

ABSTRACT

WANGWATCHARAKUL, WORAWUT. Modeling Inventory Systems with Imperfect Supply. (Under the direction of Russell E. King and Donald P. Warsing.)

We study inventory systems operating under an infinite-horizon, periodic-review base-stock control policy with stochastic demand and imperfect (i.e., less than 100% reliable) supply. We model demand using a general discrete distribution and replenishment lead time using a geometric distribution, resulting from a Bernoulli trial-based model of supply uncertainty. We develop a computational approach using a discrete time Markov process (DTMP) model to minimize the total system cost and obtain the optimal base-stock level when the backorder penalty is given. We develop a general, recursive solution for the steady state probability of each inventory level and use this to find the optimal base-stock level in this setting. Moreover, for specific demand distributions we are able to develop closed-form solutions for these outcomes.

The lead-time demand (LTD) distribution can also be obtained from these recursive equations to determine the base-stock level when a target customer service level is specified in lieu of a backorder penalty cost. We conduct extensive computational experiments to observe the robustness of various approximate solutions under two scenarios for the lead-time distribution. The first scenario assumes a geometric lead time. The second scenario considers a general lead-time distribution. We conduct computational experiments to observe the conditions in which the DTMP model performs well, including situations where the demand and the lead-time distributions are specified separately, and where the LTD distribution is given and follows either a Beta distribution or a bimodal distribution.

Finally, for a two-location inventory system consisting of a single retailer supplied by a single distributor, whose supply ultimately comes from an unreliable supplier upstream, we propose a computational approach to determine optimal or near-optimal base-stock levels at the retailer and distributor. We develop two decomposition-based approximation methods, solving the separate single-site inventory problems (distributor, retailer) sequentially, but

with different methods to compute the implied backorder penalty at the distributor that induces near-optimal base-stock levels at both locations.

Modeling Inventory Systems with Imperfect Supply

by
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DEDICATION

To my parents, my teachers, my brothers, my cousins, and my friends

BIOGRAPHY

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TABLE OF CONTENTS

LIST OF TABLES.....	ix
LIST OF FIGURES.....	x

CHAPTER 1

Introduction and motivation.....	1
References.....	5

CHAPTER 2

OPTIMAL BASE-STOCK LEVELS FOR AN INVENTORY SYSTEM WITH IMPERFECT SUPPLY.....	6
1. Introduction.....	6
2. Model.....	10
2.1 Model specification.....	10
2.2 Analytical results for general discrete demand distribution.....	13
2.2.1 Explicit forms of the steady state probability distribution	13
2.2.2 Solution procedure.....	14
2.2.3 Decomposition of the total cost.....	17
2.2.4 Performance measure of the system and its bounds.....	19
2.3 Analytical results for specific demand distributions.....	20
2.3.1 Closed-form solutions: EOQ-type customer.....	20
2.3.2 Closed-form solutions: Geometric demand.....	22
2.4 Effect of changing supply risk on the total system cost	23

3. Model with service level constraint.....	26
3.1 A procedure to determine a service-constrained optimal base-stock level.....	26
3.2 Application to lead-time demand distribution.....	27
3.3 Numerical example.....	30
4. Conclusions.....	32
References.....	34
Appendices.....	36

CHAPTER 3

COMPUTING BASE-STOCK LEVELS FROM APPROXIMATE LEAD-TIME- DEMAND DISTRIBUTIONS UNDER IMPERFECT SUPPLY.....	43
1. Introduction.....	43
2. Markov model and analysis.....	45
2.1 Model characteristic.....	45
2.2 Service-level model and service performance measures.....	48
2.3 Effect of changing penalty cost and target service level on the total system cost... 50	
3. Comparing approximation methods under geometric lead time.....	51
3.1 Approximation methods.....	53
3.2 Experiment results.....	54
4. Comparing approximation methods under general lead time.....	56
4.1 Demand and lead-time distributions specified	56
4.1.1 Data.....	57
4.1.2 LTD distribution parameters.....	58
4.1.3 Experimental results.....	61
4.2 Specified LTD distributions.....	63

4.2.1 Symmetric beta distribution.....	64
4.2.2 Right-skewed beta distribution.....	65
4.2.3 Left-skewed beta distribution.....	66
4.2.4 Bimodal lead-time demand distribution.....	67
5. Conclusions.....	67
References.....	69
Appendices.....	71

CHAPTER 4

COMPUTING BASE-STOCK LEVELS FOR A SERIAL INVENTORY SYSTEM WITH IMPERFECT SUPPLY.....	78
1. Introduction.....	79
2. Model.....	80
2.1 Model specification.....	80
2.2 System dynamics.....	82
2.3 Computing transition probabilities.....	85
2.4 Numerical experiments.....	86
2.4.1 Impact of h_2 on the optimal base-stock levels.....	88
2.4.2 Impact of α on the optimal base-stock levels.....	89
2.4.3 Impact of b_1 on the optimal base-stock levels.....	89
3. Decentralized approximation methods.....	90
3.1 Method 1: Functional form of implied backorder penalty.....	91
3.2 Method 2: Decentralized objective function.....	94
4. Experimental results.....	96
5. Conclusions.....	98
References.....	101

Appendices.....102

CHAPTER 5

Conclusions..... 106

LIST OF TABLES

CHAPTER 3

Table 1. Deviation (%) from optimal cost of approximation methods for Poisson demand	55
Table 2. CV of LTD for all combinations of demand and lead time distributions	58
Table 3. Average deviation (%) from optimal cost, symmetric LTD	64
Table 4. Average deviation (%) from optimal cost, right-skewed LTD	65
Table 5. Average deviation (%) from optimal cost, left-skewed LTD	66
Table 6. Average deviation (%) from optimal cost, bimodal LTD	67

CHAPTER 4

Table 1. Summary of demand parameters used in this experiment.....	87
Table 2. Summary of cost deviation from optimal (expressed as %) for method 1 for different demand parameters.....	94
Table 3. Summary of cost deviation from optimal (expressed as %) for method 2 for different demand parameters.....	96
Table 4. Beta demand distributions used in the experiment.....	97
Table 5. Summary of % cost deviation from the experiment.....	98

LIST OF FIGURES

CHAPTER 2

Fig. 1. The sequence of events during period t	12
Fig. 2. Inventory cost vs. base-stock level when $\alpha=0.9, h=1, b=20$	18
Fig. 3. Inventory cost vs. base-stock level: $\alpha = 0.9$ for different ratios of b/h	18
Fig. 4. $\delta(\alpha)$ vs α	24
Fig. 5. Rate of change on cost deviation vs. b for different demand distributions	25
Fig. 6. DTMP-derived LTD distribution for different demand distributions and supply risk factors.....	28
Fig. 7. Comparing the base-stock level from three methods.....	31

CHAPTER 3

Fig. 1. LTD distribution for different demand distributions and supply risk factors.....	48
Fig. 2. Optimal total cost vs backorder penalty and target service level.....	51
Fig. 3. Comparisons of the pdf and cdf of LTD distribution for various methods	52
Fig. 4. The relationship between optimal base-stock and optimal total cost for different CV of X	52
Fig. 5. Lead-time distributions used in computational experiments.....	57
Fig. 6. Gamma demand distribution used in computational experiments.....	58
Fig. 7. Comparative results across six different approximation models for the D2-L4 case	62
Fig. 8. Bimodal distributions generated from Poisson distributions with $\lambda=20$ and $\lambda=40$	64

CHAPTER 4

Fig.1. System configuration.....	80
Fig.2. Sequence of events during a period.....	82
Fig.3. Different beta demand distributions.....	87
Fig.4. Optimal base-stock levels vs. h_2 , when $b_1 = 20, \alpha = 0.85$	88
Fig.5. Optimal base-stock levels vs. α , when $h_2 = .5, b_1 = 20$	89
Fig.6. Optimal base-stock levels vs. b_1 , when $h_2 = .5, \alpha = 0.85$	90
Fig.7. Optimal node 2 penalty cost vs. Fitted node 2 penalty cost.....	92
Fig.8. Total system cost for decentralized system vs. \hat{b}_2	95
Fig.9. Beta demand distributions used in the experiment.....	97

CHAPTER 1

Introduction and motivation

In this dissertation, we study an inventory system operating under a periodic-review, base-stock control policy with stochastic demand and imperfect (i.e., less than 100% reliable) supply. Under imperfect supply, a constant nominal lead time for re-supply becomes a stochastic replenishment time, with an obvious impact on the system's ability to meet a specified customer service level. Two basic approaches to determine optimal base-stock levels are the "shortage costing approach" and the "service-level approach," the latter applying to the case for which the shortage penalty cost is not specified (Dullaert et al., 2007). We develop a computational approach to obtain the optimal base-stock level for the shortage-costing case, and also show that the method can be modified to handle the service-level case. For specific demand distributions, we are able to develop closed-form solutions for the optimal base-stock level, and we also develop expressions for upper and lower bounds on the corresponding optimal service level. To handle the service-level case, traditional methods for computing base-stock levels require the distribution of demand during the lead time be specified. When demand and lead time are stochastic, however, analytically tractable forms of the lead-time demand (LTD) distribution can be obtained from the convolution of two distributions of demand and lead time, which is viable for only few special cases. An approximation method which assumes the LTD distribution is Normally distributed has been widely used to estimate the base-stock level with central limit theorem justification (e.g., Tyworth and O'Neill, 1997). We examine the accuracy of this and several other approximation methods under both restricted and general lead time cases to observe the practical value of our model.

In Chapter 2, we consider a single inventory location and compute the optimal base-stock level under supply uncertainty for an infinite-horizon, periodic-review inventory model. We model demand using a general discrete distribution and replenishment lead time

using a geometric distribution resulting from a Bernoulli trial-based model of supply uncertainty. A discrete-time Markov process model (DTMP) is used to minimize the total system cost, comprised of inventory holding and backorder penalty costs. We present two complementary scenarios. In the first scenario, when penalty cost is specified, we develop a general, recursive solution for the steady state probability of each inventory level and use this to find the optimal base-stock level in this setting. Moreover, for specific demand distributions we are able to develop closed-form solutions for these outcomes, and we also develop expressions for upper and lower bounds on the corresponding optimal service level. We also quantify the impact of supply risk on the deviation from the optimal cost under fully reliable supply. In the second scenario, when the penalty cost is not known, we seek a base-stock level that satisfies a service-level target. This setting also assumes demand and lead time conditions that result in a variety of lead-time demand shapes. We compare the base-stock levels from our DTMP model with a model that uses the traditional assumption of Normally-distributed demand and lead time. We find that the Normal approximation consistently overestimates the optimal base-stock level up to a threshold service level, and beyond that, underestimates it, clearly based on the bimodal nature of LTD. Not only does our method generate base stock levels that more accurately determine the optimal base stock, but it is also much more flexible in that it can accommodate any discrete demand distribution.

Chapter 3 focuses on analyses of the DTMP model presented in Chapter 2 including its robustness to approximate the LTD when the lead time is not geometrically distributed. The LTD is a key component in determining the base-stock level to satisfy a given service level and, for our model, can be obtained directly from the recursive steady-state probability equations. Due to its simplicity, a classical approach to approximate the LTD distribution is to use a Normal approximation. This approach, however, yields significant cost deviations from the optimal cost in some circumstances, as demonstrated in Chopra et al. (2004). Their results indicate that the Normal approximation could lead to a misspecification of the buffer to guard against the lead-time uncertainty and misleading managerial insights on the need for safety stock. When a target customer service level is specified, we can determine the

corresponding backorder penalty, resulting from the specified service level that makes the two models equivalent. The functional relationship between a required service level and optimal total cost is investigated to observe the effect of the service level target on the total system cost.

We conduct extensive computational experiments to observe the robustness of various approximate solutions under two scenarios depending on the lead-time distribution. The first case is based upon assumptions of a geometric lead time and a discrete demand distribution, such that the DTMP model represents the real system. We investigate the quality of three approximations to the DTMP model: Normal and gamma approximations that match the first two moments of the LTD distribution, and a statistical bootstrap procedure that uses a re-sampling approach to approximate the LTD. In the second set of experiments, we consider a general situation where the lead time is not restricted to a geometric distribution. To investigate the practical value of the DTMP model, we treat it as an alternative approximation to a simulated near-optimal base-stock level. The model parameters are estimated by matching the first two or three moments with the ones from data sampled from the true, underlying LTD distribution (Lau, 1989). We perform numerical experiments where the demand distribution and the lead-time distribution are modeled individually, and when the LTD distribution is characterized by a discretized beta distribution or a bimodal distribution. The cost deviations from the simulation results are compared in both cases. In addition, we observe the conditions when the model provides a good approximation, and the influence of changing coefficient of variation of the LTD on cost deviation.

In Chapter 4, we propose a computational method to determine optimal base-stock levels in a two-location inventory system consisting of one distributor (upstream) and one retailer (downstream), with a fixed one-period lead time from distributor to retailer but an unreliable supplier to the distributor. The distributor and retailer both operate under a periodic review base-stock control policy. The customer demand is assumed to follow any discrete probability distribution. Supply uncertainty is modeled as a Bernoulli trial, with a specified probability that the total quantity on-order is filled in a period. The objective is to minimize the total system cost. The one-node system presented in Chapter 2 serves as a

building block of this two-node serial system. Since a closed-form solution appears to be unattainable, direct numerical computation is a viable approach to finding an optimal solution. When the state space of this model increases, however, computational complexity becomes an issue. In order to reduce the search space and computational time, we develop two approximation methods to find a good starting solution. The decomposition-based approximation methods are used to solve the problem sequentially by optimizing the separate single-site inventory problems (distributor, retailer) with an implied backorder penalty at the upstream node. The first method provides a functional relationship between input parameters and the implied backorder penalty. The second method numerically searches for the implied backorder penalty to minimize the total system cost for the decentralized system. The approximation solution could serve as an initial solution for a steepest descent search algorithm to obtain the optimal solution. Numerical experiments are conducted to evaluate the effectiveness of the approximation methods. The cost deviations from the optimal solutions are presented. We also observe the impact of changing input parameters such as backorder penalty and demand shape on the base-stock levels and the total system cost.

