WELL POSEDNESS FOR DAMPED SECOND ORDER SYSTEMS WITH UNBOUNDED INPUT OPERATORS*

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June, 1993

*Research supported in part by the Air Force Office of Scientific Research under grants AFOSR-90-0091 and F49620-93-1-0198, and the National Science Foundation under grant NSF INT-9015007.

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Abstract: We consider damped second order in time systems such as those arising in structures with piezoceramic actuators and sensors. These systems are naturally formulated as abstract second order systems with unbounded nonhomogeneous term. Existence, uniqueness and continuous dependence of solutions in a weak or variational setting are given. A semigroup formulation is presented and conditions under which the variational solutions and semigroup solutions are the same are discussed.

1. Introduction

In this paper we present existence, uniqueness and continuous dependence results for a class of damped second order in time partial differential equation models with input or control operators that are unbounded in the natural state space for these systems. Our efforts are prompted by the growing literature (see [BSW], [BFSS], [BS], [BWIS2] and the references therein) on smart material (materials possessing the capabilities of sensing and actuation in response to information available from sensing) structures. There are a number of smart materials currently under investigation in the research literature (electrorheological fluids, magnetostrictives, shape memory alloys) but our considerations here were prompted by the use of bonded or embedded piezoceramic patches as components in a smart material technology. These piezoceramic patches, when bonded to a structure such as a beam, plate, or curved cylindrical shell, act as an electro-mechanical transducer. When excited by an electric field, the patch induces a strain (and hence deformation or displacement) in the material to which it is bonded or in which it is embedded. The excited patch can thus be employed as an actuator for the host structure. In addition, if the host structure undergoes a deformation (either bending or extension/contraction), this produces a strain in the patch which results in a voltage across the patch that is proportional to the strain. This permits use of the patch as a mechanical sensor. These sensing/actuating functions can be combined to produce a smart or adaptive material capability for the structure to which the patch is bonded or in which it is embedded.

Mathematical models for estimation and control for such structures pose interesting conceptual, theoretical and computational challenges for scientists and engineers. The models typically involve unbounded terms in both coefficients and input terms. To

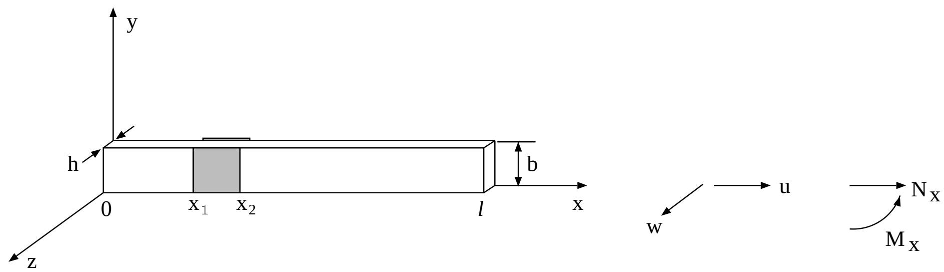

Fig. 1. *Cantilever beam with piezoceramic patches.*

illustrate this, we consider a cantilevered beam as depicted in Figure 1.

This homogeneous material beam, which we shall assume satisfies the Euler-Bernoulli hypothesis for displacements and the Kelvin-Voigt hypothesis (damping proportional to strain rate), is of length ℓ, width b, thickness h, and is fixed at $x = 0$, free at $x = \ell$. The beam is assumed to have two piezoceramic patches bonded to it (one on each side) in the region $x_1 < x < x_2$. The Young's modulus for the beam and patch are denoted by E_b and E_{pe}, respectively, the mass densities (in mass per unit volume) are given by ρ_b and ρ_{pe}, respectively, and the damping coefficients are denoted by c_{Db} and c_{Dpe}, respectively. Each patch is of thickness T and for simplicity, we ignore the bonding layer material properties and geometry.

For such a beam, subject only to forces and moments generated by actuating the patches, force and moment balancing [BSW] lead to the dynamic system of equations

$$\text{(1.1)} \qquad \begin{aligned} &\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial N_x}{\partial x} = -S_{1,2}\frac{\partial [N_x]_{pe}}{\partial x} \\ &\rho(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_x}{\partial x^2} = -\frac{\partial^2 [M_x]_{pe}}{\partial x^2}, \qquad 0 < x < \ell \end{aligned}$$

for the axial displacement $u = u(t, x)$ and the transverse displacement $w = w(t, x)$. Here

N_x and M_x are the *internal* force and moment resultants given by

$$
\begin{aligned}
N_x &= Eh(x)\frac{\partial u}{\partial x} + c_D h(x)\frac{\partial^2 u}{\partial x \partial t} \\
M_x &= EI(x)\frac{\partial^2 w}{\partial x^2} + c_D I(x)\frac{\partial^3 w}{\partial x^2 \partial t}
\end{aligned}
\tag{1.2}
$$

where

$$
\begin{aligned}
Eh(x) &= E_b hb + 2bE_{pe}T\chi_{pe}(x), \quad c_D h(x) = c_{Db}hb + 2bc_{Dpe}T\chi_{pe}(x), \\
EI(x) &= E_b\frac{h^3 b}{12} + \frac{2b}{3}E_{pe}a_3\chi_{pe}(x), \quad c_D I(x) = c_{Db}\frac{h^3 b}{12} + \frac{2b}{3}c_{Dpe}T\chi_{pe}(x)
\end{aligned}
\tag{1.3}
$$

with $\chi_{pe}(x)$ the characteristic function that is 1 for $x_1 \leq x \leq x_2$, and zero elsewhere and a_3 is a patch constant. The linear mass density ρ is given by $\rho(x) = \rho_b hb + 2b\rho_{pe}T\chi_{pe}(x)$. The indicator function $S_{1,2}$ has value 1 for $x < (x_1 + x_2)/2$, 0 for $x = (x_1 + x_2)/2$ and -1 for $x > (x_1 + x_2)/2$.

The *external* forces and moments $[N_x]_{pe}$ and $[M_x]_{pe}$ depend on the voltages supplied to each of the two patches. If the voltages are denoted by V_1 and V_2, respectively, these forces and moments are given by

$$
\begin{aligned}
[N_x]_{pe} &= \mathcal{K}_A S_{1,2}(x)\chi_{pe}(x)[V_1(t) + V_2(t)] \\
[M_x]_{pe} &= -\mathcal{K}_B \chi_{pe}(x)[V_1(t) - V_2(t)],
\end{aligned}
\tag{1.4}
$$

where $\mathcal{K}_A$, $\mathcal{K}_B$ are constants depending on the piezoceramic material properties. If the patches are excited *in-phase*, for example, with $V_1(t) = V_2(t) = V(t)$, we find

$$
\begin{aligned}
[N_x]_{pe} &= 2\mathcal{K}_A S_{1,2}(x)\chi_{pe}(x)V(t) \\
[M_x]_{pe} &= 0,
\end{aligned}
$$

resulting in axial motion only. If the patches are excited *out-of-phase*, with $V_1(t) = -V_2(t) = V(t)$, we have pure bending or transverse motion since

$$
\begin{aligned}
[N_x]_{pe} &= 0, \\
[M_x]_{pe} &= -2\mathcal{K}_B \chi_{pe}(x)V(t).
\end{aligned}
$$

Coupled to the system (1.1)–(1.4) are appropriate boundary conditions

$$
\begin{aligned}
&u(t,0)=0, \quad \frac{\partial u}{\partial x}(t,\ell)=0 \\
&w(t,0)=\frac{\partial w}{\partial x}(t,0)=0, \quad M_x(t,\ell)=\frac{\partial}{\partial x}M_x(t,\ell)=0,
\end{aligned} \tag{1.5}
$$

and initial conditions

$$
\begin{aligned}
&u(0,x)=\eta(x), \quad \frac{\partial u}{\partial t}(0,x)=\gamma(x), \\
&w(0,x)=\phi(x), \quad \frac{\partial w}{\partial t}(0,x)=\psi(x).
\end{aligned} \tag{1.6}
$$

For a beam containing a pair of identical patches which are bonded symmetrically about the middle surface, the differential equations (1.1)–(1.4), under the first order Euler-Bernoulli assumptions, describe vibrations in the axial and transverse directions that are uncoupled. If one has only a single patch bonded to the beam, or if the patches are not identical, then one obtains a set of equations for longitudinal and transverse vibrations that are *coupled* (which is not surprising since the structure consisting of beam plus patch is no longer symmetric).

To complete our illustration, let us consider the beam described above under *out-of-phase* excitation, resulting in pure bending (transverse vibrations). Using the abbreviated notation $D=\frac{\partial}{\partial x}$, we then have the model

$$
\rho w_{tt}+D^2(EID^2w+c_DID^2w_t)=-2\mathcal{K}_BD^2\chi_{pe}(x)V(t) \tag{1.7}
$$

coupled with the boundary and initial conditions of (1.5), (1.6) involving w. The coefficients EI and c_DI are discontinuous (see (1.3)) while the coefficient involving the input voltages is given by

$$
D^2\chi_{pe}(x)=D^2\{H(x-x_1)-H(x-x_2)\}=\delta'(x-x_1)-\delta'(x-x_2) \tag{1.8}
$$

where H is the Heaviside function and δ' is the "derivative" of the Dirac function with mass at $x=0$. This strong form of the equation involves irregularities which can (and have) led to computational difficulties for estimation and control efforts found in the literature. Retention of such irregular terms as the discontinuous coefficients in (1.7) and the impulse derivatives in (1.8) has been shown (see [BWIS1] and the references there) to be of great importance (indeed, essential) when using such models with experimental data from actual structures.

