EVALUATION OF CUMULATIVE PROBABILITIES FOR WISHART AND MULTIVARIATE BETA MATRICES AND THEIR LATENT ROOTS *

by

Donald St. P. Richards and Rameshwar D. Gupta

University of North Carolina and University of New Brunswick

ABSTRACT

We consider the problems of evaluating $\Pr(S > \Omega)$, when S has a Wishart or multivariate beta distribution and Ω is a given positive definite matrix. The most general results are obtained for the central distributions, while special cases are evaluated for the non-central distributions. These results are applied to obtain the distributions of ℓ_{min}, the smallest latent root of S. We also evaluate special cases of $\Pr(\Omega < S < \Lambda)$ for the central distributions and thence derive expressions for the probability that all the latent roots of S lie in a given interval.

AMS 1970 Subject Classification: Primary 62H10; Secondary 33A30.

Key Words and Phrases: Hypergeometric functions of matrix argument; Wishart; multivariate beta; latent roots; zonal polynomials.

---

*Parts of this work were done while both authors were at the University of the West Indies.

1. INTRODUCTION

In a major breakthrough for the theory of multivariate distributions, Constantine (1963) showed that the hypergeometric functions of matrix argument which were introduced by Herz (1955) could be expanded in series of zonal polynomials. Various aspects of the theory are thoroughly discussed by James (1964) and Johnson and Kotz (1972).

When S is a random matrix having the Wishart or multivariate beta distribution, Constantine (1963) evaluated $\Pr(S < \Omega)$, the probability that $\Omega - S$ is positive definite, Ω being any given positive definite matrix. More recently, Davis (1979, 1980) has extended Constantine's results to the non-central cases. Davis' results are expressed in terms of a class of polynomials having two matrix arguments which generalize the classical zonal polynomials.

In this article, we obtain expressions which allow the evaluation of the "complementary" probabilities $\Pr(S > \Omega)$. For the central Wishart matrix, the result involves Muirhead's (1970) confluent hypergeometric function of matrix argument of the second kind, while the case of the central beta matrix is settled using the corresponding result of Constantine. As an application, we obtain the distribution of $\ell_{\min}$, the smallest latent root of S. In the non-central cases, the expressions obtained involve Davis' polynomials; even so, we have been able to provide only partial solutions there.

In another context, Jack (1966) has discussed integrals of the form $\int_A^B (\det S)^{\alpha}\, dS$, where "$\int_A^B$" denotes integration over all positive definite S for which $A < S < B$. We also derive results for this problem and make applications to evaluating the probability that all the latent roots of S are contained in a given interval.

2. PRELIMINARIES

Our notation will coincide with that used by Constantine (1963) or Davis (1979, 1980). Throughout, κ, λ, ϕ, and ρ are partitions of the non-negative integers k, ℓ, $f = k + \ell$, and r respectively, while $C_\kappa(X)$ is the zonal polynomial of the symmetric $m\times m$ matrix X and corresponding to κ. Davis (1979, 1980) has introduced a class of polynomials $C_\phi^{\kappa,\lambda}(X,Y)$ of $m\times m$ symmetric matrix arguments X and Y, which are invariant under the transformation $(X,Y) \to (HXH', HYH')$, $H \in O(m)$, the group of $m\times m$ orthogonal matrices. For convenience, we list the following results, which are proven in Davis' articles:

$$C_\phi^{\kappa,\lambda}(X,X) = \theta_\phi^{\kappa,\lambda}\, C_\phi(X) \,, \tag{2.1}$$

where $\theta_\phi^{\kappa,\lambda} = C_\phi^{\kappa,\lambda}(I,I)/C_\phi(I)$;

$$C_\kappa(X)C_\lambda(Y) = \sum_{\phi\in\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X,Y) \,, \tag{2.2}$$

where $\phi \in \kappa \cdot \lambda$ denotes that the irreducible representation of $GL(m,R)$, the group of $m\times m$ real invertible matrices, indexed by 2ϕ, appears in the decomposition of the tensor product $2\kappa \otimes 2\lambda$ of the irreducible representations indexed by 2κ and 2λ;

$$C_\kappa(X)C_\lambda(X) = \sum_{\phi^*\in\kappa\cdot\lambda} g_{\kappa,\lambda}^{\phi^*} C_{\phi^*}(X) \,, \qquad g_{\kappa,\lambda}^{\phi^*} = \sum_{\phi\equiv\phi^*}(\theta_\phi^{\kappa,\lambda})^2 \,, \tag{2.3}$$