REFERENCES

Chopra. S., Reinhardt. G., & Dada. M. (2004). The effect of lead-time uncertainty on safety stocks. *Decision Sciences*, 35(1), 1–24.

Dullaert, W., Vernimmen, B., Aghezzaf, E., & Raa, B. (2007). Revisiting service-level measurement for an inventory system with different transport modes. *Transport Reviews*, 27(3), 273-283.

Lau, H.S. (1989). Toward an inventory control system under non-normal demand and lead-time uncertainty. *Journal of Business Logistics*, 10, 89–103.

Tyworth, J. E., & O’Neill, L. (1997). Robustness of the normal approximation of lead-time demand in a distribution setting. *Naval Research Logistics*, 44(2), 165-186.

CHAPTER 2

Optimal Base-stock Levels for an Inventory System with Imperfect Supply

Abstract

We develop a method to compute the optimal base-stock level under supply and demand uncertainty for an infinite-horizon, periodic-review inventory model. A discrete-time Markov process (DTMP) model is used to minimize the total system cost, comprised of inventory holding and backorder penalty costs. We present two complementary scenarios. In the first scenario, when penalty cost is specified, we model demand using a general discrete distribution and replenishment lead time using a geometric distribution, resulting from a Bernoulli trial-based model of supply uncertainty. We develop a general, recursive solution for the steady state probability of each inventory level and use this to find the optimal base-stock level in this setting. Moreover, for specific demand distributions we are able to develop closed-form solutions for these outcomes, and we also develop expressions for upper and lower bounds on the corresponding optimal service level. We quantify the impact of the level of supply uncertainty on the deviation from the optimal cost under fully reliable supply, and in the second scenario, when the penalty cost is not known, we seek a base-stock level that satisfies a service-level target. This setting also assumes demand and lead-time conditions that result in a bimodal lead-time demand distribution. In this setting, we compare the base-stock levels from our DTMP model with a model that uses the traditional assumption of Normally-distributed demand and lead time. We find that the Normal approximation consistently overestimates the optimal base-stock level up to a threshold service level, beyond which it underestimates it, which appears to be driven by the bimodal nature of lead-time demand. Not only does our method generate base stock levels that accurately determine the optimal base stock, but it is also flexible in that it can accommodate any discrete demand distribution.

1. Introduction

The inventory model with demand uncertainty and supply risks has gained considerable attention from researchers in recent years. In this paper, we study a single inventory location operating under a periodic-review, base-stock control policy with

stochastic demand and imperfect (i.e., less than 100% reliable) supply. Periodic review is practical, easy to implement, and suitable for an inventory system of a single, fast-moving product or multiple independent products transported jointly from a single supplier. The base-stock inventory control policy is widely used in practice since it is easy to implement in a periodic replenishment framework and has been proven to be optimal when set up cost is negligible (Clark and Scarf, 1960). In our model, the nominal replenishment lead time is one period, with a Bernoulli trial probability of α that the quantity on-order is filled in that one-period lead time. Under these conditions, the replenishment lead time follows a geometric distribution with parameter α . We formulate a discrete-time Markov process (DTMP) model with a finite state space to determine the optimal base-stock level in this single-site inventory system. The total cost for this system is comprised of inventory holding and backorder penalty costs. This DTMP model can be viewed as an extension of the classical newsvendor model in an infinite-horizon setting with the risk of a supply shortfall. Intuitively, more inventory needs to be carried to hedge against supply uncertainty. Our goal is to provide an approach to compute the optimal base-stock level that accounts for this supply uncertainty in addition to the uncertainty in demand. We also extend our analysis to determine the base-stock level required to satisfy a service level target in the case for which a backorder penalty is not specified.

In academic research, it is typical to assume Normally distributed demand and lead-time distributions, owing to the simplicity of convolving two Normal distributions to specify the resulting lead-time demand (LTD) distribution. The use of the Normal distribution to model customer demand is questionable in many situations, though, especially when the coefficient of variation is high. Also, a symmetric lead time distribution is difficult to justify in a realistic setting. These issues motivate us to establish a computational approach or approximation procedure to assist a decision maker in managing inventory in a timely manner under rapidly changing environment such as facing dynamic period demand distribution without imposing overly-simplified assumptions on demand and lead time. That is, we specify a model in which the demand has a general form, symmetric or skewed, and

the lead time falls in a fairly narrow range of values, possibly a multimodal distribution in the case of supply disruptions that irregularly result in long lead times.

We present two complementary scenarios. In the first scenario, when the backorder penalty cost is given, the model allows determination of the optimal base-stock level and the corresponding optimal service level to minimize the total system cost. For some specific demand distributions, a closed-form expression for the optimal base-stock level is developed, based on the steady state probability of inventory levels. We use these expressions to specify upper and lower bounds on the optimal service level. In the second scenario, where the backorder penalty cost is not known but a service level target is specified, we formulate a model to determine the base-stock level to meet this customer service level based on the approach developed for the first scenario. The LTD distribution clearly plays a key role in determining a base-stock level, safety stock, or reorder point to meet a required service level. The derivation of the LTD distribution using the convolution of demand and lead-time distribution is, however, a challenging task in general and may be possible only for particular lead time and demand distribution pairs. When lead time is deterministic, it is easy to compute the LTD distribution. When lead time is stochastic, however, the Normal approximation is extensively used to estimate the LTD distribution and the base-stock level. A distribution-free method using a convex combination of discrete lead-time probability is another approach for safety stock and reorder point determination (Tyworth and O'Neill, 1997). The DTMP model here provides the exact solution and can incorporate any discrete demand distribution and resulting LTD distribution from underlying supply interruption.

Models with supply uncertainty can be categorized into yield loss models (supply capacity uncertainty) and supply disruption models (supply availability uncertainty) (Tomlin, 2006). We concentrate on the latter, under the assumption that the supplier has infinite capacity. Most of the previous work in this regard has assumed that the inventory control policy operates under continuous review (Parlar, 1997; Arreola and DeCroix, 1998; Parlar and Perry, 1995; Mohebbi, 2003; Mohebbi, 2004; Gupta, 1996). A Poisson process is a common choice for the demand process, which is suited to slow-moving products where demand is

sporadic and not necessarily symmetric (Tersine, 1988). Moreover, the compound Poisson is generally used when a demand batches of random size occur at random inter-arrival times.

Random supply interruptions are the consequence of a diversity of situations, including equipment breakdowns, material shortages, strikes, natural disaster, and political crises (Mohebbi, 2003). The random nature of supply uncertainty has been modeled in several ways to characterize the replenishment lead time such as on/off rates. In this continuous-review framework, the status of the supply process is typically modeled as “on” (available) and “off” (unavailable) states. Transitions between these two states are typically modeled as a continuous-time Markov process to exploit the memoryless property. Arreola-Risa and De Croix (1998) use an on/off rate model to provide optimal values of (s,S) policy parameters including optimal inventory strategy with different severity of supply disruptions with partial backorder. They use a more general stockout behavior, that is, the unfilled demands become a mixture of lost sales and backorders. Gupta (1996) studies the impact of an unreliable supplier on the operating cost in a continuous review (s,Q) system. The supplier on-off periods follow an exponential distribution. Parlar and Perry (1995) incorporate two unreliable suppliers in an (s,Q) model with independent non-identical on/off periods following exponential distribution. In this paper, the inventory is reviewed periodically.

Lead time is another significant stochastic element in an inventory system. A common simplifying assumption is a constant or zero lead time due to the complexity of solving a more general system analytically. Mohebbi (2003) generalizes the (s,Q) inventory model of Gupta (1997) by allowing Erlang (E_k) distributed lead times and exponentially distributed on-off periods and provides an analytical cost-minimization model in a lost-sales environment. Mohebbi (2004) provides an exact expression of the long-run average cost rate function by extending the model in Mohebbi (2003) to incorporate a hyperexponentially-distributed lead time. He employs an alternating renewal process to model the supplier availability where the on and off periods follow general and hyperexponential distributions, respectively. In this study, we concentrate on how inventory is affected by underlying assumptions about supply uncertainty, using a lead-time model that, under certain conditions, yields closed-form solutions for base-stock levels.

In our study, we conduct numerical experiments to capture model behavior under different model parameters. The results from this single-node model allow us to gain some managerial insights and serve as a building block for considering a set of base-stock levels in more complex system configurations in multi-node supply chains such as serial systems or assembly or distribution networks.

2. Model

2.1. Model specification

Our model assumes a periodic-review inventory system with negligible fixed ordering costs. This implies that a base-stock policy is optimal (Clark and Scarf, 1960). At the beginning of each period, an order is made to the supplier to bring the inventory position back up to the base-stock level. However, the supplier is not entirely reliable. Here, we assume that the re-supply process in each period is a Bernoulli trial, meaning that with probability α the supplier delivers the current order and any accumulated backorders at the end of the current period (without capacity constraint, independent of the order size). If the nominal lead time is one period, then the supply uncertainty induces a stochastic replenishment lead time having a geometric distribution with parameter α . The customer demand in our model is i.i.d. over time, following a given discrete (or discretized) non-negative distribution. The backorder penalty cost is incurred per unit of backordered demand in any period when the demand is not fulfilled and the on-hand inventory level is negative.

Uncertainty in both demand and supply adds difficulty in determining the optimal base-stock level. To solve the model optimally in a finite-horizon framework, dynamic programming is generally used to capture the dynamic aspects of the system under uncertainty (Gallego and Özer, 2005). Here, we consider an infinite horizon model. We model the system using a DTMP in order to determine the optimal base-stock level. For some specific cases—e.g., where demand has a geometric distribution or where the customer ordering policy is EOQ with a common review period—we develop closed-form expressions

for the optimal base-stock level, the corresponding service level, and the optimal holding and backorder costs as a function of the model input parameters.

The following notation is used to describe the model.

S = base-stock level

S^* = optimal base-stock level

g = maximum backorder level

Ω = state space of the system = $-g, \dots, -1, 0, 1, \dots, S$

I_t = on-hand inventory at the end of time t ($I_t \in \Omega$)

α = probability that the total quantity on-order arrives in the current period

D_t = customer demand in period t (assumed stationary)

$p_d = P D_t = d$ for $d \in d_{\min}, \dots, d_{\max}$ ($d_{\min} \geq 0$ and $d_{\max} < \infty$)

h = holding cost per unit per period

b = backorder penalty cost per unit per period

π_i = steady-state probability of being in state i , where i = end-of-period on-hand inventory

$[i]^+ = \max(i, 0)$

$[i]^- = \max(-i, 0)$

BC = backorder cost

IC = inventory holding cost

During any period t in the infinite horizon, three events determine the state transition. Each of these events is described below and their timing is displayed in Figure 1.

1. An order is placed to bring the inventory position back up to the base-stock level S .
2. With probability α , supply arrives from the order in step 1 including any unfulfilled order(s) from previous periods. This brings the on-hand inventory back to the base-stock level S . With probability $(1 - \alpha)$, no delivery is made and the on-hand inventory stays the same as at the beginning of period t .

3. Customer demand is observed and fulfilled, up to the current inventory level. The state of the system is evaluated at this point in time.

Note that the demand is accumulated over the whole period and filled at the time of event 3. Also, the supplier incurs the inventory holding cost during the transportation. When the order arrives (event 2), the holding cost is incurred immediately after the replenishment of the customer orders (event 3).



Fig. 1. The sequence of events during period t

The objective is to minimize the long-run total system cost which consists of inventory holding cost and backorder penalty cost. The cost parameters, both holding and backorder penalty, are assumed to be stationary and independent of the current state. The decision variable is the base-stock level S . Specifically, the objective function is

$$\text{Minimize } TC(S) = \sum_{i \in \Omega} h i^+ + b i^- \pi_i$$

Here π_i is a function of demand uncertainty (i.e., the demand distribution) and supply uncertainty (i.e., the probability of replenishment success, α) and the base-stock level S . Therefore, the objective function can ultimately be represented as a function of S . The inventory position is defined as the on-hand inventory plus the on-order inventory minus backorders. The on-hand inventory level is used in the objective function to determine the total cost in each period. However, the inventory position is used to determine the order size at the end of each period.

2.2. Analytical results for general discrete demand distribution

In this section, we assume demand each period follows some arbitrary discrete distribution. On-hand inventory is subject to uncertainty in supply and is given by

$$I_{t+1} = \max \{-g, I_t + y S - I_t - D_{t+1}\},$$

where $y = 0$ w.p. $1 - \alpha$ and $y = 1$ w.p. α .

2.2.1. Explicit forms of the steady state probability distribution

The probability of transitioning from state i to state j in a period is $\phi_{ij} = P[I_{t+1} = j | I_t = i]$. The state-transition probability matrix, $P = [\phi_{ij}]$, is constructed based on the supply uncertainty parameter α and demand mass function (p_d). Given the base stock level, S , and the maximum backlog, g , the size of the P matrix is $[S + g + 1 \times S + g + 1]$. Since we concentrate on a finite number of states, g is arbitrarily assigned to the system, but chosen carefully the steady state probability of state g is reasonably close to 0.

