^2,q,n,2)$ imply the existence of a group divisible OABA with parameters $b = \gamma n^2 m(m-1)$, $k = \min(p,q)$, $v = mn$, $r = \gamma n(m-1)$, $\lambda_1 = 0$ and $\lambda_2 = \gamma$.

Proof: This is an analogue of Corollary 3.3.5, where a TA is used instead of a SBA.

Theorem 5.3.3: A GDOABA with parameters $b = 6n^2$, $k = 3$, $v = 3n$, $r = 2n$, $\lambda_1 = 0$, $\lambda_2 = 1$ can always be constructed.

Proof: Let the groups of the GD$(3,n)$ association scheme be $G_i = \{t_{i1},...,t_{in}\}$, $i = 1,2,3$.

Using the same notations given in the proof of Theorem 3.3.3, the

GDOABA is obtained by juxtaposition of the GDABA given in Theorem 3.3.3 to the GDABA:

$$\begin{bmatrix} \vec{t}_{11} \dots \vec{t}_{1n} & \vec{t}_{21} \dots \vec{t}_{2n} & \vec{t}_{31} \dots \vec{t}_{3n} \\ t_{3\cdot} \dots t_{3\cdot} & t_{1\cdot} \dots t_{1\cdot} & t_{2\cdot} \dots t_{2\cdot} \\ t^0_{2\cdot} \dots t^{n-1}_{2\cdot} & t^0_{3\cdot} \dots t^{n-1}_{3\cdot} & t^0_{1\cdot} \dots t^{n-1}_{1\cdot} \end{bmatrix}.$$

Example 5.3.2: A GDOABA$(24, 3, 6, 4; 0, 1)$ is:

$$\begin{matrix} 0\,0 & 3\,3 & 1\,1 & 4\,4 & 2\,2 & 5\,5 & 0\,0 & 3\,3 & 1\,1 & 4\,4 & 2\,2 & 5\,5 \\ 1\,4 & 1\,4 & 2\,5 & 2\,5 & 0\,3 & 0\,3 & 2\,5 & 2\,5 & 0\,3 & 0\,3 & 1\,4 & 1\,4 \\ 2\,5 & 5\,2 & 0\,3 & 3\,0 & 1\,4 & 4\,1 & 1\,4 & 4\,1 & 2\,5 & 5\,2 & 0\,3 & 3\,0 \end{matrix}.$$

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