where $\sum_{\phi^*\in\kappa\cdot\lambda}$ denotes that the sum is over the inequivalent representations $2\phi^*$ appearing in $2\kappa \otimes 2\lambda$, while $\sum_{\phi\equiv\phi^*}$ denotes summation over all representations equivalent to $2\phi^*$ in $2\kappa \otimes 2\lambda$;

$$(2.4)\qquad \int_{S>0} \exp(-\mathrm{tr}\ RS)(\det S)^{t-p} C_\phi^{\kappa,\lambda}(ASA', B)dS$$
$$= \Gamma_m(t)(t)_\kappa\ (\det R)^{-t} C_\phi^{\kappa,\lambda}(AR^{-1}A', B)\ ,$$

where $\mathrm{Re}(R) > 0$, $\mathrm{Re}(t) > p - 1$;

$$(2.5)\qquad C_\phi(X+Y) = \sum_{\kappa,\lambda(\phi\in\kappa\cdot\lambda)} \sum_{\phi'\equiv\phi} \binom{f}{k} \theta_{\phi'}^{\kappa,\lambda} C_{\phi'}^{\kappa,\lambda}(X,Y)\ ;$$

$$(2.6)\qquad \int_0^X \exp(-\mathrm{tr}\ AS)(\det S)^{t-p} C_\lambda(BS)dS$$
$$= \{\Gamma_m(t)\Gamma_m(p)/\Gamma_m(t+p)\}(\det X)^t$$
$$\cdot \sum_{k=0}^\infty \sum_{\kappa;\ \phi\in\kappa\cdot\lambda} (t)_\phi\ \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(-AX, BX)/k!(t+p)_\phi\ .$$

3. EVALUATION OF $\Pr(S > \Omega)$ FOR THE CENTRAL DISTRIBUTIONS

For the Wishart matrix, we wish to evaluate

$$(3.1)\qquad \int_{S>\Omega} (\det S)^{t-p} \exp(-\mathrm{tr}\ RS)dS\ ,$$

where $\mathrm{Re}(t) > p - 1$, $\mathrm{Re}(R) > 0$. Making the transformation $S \to \Omega^{\frac{1}{2}} S \Omega^{\frac{1}{2}} + \Omega$, then (3.1) becomes

$$(3.2)\qquad (\det \Omega)^t \exp(-\mathrm{tr}\ R\Omega)\int_{S>0} \det(I+S)^{t-p} \exp(-\mathrm{tr}\ \Omega^{\frac{1}{2}} R \Omega^{\frac{1}{2}} S)dS$$
$$= \Gamma_m(p)(\det \Omega)^t \exp(-\mathrm{tr}\ R\Omega)\Psi(p,\ p+t;\ R\Omega)\ ,$$

the equality following from Muirhead's (1970) definition of the function Ψ.

For the multivariate beta matrix, the integral to be evaluated is

$$(3.3)\qquad \int_{\Omega}^{I}(\det S)^{t-p}\det(I-S)^{u-p}\,dS\ ,$$

where $0 < \Omega < I$, $\mathrm{Re}(t) > p-1$, $\mathrm{Re}(u) > p-1$. Making the transformation $S \to I - S$, (3.3) becomes

$$(3.4)\qquad \int_{0}^{I-\Omega}(\det S)^{u-p}\det(I-S)^{t-p}\,dS$$

$$= \Gamma_m(u)\Gamma_m(p)\det(I-\Omega)^{u}\ {}_2F_1(u,\ -t+p;\ u+p;\ I-\Omega)/\Gamma_m(u+p)$$

by an application of Constantine ((1963), eq. (61)).

If we substitute $(t,R,\Omega) \to (\tfrac{1}{2}n,\ \tfrac{1}{2}\Sigma^{-1},\ \ell I)$ in (3.2) and multiply by the appropriate normalizing constant, we deduce that if S has the $W_m(\Sigma,n)$ distribution, then

$$(3.5)\qquad \Pr(\ell_{min} > \ell) = (\Gamma_m(p)/2^{\frac{1}{2}mn}\,\Gamma_m(\tfrac{1}{2}n)(\det\Sigma)^{\frac{1}{2}n})\ell^{\frac{1}{2}mn}$$

$$\cdot \exp(-\tfrac{1}{2}\ell\ \mathrm{tr}\ \Sigma^{-1})\Psi(p,\ p+\tfrac{1}{2}n;\ \tfrac{1}{2}\ell\Sigma^{-1})\ ,$$