Assume that the current state is i ($I_t = i$). If the supplier is able to ship, then after satisfying customer demand $I_{t+1} \in S, S-1, S-2, \dots, S-d_{\max}$ with probabilities p_0, p_1 , and $p_2, \dots, p_{d_{\max}}$, respectively. On the other hand, if the supplier is unable to ship, then $I_{t+1} \in i, i-1, i-2, \dots, -g-1, -g$ with probabilities $p_0, p_1, p_2, \dots, p_{g-1}, \sum_{d \geq g} p_d$, respectively. Based upon this, the state transition probabilities are

$$\phi_{ij} = \begin{cases} \alpha p_{S-j} & i < j \\ (1-\alpha)p_0 + \alpha p_{S-j} & i = j \\ (1-\alpha)p_{i-j} + \alpha p_{S-j} & i > j > -g \\ \alpha [1 - \sum_{k=0}^{S+g-1} p_k] + (1-\alpha)[1 - \sum_{k=0}^{i+g-1} p_k] & j = -g \end{cases}$$

Based on the particular operating characteristics of the system, the steady-state probability vector can be derived without explicitly solving the system of linear equations. Under steady-state conditions, $\pi P = \pi$. The right-most column of the P matrix is used to obtain

$$\pi_s = \alpha p_0 \left[\pi_{-g} + \pi_{-g-1} + \dots + \pi_{s-1} \right] + p_0 \pi_s,$$

which yields

$$\pi_s = \frac{\alpha p_0}{1 - (1 - \alpha) p_0} \quad (1)$$

We can use expression (1) to solve the remaining column-wise terms of the system $\pi P = \pi$, ultimately yielding a recursive expression,

$$\pi_{s-d} = \frac{\alpha p_d + (1 - \alpha) \sum_{k=1}^d p_{d-k+1} \pi_{s-k+1}}{1 - (1 - \alpha) p_0} \quad d = 1, 2, \dots, d_{\max} \quad (2)$$

As we show in subsequent section, for some special cases of the demand distribution, we can derive a closed-form solution of the steady state distribution of the DTMP.

2.2.2. Solution procedure

A procedure to find the base-stock level to minimize the total system cost starts from considering the objective function with respect to the base-stock level S , specifically

$$\begin{aligned} \text{TC}(S) &= \sum_{i \in \Omega} h i^+ + b i^- \pi_i \\ &= \sum_{i \in \Omega} h i + h + b i^- \pi_i \\ &= \sum_{i \in \Omega} h + b i^+ - b i \pi_i \end{aligned}$$

Proposition 1 and Proposition 2 provide a convexity proof of the objective function including the optimality condition for the base-stock level S as follows.

Proposition 1: The objective function $TC(S)$ is a convex function of S .

Proof.

$TC(S)$ is convex if and only if the second difference of the function is non-negative for all S .

The second difference of $TC(S)$ is

$$\begin{aligned}\Delta^2 TC J &= TC J+2 - 2TC J+1 + TC(J) \\ &= [TC J+2 - TC J+1] - [TC J+1 - TC J].\end{aligned}$$

Substituting into the expression for $TC(S)$, above, we obtain

$$\begin{aligned}\Delta^2 TC J &= \left[-b + b+h \sum_{d=0}^{J+1} \pi_{S-d} \right] - \left[-b + b+h \sum_{d=0}^J \pi_{S-d} \right] \\ &= (b+h)\pi_{S-J-1},\end{aligned}$$

which, by definition, is non-negative. \square

Proposition 2: Base-stock level S is optimal if

$$\sum_{d=0}^S \pi_{S-d} \geq \frac{b}{b+h} \quad \text{and} \quad \sum_{d=0}^{S-1} \pi_{S-d} \leq \frac{b}{b+h}.$$

Proof.

The difference of total cost of two adjacent base-stock levels, $\Delta TC(J)$, is used to obtain the optimal base-stock level. For any non-negative integer J ,

$$TC J = h \left[Jp_S + (J-1)p_{S-1} + \cdots + p_{S-J+1} \right] + b p_{S-J-1} + 2p_{S-J-2} + 3p_{S-J-3} + \cdots$$

and therefore,

$$TC J+1 = h \left[(J+1)p_S + J p_{S-1} + \cdots + p_{S-J} \right] + b p_{S-J-2} + 2p_{S-J-3} + 3p_{S-J-4} + \cdots .$$

Computing the first difference,

$$\begin{aligned}
\Delta TC(J) &= TC(J+1) - TC(J) = h \sum_{d=0}^J \pi_{S-d} - b \sum_{d=J+1}^{\infty} \pi_{S-d} \\
&= h \sum_{d=0}^J \pi_{S-d} - b \left[1 - \sum_{d=0}^J \pi_{S-d} \right] \\
&= -b + (b+h) \sum_{d=0}^J \pi_{S-d} .
\end{aligned}$$

Since $\sum_{d=0}^J \pi_{S-d}$ is monotonically non-decreasing in J and the objective function is convex, the optimal base-stock level S^* can be found as the first value that results in the first positive value of $\Delta TC(J)$. In other words, the optimal solution can be characterized using a critical fractile approach. Therefore, S is an optimal base-stock level if

$$\sum_{d=0}^S \pi_{S-d} \geq \frac{b}{b+h} \quad \text{and} \quad \sum_{d=0}^{S-1} \pi_{S-d} \leq \frac{b}{b+h} . \quad \square$$

The convexity of the objective function and the optimal condition in Proposition 2 can be used to develop a procedure to find the optimal base-stock level (S^*), as follows:

1. Calculate the steady-state probabilities from the recursive equations (1) and(2).
2. S^* is given by

$$S^* = \arg \min_{J \in 0,1,2,\dots} \left\{ J : \sum_{d=0}^J \pi_{S-d} \geq \frac{b}{b+h} \right\} .$$

From these results, we observe the following:

1. The above procedure can be applied to any discrete demand distribution with non-negative values.
2. The components of the vector $\boldsymbol{\pi} = [\pi_S, \pi_{S-1}, \dots, \pi_{-g}]$ are independent of the base-stock level, but each is a function of α and the demand distribution p_d .

Given the above observations, if we know p_d and α , we can compute the optimal base-stock level by comparing the summation of steady-state probabilities with a ratio of system costs b and h . Moreover, we can state the following proposition.

Proposition 3: If $\sum_{d=0}^S \pi_{S-d} = \frac{b}{b+h}$, there is a pair of optimal base-stock levels, S and $S+1$.

Proof:

From the proof of Proposition 2, we know that the difference of total cost of two adjacent base-stock levels S and $S+1$ is

$$\text{TC}(S+1) - \text{TC}(S) = -b + (b+h) \sum_{d=0}^S \pi_{S-d}.$$

If $\sum_{d=0}^S \pi_{S-d} = \frac{b}{b+h}$, then by Proposition 2, $S = S^*$ is an optimal base-stock level, but the first-difference expression above also implies that $\text{TC}(S+1) - \text{TC}(S) = 0$, meaning that $\text{TC}(S+1) = \text{TC}(S)$, meaning in turn that both S and $S+1$ are optimal. \square

2.2.3. Decomposition of the total cost

As mentioned earlier, the total cost for our inventory system model consists of inventory holding cost (IC) and backorder cost (BC). We can observe how those elements contribute to the total cost as the base-stock level increases. The graph below illustrates the total cost where demand has a discretized Normal distribution truncated at zero with the mean of 20 and standard deviation of 6 under supply risk $\alpha = 0.9$. The holding cost (h) is set arbitrarily to $h = 1$ and the backorder cost $b = 20$.

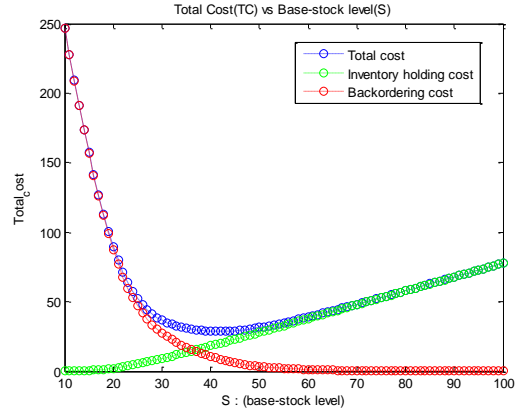


Fig. 2. Inventory cost vs. base-stock level when $\alpha = 0.9$, $h = 1$, and $b = 20$

The inventory holding cost monotonically increases with the base-stock level. In contrast, the backordering cost monotonically decreases with the base-stock level. Thus, the total cost must be convex in S as depicted in Figure 2. As shown in Proposition 2, the optimal base-stock level depends mainly on the cost ratio, b/h . Figure 3 exhibits the shape of total cost for several values of cost ratio (b/h). The larger the cost ratio (b/h), the larger the minimum point on the total cost curve, and therefore the larger the optimal base-stock level (S^*).

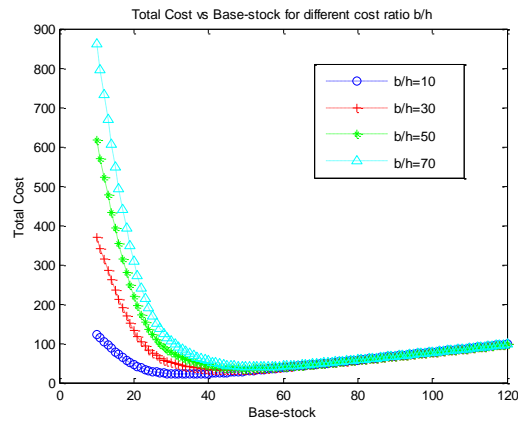


Fig. 3. Inventory cost vs. base-stock level: $\alpha = 0.9$ for different ratios of b/h

2.2.4. Performance measure of the system and its bounds

We define the service level, γ , as the probability of not stocking out in any period. When a stockout occurs, the on-hand inventory at the end of the period is negative.

Therefore, the summation of all non-negative states after the demand fulfillment determines the service level (i.e., the probability of non-negative stock) for the system, specifically

$$\gamma^* = \sum_{d=0}^{S^*} \pi_{S^*-d},$$

where S^* is the optimal base-stock level and γ^* is the corresponding optimal service level.

From Proposition 2, $\sum_{d=0}^{S^*} \pi_{S^*-d} \geq b / (b+h)$. Thus, a lower bound on the optimal service level

is $\underline{\gamma}^* = b / (b+h)$. As stated in Proposition 3, if $\sum_{d=0}^S \pi_{S-d} = b / (b+h)$, there are two optimal base-stock levels, S and $S+1$. Nevertheless, base-stock level $S+1$ has a service level at least as high as, if not higher than, that of base-stock level S . From Proposition 2,

$\sum_{d=0}^{S^*} \pi_{S-d} \geq b / (b+h)$ and $\sum_{d=0}^{S^*-1} \pi_{S-d} \leq b / (b+h)$. An upper bound on the optimal service level is therefore $b / (b+h) + \max_{d \in \{d_{\min}, \dots, d_{\max}\}} \pi_{S-d}$. From equation (2), $\max_d \{\pi_{S-d}\} \leq \max_d \{p_d\}$

for any demand distribution and α . Hence, a more readily computed upper bound is

$$\bar{\gamma}^* = \min \left\{ 1, \frac{b}{b+h} + \max_d (p_d) \right\}.$$

Note that the upper and lower bounds of γ^* are independent of the supplier uncertainty factor α , but depend on the cost parameters and demand distribution. To gather some insight on how the cost and demand parameters affect the bounds on the optimal service level, we conducted numerical experiments to observe the impact of the backorder penalty cost (b) on the optimal service level for different values of the supplier uncertainty factor (α) and evaluated how tight the lower bound was for different values of the coefficient of variation (CV) of the demand distribution. The empirical results shown in Appendix A indicate that when the penalty cost is high, the bounds on γ^* are very close for three different values of α .

Hence, α has less impact on γ^* when the backorder penalty is high, irrespective of the demand distribution. Moreover, the upper and lower bounds of γ^* are tighter when the CV of the demand distribution is large or where $\max_d(p_d)$ is comparatively low.

2.3. Analytical results for specific demand distributions

For the general demand distribution case, the steady state probability distribution of the inventory level is determined by recursive equations (1) and (2). The optimal base-stock level is obtained from the simple procedure described in Section 2.2.2 above. However, for some specific demand distributions, we are able to derive closed-form solutions for the steady state probability, the optimal base-stock level, the corresponding optimal service level and the holding and backorder costs of the system.

2.3.1. Closed-form solutions: EOQ-type customer

In this section, we consider an intermittent and lumpy demand pattern. In a business-to-business setting, the customer demand distribution may actually be the result of an ordering policy. Here we assume the customer ordering policy is of EOQ-type. Specifically, at the end of each period the customer evaluates its inventory position and if it is below its system reorder point, the customer places a replenishment order for q units. Since the EOQ-type policy generates a fixed order quantity, the customer order would be either exactly q units or zero in each period. Customer demand is therefore discretely distributed with non-zero p_0 and p_q , i.e., a $(q,0)$ demand distribution, such that

$$D_t = \begin{cases} 0 & \text{w.p. } p_0 \\ q & \text{w.p. } p_q, p_0 + p_q = 1. \end{cases}$$

Based on the recursive equations of the steady-state probability in the general case, the long-run probability of each level of on-hand inventory can be expressed as

$$\pi_i = \begin{cases} \frac{\alpha p_0}{1 - (1 - \alpha)p_0} & \text{for } i = S \\ \frac{\alpha(1 - \alpha)^{d-1} p_q^d}{1 - (1 - \alpha)p_0} & \text{for } i = S - qd; d = 1, 2, \dots, d_{\max} \\ 0 & \text{otherwise.} \end{cases}$$

Based on the explicit form of the steady state probabilities, and using the procedure to find the optimal base-stock level in the general case, the closed-form solution for the optimal base-stock level is

$$S^* = q \cdot \left\lceil \frac{\log \left(1 - \left[\left(\frac{b}{b+h} - \frac{\alpha p_0}{A} \right) \frac{A^2}{\alpha p_q} (1-a) \right] \right)}{\log(a)} \right\rceil,$$

where $A = 1 - (1 - \alpha)p_0$ and $a = (1 - \alpha)p_q/A$. The derivation is shown in Appendix B.