From a mathematical perspective, the obvious first step in treating equations such as (1.7), (1.8) is to write it in weak form

$$
\begin{aligned}
&\int_0^\ell \left\{\rho w_{tt}\phi + (EID^2 w + c_D I D^2 w_t) D^2\phi\right\} dx \\
&\qquad = \left(\int_0^\ell -2\mathcal{K}_B \chi_{pe} D^2\phi dx\right) V(t) = \left(\int_{x_1}^{x_2} -2\mathcal{K}_B D^2\phi dx\right) V(t)
\end{aligned}
\tag{1.9}
$$

for some ϕ in a appropriate class of "test" functions. It is damped second order systems such as these that motivated the theoretical developments presented in this paper.

We consider in subsequent sections a general class of systems that include models such as (1.7), (1.8) as special cases. In Section 2 below, we consider a weak or variational form of these systems, a form that is most useful for development of identification methods and general computational methods (e.g., finite elements). For feedback control theoretic developments (e.g., an LQR theory), it is most advantageous to have a semigroup formulation (with the usual variation-of-parameters representation of solutions to forced systems). This approach is developed in Section 3 while questions related to the equivalence of the weak formulation and semigroup formulation are addressed in Section 4. In Section 5 we give a brief presentation of a number of damping models and systems which can be treated within the framework of this paper. We also conclude with brief remarks on related well-posedness results to be found in the research literature.

2. Variational or Weak Form

Motivated by the examples given in Section 1, we consider the abstract second order (in time) system

$$
(W1)\qquad
\begin{aligned}
&\ddot{y}(t) + A_2\dot{y}(t) + A_1 y(t) = f(t) \quad \text{in } V^*,\\
&y(0) = y_0,\\
&\dot{y}(0) = y_1.
\end{aligned}
$$

More precisely, let V and H be complex Hilbert spaces forming a Gelfand triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ with pivot space H and duality pairing $\langle\, , \rangle_{V^*,V}$ (see [T,p.25], [W, p.165, 261], [BKW, p.43–44]). That is, V is continuously and densely embedded in H with $|\phi|_H \leq c|\phi|_V$, and we identify H^* with H through the Riesz map. Here V^*, H^* are the (conjugate) dual spaces to V, H respectively (i.e. the spaces of continuous, conjugate linear functionals). The duality pairing $\langle\, , \rangle_{V^*,V}$ is the extension by continuity of the inner product $\langle\, , \rangle_H$ from $V \times H$ to $V^* \times H$. Hence, elements $v^* \in V^*$ have the representation $v^*(v) = \langle v^*, v\rangle_{V^*,V}$. We assume that in this context the operators A_1 and A_2 of (W1) are defined via sesquilinear forms. Specifically, assume $\sigma_1 : V \times V \to$ is a sesquilinear form on V that is symmetric, continuous and V-elliptic. That is, σ_1 satisfies

(H1) For all ϕ, $\psi \in V$, $\sigma_1(\phi,\psi) = \overline{\sigma_1(\psi,\phi)}$.

(H2) There exists a constant c_1 such that for all ϕ, $\psi \in V$

$$|\sigma_1(\phi,\psi)| \leq c_1|\phi|_V|\psi|_V.$$

(H3) There exists a positive constant k_1 such that for all $\phi \in V$

$$\text{Re}\ \sigma_1(\phi,\phi) = \sigma_1(\phi,\phi) \geq k_1|\phi|_V^2.$$

We note that hypotheses (H1) implies that $\sigma_1(\phi,\phi)$ is real for all $\phi \in V$. Under these assumptions we have that there exists $A_1 \in \mathcal{L}(V, V^*)$ such that $\sigma_1(\phi,\psi) = \langle A_1\phi, \psi\rangle_{V^*,V}$ for all $\phi, \psi \in V$.

To allow for a wide class of damping operators of physical interest, we introduce a sesquilinear form σ_2 defined on a Hilbert space V_2 that can be the same as V or the same as H or somewhere between V and H. Specifically, assume that V_2 is a complex Hilbert space satisfying $V \subset V_2 \subset H$. We may consider V_2, H in a Gelfand setting with duality pairing $\langle\, ,\rangle_{V_2^*,V_2}$ and may write $V \hookrightarrow V_2 \hookrightarrow H \cong H^* \hookrightarrow V_2^* \hookrightarrow V^*$. We note that it is readily seen that $\langle \phi,\psi\rangle_{V^*,V} = \langle\phi,\psi\rangle_{V_2^*,V_2}$ if $\phi \in V_2^*$, $\psi \in V$ and these quantities both reduce to $\langle\phi,\psi\rangle_H$ if $\phi \in H$.

We assume that $\sigma_2 : V_2 \times V_2 \to$ satisfies

(H4) There exists a constant c_2 such that for all ϕ, $\psi \in V_2$

$$|\sigma_2(\phi,\psi)| \le c_2 |\phi|_{V_2} |\psi|_{V_2}.$$

(H5) There exist constants $k_2 > 0$, $\lambda_0 \ge 0$ such that for all $\phi \in V_2$

$$\operatorname{Re}\sigma_2(\phi,\phi) + \lambda_0 |\phi|_H^2 \ge k_2 |\phi|_{V_2}^2.$$

Again, under these assumptions we have existence of $A_2 \in \mathcal{L}(V_2, V_2^*)$ such that $\sigma_2(\phi,\psi) = \langle A_2\phi,\psi\rangle_{V_2^*,V_2}$ for all ϕ, $\psi \in V_2$. In subsequent discussions, we shall need some regularity on f in (W1). To that end we assume

(H6) The input function f satisfies $f \in L_2((0,T), V_2^*)$.

Given the above hypotheses and considerations, we thus consider the weak or variational form of our system given by

$$(W2) \qquad \begin{aligned} &\langle \ddot{y}(t), \phi \rangle + \sigma_2(\dot{y}(t), \phi) + \sigma_1(y(t), \phi) = \langle f(t), \phi \rangle \quad \forall \phi \in V, \\ &y(0) = y_0, \\ &\dot{y}(0) = y_1. \end{aligned}$$

Thus, (W1) and (W2) are equivalent equations if we interpret $\langle \, , \rangle$ as $\langle \, , \rangle_{V^*,V}$ and note that $\langle f, \phi \rangle_{V^*,V} = \langle f, \phi \rangle_{V_2^*,V_2}$ since $f \in L_2((0,T), V_2^*)$. We can then establish the following fundamental existence results.

Theorem 2.1. *Suppose that σ_1, σ_2 and f satisfy (H1)–(H6) and that $y_0 \in V$, $y_1 \in H$. Then there exists a unique solution y of (W2) (equivalently (W1)) with $y \in L_2((0,T), V)$, $\dot{y} \in L_2((0,T), V_2)$ and $\ddot{y} \in L_2((0,T), V^*)$. Moreover, solutions of (W2) depend continuously on the data (y_0, y_1, f) in the sense that the map*

$$(y_0, y_1, f) \to (y, \dot{y})$$

is continuous from $V \times H \times L_2((0,T), V_2^*)$ to $L_2((0,T), V) \times L_2((0,T), V_2)$.

Before sketching the proof of this theorem (the analogous arguments for undamped systems are rather standard [L, p.272–278], [W, p.439–442] – and we sketch the arguments here for the sake of completeness and to indicate how the damping sesquilinear form is treated), we remark that a number of practically important examples (e.g., Euler-Bernoulli or Timoshenko beams, Love-Kirchoff or Mindlin plates with various damping models – Kelvin-Voigt, viscous or air, square-root or "structural", spatial hysteresis) can be treated within this framework. These examples will be discussed in more detail in a subsequent section.

Proof: Let $\{w_i\}_{i=1}^{\infty}$ a linear independent total subset of V. For each m, let $V^m =$ span $\{w_1, \ldots, w_m\}$ and let $y_{0m}, y_{1m} \in V^m$ be chosen so that $y_{0m} \to y_0$ in V, $y_{1m} \to y_1$ in

H as $m \to \infty$. Let $y_m(t) \equiv \sum_{i=1}^m \eta_{im}(t) w_i$ be the unique solution to the m dimensional linear system

$$
(2.1) \qquad \begin{aligned}
&\langle \ddot{y}_m(t), w_j\rangle_H + \sigma_2(\dot{y}_m(t), w_j) + \sigma_1(y_m(t), w_j) = \langle f(t), w_j\rangle_{V_2^*,V_2} \\
&y_m(0) = y_{0m} \qquad\qquad\qquad\qquad j = 1, 2, \ldots, m, \\
&\dot{y}_m(0) = y_{1m}.
\end{aligned}
$$

Multiplying the equation in (2.1) by $\dot{\bar{\eta}}_{jm}(t)$ and summing over j, we obtain

$$
\langle \ddot{y}_m(t), \dot{y}_m(t)\rangle_H + \sigma_2(\dot{y}_m(t), \dot{y}_m(t)) + \sigma_1(y_m(t), \dot{y}_m(t)) = \langle f(t), \dot{y}_m(t)\rangle_{V_2^*,V_2}.
$$

But since $\frac{d}{dt}\sigma_1(y_m(t), y_m(t)) = 2\,\mathrm{Re}\sigma_1(y_m(t), \dot{y}_m(t))$, this equality implies

$$
\begin{aligned}
\frac{d}{dt}\left\{|\dot{y}_m(t)|_H^2 + \sigma_1(y_m(t), y_m(t))\right\} &+ 2\,\mathrm{Re}\sigma_2(\dot{y}_m(t), \dot{y}_m(t)) \\
&= 2\,\mathrm{Re}\langle f(t), \dot{y}_m(t)\rangle_{V_2^*,V_2}.
\end{aligned}
$$

Upon integrating this equality, we obtain

$$
\begin{aligned}
|\dot{y}_m(t)|_H^2 &+ \sigma_1(y_m(t), y_m(t)) + \int_0^t 2\,\mathrm{Re}\sigma_2(\dot{y}_m(s), \dot{y}_m(s))ds \\
&= |\dot{y}_m(0)|_H^2 + \sigma_1(y_m(0), y_m(0)) + \int_0^t 2\,\mathrm{Re}\langle f(s), \dot{y}_m(s)\rangle_{V_2^*,V_2} ds.
\end{aligned}
$$