$\ell > 0$. Similarly, the substitutions $(t,u,\Omega) \to (\tfrac{1}{2}n,\ \tfrac{1}{2}q,\ \ell I)$ in (3.4) show that if S has the $B_m(\tfrac{1}{2}n,\ \tfrac{1}{2}q)$ distribution, then for $0 < \ell < 1$,

$$(3.6)\qquad \Pr(\ell_{min} > \ell) = \{\Gamma_m(p)\Gamma_m(\tfrac{1}{2}n+\tfrac{1}{2}q)/\Gamma_m(\tfrac{1}{2}n)\Gamma_m(\tfrac{1}{2}q+p)\}$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}q,\ -\tfrac{1}{2}n+p;\ \tfrac{1}{2}q+p;\ (1-\ell)I)\ .$$

We remark that (3.5) appears to be new (cf. John (1963), Muirhead (1975)), while (3.6) was previously obtained by Sugiyama (1967). For $m = 2$, Gupta and Richards (1980) have expanded (3.5) in a series of Ψ-functions of scalar argument.

4. EVALUATION OF $\Pr(S > \Omega)$ FOR THE NON-CENTRAL DISTRIBUTIONS

In the Wishart case, we need to evaluate

$$(4.1)\qquad \int_{S>\Omega} (\det S)^{t-p} \exp(-\operatorname{tr} RS) C_\phi(TS)\,dS \ ,$$

where $\operatorname{Re}(t) > p - 1$, $\operatorname{Re}(R) > 0$, and T is arbitrary symmetric. Under the transformation $S \to \Omega^{\frac{1}{2}} S \Omega^{\frac{1}{2}} + \Omega$, (4.1) becomes

$$(4.2)\qquad (\det \Omega)^t \exp(-\operatorname{tr} R\Omega) \cdot \int_{S>0} \det(I+S)^{t-p} \exp(-\operatorname{tr} \Omega^{\frac{1}{2}} R \Omega^{\frac{1}{2}} S) C_\phi(\Omega^{\frac{1}{2}} T \Omega^{\frac{1}{2}} S + T\Omega)\,dS \ ,$$

and on using (2.5), the integral in (4.2) becomes

$$(4.3)\qquad \sum_{\kappa,\lambda(\phi\in\kappa\cdot\lambda)} \sum_{\phi'\equiv\phi} \binom{f}{k} \theta_{\phi'}^{\kappa,\lambda} \cdot \int_{S>0} \det(I+S)^{t-p} \exp(-\operatorname{tr} \Omega^{\frac{1}{2}} R \Omega^{\frac{1}{2}} S) C_{\phi'}^{\kappa,\lambda}(\Omega^{\frac{1}{2}} T \Omega^{\frac{1}{2}} S,\ \Omega T)\,dT \ .$$

We have been able to evaluate the integral in (4.3) only when $t = p$, in which case it is then computed using (2.4). After substituting into (4.3), we finally deduce that when $t = p$, the integral (4.1) equals

$$(4.4)\qquad \Gamma_m(p)(\det \Omega R^{-1})^p \exp(-\operatorname{tr} R\Omega) \cdot \sum_{\kappa,\lambda(\phi\in\kappa\cdot\lambda)} \sum_{\phi'\equiv\phi} \binom{f}{k} (p)_\kappa\ \theta_{\phi'}^{\kappa,\lambda} C_{\phi'}^{\kappa,\lambda}(TR^{-1},\ T\Omega) \ .$$

The corresponding integral for the beta matrix is

$$(4.5)\qquad \int_\Omega^I (\det S)^{t-p} \det(I-S)^{u-p}\, C_\lambda(TS)\,dS \ ,$$

where $0 < \Omega < I$, $\operatorname{Re}(t) > p - 1$, $\operatorname{Re}(u) > p - 1$, and T is arbitrary symmetric. Transforming $S \to I - S$ and expanding the term $\det(I-S)^{u-p}$ in a series of zonal polynomials, (4.5) then becomes

$$(4.6)\qquad \sum_{k=0}^{\infty} \sum_{\kappa}((-t+p)_{\kappa}/k!)\int_{0}^{I-\Omega}(\det S)^{u-p}\, C_{\kappa}(S)C_{\lambda}(T(I-S))dS$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa}((-t+p)_{\kappa}/k!)\sum_{\phi\in\kappa\cdot\lambda} \theta_{\phi}^{\kappa,\lambda}$$

$$\cdot \int_{0}^{I-\Omega}(\det S)^{u-p}\, C_{\phi}^{\kappa,\lambda}(S,\, T(I-S))dS\ ,$$