When the optimal base-stock level S^* is known, the optimal service level γ^* is represented as a function of the demand probabilities and the supply uncertainty factor α , yielding

$$\gamma^* = \frac{\alpha p_0}{A} + \frac{\alpha}{(1 - \alpha)A} \left(\frac{a(1 - a^n)}{1 - a} \right),$$

where $n = S^*/q$ and A and a are defined as above.

The decomposition of the total cost in section 2.2.3 shows that the optimal base-stock level balances the inventory holding and backorder costs. Similarly, closed-form expressions for holding and backorder costs can be computed as functions of S , α , p_0 and p_q , yielding

$$IC = h \cdot \left[\frac{S^* \alpha p_0}{A} + \frac{S \alpha}{(1 - \alpha)A} \left(\frac{a(1 - a^n)}{1 - a} \right) - \frac{q\alpha}{(1 - \alpha)A} \left[\frac{a(1 - a^n) - na^{n+1}(1 - a)}{(1 - a)^2} \right] \right]$$

and

$$BC = \frac{bq\alpha(1-\alpha)^n p_q^{n+1}}{A^{n+2}(1-a)^2},$$

where A and a are as defined above.

The following properties can be observed from the expressions derived above. It is obvious from the closed-form expression that the optimal base-stock level is always an integer multiple of q as determined by the input parameters. We observe the impact of each parameter independently by holding all other parameters fixed. The results are consistent with our intuition. That is, the optimal base-stock level is high when α is low, p_q/p_0 is high, and b/h is high. However, α has more influence on the optimal base-stock level as p_q/p_0 increases. As the value of α decreases, however, the cost ratio b/h has a more pronounced impact on the optimal base-stock level. The graphs from these experiments are shown in Appendix C.

2.3.2. Closed-form solutions: Geometric demand

In this section, we observe the system behavior when the customer demand follows a geometric distribution, $D_i \sim \text{Geo}(p)$, meaning that $p_d = p(1-p)^d$, $d = 0, 1, \dots, d_{\max}$.

Closed-form solutions can be obtained for the steady state probability distribution, optimal base-stock level, optimal service level, and optimal inventory holding and backorder costs.

Based on the recursive equations of the steady-state probability in the general case, the long-run probability of each level of on-hand inventory can be expressed as

$$\pi_{S-d} = \frac{\alpha p(1-p)^d}{1-(1-\alpha)p^{d+1}} \quad \forall d = 0, 1, \dots, d_{\max}$$

Based on the explicit form of steady state probabilities, and using the procedure to find the optimal base-stock level in the general case, the closed-form solution for the optimal base-stock level is

$$S^* = \left\lceil \frac{\log \left(1 - \left[\left(\frac{b}{b+h} - \frac{\alpha p}{A} \right) \frac{A^2}{\alpha p(1-p)} (1-a) \right] \right)}{\log(a)} \right\rceil,$$

where $A = 1 - (1 - \alpha)p$ and $a = 1 - p/A$. The derivation is shown in Appendix D.

The corresponding optimal service level is

$$\gamma^* = \frac{\alpha p}{A} \left(\frac{1 - a^{S^*+1}}{1 - a} \right).$$

Closed-form expressions for holding and backorder costs can be computed as functions of S , α , p_0 and p_q , yielding

$$IC = \frac{\alpha h p}{A} \cdot \left[S \left(\frac{1 - a^{S+1}}{1 - a} \right) - \left[\frac{a(1 - a^S) - (1 - a)S a^{S+1}}{(1 - a)^2} \right] \right]$$

and

$$BC = \frac{\alpha b p a^{S+1}}{A(1 - a)^2}.$$

2.4 Effect of changing supply risk on the total system cost

For managing supply risk, it is beneficial to gain information on the cost impact of unreliable supply. This can help the inventory system manager to determine if a supplier that provides more reliability with higher cost may be preferred over a less reliable but cheaper supplier. Based upon our experiments, we approximate the functional relationship between the supply risk factor (α) and fraction cost deviation (δ) from the situation where the supply is fully reliable, specifically

$$\delta(\alpha) = \frac{TC(S_\alpha^*) - TC(S_{\alpha=1}^*)}{TC(S_{\alpha=1}^*)}, \quad 0 \leq \alpha \leq 1. \quad (3)$$

We observe that the functional form of the relationship is consistent across different demand distributions and mean values. Figure 4 depicts this functional relationship for different backorder penalty costs (b) when demand follows a Poisson distribution with $\lambda=20$.

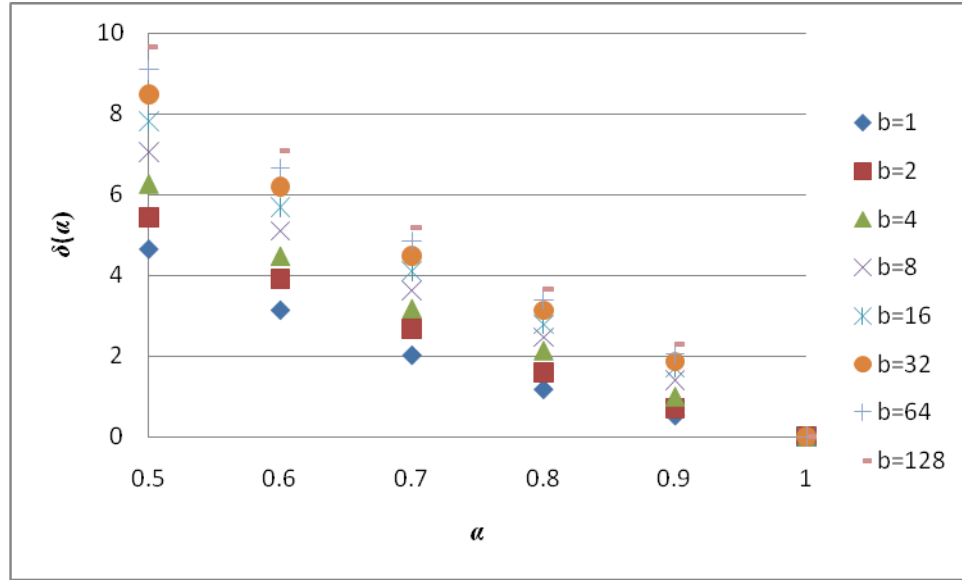


Fig. 4. $\delta(\alpha)$ vs α

For a practical range of α values, $0.5 < \alpha < 1$, it appears that $\delta(\alpha)$ is linear. For $b=16$, for example, a linear regression yields $\hat{\delta}(\alpha) = 14.89 - 14.94\alpha$, with $R^2 = 98.67\%$ and the linearity test shows a p -value of 0.000042. From this regression equation, therefore, a decrease of α by 0.01 increases the cost deviation from the perfect-supply case by 0.1494, or approximately 15%. By manipulating equation (3), the total system cost can also be expressed in terms of the total system cost under fully reliable supply as

$$TC(S_\alpha^*) = [1 - (1 - \alpha)^m] \cdot TC(S_{\alpha=1}^*), \quad (4)$$

where m is the slope of the linear regression line. In this case, for example,

$$TC S_{\alpha=0.9}^* = 2.49 TC S_{\alpha=1}^* \quad \text{and} \quad TC S_{\alpha=0.8}^* = 3.99 TC S_{\alpha=1}^* .$$

However, the rate of change of the cost deviation—i.e., the slope m of the linear regression equation between δ and α —depends on the penalty cost and the demand distribution. Figure 5 depicts the regression relationship between b , m , and CV for different demand distributions—Poisson, Normal, and Gamma.

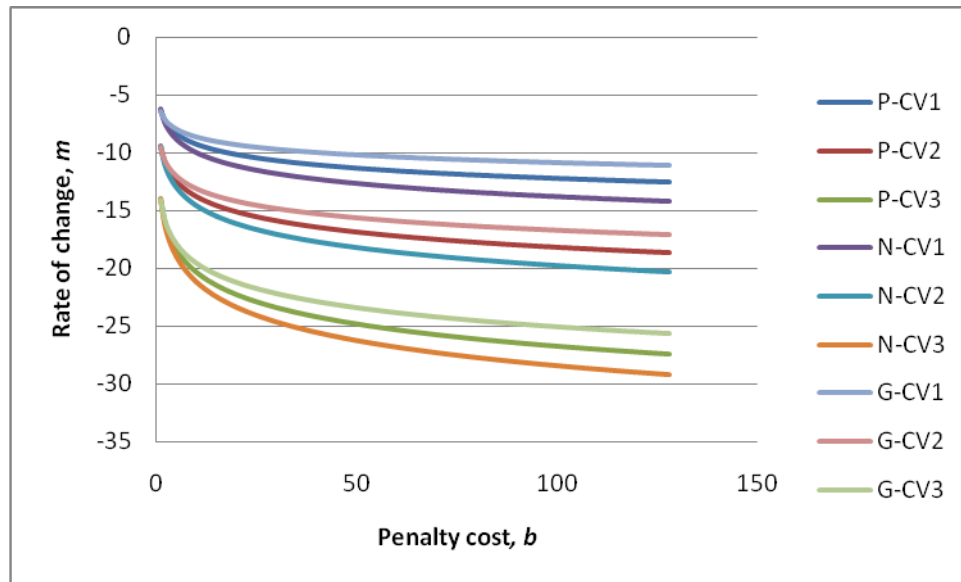


Fig. 5. Rate of change on cost deviation vs. b for different demand distributions where P=Poisson demand, N=Normal demand, G=Gamma demand and CV1=0.316, CV2=0.223, CV3=0.158.

From Figure 5, it is clear that the rate of change of the cost deviation δ has a strong logarithmic relationship with b . For example, $m = -9.5452 - 1.878 \ln b$ with $R^2 = 99.58\%$ when demand has a Poisson distribution with $\lambda=20$ and CV=0.223. Therefore, we can approximate the rate of change for any value of b via the relationship. Furthermore, the rate of change of the cost deviation can be generalized across our computational experiments as a function of b , CV and skewness (Ψ) of the period demand and can be expressed as

$$m = (-21.04 - 0.31 \Psi + 48 CV) + (-4.2 + 1.475 \Psi + 7.98 CV) \ln(b). \quad (5)$$

Thus, we can approximate the total system cost—through a combination of expressions (4) and (5)—as a function of the total system cost under fully reliable supply when the CV and skewness of the period demand are known for a specified value of backorder penalty cost.

3. Model with service level constraint

In the previous model, we assume that the backorder penalty cost is given, and therefore that the optimal base-stock level can be computed directly in a way that accounts for the demand and supply uncertainty parameters. Based on the optimal base-stock level, the corresponding optimal customer service level can be determined. In some situations, the objective is to meet or exceed a customer service target at minimum total system cost. Thus, a minimum customer service level constraint is specified. In such situations, the previous model can be modified to satisfy a service level constraint by minimizing only the inventory holding cost. This model is more realistic because it is often difficult to determine a reasonable backorder penalty cost. In this section, we develop a computational approach to determine the base-stock level to satisfy a given service level.

3.1. A procedure to determine a service-constrained optimal base-stock level

When there is no supply risk (i.e., no uncertainty in lead time), the DTMP model behaves the same as the classical newsvendor model, and the steady-state probabilities follow the demand distribution, i.e., $\pi_{s-d} = p_d$. The cumulative distribution of the steady-state probabilities is the key element to determining the base-stock level using a critical fractile approach. With supply uncertainty, the values of the steady-state probabilities decay as a function of the supply risk factor, α . Since the total cost is a convex function of S , the procedure developed earlier in this paper to find the optimal base stock level is modified slightly from the one when the penalty cost is known, as follows:

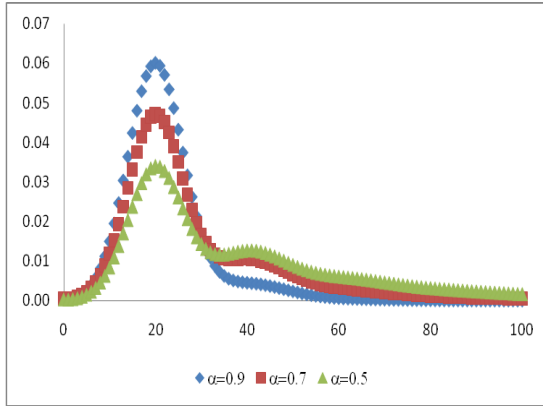
1. Calculate the steady-state probabilities from the recursive equations (1) and(2).
2. Find the smallest non-negative value of S such that is at least equal to the target service level (γ_{tgt}), i.e.,

$$S^* = \arg \min_{S \in 0,1,2,\dots} \left\{ S : \sum_{d=0}^S \pi_{S-d} \geq \gamma_{\text{tgt}} \right\}.$$

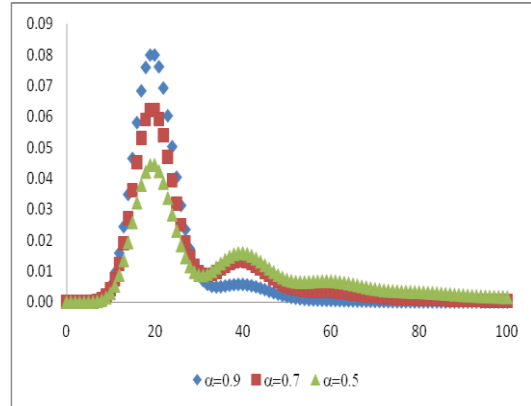
The procedure above is optimal for any discrete demand distribution with geometrically distributed lead time. Below, we compare our method with classical methods that rely on the LTD distribution to determine the safety stock and order-up-to level in a system subject to demand and lead-time uncertainties.