Using (H3), (H5) and for arbitrary $\epsilon > 0$ the inequality

$$
(2.2) \qquad |\langle f(s), \dot{y}_m(s)\rangle_{V_2^*,V_2}| \le \frac{1}{4\epsilon}|f(s)|_{V_2^*}^2 + \epsilon|\dot{y}_m(s)|_{V_2}^2,
$$

in the above equality, we find

$$
(2.3) \qquad \begin{aligned}
&|\dot{y}_m(t)|_H^2 + k_1|y_m(t)|_V^2 + \int_0^t 2(k_2 - \epsilon)|\dot{y}_m(s)|_{V_2}^2 ds \\
&\qquad \le |y_{1m}|_H^2 + c_1|y_{0m}|_V^2 + 2\lambda_0\int_0^t |\dot{y}_m(s)|_H^2 ds + \int_0^t \frac{1}{2\epsilon}|f(s)|_{V_2^*}^2 ds.
\end{aligned}
$$

Recalling that $y_{1m} \to y_1$ in H, $y_{0m} \to y_0$ in V (and thus $\{y_{1m}\}, \{y_{0m}\}$ are H and V bounded) and fixing ϵ such that $2(k_2 - \epsilon) = \delta > 0$, we conclude that the estimate (2.3) can be replaced by (for m sufficiently large)

$$
\begin{aligned}
(2.4) \qquad & |\dot{y}_m(t)|_H^2 + k_1|y_m(t)|_V^2 + \delta\int_0^t |\dot{y}_m(s)|_{V_2}^2 ds \\
& \qquad \le (M+1) + 2\lambda_0 \int_0^t |\dot{y}_m(s)|_H^2 ds
\end{aligned}
$$

where

$$M \equiv |y_1|_H^2 + c_1|y_0|_V^2 + \int_0^t \frac{1}{2\epsilon}|f(s)|_{V_2^*}^2 ds.$$

It follows that $\{\dot{y}_m\}$ is bounded in $C((0,T),H)$ – we obtain this by ignoring the second and third terms in (2.4) and using Gronwall's inequality. Then using (2.4) again, knowing that $\{\dot{y}_m\}$ is bounded this way, we may conclude that $\{y_m\}$ is $C((0,T),V)$ bounded. It also follows from (2.4) and the fact that $\{\dot{y}_m\}$ is $C((0,T),H)$ bounded that $\{\dot{y}_m\}$ is $L_2((0,T),V_2)$ bounded. Hence we may find a subsequence $\{y_{m_k}\}$ and limit functions $y \in L_2((0,T),V)$ and $\hat{y} \in L_2((0,T),V_2)$ such that

$$
\begin{aligned}
(2.5) \qquad & y_{m_k} \to y \ \text{ weakly in } L_2((0,T),V) \\
& \dot{y}_{m_k} \to \hat{y} \ \text{ weakly in } L_2((0,T),V_2).
\end{aligned}
$$

But we have that for $t \in [0,T)$

$$(2.6) \qquad y_{m_k}(t) = y_{m_k}(0) + \int_0^t \dot{y}_{m_k}(s)ds$$

in the V (and hence V_2 and H) sense. Moreover, $y_{m_k}(0) = y_{0m_k} \to y_0$ in the V and hence V_2 sense whereas for each t, $\int_0^t \dot{y}_{m_k}(s)ds \to \int_0^t \hat{y}(s)ds$ weakly in V_2. Hence, taking the limit in the weak V_2 sense in (2.6) we obtain

$$y(t) = y_0 + \int_0^t \hat{y}(s)ds \qquad \text{for } t \in [0,T).$$

in the V_2 sense. Thus, $\dot{y}(t)$ exists a.e. in the V_2 sense with $\dot{y} = \hat{y} \in L_2((0,T),V_2)$ and $y(0) = y_0$. It remains to argue that y is indeed a solution to (W2).

To this end, we return to (2.1) and let $\psi \in C^1[0,T]$ with $\psi(T) = 0$ be arbitrarily chosen. Put $\psi_j(t) \equiv \psi(t)w_j$ where the $\{w_j\}$ are the same as in (2.1). Multiplying (2.1) by $\psi(t)$ and fixing $j < m$, we have, upon integration,

$$\int_0^T \{\langle \ddot{y}_m(t), \psi_j(t)\rangle_H + \sigma_2(\dot{y}_m(t), \psi_j(t)) + \sigma_1(y_m(t), \psi_j(t))\}\, dt$$
$$= \int_0^T \langle f(t), \psi_j(t)\rangle_{V_2^*, V_2} dt.$$

Integrating by parts in the first term, using the convergences of (2.5), the facts that $\sigma_2(\cdot\,, \psi_j(t)) \in V_2^*$, $\sigma_1(\cdot\,, \psi_j(t)) \in V^*$ for each t, and taking subsequential limits as $m = m_k \to \infty$ in this equation, we obtain

$$\begin{aligned}\int_0^T \Big\{\langle -\dot{y}(t), \dot{\psi}_j(t)\rangle_H + \sigma_2(\dot{y}(t), \psi_j(t)) + \sigma_1(y(t), \psi_j(t))\Big\}\, dt \\ = \int_0^T \langle f(t), \psi_j(t)\rangle_{V_2^*, V_2} dt + \langle y_1, \psi_j(0)\rangle_H\end{aligned} \tag{2.7}$$

for each j. Recalling that $\psi_j(t) = \psi(t)w_j$ and further restricting ψ so that $\psi \in C_0^\infty(0,T)$, we obtain from (2.7)

$$\int_0^T \dot{\psi}(t)\langle -\dot{y}(t), w_j\rangle_H dt \quad + \int_0^T \psi(t)\Big\{\sigma_2(\dot{y}(t), w_j) + \sigma_1(y(t), w_j) - \langle f(t), w_j\rangle_{V_2^*, V_2}\Big\}\, dt$$
$$= 0, \quad \text{for each } w_j.$$

This implies for each w_j

$$\frac{d}{dt}\langle \dot{y}(t), w_j\rangle_H + \sigma_2(\dot{y}(t), w_j) + \sigma_1(y(t), w_j) = \langle f(t), w_j\rangle_{V_2^*, V_2}. \tag{2.8}$$

Since $\{w_j\}$ is total in V we thus have that $\ddot{y} \in L_2((0,T), V^*)$ and for all $\phi \in V$

$$\langle \ddot{y}(t), \phi\rangle_{V^*, V} + \sigma_2(\dot{y}(t), \phi) + \sigma_1(y(t), \phi) = \langle f(t), \phi\rangle_{V_2^*, V_2}$$

which is the equation in (W2).

We already have $y(0) = y_0$ and to argue $\dot{y}(0) = y_1$, we return to (2.7) which holds for all $\psi_j(t) = \psi(t)w_j$, $\psi \in C^1[0,T]$, $\psi(T) = 0$. Integrating by parts in the first term in (2.7) and using (2.8), we obtain

$$\langle -\dot{y}(t), \psi_j(t)\rangle_H \Big|_{t=0}^{t=T} = \langle y_1, \psi_j(0)\rangle_H$$

or

$$\langle \dot{y}(0), w_j\rangle_H \psi(0) = \langle y_1, w_j\rangle \psi(0) \text{ for all } j.$$

From this it follows that $\dot{y}(0) = y_1$ and thus y is indeed a solution to the initial value problem (W2).

Continuous dependence of these solutions on the data (y_0, y_1, f) follows readily from some of the estimates used above in establishing existence. Recalling (2.3), we see that this estimate implies

$$|\dot{y}_m(t)|_H \le K_m + 2\lambda_0 \int_0^t |\dot{y}_m(s)|_H^2 ds$$

where $K_m \equiv |y_{1m}|_H^2 + c_1|y_{0m}|_V^2 + \frac{1}{2\epsilon}|f|_{L_2((0,T),V_2^*)}$. Thus, using Gronwall's inequality we obtain

$$|\dot{y}_m(t)|_H \le K_m e^{2\lambda_0 t}.$$

But use of this in (2.3) yields

$$\begin{aligned} k_1|y_m(t)|_V^2 + \delta \int_0^t |\dot{y}_m(s)|_{V_2}^2 ds &\le K_m + K_m^2\, 2\lambda_0 \int_0^t e^{4\lambda_0 s} ds \\ &\le K_m C_1 \end{aligned}$$

where C_1 is a constant. Integrating over $(0,T)$ we obtain

$$k_1|y_m|_{L_2((0,T),V)}^2 + \delta T|\dot{y}_m|_{L_2((0,T),V_2)}^2 \le K_m C_2$$

where again C_2 is a constant. Recalling that $y_{1m} \to y_1$ in H, $y_{0m} \to y_0$ in V and using weak lower semicontinuity of norms, along with the weak convergences in (2.5), we may take limits in this last inequality to obtain

$$\begin{aligned} k_1|y|_{L_2((0,T),V)}^2 &+ \delta T|\dot{y}|_{L_2((0,T),V_2)}^2 \\ &\le \left\{|y_1|_H^2 + c_1|y_0|_V^2 + \frac{1}{2\epsilon}|f|_{L_2((0,T),V_2^*)}^2\right\} C_2. \end{aligned} \tag{2.9}$$

Since the mapping $(y_0, y_1, f) \to (y, \dot{y})$ is linear, this estimate yields the desired continuity statement of the theorem.