the equality following from (2.2). We have been able to solve this last integral only when $T = I$. If we use eq. (6.6) in Davis (1980) to expand $C_{\phi}^{\kappa,\lambda}(A,\, I+B)$ as a linear combination of the $C_{\tau}^{\kappa,\rho}(A,B)$, then one expression can be derived. A nicer result is obtained if we retreat to the left-hand side of (4.6); expanding the term $C_{\lambda}(I-S)$ as in Constantine (1966) (cf. Bingham (1974), Richards (1981)), and using (2.3), the integral on the left in (4.6) is seen to equal

$$(4.7)\qquad \sum_{r=0}^{\ell} \sum_{\rho}((-1)^{r} \tbinom{\lambda}{\rho}C_{\lambda}(I)/C_{\rho}(I))$$

$$\cdot \sum_{\phi^{*}\in\kappa\cdot\rho} g_{\kappa,\rho}^{\phi^{*}} \int_{0}^{I-\Omega} (\det S)^{u-p}\, C_{\phi^{*}}(S)dS\ .$$

The integral is easily evaluated by changing variables $S \to (I-\Omega)^{\frac{1}{2}} S(I-\Omega)^{\frac{1}{2}}$ and applying Constantine's (1963) eq. (22). Consequently, when $T = I$, (4.5) equals

$$(4.8)\qquad (\Gamma_{m}(u)\Gamma_{m}(p)/\Gamma_{m}(u+p))\det(I-\Omega)^{u}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{r=0}^{\ell} \sum_{\rho} \sum_{\phi^{*}\in\kappa\cdot\rho}(-1)^{r} \tbinom{\lambda}{\rho}(-t+p)_{\kappa}\, (u)_{\phi^{*}}\, g_{\kappa,\rho}^{\phi^{*}}$$

$$\cdot\ (C_{\lambda}(I)C_{\phi^{*}}(I-\Omega)/k!(u+p)_{\phi^{*}}\, C_{\rho}(I))\ .$$

5. EVALUATION OF $\Pr(\Omega < S < \Lambda)$ FOR THE CENTRAL DISTRIBUTIONS

In this section, we shall assume that $0 < \Omega < \Lambda < 2\Omega$.

To evaluate

$$(5.1) \qquad \int_{\Omega}^{\Lambda} (\det S)^{t-p} \exp(-\mathrm{tr} RS)\, dS, \quad \mathrm{Re}(t) > p-1 ,$$

we transform $S \to (\Lambda-\Omega)^{\frac{1}{2}} S (\Lambda-\Omega)^{\frac{1}{2}} + \Omega$, and after simplifying, (5.1) becomes

$$(5.2) \qquad \exp(-\mathrm{tr} R\Omega)(\det\Omega)^{t-p} \det(\Lambda-\Omega)^{p} \int_0^I \det(I+TS)^{t-p} \exp(-\mathrm{tr} US)\, dS$$

where $T = (\Lambda-\Omega)^{\frac{1}{2}}\Omega^{-1}(\Lambda-\Omega)^{\frac{1}{2}}$, and $U = (\Lambda-\Omega)^{\frac{1}{2}} R (\Lambda-\Omega)^{\frac{1}{2}}$. Since $\Lambda < 2\Omega$, then $T < I$; thus, expanding the term $\det(I+TS)^{t-p}$ in a zonal polynomial series and using (2.6), the final result is

$$(5.3) \qquad (\Gamma_m(p)^2/\Gamma_m(2p))(\det\Omega)^{t-p}\det(\Lambda-\Omega)^{p}\exp(-\mathrm{tr} R\Omega)$$

$$\sum_{\ell=0}^{\infty}\sum_{\lambda}\sum_{k=0}^{\infty}\sum_{\kappa;\phi\in\kappa\cdot\lambda}((-t+p)_{\lambda}(p)_{\phi}\theta_{\phi}^{\kappa,\lambda}/\ell!k!(2p)_{\phi})\; C_{\phi}^{\kappa,\lambda}(-U,T) .$$

In the special case when $R = 0$, it follows from (5.2) and Herz (1955), eq. (2.9), or alternatively (but less directly) from various results of Davis (1979, 1980) that (5.3) simplifies to

$$(5.4) \qquad (\Gamma_m(p)^2/\Gamma_m(2p))(\det\Omega)^{t-p}\det(\Lambda-\Omega)^{p}$$

$$\cdot\ {}_2F_1(p,-t+p;2p;-T),$$

which is the solution to Jack's (1966) problem. We note that (5.3) can be further simplified using the Kummer transformation (cf. James (1964), eq. (49)).