3.2. Application to lead-time demand distribution

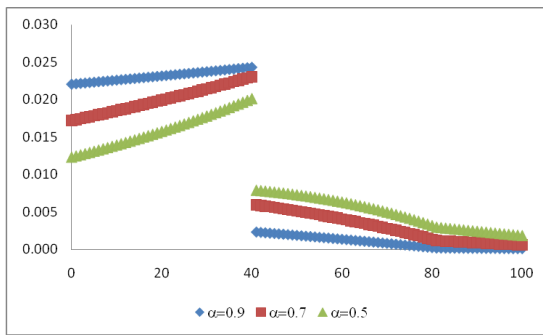
The probability of demand during lead time being equal to d units can be characterized by the steady-state probability (π_{S-d}) of having on-hand inventory of $S-d$ at the end of a period where nominal lead time is one period and all outstanding orders are supplied with probability α . In other words, π_{S-d} provides the LTD distribution. Figure 6 illustrates the LTD distribution for different period demand distributions and several values of the supply risk factor α . As indicated in the figure, changes in the supply risk factor results in different shapes of the lead-time distribution.



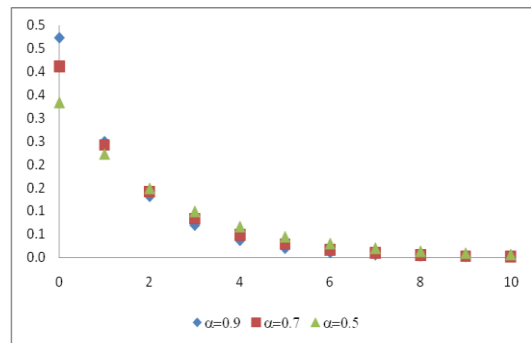
(a) $D \sim \text{Normal}(\mu=20, \sigma=6)$



(b) $D \sim \text{Poisson}(\lambda=20)$



(c) $D \sim \text{Uniform}(0,40)$



(d) $D \sim \text{Geo}(p=0.5)$

Fig. 6. DTMP-derived LTD distribution for different demand distributions and supply risk factors.

To determine the base-stock level or order-up-to level to meet a target customer service level under arbitrary or unspecified demand and lead-time distributions, a common approach is to assume that the LTD is Normally distributed (Lau and Lau, 2003). Another approach is to use a convex-combination of the conditional probability distribution of demand for each discrete lead-time value (Tyworth and O'Neill, 1997). The more general approach that specifies the LTD distribution from the convolution of the demand and lead-time distributions is analytically tractable for only few special cases. Using the Central Limit

Theorem as justification, other researchers have used a Normal distribution to approximate the LTD distribution and compute base-stock levels (e.g., Tyworth and O'Neill, 1997).

The effectiveness of using a Normal distribution to approximate the LTD distribution has been an active area of debate in the inventory literature. Some authors, such as Eppen and Martin (1988), have questioned the effectiveness of the Normal approximation. They reveal the errors in estimating the probability of stocking out using the Normal approximation to determine the safety stock level. Similarly, Chopra et al. (2004), demonstrate that Normal approximation can lead to poorly specified inventory levels, especially when the CV of period demand is high or when the lead time follows a skewed (e.g., gamma) distribution. These errors could lead to misleading managerial insights regarding how to specify buffers to guard against the combined effects of demand and lead-time uncertainty. On the other hand, Tyworth and O'Neill (1997) conduct empirical experiments to examine the robustness of the Normal approximation of the LTD distribution for fast-moving products in seven major industries. They conclude that the Normal approximation is robust with respect to logistics system cost, total costs, and fill rate.

Based on the overview found in Tyworth and O'Neill (1997) we describe the various approximations of LTD that we use in our computational comparison below. Where appropriate, we retain the notation developed in Section 2 above, with additional notation as follows:

X = Lead-time demand random variable

μ_X = Mean of LTD distribution

σ_X = Standard deviation of LTD distribution

μ_D = Mean per-period demand; $\mu_D = \sum_{d=d_{\min}}^{d_{\max}} p_d \cdot d$

σ_D = Standard deviation of per-period demand: $\sigma_D = \sqrt{\sum_{d=d_{\min}}^{d_{\max}} p_d \cdot d^2 - \mu_D^2}$

L = Lead-time random variable

T = Set of lead-time values, $t \in T$ and $P_t = P(L = t)$

μ_L = Mean lead time (periods)

σ_L = Standard deviation of the lead time (periods)

\hat{S} = Computed approximate base-stock level

Normal approximation: Assume that LTD X is Normally distributed. The mean and standard deviation of X can be obtained from the first two moments of demand and lead time, and the base-stock level to meet a required service level can be calculated as follows:

$$\mu_X = \mu_D \mu_L$$

$$\sigma_X = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2}$$

$$\hat{S} = \mu_X + k \sigma_X, \text{ where } k = \Phi^{-1} \gamma_{\text{tgt}} \text{ and } \Phi \text{ is the standard normal cdf.}$$

Convex combination of conditional probability: This is a distribution-free method, with no assumption required for the shape of the convolution of the demand and lead-time distributions. The procedure was developed by Bank and Fabrycky (1987) and Eppen and Martin (1988). It uses a convex combination of the conditional probability distributions of demand during the discrete values of lead time. However, its use is restricted to the cases where the period demand distribution is Poisson, Gamma, Normal, or Exponential. When demand is Normally distributed and lead time is discrete, the base-stock level to meet a specific service level can be calculated as follows:

$$\mu_X = \mu_{D_t} = t \mu_D, \quad \forall t \in T$$

$$\sigma_X = \sigma_{D_t} = \sqrt{t} \sigma_D, \quad \forall t \in T$$

$$\hat{S} = \min_{S \in \{0,1,2,\dots\}} S : \sum_{t \in T} P_t \cdot \mathbf{P} \{ D_t \leq S \} \geq \gamma_{\text{tgt}} .$$

3.3. Numerical example

We compare the approaches discussed above with the DTMP model, under the restriction that demand has a discrete distribution and lead time has a geometric distribution. A discretized Normal distribution is utilized to represent the customer demand and truncated at 0 and $\mu+5\sigma$. Figure 7 shows the resulting base-stock levels for the three methods for the case with $\mu = 20$, $\sigma = 6$, and a nominal lead time of one period with supply risk factor $\alpha = 0.9$.

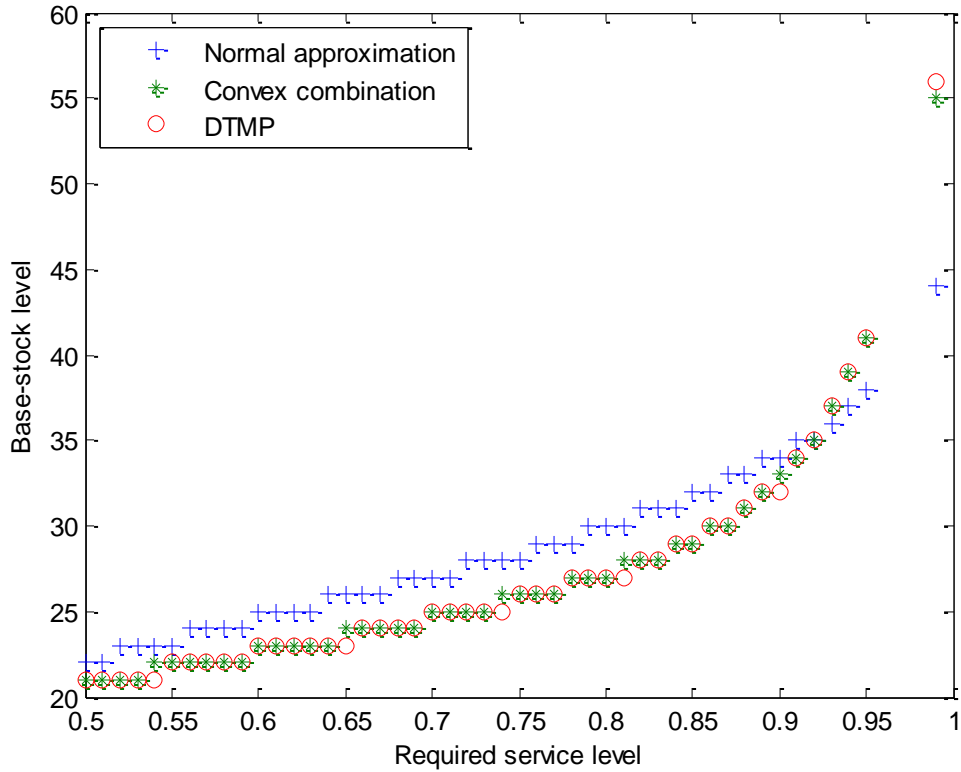


Fig. 7. Comparing the base-stock level from three methods— $\mu_D = 20$, $\sigma_D = 6$, and $L \sim \text{Geo } \alpha = 0.9$

For this system, the replenishment lead time has a geometric distribution, and therefore the DTMP model generates the optimal base-stock level for each required service level. The computed base-stock levels in Figure 5 are similar to the ones from the convex combination method with some slight deviations. The Normal approximation method overestimates the base-stock levels at low target service levels until a value of approximately 0.92, after which it underestimates the optimal base-stock level.

For these experimental conditions, with geometric lead time and discretized Normal demand, the DTMP model provides the optimal base-stock level. The DTMP model is, however, also advantageous when demand is not Normally distributed since it can accommodate any discrete demand distribution. In a more general setting when lead time is

not geometrically distributed, the effectiveness of the DTMP model to approximate the base-stock level is a question we address in follow-on research.

4. Conclusions

We propose a single-location inventory model under supply uncertainty. The model operates under base-stock control in a periodic-review framework. The supply uncertainty is modeled as a Bernoulli trial, with a specified probability α that the total quantity on-order is fulfilled at the end of the current period. The nominal lead time is one period. The replenishment lead time therefore follows a geometric distribution with parameter α . Thus, our model can be viewed as an extension of the newsvendor model with supply risk. A discrete-time Markov process (DTMP) model is introduced to solve the problem analytically. The steady state distribution of the on-hand inventory can be calculated from recursive equations given the demand distribution parameters and α . We find that a critical fractile approach may be used to obtain the base-stock level that minimizes the total system cost. For some specific demand distributions, we are able to compute the optimal base-stock level—and the optimal service level and total system cost—in closed-form. We also provide upper and lower bounds on the corresponding optimal service level γ^* . Empirical analysis indicates that α has less impact on γ^* when the backorder penalty cost is high, irrespective of the demand distribution, and that the upper and lower bounds on γ^* are tighter when the coefficient of variation of the demand is large. We quantify the impact of the supply uncertainty on the cost deviation from the optimal cost under fully reliable supply. We also present an approximate logarithmic relationship between the rate of change on the cost deviation and the penalty cost.

Computing the optimal service level using the procedure described above requires knowledge of the holding cost h and the backorder penalty cost b . However, in many real-world situations, the backorder penalty cost may be difficult to estimate. For these cases, we proposed a second model that allows us to compute the base-stock level to meet a target customer service level at minimum cost. The classical approach to this problem relies on specifying the LTD distribution, which may be difficult to do, especially if one must

convolve generally specified lead time and demand distributions. A popular approximate method is to assume a Normally distributed LTD distribution. Our approach, however, is to modify our first (DTMP) model to compute the optimal base-stock level. Our analysis shows that a Normal approximation to the base-stock level can deviate significantly from the optimal result across a wide range of target service levels.

The results from this single-site inventory model allow us to gain important managerial insights that serve as a building block for considering base-stock levels in more complex, multi-node supply chains such as serial systems or assembly or distribution networks. Furthermore, we seek to modify our model to handle more general lead-time distributions rather than the restrictive geometric lead-time distributions considered in this research. Due to the complexity of computing optimal inventory levels in such complex, generally-specified systems, determining good policies for the management of inventories in these types of realistic systems is an active area of research.