Finally, we turn to uniqueness of solutions of (W2), observing that (2.9) shows that solutions constructed through the above limiting procedure are, of course, unique. However, we must argue that *any* two solutions corresponding to the same data y_0, y_1, f are the same. It suffices to argue that only the trivial solution $y \equiv 0$ of (W2) can result from data $y_0 = 0,\ y_1 = 0,\ f = 0$. Again we follow the arguments of Lions [L, p.272–278], [W, p.437–439]. Let y be a solution of (W2) corresponding to $y_0 = 0,\ y_1 = 0,\ f = 0$ and define for s fixed but arbitrary in $(0, T)$

$$\psi(t) = \begin{cases} -\int_t^s y(\xi)d\xi & t < s \\ 0 & t \geq s \end{cases}$$

so that $\psi(T) = 0$. It is readily argued that $\psi(t) \in V$ for each t and hence we may choose $\phi = \psi(t)$ in (W2) to obtain

$$\langle \ddot{y}(t), \psi(t) \rangle_{V^*,V} + \sigma_2(\dot{y}(t), \psi(t)) + \sigma_1(y(t), \psi(t)) = 0. \tag{2.10}$$

Since $\dot{\psi}(t) = y(t)$ for *a.e.* $t < s$, we have

$$\begin{aligned} &\int_0^s \{\langle \ddot{y}(t), \psi(t) \rangle_{V^*,V} + \langle \dot{y}(t), y(t) \rangle_{V^*,V}\}\, dt \\ &\qquad = \int_0^s \frac{d}{dt} \langle \dot{y}(t), \psi(t) \rangle_{V^*,V}\, dt = \langle \dot{y}(t), \psi(t) \rangle_{V^*,V} \Big|_{t=0}^{t=s} = 0 \end{aligned}$$

Integrating (2.10) and using this last identity, we obtain

$$\int_0^s \{\langle \dot{y}(t), y(t) \rangle_{V^*,V} - \sigma_2(\dot{y}(t), \psi(t)) - \sigma_1(y(t), \psi(t))\}\, dt = 0$$

which can be equivalently written as

$$\int_0^s \frac{d}{dt} \left\{ |y(t)|_H^2 - \sigma_1(\psi(t), \psi(t)) \right\} dt = 2\,\mathrm{Re} \int_0^s \sigma_2(\dot{y}(t), \psi(t)) dt.$$

This yields (since $\psi(s)=0$ and $y(0)=0$)

$$|y(s)|_H^2+\sigma_1(\psi(0),\psi(0))=2\,\mathrm{Re}\int_0^s\sigma_2(\dot y(t),\psi(t))dt. \tag{2.11}$$

But $\frac{d}{dt}\sigma_2(y(t),\psi(t))=\sigma_2(\dot y(t),\psi(t))+\sigma_2(y(t),y(t))$ so that

$$\begin{aligned}\int_0^s\sigma_2(\dot y(t),\psi(t))dt &= -\int_0^s\sigma_2(y(t),y(t))dt+\sigma_2(y(t),\psi(t))\Big|_{t=0}^{t=s}\\ &= -\int_0^s\sigma_2(y(t),y(t))dt\end{aligned}$$

since $\psi(s)=0,\ y(0)=0$. Using this in (2.11), we have

$$|y(s)|_H^2+\sigma_1(\psi(0),\psi(0))=2\,\mathrm{Re}\int_0^s-\sigma_2(y(t),y(t))dt.$$

But, recalling (H3) and (H5), we thus obtain

$$|y(s)|_H^2+k_1|\psi(0)|_V^2\le 2\lambda_0\int_0^s|y(t)|_H^2dt$$

or,

$$|y(s)|_H^2\le 2\lambda_0\int_0^s|y(t)|_H^2dt$$

for s arbitrary in $(0,T)$. Once again, we may use Gronwall's inequality and conclude that $y(t)=0$ for $t\in(0,T)$.

Remark 2.1. *In actual fact, one can strengthen the result of Theorem 2.1 to conclude that $y\in C((0,T),V)$ and $\dot y\in C((0,T),H)$ – (compare with [L,p.273], [LM,Chap.3]). To see this, we first consider the situation where $(y(0),\dot y(0))=(y_0,y_1)=(0,0)$. For $h>0$ define*

$$y_h(t)=\frac{1}{h}\int_{-h}^0 y(t+s)ds,\qquad f_h(t)=\frac{1}{h}\int_{-h}^0 f(t+s)ds$$

where we take $y(s)=y(0)=0,\ f(s)=0$ for $s<0$. Then $y_h\in C^1((0,T),V)\cap H^2((0,T),H)$. From (W2) we have

$$\langle\ddot y_h(t),\phi\rangle+\sigma_2(\dot y_h(t),\phi)+\sigma_1(y_h(t),\phi)=\langle f_h(t),\phi\rangle \tag{2.12}$$

for all $\phi \in V$ *since*

$$\frac{1}{h}\int_{-h}^{0} \ddot{y}(t+s)ds = \ddot{y}_h(t)$$

$$\frac{1}{h}\int_{-h}^{0} \dot{y}(t+s)ds = \dot{y}_h(t).$$

Let Z *be the Banach space defined by*

$$Z = \{z \in C((0,T),V) \mid \dot{z} \in C((0,T),H) \cap L_2((0,T),V_2)\}$$

equipped with norm

$$|z| = \sup_{t\in[0,T]} \{|z(t)|_V + |\dot{z}(t)|_H\} + \left(\int_0^T |\dot{z}(s)|_{V_2}^2 ds\right)^{1/2}.$$

Then it is readily seen that $\{y_h\}_{h>0}$ *is Cauchy in* Z*. Indeed, for* h, $k > 0$ *we have from (2.12)*

$$\langle \ddot{y}_h(t) - \ddot{y}_k(t), \phi\rangle + \sigma_2(\dot{y}_h(t) - \dot{y}_k(t), \phi) + \sigma_1(y_h(t) - y_k(t), \phi)$$
$$= \langle f_h(t) - f_k(t), \phi\rangle \qquad \text{for all } \phi \in V.$$

Putting $\phi = \dot{y}_h(t) - \dot{y}_k(t)$*, we obtain (arguing as before – see (2.3))*

$$|\dot{y}_h(t) - \dot{y}_k(t)|_H^2 + k_1|y_h(t) - y_k(t)|_V^2 + \delta \int_0^t |\dot{y}_h(s) - \dot{y}_k(s)|_{V_2}^2 ds$$
$$\leq 2\lambda_0 \int_0^t |\dot{y}_h(s) - \dot{y}_k(s)|_H^2 ds + \frac{1}{2\epsilon}\int_0^t |f_h(s) - f_k(s)|_{V_2^*}^2 ds.$$

Since $\{f_h\}_{h>0}$ *is Cauchy in* $L_2((0,T),V_2^*)$*, our claim follows from this inequality combined with Gronwall's inequality. But since* Z *is complete, the limit* y *of* $\{y_h\}$ *as* $h \to 0^+$ *is in* Z*.*

The case that $y \in Z$ *for nontrivial* $(y_0, y_1) \in V \times H$ *and* $f \equiv 0$ *follows from Theorems 3.1 and 4.1 presented below. Moreover, the arguments above yield the stronger continuous dependence result which we state as a corollary.*

Corollary 2.1 *The solutions of (W2) guaranteed by Theorem 2.1 satisfy*

$$|\dot{y}(t)|_H^2 + k_1|y(t)|_V^2 + \delta \int_0^t |\dot{y}(s)|_{V_2}^2 ds$$
$$\leq |y_1|_H^2 + k_1|y_0|_V^2 + 2\lambda_0 \int_0^t |\dot{y}(s)|_H^2 ds + \frac{1}{2\epsilon}\int_0^t |f(s)|_{V_2^*}^2 ds$$

for $(y_0, y_1) \in V \times H$ and $f \in L_2((0,T), V_2^*)$. Hence the map $(y_0, y_1, f) \to y$ is continuous from $V \times H \times L_2((0,T), V_2^*)$ to Z.

Remark 2.2. *In the event that $k_2 = 0$ in (H5) is the best lower bound available, the arguments and results above are valid with the following modifications. One requires $f \in L_2((0,T), H)$ and replaces the V_2^* and V_2 norms in the estimates (H4) and (2.2) by the H norm ($\epsilon = \frac{1}{2}$ can be chosen then in (2.2)). One does not obtain $L_2((0,T), V_2)$ boundedness for $\{\dot{y}_m\}$, only $C((0,T), H)$ boundedness. Thus the convergence of $\dot{y}_{m_k}$ in (2.5) is weak in $L_2((0,T), H)$. One then obtains only that $\dot{y} \in L_2((0,T), H)$ in the statement of the theorem. Essentially, one replaces V_2 by H in the statement and hypotheses of the theorem and one obtains the same results as in the case of no damping ($\sigma_2 = 0$) given in [L], [W].*

3. Semigroup Formulation

In this section we turn to a semigroup formulation for solutions of (W1), or equivalently (W2) under the same assumptions (H1)–(H6) of the previous section. As a first step we rewrite (W1) in first order form in the variables $w = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ on the state space $\mathcal{H} = V \times H$. We then establish that the resulting operator generates a C_0-semigroup on $\mathcal{H}$. We define the space $\mathcal{V} = V \times V$ and note that under the Gelfand triple formulation of the previous section, we also have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ where $\mathcal{V}^* = V \times V^*$. We define a sesquilinear form $\sigma : \mathcal{V} \times \mathcal{V} \to$ by

$$
\begin{aligned}
\sigma(\Phi, \Psi) &= \sigma\left((\phi_1, \phi_2),\ (\psi_1, \psi_2)\right) \\
&= -\langle \phi_2, \psi_1 \rangle_V + \sigma_1(\phi_1, \psi_2) + \sigma_2(\phi_2, \psi_2)
\end{aligned} \tag{3.1}
$$

for $\Phi = (\phi_1, \phi_2),\ \Psi = (\psi_1, \psi_2) \in \mathcal{V}$. Then (W2) can be rewritten for $w(t) = (y(t), \dot{y}(t))$ as

$$
\begin{aligned}
\langle \dot{w}(t), \chi \rangle_{\mathcal{V}^*, \mathcal{V}} + \sigma(w(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{V}^*, \mathcal{V}} &\text{ for } \chi \in \mathcal{V} \\
w(0) = w_0 = (y_0, y_1)&
\end{aligned} \tag{3.2}
$$

where $F(t) = (0, f(t))$. This is formally equivalent to the strong form of the equation given by (we won't distinguish between a two vector and its transpose in this section)

$$
\begin{aligned}
\dot{w}(t) &= \mathcal{A} w(t) + F(t) \\
w(0) &= w_0 = (y_0, y_1)
\end{aligned} \tag{3.3}
$$

where $\mathcal{A}$ is given by

$$
\text{dom } \mathcal{A} = \{\chi = (\phi, \psi) \in \mathcal{H} \mid \psi \in V \text{ and } A_1\phi + A_2\psi \in H\} \tag{3.4}
$$

and $\mathcal{A}\chi = (\psi, -A_1\phi - A_2\psi)$, or (in matrix operator form)

$$
\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_1 & -A_2 \end{bmatrix}. \tag{3.5}
$$