The substitutions $(t,R,\Omega,\Lambda) \to (\frac{1}{2}n, \frac{1}{2}\Sigma^{-1}, \ell_1 I, \ell_2 I)$ where $0 < \ell_1 < \ell_2 < 2\ell_1$, in (5.3), lead to an evaluation of the probability that all the roots of a Wishart matrix lie in (ℓ_1,ℓ_2). For the beta matrix, we evaluate

(5.5) $$\int_\Omega^\Lambda (\det S)^{t-p}\det(I-S)^{u-p}dS,$$

where $Re(t) > p-1$, $Re(u) > p-1$. Again transforming $S \to (\Lambda-\Omega)^{\frac{1}{2}}S(\Lambda-\Omega)^{\frac{1}{2}}+\Omega$, then (5.5) becomes

(5.6) $$\det(\Lambda-\Omega)^p\det(I-\Omega)^{t+u-2p} \cdot \int_0^I \det(I+TS)^{t-p}\det(I-US)^{u-p}dS,$$

where $T = (\Lambda-\Omega)^{\frac{1}{2}}\Omega^{-1}(\Lambda-\Omega)^{\frac{1}{2}}$, and $U = (\Lambda-\Omega)^{\frac{1}{2}}(I-\Omega)^{-1}(\Lambda-\Omega)^{\frac{1}{2}}$. Expanding both determinants in the integrand, the integral in (5.6) becomes

(5.7) $$\sum_{k,\ell=0}^{\infty}\sum_{\kappa,\lambda}((-1)^k(-t+p)_\kappa(-u+p)_\lambda/k!\ell!) \cdot \int_0^I C_\kappa(TS)C_\lambda(US)dS$$

Using (2.2) and Constantine (1963), eq. (22), to compute the latter integral, the final result is

(5.8) $$\det(\Lambda-\Omega)^p\det(I-\Omega)^{t+u-2p}(\Gamma_m(p)^2/\Gamma_m(2p)) \cdot \sum_{k,\ell,\kappa,\lambda}((-1)^k(-t+p)_\kappa(-u+p)_\lambda/k!\ell!) \cdot \sum_{\phi\epsilon\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}((p)_\phi)^2C_\phi^{\kappa,\lambda}(T,U)/(2p)_\phi\ .$$

It is clear that (5.8) can be used to find the probability that all the roots of a beta matrix lie in the interval (ℓ_1,ℓ_2), $\ell_1 < \ell_2 < 2\ell_1$.

At this time it is not evident how the condition $\Lambda < 2\Omega$ can be removed; moreover, the extension of these results to the non-central cases seems to be a problem worthy of the name recondite.

Acknowlegdment. The authors wish to thank Professor A. W. Davis who kindly provided them with preprints of his articles, and careful comments on the reults presented here.

REFERENCES

[1] Bingham, C. (1974) An identity involving partitional generalized binomial coefficients. *J. Multiv. Anal.*, *4*, 210-223.

[2] Constantine, A. G. (1963) Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.*, *34*, 1270-1285.

[3] Constantine, A. G. (1966) The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.*, *37*, 215-225.

[4] Davis, A. W. (1979) Invariant polynomials with two matrix arguments extending the zonal polynomials: applications to multivariate distribution theory. *Ann. Inst. Statist. Math.*, *31*, 465-485.

[5] Davis, A. W. (1980) Invariant polynomials with two matrix arguments extending the zonal polynomials. *Multivariate Analysis-V* (ed. P.R. Krishnaiah), 287-299.

[6] Gupta, R. D. and Richards, D. St. P. (1980) Series expansions for a hypergeometric function of matrix argument. Submitted for publication.

[7] Herz, C. S. (1955) Bessel functions of matrix argument. *Ann. of Math.*, *61*, 474-523.

[8] Jack, H. (1966) A matrix analogue of the integral $\int_a^b x^\alpha dx$. *Proc. Roy. Soc. Edin.*, A, *67*, 205-214.

[9] James, A. T. (1964) Distribution of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.*, *35*, 475-501.

[10] John, S. (1963) A tolerance region for multivariate normal distribution. *Sankhya*, Series A, *25*, 363-368.

[11] Johnson, N. L. and Kotz, S. (1972) *Distributions in Statistics: Continuous Multivariate Distributions*, Wiley, New York.

[12] Muirhead, R. J. (1970) Asymptotic distributions of some multivariate tests. *Ann. Math. Statist.*, *41*, 1002-1010.

[13] Richards, D. St. P. (1981) Differential operators associated with zonal polynomials, I, II. *Ann. Inst. Statist. Math.*, to appear.

[14] Sugiyama, T. (1967) Distribution of the largest latent root and the smallest latent root of the generalized B statistic and F statistics in multivariate analysis. *Ann. Math. Statist.*, *38*, 1152-1159.