REFERENCES

- Arreola-Risa, A. and DeCroix, G. A. (1998). Inventory management under random supply disruptions and partial backorders. *Naval Research Logistics*, 45(7), 687-703.
- Banks J. and Fabrycky W. J.(1987). Procurement and inventory systems analysis (Englewood Cliffs, N.J.: Prentice-Hall, 129-139.
- Chopra, S., Reinhardt, G., and Dada, M. (2004). The effect of lead-time uncertainty on safety stocks. *Decision Sciences*, 35(1), 1–24.
- Clark, A. J. and Scarf, H. (1960). Optimal policies for a multi-echelon inventory problem. *Management Science*, 6, 475–490.
- Eppen, G. D. and Martin, R. K. (1988). Determining safety stock in the presence of stochastic lead-time and demand. *Management Science*, 34(11), 1380-1390.
- Gallego, G. and Özer, Ö. (2005). A new algorithm and a new heuristic for serial supply systems. *Operations Research Letters*, 33(4), 349-362.
- Gupta, D. (1996). The (Q,r) inventory system with an unreliable supplier. *INFOR*, 34(2), 59-76.
- Lau, H. S., and Lau, A. H. L. (2003). Nonrobustness of the normal approximation of lead-time demand in a (Q, R) system. *Naval Research Logistics*, 50(2), 149-166.
- Mohebbi, E. (2003). Supply interruptions in a lost-sales inventory system with random lead-time. *Computers and Operations Research*, 30(3), 411-426.
- Mohebbi, E. (2004). A replenishment model for the supply-uncertainty problem. *International Journal of Production Economics*, 87(1), 25-37. doi:10.1016/S0925-5273(03)00098-7

Parlar, M. (1997). Continuous-review inventory problem with random supply interruptions. *European Journal of Operational Research*, 99(2), 366-385.

Parlar, M. and Perry D. (1995). Optimal (Q, r, T) policies in deterministic and random yield models with uncertain future supply. *European Journal of Operational Research*, 84, 431–443.

Sherbrooke, C. (2004). Optimal inventory modeling of systems: multi-echelon techniques, second ed. Springer-Verlag, New York, LLC.

Tersine, R. J. (1988). Principles of inventory and materials management, New York: North-Holland.

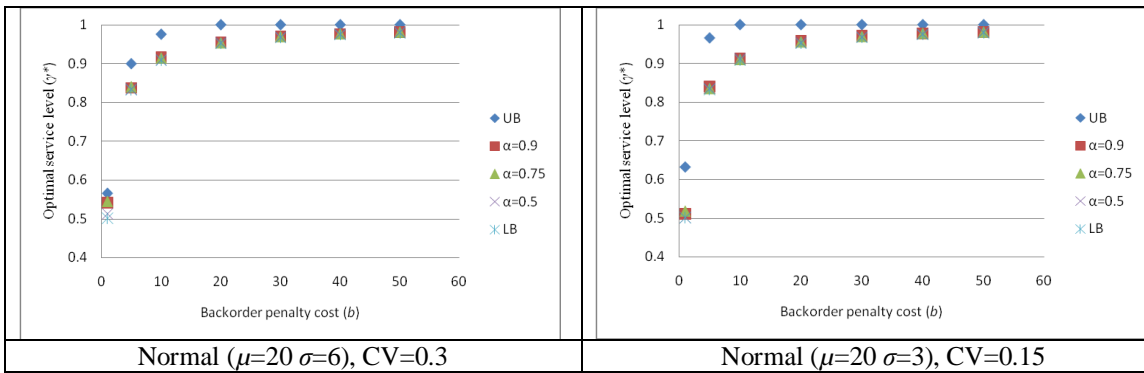
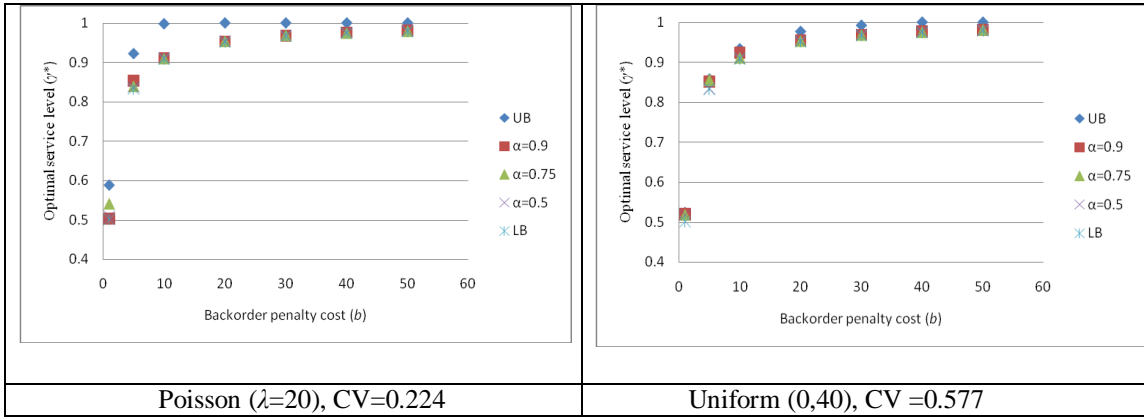
Tomlin B. (2006). On the value of mitigation and contingency strategies for managing supply chain disruption risks. *Management Science*, 52(5), 639-657.

Tyworth, J. E. and O'Neill, L. (1997). Robustness of the normal approximation of lead-time demand in a distribution setting. *Naval Research Logistics*, 44(2), 165-186.

APPENDICES

APPENDIX A

Impact of demand distribution, backorder penalty cost and α on optimal service levels



APPENDIX B

Derivation of optimal base-stock level for $(q,0)$ discrete demand distribution

From Proposition 2, base-stock level S is optimal if $\sum_{d=0}^S \pi_{S-d} \geq \frac{b}{b+h}$ and $\sum_{d=0}^{S-1} \pi_{S-d} \leq \frac{b}{b+h}$.

Let S^* be the optimal base-stock with $S^* = nq$ where n is a positive integer

$$\text{Therefore, } \sum_{d=0}^{S^*} \pi_{S-d} \geq \frac{b}{b+h}$$

Let $A = 1 - (1 - \alpha)p_0$ and substituting π_{S-d} from the closed-form expression (2)

$$\begin{aligned} \frac{\alpha p_0}{A} + \sum_{d=1}^n \frac{\alpha(1-\alpha)^{d-1} p_q^d}{A^{d+1}} &\geq \frac{b}{b+h} \\ \frac{\alpha p_0}{A} + \frac{\alpha p_q}{A^2} \left[1 + \frac{(1-\alpha)p_q}{A} + \left[\frac{(1-\alpha)p_q}{A} \right]^2 + \dots + \left[\frac{(1-\alpha)p_q}{A} \right]^{n-1} \right] &\geq \frac{b}{b+h} \end{aligned}$$

Given $a = \frac{(1-\alpha)p_q}{A}$ and substituting into the above yields

$$\begin{aligned} \frac{\alpha p_0}{A} + \frac{\alpha p_q}{A^2} \left[\frac{1-a^n}{1-a} \right] &\geq \frac{b}{b+h} \\ \frac{1-a^n}{1-a} &\geq \left[\frac{b}{b+h} - \frac{\alpha p_0}{A} \right] \frac{A^2}{\alpha p_q} \\ 1-a^n &\geq \left[\frac{b}{b+h} - \frac{\alpha p_0}{A} \right] \frac{(1-a)A^2}{\alpha p_q} \\ a^n &\geq 1 - \left[\frac{b}{b+h} - \frac{\alpha p_0}{A} \right] \frac{(1-a)A^2}{\alpha p_q} \\ n \log(a) &\geq \log \left[1 - \left[\frac{b}{b+h} - \frac{\alpha p_0}{A} \right] \frac{(1-a)A^2}{\alpha p_q} \right] \end{aligned}$$

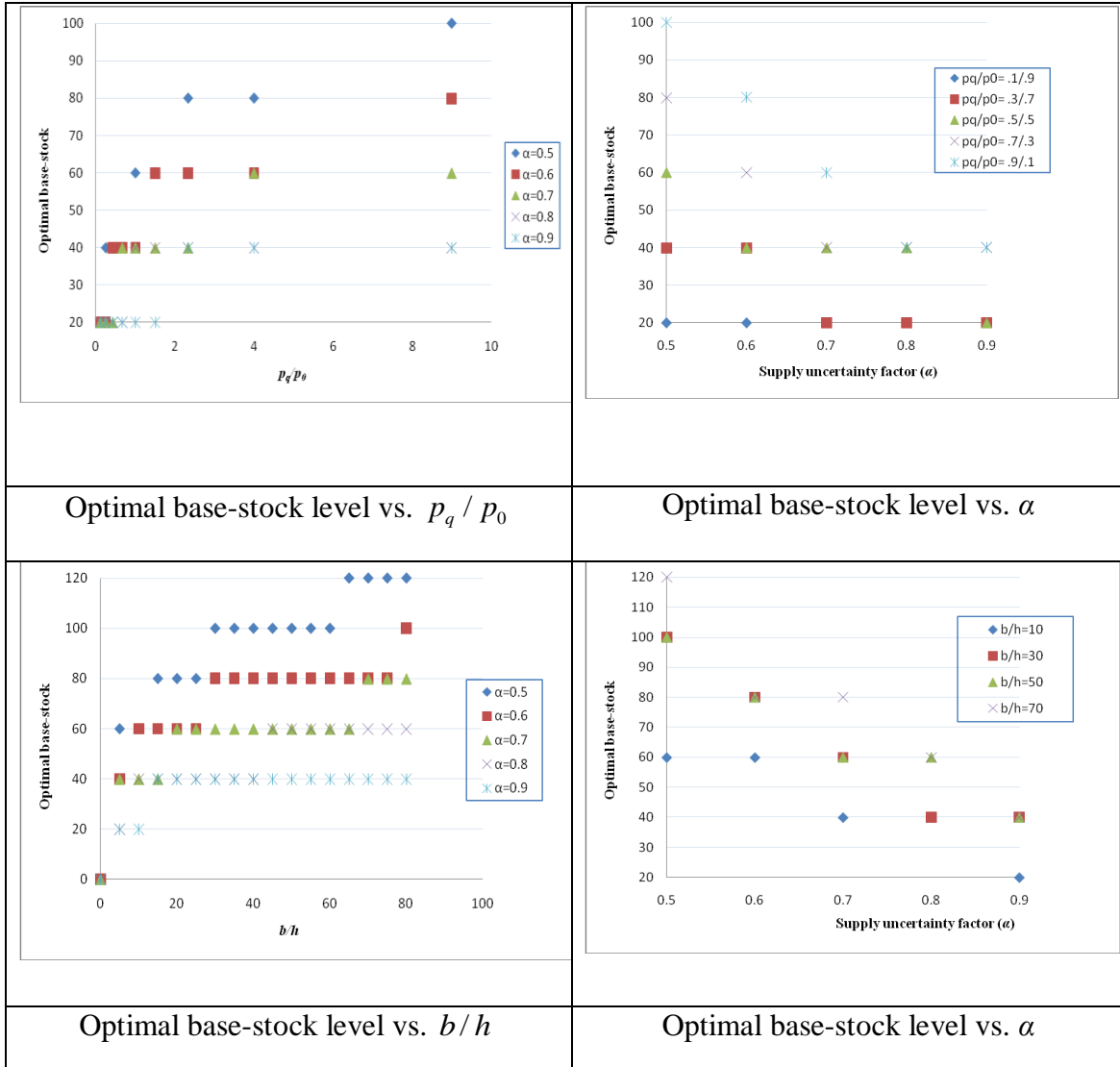
$$n \geq \frac{\log\left(1 - \left[\left(\frac{b}{b+h} - \frac{\alpha p_0}{A}\right) \frac{A^2}{\alpha p_q} (1-a)\right]\right)}{\log(a)}$$

Since, from Proposition 2, $\sum_{d=0}^{s^*-1} \pi_{s-d} \leq \frac{b}{b+h}$, S^* is the first value of S that satisfies the inequality and n is the smallest integer that satisfies the inequality above, therefore

$$S^* = q \cdot \left\lceil \frac{\log\left(1 - \left[\left(\frac{b}{b+h} - \frac{\alpha p_0}{A}\right) \frac{A^2}{\alpha p_q} (1-a)\right]\right)}{\log(a)} \right\rceil$$

APPENDIX C

Impact of system parameters on optimal base-stock levels



APPENDIX D

Derivation of optimal base-stock level for geometric demand

From Proposition 2, base-stock level S is optimal if $\sum_{d=0}^S \pi_{S-d} \geq \frac{b}{b+h}$ and $\sum_{d=0}^{S-1} \pi_{S-d} \leq \frac{b}{b+h}$.