We note that $\mathcal{A}$ is the negative of the restriction to dom $\mathcal{A}$ of the operator $\tilde{\mathcal{A}} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ defined by $\sigma(\Phi, \Psi) = \langle \tilde{\mathcal{A}}\Phi, \Psi \rangle_{\mathcal{V}^*, \mathcal{V}}$ so that $\sigma(\Phi, \Psi) = \langle -\mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}}$ for $\Phi \in$ dom $\mathcal{A}, \Psi \in \mathcal{V}$. Under (H1)–(H3), it is readily observed that (recall that $\sigma_1(\phi, \phi)$ is real)

$$
k_1 |\phi|_V^2 \leq \sigma_1(\phi, \phi) \leq c_1 |\phi|_V^2
$$

so that σ_1 and the V inner product are equivalent. We may thus define V_1 as the set V taken with σ_1 as inner product, i.e., $\langle \phi, \psi \rangle_{V_1} = \sigma_1(\phi, \psi)$, obtaining a space that is setwise equal and topologically equivalent to V. We shall argue that $\mathcal{A}$ generates a

C_0-semigroup in $\mathcal{H}_1 = V_1 \times H$ and hence also in $\mathcal{H}$. In the space $\mathcal{H}_1$ the operator $\mathcal{A}$ is associated with a $\mathcal{V}_1 = V_1 \times V_1$ sesquilinear form $\sigma^{(1)}$

$$\sigma^{(1)}(\Phi, \Psi) = \langle \mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}_1}, \;\; \Phi \in \text{ dom } \mathcal{A}, \;\; \Psi \in \mathcal{V}_1.$$

To argue that $\mathcal{A}$ is an infinitesimal generator, we use the Lumer-Phillips theorem [P,p.14]. In the Hilbert space $\mathcal{H}_1$ it suffices to argue that $\mathcal{A} - \lambda$ is dissipative and that the range of $\lambda - \mathcal{A}$, $\mathcal{R}(\lambda - \mathcal{A})$, is all of $\mathcal{H}_1$ for some $\lambda > 0$.

Dissipativeness follows immediately from the definition of $\mathcal{A}$, (H1) and (H5) since

$$\begin{aligned} \text{Re } \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}_1} &= \text{Re } \{\sigma_1(\phi_2, \phi_1) - \sigma_1(\phi_1, \phi_2) - \sigma_2(\phi_2, \phi_2)\} \\ &= \text{Re } \{\overline{\sigma_1(\phi_2, \phi_1)} - \sigma_1(\phi_1, \phi_2) - \sigma_2(\phi_2, \phi_2)\} \\ &\leq \lambda_0 |\phi_2|_H^2 - k_2 |\phi_2|_{V_2}^2 \leq \lambda_0 |\Phi|_{\mathcal{H}_1}^2 \end{aligned} \tag{3.6}$$

where $\lambda_0 \geq 0$ is as in (H5). Thus Re $\langle (\mathcal{A} - \lambda_0)\Phi, \Phi \rangle \leq 0$ and it remains to argue the range condition.

We wish to argue that for some $\lambda > 0$, the range of $\lambda - \mathcal{A}$ is $\mathcal{H}_1$. Thus given $\xi = (\eta, \zeta) \in \mathcal{H}_1$, we wish to establish solvability of $(\lambda - \mathcal{A})\chi = \xi$ for $\chi = (\phi, \psi) \in$ dom $\mathcal{A}$. But this equation is equivalent to

$$\begin{aligned} \lambda\phi - \psi &= \eta \\ \lambda\psi + A_1\phi + A_2\psi &= \zeta. \end{aligned} \tag{3.7}$$

If we formally solve the first equation for $\psi = \lambda\phi - \eta$ and substitute this into the second equation, we obtain

$$\lambda^2\phi + A_1\phi + \lambda A_2\phi = \zeta + \lambda\eta + A_2\eta \tag{3.8}$$

which must be solved for $\phi \in V_1$ (and then ψ defined by $\psi = \lambda\phi - \eta$ will also be in V_1, with $A_1\phi + A_2\psi$ in H). These formal calculations suggest that we define for $\lambda > 0$ the sesquilinear form on $V \times V$ given by

$$\sigma_\lambda(\phi, \psi) = \lambda^2 \langle \phi, \psi \rangle_H + \sigma_1(\phi, \psi) + \lambda\sigma_2(\phi, \psi).$$

Since $\sigma_1,\ \sigma_2$ satisfy (H3), (H5) we have

$$\begin{aligned}\text{Re }\sigma_\lambda(\phi,\phi) &= \lambda^2|\phi|_H^2+\text{Re }\sigma_1(\phi,\phi)+\lambda\text{Re }\sigma_2(\phi,\phi)\\ &\geq \lambda^2|\phi|_H^2+k_1|\phi|_V^2+\lambda(k_2|\phi|_{V_2}^2-\lambda_0|\phi|_H^2)\\ &= \lambda(\lambda-\lambda_0)|\phi|_H^2+k_1|\phi|_V^2+\lambda k_2|\phi|_{V_2}^2\\ &> k_1|\phi|_V^2\end{aligned}$$

for $\lambda \geq \lambda_0 \geq 0$. Hence σ_λ is V-elliptic for $\lambda \geq \lambda_0$ and thus (3.8) is solvable for $\phi \in V$. It follows, as noted above, that (3.7) is thus solvable for $(\phi,\psi) \in \text{dom } \mathcal{A}$ and hence $\mathcal{R}(\lambda-\mathcal{A})=\mathcal{H}_1$. Thus for $\lambda \geq \lambda_0,\ \mathcal{A}-\lambda$ generates a contraction C_0-semigroup on $\mathcal{H}_1$ and hence we have established the following.

Theorem 3.1. *Under hypotheses (H1)–(H5) on $\sigma_1,\ \sigma_2$, the operator $\mathcal{A}$ defined in (3.4), (3.5) generates a C_0-semigroup $S(t)$ on $\mathcal{H}=V\times H$ which satisfies $|S(t)|_{\mathcal{H}_1}\leq e^{\lambda t}$ for any $\lambda \geq \lambda_0$.*

As usual, we can use this semigroup to define mild solutions for (3.3). For $w_0=(y_0,y_1)\in\mathcal{H}$ and $F=(0,f)$ with $f\in L_2((0,T),H)$ the "variation-of-parameters" representation

$$w(t)=S(t)w_0+\int_0^t S(t-s)F(s)ds \tag{3.9}$$

defines a mild solution for (3.3) which for $w_0 \in \text{dom } \mathcal{A}$ and $f\in C^1([0,T],H)$ is the unique strong solution [T,p.64] to (3.2) and hence to (W2). Thus this solution must be the same as the weak or variational solution guaranteed by Theorem 1 of Section 2 whenever $w_0=(y_0,y_1)$ is in dom $\mathcal{A}$ and $f\in C^1((0,T),H)$.

We wish, of course, to extend the formula (3.9) and the concept of mild solution to allow for $f\in L_2((0,T),V^*)$ and to establish equivalence to weak solutions in this case.

We consider first the case for σ_2 V-elliptic and V-continuous. That is, we take $V_2 = V$ in hypotheses (H4) and (H5); call these (H4′), (H5′). That is,

(H4′) The form σ_2 satisfies (H4) with V_2 replaced by V.

(H5′) The form σ_2 satisfies (H5) with V_2 replaced by V.

Under these assumptions, consider the operator $\tilde{A}$ (which is an extension of $\mathcal{A}$ from dom $\mathcal{A}$ to $\mathcal{V}_1 = V_1 \times V_1$) given by

$$\begin{aligned}\sigma^{(1)}(\Phi, \Psi) &\equiv -\sigma_1(\phi_2, \psi_1) + \sigma_1(\phi_1, \psi_2) + \sigma_2(\phi_2, \phi_2) \\ &= \langle \tilde{A}\Phi, \Psi \rangle_{\mathcal{V}_1^*, \mathcal{V}_1}.\end{aligned}$$

Note that $\sigma^{(1)}$ is just σ of (3.1) where we use the σ_1 inner product with V, i.e. V_1. The arguments in (3.6) yield immediately that

$$\begin{aligned}\text{Re } \sigma(\Phi, \Phi) &\geq k_2|\phi_2|_V^2 - \lambda_0|\phi_2|_H^2 \\ &\geq k_2\left(|\phi_1|_V^2 + |\phi_2|_V^2\right) - k_2|\phi_1|_V^2 - \lambda_0|\phi_2|_H^2 \\ &\geq \tilde{k}_2|\Phi|_{\mathcal{V}_1}^2 - \tilde{\lambda}_0|\Phi|_{\mathcal{H}_1}^2\end{aligned}$$

where $\tilde{k}_2$ and $\tilde{\lambda}_0$ are positive constants. Thus $\sigma(\cdot, \cdot) - \tilde{\lambda}_0\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ is a $\mathcal{V}_1$ elliptic sesquilinear form. That σ is $\mathcal{V}_1$ continuous also follows readily from (H2) and (H4′). Thus we have [T,p.76], [S,p.99] the following results.