Let S^* be the optimal base-stock

$$\sum_{d=0}^{S^*} \pi_{S^*-d} \geq \frac{b}{b+h}$$

Let $A = 1 - (1 - \alpha)p$ and substituting π_{S-d} from the closed-form expression (2)

$$\begin{aligned} \frac{\alpha p}{A} + \sum_{d=1}^{S^*} \frac{\alpha p(1-p)^d}{A^{d+1}} &\geq \frac{b}{b+h} \\ \frac{\alpha p}{A} + \frac{\alpha p(1-p)}{A^2} \left[1 + \frac{(1-p)}{A} + \frac{(1-p)^2}{A^2} + \dots + \frac{(1-p)^{S^*-1}}{A^{S^*-1}} \right] &\geq \frac{b}{b+h} \end{aligned}$$

Given $a = \frac{1-p}{A}$ and substituting into the above yields

$$\begin{aligned} \frac{\alpha p}{A} + \frac{\alpha p(1-p)}{A^2} \left[\frac{1-a^{S^*}}{1-a} \right] &\geq \frac{b}{b+h} \\ \frac{1-a^{S^*}}{1-a} &\geq \left[\frac{b}{b+h} - \frac{\alpha p}{A} \right] \frac{A^2}{\alpha p(1-p)} \\ 1-a^{S^*} &\geq \left[\frac{b}{b+h} - \frac{\alpha p}{A} \right] \frac{(1-a)A^2}{\alpha p(1-p)} \\ a^{S^*} &\geq 1 - \left[\frac{b}{b+h} - \frac{\alpha p}{A} \right] \frac{(1-a)A^2}{\alpha p(1-p)} \\ S^*[\log(a)] &\geq \log \left[1 - \left[\frac{b}{b+h} - \frac{\alpha p}{A} \right] \frac{(1-a)A^2}{\alpha p(1-p)} \right] \end{aligned}$$

Since $\sum_{d=0}^{S^*-1} \pi_{S^*-d} \leq \frac{b}{b+h}$

$$S^* = \left[\frac{\log \left(1 - \left[\left(\frac{b}{b+h} - \frac{\alpha p}{A} \right) \frac{A^2}{\alpha p(1-p)} (1-a) \right] \right)}{\log(a)} \right]$$

CHAPTER 3

Computing Base-stock Levels from Approximate Lead-time Demand Distributions under Imperfect Supply

Abstract

We consider an infinite-horizon, periodic-review, inventory model under a base-stock control policy and supply uncertainty. A discrete-time Markov process (DTMP) model is used to obtain the optimal base-stock level when the backorder penalty and inventory holding costs are given. Under these assumptions, the lead-time demand (LTD) distribution can be obtained from recursive equations. This distribution is significant in determining the base-stock level to satisfy a target service level. We conduct experiments to observe the quality of various approximate solutions under two scenarios, when lead time has a geometric distribution and when lead time has any distribution. In the first scenario, we observe the quality of the classical approximations of LTD, Normal and gamma distributions, approximating the optimal base-stock level by matching the first two moments of the LTD distribution obtained from the DTMP model. In the second scenario, we consider a general situation where the lead-time distribution is not restricted to a geometric distribution, and simulation is used to generate the LTD distribution. In this case, the DTMP model is used as an alternative LTD approximation to determine the base-stock level that meets the target service level. We perform experiments where the demand distribution and the lead-time distribution are specified individually, and where the LTD distribution is specified by a discretized beta distribution or a bimodal distribution. We assess the quality of the approximate solutions by computing their cost deviations from the simulation results.

1. Introduction

Two basic approaches to determine optimal base-stock levels in inventory systems are the “shortage-costing approach” and the “service-level approach,” the latter applying to the case for which a shortage penalty cost is not specified, or cannot be determined (Dullaert et al., 2007). Specifying the lead-time demand (LTD) distribution is a crucial element to determining the base-stock level or safety stock required to meet a specified service level. When the backorder penalty is given, the LTD is used to formulate a stochastic optimization problem to compute the optimal base-stock level. The service level approach relies on

determining an appropriate fractile of the specified or computed LTD distribution. In general, there are two methods to model the LTD distribution (Vernimmen, 2008). The first method estimates the LTD directly using empirical data. The second method models the demand and lead-time distributions separately and constructs the compound distribution from the two. Obtaining the lead-time demand distribution from the convolution of the demand and lead-time distributions is analytically tractable for only few special cases, though. Accordingly, the Normal distribution has been widely used to characterize the LTD distribution and estimate the base-stock level, typically with central limit theorem justification (Tyworth and O'Neill, 1997).

There is disagreement over the effectiveness of using a Normal approximation of the LTD distribution. For fast-moving products, Tyworth and O'Neill (1997) conduct empirical experiments to examine the robustness of the Normal approximation of the LTD distribution in seven major industries. They conclude that the Normal approximation is robust with respect to logistics system cost, total costs, and fill rate. In contrast, though, several authors have demonstrated that the error of using approximation models could be substantial. Eppen and Martin (1988) reveal the errors in estimating the probability of stocking out from using the Normal approximation to determine the safety stock level. Chopra et al. (2004) demonstrate the errors that could stem from using a Normal approximation, leading to misleading managerial insights on setting inventory buffers to guard against the lead-time uncertainty. Lau and Lau (2003) demonstrate that another drawback of the Normal distribution is that it can be used only with low coefficient of variation (CV) to avoid negative demand values. Hence, they use a beta distribution to represent the LTD distribution in a (Q,R) inventory system, and they show that the Normal approximation of LTD is not robust in some situations and results in a substantial cost penalty even in the case of constant lead time. A review of related studies on the effects of using an approximated LTD distribution on the system total cost can be found in Lau and Lau (2003).

In this paper, we consider the case where a target customer service level is specified, and we compute the corresponding backorder penalty that generates equivalent outcomes between the service-level and shortage-costing approaches. We also investigate the

functional relationship between the target customer service level and the optimal total system cost, and we quantify the impact of supply risk on the deviation from the system cost under fully reliable supply. We conduct extensive computational experiments to observe the quality of various approximate solutions under two scenarios depending on the lead-time distribution. The first scenario assumes a geometric lead time and a discrete demand distribution, such that a discrete-time Markov process (DTMP) model represents the real system. We investigate the quality of three approximations to this DTMP model: Normal and gamma approximations that match the first two moments of the LTD distribution, and a statistical bootstrap procedure (Fricker and Goodhart, 2000) that uses a re-sampling approach to approximate the LTD.

In the second set of experiments, we consider a general lead-time distribution, meaning that the DTMP model serves as an alternative approximation to those considered in the first scenario. The DTMP model parameters are estimated by matching the first two or three moments of data sampled from the true, underlying LTD distribution. We perform some experiments where the demand distribution and the lead-time distribution are specified individually, and when the LTD distribution is characterized by a discretized beta distribution or a bimodal distribution. Simulations are conducted to obtain the LTD distribution and the corresponding base-stock level to satisfy the required service level. For both scenarios, we compare the cost deviation between the approximate and simulation-generated results. In addition, we observe the conditions when the DTMP model provides a good approximation, and the influence of different values of the CV of the LTD distribution on this cost deviation.

2. Markov Model and analysis

2.1. Model characteristics

We consider inventory control under a base-stock policy. The inventory level is monitored under periodic review. At the beginning of each period, an ordering decision to the supplier is determined to bring the inventory position back up to the base-stock level. The

supplier is not entirely reliable, and in each period the supplier is able to deliver an order according to a Bernoulli trial with probability α that the current order and any accumulated backorders are replenished in that period. We assume a nominal lead-time of one period, independent of the order size. This supply uncertainty induces a stochastic replenishment lead time having a geometric distribution, where the replenishment time is the nominal lead time plus the expected delay time due to supply uncertainty.

The customer demand is i.i.d. over time following some given discrete or discretized non-negative distribution. Backorders are allowed with a penalty cost when the demand is higher than the on-hand inventory. Furthermore, there are no scale economies in ordering. We model the system using a DTMP in order to determine the optimal base-stock level. The following notation is used to describe the model:

S = base-stock level

S^* = optimal base-stock level

π_i = steady-state probability of being in state i , where i is end-of-period on-hand inventory

α = supply risk factor (probability that total quantity on-order arrives in the current period)

h = holding cost per unit per period

b = backorder penalty cost per unit per period

D = random variable denoting per-period demand in any period t

p_d = probability of a demand of d units in a period = $P(D = d)$

TC^* = optimal total cost

$[i]^+ = \max(i, 0)$

$[i]^- = \max(-i, 0)$

The steady state probabilities are a function of α and p_d and can be represented as recursive equations. The derivation is presented in Wangwacharakul et al. (2009), namely

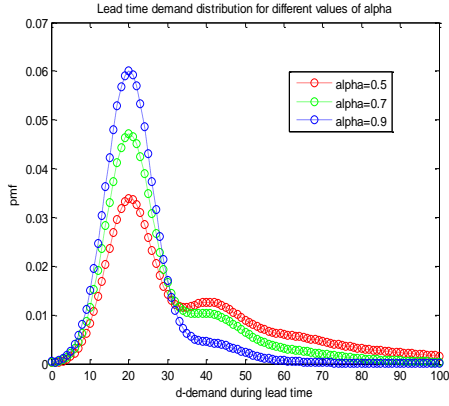
$$\pi_s = \frac{\alpha p_0}{1 - (1 - \alpha)p_0}$$

and

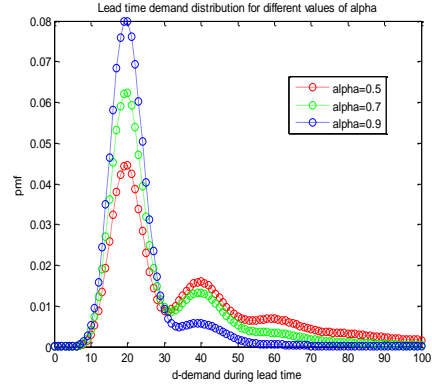
$$\pi_{s-d} = \frac{\alpha p_d + (1 - \alpha) \sum_{k=1}^d p_{d-k+1} \pi_{s-k+1}}{1 - (1 - \alpha)p_0} \quad \forall d = 1, 2, 3, \dots$$

The probability of having demand during lead time (i.e., lead-time demand, LTD) of d units can be characterized by the steady-state probability (π_{s-d}) of having on-hand inventory of $S - d$ at the end of a period. (See Wangwatcharakul et al., 2009, for the discussion that supports this equivalence.)

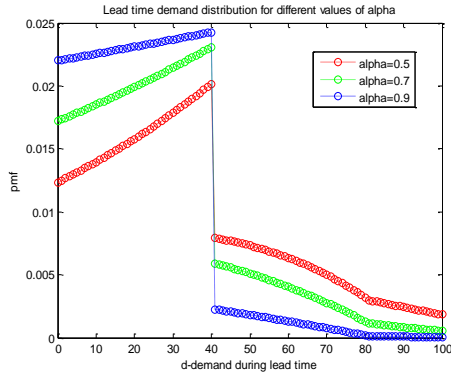
Therefore, the DTMP model provides the LTD distribution via the π_{s-d} distribution when demand has any discrete distribution and lead time has a geometric distribution with parameter α . Figure 1 illustrates the LTD distributions for different demand distributions and several values of the supply risk factor α .



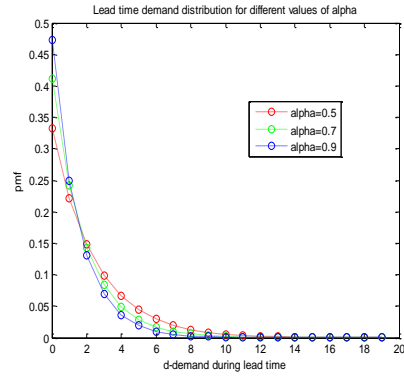
(a) Normal: $D \sim N \quad \mu = 20, \sigma = 6$



(b) Poisson: $D \sim \text{Poisson} \quad \lambda = 20$



(c) Uniform: $D \sim U \quad 0, 40$



(d) Geometric: $D \sim \text{Geo} \quad p = 0.5$

Fig. 1. LTD distribution for different demand distributions and supply risk factors

2.2. Service-level model and service performance measures

From the DTMP model in Wangwacharakul et al. (2009), when the backorder penalty cost (b) is given, the total system cost consists of inventory holding and backorder penalty costs and can be represented in terms of π_{S-d} as

$$TC(S) = h \sum_{d=0}^{S-1} (S-d) \pi_{S-d} + b \sum_{d=S+1}^{\infty} (d-S) \pi_{S-d} .$$

When a target service level (γ_{igt}) is specified, we assume that a backorder penalty cost is not, or cannot be, specified, such that the total system cost is represented only by the inventory holding cost. Thus, the service-level problem to determine the optimal base-stock level (S^*)—i.e., the minimum base-stock level that satisfies the service level target γ_{igt} —can be specified as follows:

$$\begin{aligned} \text{Minimize } TC(S) &= h \sum_{d=0}^{S-1} (S-d)\pi_{S-d} \\ \text{subject to } \sum_{d=0}^S \pi_{S-d} &\geq \gamma_{igt}. \end{aligned}$$

Two common criteria to measure customer service performance are service level— γ in our model—and fill rate, which we denote by β . Service level is defined as the probability of not stocking out in a period and fill rate is the fraction of customer demand immediately met by on-hand stock. Dullaert et al. (2007) present several applications in which the service level measure provides a more practical approach than using the fill rate to set safety stock. Accordingly, we use the service level measure criterion, probability of not stocking out in a period, in our analysis. Moreover, as shown in Wangwatcharakul et al. (2009), the optimal service level, $\gamma(S^*)$, for the DTMP model is such that

$$\gamma(S^*) \geq \frac{b}{b+h}.$$

Thus, we can easily compute the lower bound of the optimal service level when b is given.

In contrast, if γ_{igt} is specified, we can obtain the exact value of b associated with it, namely

$$b = \frac{h \cdot \gamma_{igt}}{1 - \gamma_{igt}}.$$

Moreover, the relationship between a given backorder penalty (b) and the target service level can be obtained from the optimality condition in the DTMP model in Wangwatcharakul et al. (2009), which is

$$S^* = \left\{ S : F(S-1) \leq \frac{b}{b+h}, F(S) \geq \frac{b}{b+h} \right\},$$

where X denotes the random variable for lead-time demand and $F(X)$ denotes its cumulative distribution.

In addition, for a given base-stock S , we can compute the fill rate as a function of the expected shortage per period, i.e.,

$$\beta(S) = 1 - \frac{ES(S)}{E(D)},$$

where $ES(S)$ is the expected shortage per period when base-stock level is S and $E(D)$ is the expected demand per period. For our DTMP model, this can be computed as

$$\beta(S) = 1 - \frac{\sum_{i=1}^{\infty} i\pi_{(-i)}}{E(D)}.$$

2.3. Effect of changing penalty cost and target service level on the total system cost

In the previous section, we match the service level constraint with the backorder penalty in the optimization model. Since $TC(S)$ is a non-linear function of S , the nature of the relationship between $TC(S^*)$ and b is complex. Figure 2 illustrates the relationship between the optimal total system cost, the backorder penalty cost, and the service level when demand has a Poisson distribution with parameter $\lambda = 20$ and when the supply risk factor is $\alpha=0.9$. The shape of this graph is consistent with those for other demand distributions such as uniform, gamma, and Normal at all levels of supply risk we studied. The total system cost grows exponentially as the service level increases. Therefore, the marginal system cost of increasing the service level is much higher for high target service levels than for lower ones.