Theorem 3.2. *Under hypotheses (H1)–(H3), (H4′), (H5′) on σ_1, σ_2, the operator $\tilde{\mathcal{A}}$ with domain $\mathcal{V}_1$ is an extension of $\mathcal{A}$ and generates an analytic semigroup $\tilde{S}(t)$ on $\mathcal{H}_1$ and $\mathcal{V}_1^*$ (and hence $\mathcal{V}^*$). We have $\tilde{S}(t) = S(t)$ on $\mathcal{H}$ and (3.9) can be extended to define mild solutions for (3.3) whenever $w_0 \in \mathcal{V}^*$ and $F = (0, f)$ with $f \in L_2((0,T), V^*)$.*

For the general case of (H4), (H5) for σ_2 with V_2 V, the concept of mild solution (i.e. a variation of parameters representation) is somewhat more delicate. We still desire however to have a concept of mild solution whenever $f \in L_2((0,T), V_2^*)$. To this end we must extend the semigroup $S(t)$ on $\mathcal{H}$ of Theorem 3.1 to a larger space $\mathcal{W}$ where $\mathcal{H} \subset \mathcal{W}$ as well as $\{0\} \times V_2^* \subset \mathcal{W}$. This space $\mathcal{W}$ will be chosen as $\mathcal{W} = \mathcal{Y}^*$ where $\mathcal{Y} = [\text{dom } \mathcal{A}^*]$ will be carefully defined below and, hence, before carrying out this extension, it is useful to characterize in some detail the adjoint $\mathcal{A}^*$ of the operator $\mathcal{A}$ of (3.4), (3.5). We compute and use the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ in the space $\mathcal{H}_1 = V_1 \times H$. We shall find

Lemma 3.1. *The adjoint $\mathcal{A}^*$ of $\mathcal{A}$ in $\mathcal{H}_1$ is given by*

$$\text{(3.10)} \qquad \textit{dom } \mathcal{A}^* = \{\Psi = (\psi_1, \psi_2) \in \mathcal{H} \mid \psi_2 \in V,\ A_1^*\psi_1 - A_2^*\psi_2 \in H\}$$

and

$$\text{(3.11)} \qquad \mathcal{A}^*\Psi = (-\psi_2,\ A_1^*\psi_1 - A_2^*\psi_2)$$

or, in matrix operator form,

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I \\ A_1^* & -A_2^* \end{bmatrix}.$$

Proof. From the definition of the adjoint we may characterize dom $\mathcal{A}^*$ and $\mathcal{A}^*$ by

$$\text{dom } \mathcal{A}^* = \{\Psi = (\psi_1, \psi_2) \in \mathcal{H} \mid \Phi \to (\mathcal{A}\Phi)(\Psi) \text{ is continuous on dom } \mathcal{A}\}$$

and

$$\langle \mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}_1} = \langle \Phi, \mathcal{A}^*\Psi \rangle_{\mathcal{H}_1}$$

for all $\Phi = (\phi_1, \phi_2) \in$ dom $\mathcal{A}$ and $\Psi = (\psi_1, \psi_2) \in$ dom $\mathcal{A}^*$. From the definition of $\mathcal{A}^*$ we see that $\Psi \in$ dom $\mathcal{A}^*$ if and only if there exists $\Gamma = (\gamma_1, \gamma_2) \in \mathcal{H}_1 = V_1 \times H$ such that

$$\langle \mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}_1} = \langle \Phi, \Gamma \rangle_{\mathcal{H}_1}$$

for all $\Phi \in \text{dom } \mathcal{A}$. Thus we have the defining condition

$$(3.12) \qquad \langle \phi_2, \psi_1 \rangle_{V_1} + \langle -A_1\phi_1 - A_2\phi_2, \psi_2 \rangle_H = \langle \phi_1, \gamma_1 \rangle_{V_1} + \langle \phi_2, \gamma_2 \rangle_H$$

for all $(\phi_1, \phi_2) \in \text{dom } \mathcal{A}$, $(\psi_1, \psi_2) \in \text{dom } \mathcal{A}^*$. Observing that (3.12) must hold for $(\phi_1, \phi_2) \in \mathcal{H}_1$ with $\phi_2 = 0$, $A_1\phi_1 \in H$, we find

$$\langle -A_1\phi_1, \psi_2 \rangle_H = \langle \phi_1, \gamma_1 \rangle_{V_1} = \langle A_1\phi_1, \gamma_1 \rangle_H$$

for all $\phi_1 \in V$ with $A_1\phi_1 \in H$, or

$$\langle A_1\phi_1, -\psi_2 - \gamma_1 \rangle_H = 0.$$

Since the set of all such ϕ_1 is dense in H and since A_1 is V elliptic, this implies that $\gamma_1 = -\psi_2$. Since $\gamma_1 \in V$, we find that $\psi_2 \in V$.

Using $\gamma_1 = -\psi_2$ in (3.12), we obtain

$$\langle A_1\phi_2, \psi_1 \rangle_H + \langle -A_2\phi_2, \psi_2 \rangle_H = \langle \phi_2, \gamma_2 \rangle_H$$

for all $(\phi_1, \phi_2) \in \text{dom } \mathcal{A}$. But this establishes that the map $\phi_2 \to \langle A_1\phi_2, \psi_1 \rangle_H + \langle -A_2\phi_2, \psi_2 \rangle_H$ is continuous in H for $(\phi_1, \phi_2) \in \text{dom } \mathcal{A}$, which implies that $A_1^*\psi_1 - A_2^*\psi_2 \in H$ and that

$$\langle \phi_2, A_1^*\phi_1 - A_2^*\psi_2 \rangle_H = \langle \phi_2, \gamma_2 \rangle_H$$

for all $\phi_2 \in V$. It follows that $\gamma_2 = A_1^*\psi_1 - A_2^*\psi_2$ and hence dom $\mathcal{A}^*$ and $\mathcal{A}^*$ are as stated in the lemma.

We note that since $\psi_2 \in V$ for any $\Psi = (\psi_1, \psi_2)$ in dom $\mathcal{A}^*$, we have dom $\mathcal{A}^* \subset V \times V = \mathcal{V}$. We next define an inner product (equivalent to a graph norm for $\mathcal{A}^*$) under which dom $\mathcal{A}^*$ will be a Hilbert space $\mathcal{Y}$ contained in, but not densely embedded in $\mathcal{V}$.

For our extension formulation, we use "extrapolation space" techniques that are similar to those used by Weissler [Wi], DaPrato and Grisvard [DG], Haraux [H], and numerous other investigators in recent years. The form of the results used here follows the ideas of Haraux in [H]. We summarize the necessary material, referring the reader to [H], [BKW] for more specific details.

Let $\lambda > \lambda_0$ be fixed so that $\lambda - \mathcal{A}^*$ is invertible in $\mathcal{H}_1$. We define an inner product on dom $\mathcal{A}^*$ by

$$\langle \Phi, \Psi \rangle_{\mathcal{Y}} \equiv \langle (\lambda - \mathcal{A}^*)\Phi, (\lambda - \mathcal{A}^*)\Psi \rangle_{\mathcal{H}_1} \tag{3.13}$$

and a norm $|\Phi|^2_{\mathcal{Y}} = \langle \Phi, \Phi \rangle_{\mathcal{Y}}$. We define $\mathcal{Y} \equiv [\text{dom } \mathcal{A}^*]$ as the space consisting of dom $\mathcal{A}^*$ taken with this inner product and norm. It can be shown (e.g., see Lemma 3.1 of [BKW]) that this norm is equivalent to the graph norm of $\mathcal{A}^*$, that is, for some constants C_1, C_2 we have

$$C_1|\Phi|_{\mathcal{Y}} \le |\Phi|_{\mathcal{H}_1} + |\mathcal{A}^*\Phi|_{\mathcal{H}_1} \le C_2|\Phi|_{\mathcal{Y}} \tag{3.14}$$

and thus the embedding $\mathcal{Y} \hookrightarrow \mathcal{H}_1$ is continuous. Since $\mathcal{A}^*$ (which also generates a C_0-semigroup on $\mathcal{H}_1$) is a closed operator, we have that $\mathcal{Y}$ is closed and hence $\mathcal{Y} = [\text{dom } \mathcal{A}^*]$ is a Hilbert space which is dense in $\mathcal{H}_1$. Moreover $\mathcal{Y} \hookrightarrow \mathcal{H}_1 = \mathcal{H}_1^* \hookrightarrow \mathcal{Y}^*$ forms a Gelfand triple, where $\mathcal{Y}^*$ is the conjugate dual of $\mathcal{Y}$. Define $\mathcal{W} = \mathcal{Y}^*$ so that $\mathcal{Y} \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{W}$ but while $\mathcal{Y} \subset \mathcal{V}$, we do not have $\mathcal{Y} \hookrightarrow \mathcal{V}$ and hence do not obtain $\mathcal{V}^* \hookrightarrow \mathcal{Y}^* = \mathcal{W}$. (If we had $\mathcal{A}^*$ satisfying a $\mathcal{V}$ ellipticity condition, then the embedding would be dense and continuous. However, that is not the situation in the general case.) As we shall see, we can argue that $\mathcal{V}^* = V \times V^* \subset \mathcal{W}$ and $|\cdot|_{\mathcal{W}} \le C_3|\cdot|_{\mathcal{V}^*}$, which suffices for our purposes. It is not difficult to argue (see [BKW]) that $\mathcal{W}$ is the completion of $\mathcal{H}_1$ in the norm $|w|_{\mathcal{W}} \equiv |R_\lambda(\mathcal{A})w|_{\mathcal{H}_1}$.

We next extend the operator $\mathcal{A}$ defined on dom $\mathcal{A} \subset \mathcal{H}_1$ to all of $\mathcal{H}_1$. This extension $\widehat{\mathcal{A}}$ will be the generator of a semigroup $\widehat{S}(t)$ on $\mathcal{W}$ that is an extension of $S(t)$. This extension will be achieved through a sesquilinear form $\widehat{\sigma}$ defined on $\mathcal{H}_1 \times \mathcal{Y}$ by

$$\widehat{\sigma}(\Phi, \Psi) \equiv \langle \Phi, \mathcal{A}^* \Psi \rangle_{\mathcal{H}_1}, \qquad \Phi \in \mathcal{H}_1, \;\; \Psi \in \mathcal{Y}. \tag{3.15}$$

From (3.14) we have immediately that

$$|\widehat{\sigma}(\Phi, \Psi)| \leq |\Phi|_{\mathcal{H}_1} |\mathcal{A}^* \Psi|_{\mathcal{H}_1} \leq C_2 |\Phi|_{\mathcal{H}_1} |\Psi|_{\mathcal{Y}}$$

which implies that $\Psi \to \widehat{\sigma}(\Phi, \Psi)$ is in $\mathcal{Y}^* = \mathcal{W}$ for each $\Phi \in \mathcal{H}_1$. Hence there exists $\widehat{\mathcal{A}} : \mathcal{H}_1 \to \mathcal{W}$ such that $(\widehat{\mathcal{A}}\Phi)(\Psi) = \widehat{\sigma}(\Phi, \Psi)$, i.e.

$$(\widehat{\mathcal{A}}\Phi)(\Psi) = \langle \Phi, \mathcal{A}^* \Psi \rangle_{\mathcal{H}_1} \tag{3.16}$$

for $\Phi \in \mathcal{H}_1, \;\; \Psi \in \mathcal{Y}$. Then we have for $\Phi \in$ dom $\mathcal{A}, \;\; \Psi \in \mathcal{Y}$

$$(\widehat{\mathcal{A}}\Phi)(\Psi) = \langle \Phi, \mathcal{A}^* \Psi \rangle_{\mathcal{H}_1} = \langle \mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}_1} = (\mathcal{A}\Phi)(\Psi)$$

so that $\widehat{\mathcal{A}}$ is indeed an extension of $\mathcal{A}$ from dom $\mathcal{A}$ to $\mathcal{H}_1$.

In [BKW], generic arguments are given (see Lemma 3.2, Theorem 3.1 of that paper) to show that for operators $\mathcal{A}, \mathcal{A}^*$ and $\widehat{\mathcal{A}}$ as defined above, we have that $\widehat{\mathcal{A}} - \lambda$ is dissipative in $\mathcal{W}$, with dom $\widehat{\mathcal{A}} = \mathcal{H}_1$ and that $\mathcal{R}(\lambda - \widehat{\mathcal{A}}) = \mathcal{W}$. Moreover, $\widehat{\mathcal{A}}$ is the infinitesimal generator of a C_0-semigroup $\widehat{S}(t)$ on $\mathcal{W}$ that is an extension of $S(t)$ from $\mathcal{H}_1$ to $\mathcal{W}$. Hence, we have an extension of the representation (3.9) given by

$$\widehat{w}(t) = \widehat{S}(t) w_0 + \int_0^t \widehat{S}(t-s) F(s) ds \tag{3.17}$$

which is valid for $w_0 \in \mathcal{W}$ and $F \in L_2((0,T), \mathcal{W})$. In general the space $\mathcal{W}$ is difficult to characterize exactly. As mentioned earlier, it suffices for our purposes to argue that $\mathcal{V}^* = V \times V^*$ is contained in $\mathcal{W}$ and $|\cdot|_{\mathcal{W}} \leq C_3 |\cdot|_{\mathcal{V}^*}$.

Lemma 3.2. *Under (H1)–(H5), we have* $\mathcal{V}^* = V \times V^* \subset \mathcal{W} = \mathcal{Y}^* = [\mathrm{dom}\ \mathcal{A}^*]^*$ *and for some constant* $C_3 > 0$

$$|\Phi|_{\mathcal{W}} \leq C_3 |\Phi|_{\mathcal{V}^*} \text{ for all } \Phi \in \mathcal{V}^*. \tag{3.18}$$

Proof. First note that $\mathcal{Y} \subset \mathcal{V}$ and from (3.14), (3.11) and (H3) we have for $\Psi = (\psi_1, \psi_2) \in \mathcal{Y}$.

$$\begin{aligned} C_2|\Psi|_{\mathcal{Y}} \geq |\Psi|_{\mathcal{H}_1} + |\mathcal{A}^*\Psi|_{\mathcal{H}_1} &\geq |\psi_1|_{V_1} + |\psi_2|_{V_1} \\ &\geq \sqrt{k_1}\left(|\psi_1|_V + |\psi_2|_V\right) \geq \tilde{k}|\Psi|_{\mathcal{V}}. \end{aligned}$$

Hence, for some constant $C_3 > 0$ we have for all $\Psi \in \mathcal{Y}$

$$|\Psi|_{\mathcal{V}} \leq C_3 |\Psi|_{\mathcal{Y}}. \tag{3.19}$$

Now suppose $\Lambda \in \mathcal{V}^* = V \times V^*$. Then for all $\Psi \in \mathcal{V}$ we have $|\Lambda(\Psi)| \leq |\Lambda|_{\mathcal{V}^*}|\Psi|_{\mathcal{V}}$. In particular, if $\Psi \in \mathcal{Y} \subset \mathcal{V}$ we then find using (3.19)

$$|\Lambda(\Psi)| \leq C_3 |\Lambda|_{\mathcal{V}^*}|\Psi|_{\mathcal{Y}}$$

and thus $\Lambda \in \mathcal{Y}^* = \mathcal{W}$ with $|\Lambda|_{\mathcal{W}} \leq C_3|\Lambda|_{\mathcal{V}^*}$. That is, $\mathcal{V}^* \subset \mathcal{W}$ with (3.18) holding.

Remark 3.1. *Under (H4′), (H5′), we have that* dom $\mathcal{A}^*$ *is dense in* $\mathcal{V}$ *(since* $\mathcal{A}^*$ *generates an analytic semigroup on* $\mathcal{V}$ *[T, p.28]) and hence* $\mathcal{Y} \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^* \hookrightarrow \mathcal{W}$.

As a consequence of this lemma and the considerations preceding it, we have the following extension results.

Theorem 3.3. *Under hypotheses (H1)–(H5) on* σ_1, σ_2, *the operator* $\hat{\mathcal{A}}$ *with domain* $\mathcal{H}_1$ *defined in (3.16) is an extension of* $\mathcal{A}$ *and generates a* C_0*-semigroup* $\hat{S}(t)$ *on* $\mathcal{W} = \mathcal{Y}^* = [\ \mathrm{dom}\ \mathcal{A}^*]^*$ *with* $\hat{S}(t) = S(t)$ *on* $\mathcal{H}$. *Moreover,* $\mathcal{V}^* \subset \mathcal{W}$ *and (3.9) can be extended so* *that (3.17) defines a mild solution $\hat{w}$ for (3.3) whenever $w_0 \in \mathcal{W}$ and $F = (0, f)$ with $f \in L_2((0,T), V^*)$.*

4. Equivalence of Solution Formulations

It is of interest to ascertain under what conditions on problem data, i.e. $w_0 = (y_0, y_1)$, f, σ_1, σ_2, one has equivalence between solutions of (W1) given in terms of a semigroup formulation such as those in Section 3 and solutions obtained from a variational formulation as given in Section 2. We first consider the case for $f \in L_2((0,T), H)$ and $w_0 = (y_0, y_1) \in \mathcal{H}$ under the general conditions (H1)–(H5) on σ_1, σ_2.

Let $w_{sg} = w(w_0, f)$ denote the semigroup solution to (3.3) given by (3.9) and guaranteed by Theorem 3.1 and let $w_{var} = (y(w_0, f),\ \dot{y}(w_0, f))$ denote the weak or variational solution guaranteed by Theorem 2.1 corresponding to initial data $w_0 = (y_0, y_1)$ and nonhomogeneous term f. Then we have:

Theorem 4.1. *Let (H1)–(H5) hold and suppose $w_0 = (y_0, y_1) \in \mathcal{H} = V \times H$ and $f \in L_2((0,T), H)$. Then $w_{sg}(w_0, f) = w_{var}(w_0, f)$.*

Proof. First suppose that $w_0 \in \text{dom}\ \mathcal{A} \subset \mathcal{V}$ which is dense in $\mathcal{H}$ and $f \in C^1((0,T), H)$ which is dense in $L_2((0,T), H)$. By standard results (e.g. see [P, Corr. 2.11] or [BK, Theor. 1.11]), in this case the mild solution $w_{sg} = w(w_0, f)$ is the unique strong solution to (3.3), or equivalently (3.2) (which is the same as (W2)) and, by the uniqueness statement in Theorem 2.1, must agree with $w_{var}(w_0, f)$ of Theorem 2.1. Hence $w_{sg} = w_{var}$ in this case. Recall that $(w_0, f) \to w_{var}(w_0, f)$ is continuous from $\mathcal{H} \times L_2((0,T), H)$ to $L_2((0,T), \mathcal{H})$ while it is readily seen from (3.9) that $(w_0, f) \to w_{sg}(w_0, f)$ is continuous in the same sense. Thus, the continuous functions w_{sg} and w_{var} agree on the dense set (dom $\mathcal{A}$) $\times C^1((0,T),H)$ and the desired result follows immediately.

We next consider the special case where σ_2 is V-elliptic and V-continuous (i.e. (H4$'$), (H5$'$) hold). As before, let w_{var} denote the solution guaranteed by Theorem 2.1. Let w_{sg} denote the mild solution given by (3.9) with the semigroup now defined on $\mathcal{V}^*$ as guaranteed in Theorem 3.2. Then we have:

Theorem 4.2. *Under hypotheses (H1)–(H3), (H4$'$), (H5$'$), we have $w_{sg}(w_0, f) = w_{var}(w_0, f)$ for $w_0 \in \mathcal{H} = V \times H$ and $f \in L_2((0,T), V^*)$.*

Proof. From Theorem 4.1, the solutions agree for $w_0 \in \mathcal{H}$ and $f \in L_2((0,T),H)$, where $L_2((0,T),H)$ is dense in $L_2((0,T),V^*)$. Moreover $(w_0, f) \to w_{sg}(w_0, f)$ is obviously continuous from $\mathcal{H} \times L_2((0,T),V^*)$ to $L_2((0,T),\mathcal{V}^*)$ and $(w_0, f) \to w_{var}(w_0, f)$ is continuous from $\mathcal{H} \times L_2((0,T),V^*)$ to $L_2((0,T),V) \times L_2((0,T),V)$ and hence to $L_2((0,T),\mathcal{V}^*)$, $\mathcal{V}^* = V \times V^*$. Again the result follows immediately.

Finally we turn to the general case to establish general equivalence of mild solutions and variational or weak solutions to (3.3). Let $\hat{w} = \hat{w}(w_0, f)$ denote the mild solution given by (3.17) for $w_0 \in \mathcal{H} = V \times H$ and $F = (0, f)$ with $f \in L_2((0,T), V_2^*)$.

Theorem 4.3. *Under hypotheses (H1)–(H5), we have $\hat{w}(w_0, f) = w_{var}(w_0, f)$ for $w_0 \in \mathcal{H} = V \times H$ and $f \in L_2((0,T), V_2^*)$.*

Proof. From Theorem 4.1 we have the equivalence for $w_0 \in \mathcal{H}$ and $f \in L_2((0,T),H)$ where $L_2((0,T),H)$ is dense in $L_2((0,T),V_2^*)$. We recall from Theorem 2.1 that $(w_0, f) \to w_{var}(w_0, f)$ is continuous from $\mathcal{H} \times L_2((0,T),V_2^*)$ to $L_2((0,T),\mathcal{V}_2)$ and hence to $L_2((0,T),\mathcal{W})$ since $\mathcal{V}_2 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{W}$, where $\mathcal{V}_2 = V \times V_2$. From (3.17) it is obvious that $(w_0, f) \to \hat{w}(w_0, f)$ is continuous from $\mathcal{H} \times L_2((0,T),\mathcal{W})$ to $L_2((0,T),\mathcal{W})$ and hence from $\mathcal{H} \times$

$L_2((0,T),\{0\}\times V_2^*)$ to $L_2((0,T),\mathcal{W})$ since (3.18) implies $|\Phi|_{\mathcal{W}} \le \tilde{C}_3|\Phi|_{\mathcal{V}_2^*}$ for some $\tilde{C}_3$ because $\mathcal{V}_2^* \hookrightarrow \mathcal{V}^*$. The desired result once again follows from equivalence on the dense subset $(w_0, f) \in \mathcal{H} \times L_2((0,T),H)$ of $\mathcal{H} \times L_2((0,T),V_2^*)$ and the continuity statements.

We remark, that of course $w_{sg}(w_0, f)$ of Theorem 4.2 is the same as $\hat{w}(w_0, f)$ for $f \in L_2((0,T),V_2^*)$ (and both equal w_{var}) under (H1)–(H3), (H4′), (H5′). Indeed, in this case the semigroups $\tilde{S}(t)$ of Theorem 3.2 and $\hat{S}(t)$ of Theorem 3.3 are the same on $\mathcal{V}^*$ and thus $\hat{S}(t)$ is an extension of $\tilde{S}(t)$ from $\mathcal{V}^*$ to $\mathcal{W}$.

5. Examples and Concluding Remarks

We briefly indicate a number of the wide variety of structural vibration examples that can be treated in the context of the framework presented in this paper. We consider first the cantilevered Euler-Bernoulli beam with a piezoceramic patch pair as described in Section 1. We consider only the transverse vibrations, i.e., equation (1.7) along with the cantilever boundary conditions. Defining $H = L_2(0,\ell)$ and

$$V = H_L^2(0,\ell) = \{\phi \in H_2(0,\ell) \mid \phi(0) = D\phi(0) = 0\}$$

and

$$\sigma_1(\phi,\psi) = \langle EID^2\phi,\, D^2\psi\rangle_H,$$

with EI given in (1.3), we readily see that (H1)–(H3) are satisfied. For the *Kelvin-Voigt* damping of this model, we define

$$V_2 = V = H_L^2(0,\ell) \text{ and } \sigma_2(\phi,\psi) = \langle c_D I D^2\phi,\, D^2\psi\rangle_H,$$

with $c_D I$ given in (1.3). Then (H4)–(H5) is easily verified. Finally for $f(t,x) \equiv -2\mathcal{K}_B D^2\chi_{pe}(x)u(t)$, we see that for $u \in L_2(0,T)$ we have $f \in L_2((0,T),V^*)$. This follows since the derivatives $D^2\chi_{pe}$ (in the sense of distributions) are in $V^* = (H_L^2(0,\ell))^*$.

That is, for $\phi \in V = H_L^2(0,\ell)$

$$(D^2\chi_{pe})(\phi) = \langle \chi_{pe}, D^2\phi\rangle_H \leq |\chi_{pe}|_H \, |D^2\phi|_H$$
$$\leq \sqrt{x_2 - x_1}|\phi|_V^2.$$

Hence (H6) is satisfied for this example and the results of Theorems 2.1, 3.2 and 4.2 are applicable.

We remark that if EI and $c_D I$ were constant, the operator $\mathcal{A}$ of (3.5) would reduce to the usual one, i.e. $\mathcal{A}$ given by (3.5) on dom $\mathcal{A} = (H^4(0,\ell) \cap H_L^2(0,\ell)) \times (H^4(0,\ell) \cap H_L^2(0,\ell))$.

Other damping models may be used with examples involving cantilevered Euler-Bernoulli beams. We list several. (The spaces H and V and the sesquilinear form σ_1 remain as above in all these examples.)

Viscous damping. This damping model, also called air damping, involves a velocity proportional damping term with damping coefficient $\gamma \in L_\infty(0,\ell)$. We choose $V_2 = H$ and

$$\sigma_2(\phi,\psi) = \langle \gamma\phi, \psi\rangle_H.$$

Spatial hysteresis damping. The model, explained in some detail in [Ru], has been shown to be appropriate for composite material beams where graphite fibers are embedded in an epoxy matrix [BWIC]. The damping sesquilinear form can be given in terms of a compact operator G on $L_2(0,\ell)$ defined by

$$(G\phi)(x) = \int_0^\ell \gamma(x,y)\phi(y)\,dy$$

where γ is a symmetric, non-negative kernel in $L_\infty((0,\ell)\times(0,\ell))$. The space V_2 is chosen as $H^1(0,\ell)$ and

$$\sigma_2(\phi,\psi) = \langle (\nu I - G)D\phi,\ D\psi\rangle_H$$

where $\nu(x) = \int_0^\ell b(x,y)\,dy$.

For a fixed end beam (i.e. $w(t,0) = Dw(t,0) = w(t,\ell) = Dw(t,\ell) = 0$), all of the above damping models may be used also. In this case, we choose $H = L_2(0,\ell)$ again, but $V = H_0^2(0,\ell) = \{\phi \in H^2(0,\ell) \mid \phi,\ D\phi$ vanish at $x = 0,\ \ell\}$ so that $V^* = H^{-2}(0,\ell)$. The sesquilinear form σ_1 is as before and the damping forms as defined above, except we replace $H_L^2(0,\ell)$ by $H_0^2(0,\ell)$ throughout. In addition to Kelvin-Voigt, viscous, and spatial hysteresis damping models for this example, one can also consider so-called "structural" (square-root or $A^{1/2}$) damping. In this case we choose $V = H_0^2(0,\ell)$ of course, $V_2 = H_0^1(0,\ell) = \{\phi \in H^1(0,\ell) \mid \phi(0) = \phi(\ell) = 0\}$ and

$$\sigma_2(\phi,\psi) = \langle cD\phi,\ D\psi\rangle_H$$

where $c \in L_\infty(0,\ell)$.

We can also include other examples in our framework: Timoshenko beams ([Wa], [BWI]), 2-D plates and grid structures [Re], [BR1], [BR2], [BCR], acoustic/structural models [BFSS], [BS], [BSW], as well as multiple component structures [K1], [K2].

We conclude our presentation with several remarks on related research literature. The well-posedness literature on second order partial differential equations (PDE) in both variational and semigroup formulations is rather extensive, dating back to the early work of Lions (see [L], [LM] and the references there to other earlier efforts) in the late 60's to the more recent treatments by Showalter [S], Tanabe [T] and Wloka [W]. While results are scattered, much of the pieces of results presented in this paper can be found in one form or another in earlier publications. Our well-posedness treatment for systems with general damping sesquilinear forms is new and the results here for these systems cannot, to our knowledge, be found in this form elsewhere. Most authors consider undamped [L], [W] second order hyperbolic systems or assume damping sufficiently strong so as to make the system essentially parabolic in nature. In recent years, owing in part to increased computational capabilities, serious interest in and efforts on use of PDE models in complex structures (for parameter estimation, system identification, and control) have substantially increased. All experimental structures and data for them exhibit damping of some form and thus, in model fitting and control design, it is important to include damping in mathematical models which are supposed to describe these structures.

To the best of our knowledge, Theorems 2.1 and 4.1 (equivalence for $f \in L_2((0,T),H)$) are new and give results for general (including weak but nontrivially) damped systems with input or inhomogeneous term bounded in the state space.

Theorems 3.2 and 4.2, the case for strong damping (σ_2 is V elliptic and V continuous) can be obtained directly from Tanabe's results by rewriting the second order system as a first order system with $\mathcal{V}$ elliptic form (as we did in Section 3 in obtaining Theorem 3.2) and then appealing to Prop. 5.5.1, p. 153 of [T].

The results of Section 3 on applying Haraux's techniques for the treatment (involving (3.17), Lemma 3.2 and Theorem 3.3) of weakly damped second order systems appear to be new as are the equivalence results of Theorem 4.3.

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