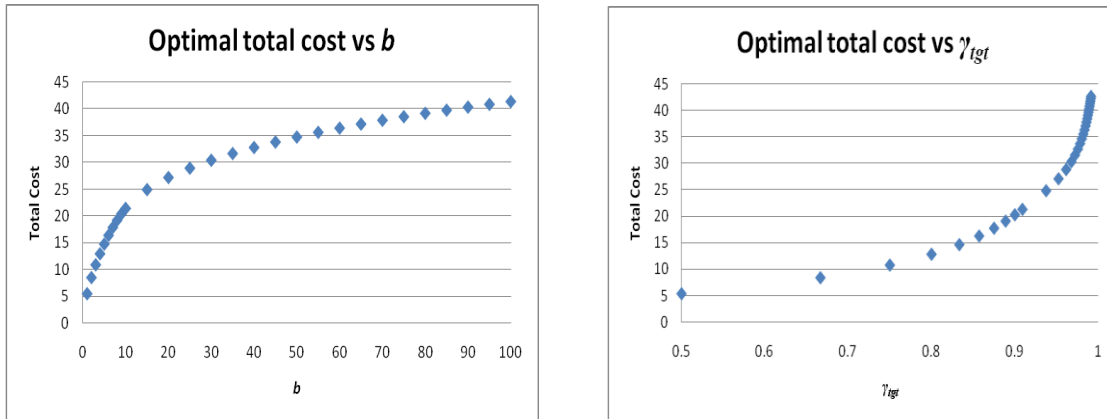


Fig. 2. Optimal total cost vs backorder penalty and target service level

3. Comparing approximation methods under geometric lead time

In this section, we observe the quality of Normal and gamma distribution-based approximations of the LTD distribution when demand follows any discrete distribution and lead time follows a geometric distribution. In this case, the exact LTD distribution (X) can be obtained directly from the distribution of π_{S-d} from the DTMP model presented in the preceding section. The cumulative distribution of X , which ultimately determines the base-stock level, is compared with those of the approximation models. Since the base-stock level for a service-constrained model is computed directly as a fractile of the LTD distribution, differences in the cdf of LTD will be reflected in different base-stock levels. The base-stock deviations can be illustrated visually through the cumulative distribution of X based on the critical fractile approach as in Figure 3.

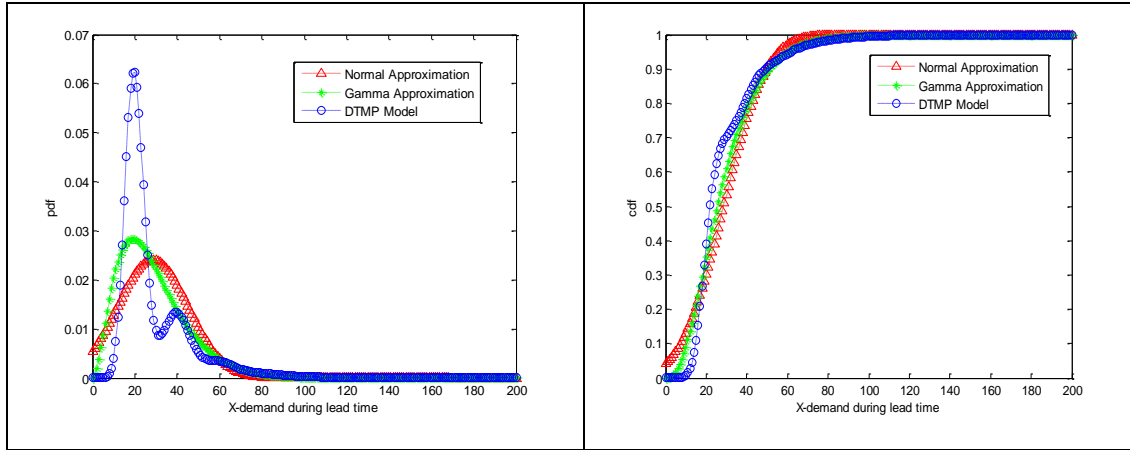


Fig.3. Comparisons of the pdf and cdf of LTD distribution for various methods

The relationship between base-stock level and total cost is significant in determining the cost deviation from the optimal cost when we wrongly estimate the optimal base-stock level. The cdf's shown in Figure 3 do not necessarily lead to a straightforward difference in the system costs. As the target service level changes, the optimal base stock level changes, and the overall system cost changes in a non-linear fashion as a function of the optimal base-stock level. Since the supply risk impacts the CV of the lead time and the LTD distribution directly, when α decreases, the mean and variance of the lead time increase as well as the CV. With higher uncertainty in supply risk (lower α), the relationship between the optimal base-stock level and optimal system cost becomes more linear as depicted in Figure 4.

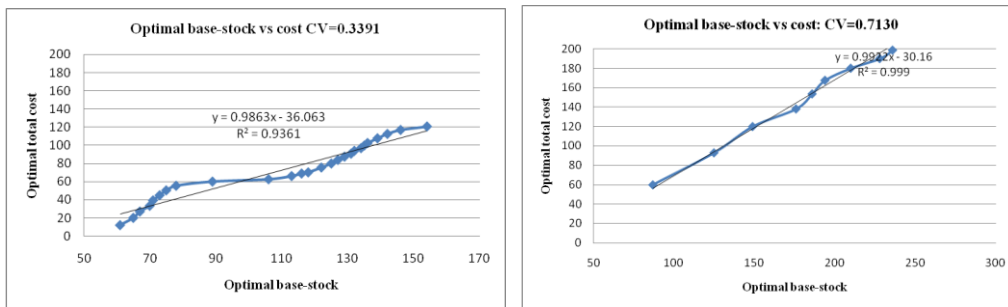


Fig. 4. The relationship between optimal base-stock and optimal total cost for different CV of X

With the combination of the base-stock deviation and the total cost difference, we obtain the cost deviation from the optimal cost as a performance measure of the approximation methods in the following experiment.

3.1. Approximation methods

In this section, we compare the LTD distribution obtained from the DTMP model with three approximations, a Normal and gamma fit to the LTD distributions, and a bootstrap procedure that is used to compute the LTD distribution by jointly sampling from the lead-time and demand distributions. We observe the quality of the approximations by comparing the total system costs that result from the approximations to the costs generated by the base-stock level computed from the actual underlying LTD distribution.

Vernimmen et al. (2008) provide an extensive overview of the distributions used in the academic literature to model demand, lead time and LTD distributions. The common distributions to model the period demand are Poisson, Normal, or gamma distributions. In reality, we can fit the LTD distribution directly from historical data. However, this approach depends on the availability of empirical data and requires a considerable amount of data to estimate the model parameters in a statistically accurate way.

There are several methods to fit the LTD distribution from empirical data such as moment-matching or maximum likelihood estimation. The most common method is to assume that the LTD is Normally distributed. As indicated in Keaton (1995), however, the LTD distribution tends to be skewed. Thus, a gamma distribution is an effective alternative to the Normal distribution, with the advantage of providing a more flexible shape than the symmetric shape of the Normal.

Normal and gamma distributions are used as parametric approximation models in this experiment. The model parameters are determined by matching the first two moments of the approximating distribution with the corresponding moments from the LTD distribution generated from the DTMP model. For the Normal distribution, this is straightforward. For the gamma distribution, the shape and scale parameters are computed as $k = \mu/\sigma^2$ and

$\theta = \mu^2 / \sigma^2$ where μ and σ^2 are the first and second moments from the DTMP-generated LTD distribution.

Bootstrap sampling is a non-parametric resampling method which is valuable when the exact distribution of interest is not known and the observed data is limited. Bookbinder and Lordahl (1989) used this technique to estimate the total system cost and conclude that it outperforms a Normal approximation. In this method, the empirical distribution is constructed from historical data. Each sample is assigned with equal probability and the resampling procedures described below are performed with replacement from the group.

We utilize a Bootstrap procedure that is similar to the one used in Fricker (2000), which consists of the following steps:

1. Resample with replacement from the observed demand data to generate m bootstrap demand observations ($m = 100$).
2. Resample with replacement from the observed lead-time data to generate n bootstrap lead-time observations ($n = 200$)
3. For each bootstrap lead time observation l sample demand randomly l times from the set of demand observations generated in step 1.
4. Repeat steps 2 and 3 r times ($r = 100$)
5. Determine the base-stock level associated with a required service level or backorder penalty cost based on the LTD distribution created in steps 3 and 4.

3.2. Experimental results

Through a series of experiments, we observe the quality of Normal, gamma, and bootstrap approximations to the base-stock levels provided by the DTMP model. The model inputs are as follows:

- The demand distribution is either Poisson, discretized Normal, discrete Uniform, or discretized gamma, all with a mean of 20.
- The supply risk factor α is chosen from $\alpha \in \{0.5, 0.7, 0.9\}$. The level of supply risk also affects the CV of the LTD distribution. The lower the supply risk factor, the higher the

CV of the LTD distribution. (Similarly, Tyworth and O’Neill (1997) indicate that the CV of the LTD is a significant factor for the Normal approximation to perform well.)

- The backorder penalty cost is chosen from $b \in \{1, 2, 4, 8, 16, 32, 64, 128\}$. The corresponding customer service levels are $\gamma = 50\%, 66.67\%, 80\%, 88.89\%, 94.12\%, 96.97\%, 98.46\%$, and 99.22% , respectively.

We use the percentage deviation from the optimal cost as the measure of the performance of the approximation methods, computed as

$$\Delta_{\%} = \frac{TC(S) - TC(S^*)}{TC(S^*)} \cdot 100\%,$$

where S is the base-stock level from the approximation method in question.

Table 1. Deviation (%) from optimal cost of approximation methods for Poisson demand with $\lambda=20$

b	$\alpha=0.9, CV=0.381$			$\alpha=0.7, CV=0.579$			$\alpha=0.5, CV=0.725$		
	N	G	B	N	G	B	N	G	B
1	8.53	3.05	3.05	14.30	5.04	1.05	5.81	0.89	0.00
2	9.53	4.80	4.80	6.61	2.97	1.32	5.72	0.76	0.39
4	9.37	6.10	3.32	1.51	0.27	0.78	1.76	0.33	2.28
8	1.57	1.57	14.01	0.14	0.43	0.84	0.04	0.04	0.66
16	1.89	0.89	8.33	1.43	0.19	0.04	1.20	0.00	0.88
32	8.41	3.33	3.52	7.10	0.60	2.39	6.61	0.26	8.37
64	20.71	4.74	0.83	17.37	2.56	9.33	17.87	0.69	22.56
128	39.50	10.54	0.00	37.91	4.28	21.75	35.91	1.62	48.35

N-Normal approximation, G-Gamma approximation, B-Bootstrap procedure

Table 1 shows the results when the demand follows Poisson distribution. The results for other demand distributions can be found in Appendix A. The overall results appear to be insensitive to the demand distribution type. The observations from the rest of experiments are as follows:

1. The ranking across all approximation methods is consistent for all values of α and most demand distributions.
2. For the Normal approximation, the higher CV of LTD results in higher cost deviation. When demand has Poisson distribution, however, the opposite is true.
3. The gamma approximation improves with higher CV, both overall and relative to the Normal and bootstrap approximations.
4. The variance of the cost outcomes from the bootstrap procedure is high, making comparative analysis difficult. Therefore, for this approximation method, it is difficult to generalize the cost deviation pattern and the impact of b and CV on the base-stock levels and system costs.
5. In general, the approximations perform best for moderate service level ($b = 8$, which is equivalent to $\gamma_{\text{tgt}} = 89\%$).

4. Comparing approximation methods under general lead time

As indicated in Vernimmen et al. (2008), there are two general methods to model the LTD distribution. The first method uses empirical observations of the LTD directly. The second method models the demand and lead-time distributions separately and constructs the compound distribution from the two. We consider the quality of using the DTMP model to approximate the LTD from both methods to observe the conditions where the approximation performs well and where it does not.

4.1. Demand and lead-time distributions specified

In this section, we assume that the demand and lead-time distributions are known and specified in each case, and that the LTD distributions are constructed from the compound distribution of these separately modeled demand and lead-time distributions. The DTMP model to determine the base-stock level to meet a required customer service level provides the LTD via the π_{s-d} distribution when demand has any discrete distribution. However, a

restrictive assumption of the model is that the lead time follows a geometric distribution. Through a set of numerical experiments, we explore the limitations of this restriction.

4.1.1. Data

Since demand distributions may be asymmetric, we model demand using a discretized gamma distribution due to its flexibility in representing a wide range of skewed distribution shapes. For example, as discussed in Tyworth et al. (1996) and Keaton (1995), gamma distributions are suitable for modeling slow-moving items with unimodal, right-skewed shapes.

Lead time is modeled by a discrete distribution. We use four different lead time distributions as shown in Figure 5. L3 and L4 are taken from empirical data obtained from the food and automobile industries, respectively, by Tyworth and O’Neill (1997). L1 and L2 are arbitrarily generated to reflect cases we believe to be representative of reasonable real situations—i.e., when lead time is most likely to be one or two days, but may extend a few days beyond that, or when lead time is equally likely to be any value in a given range.

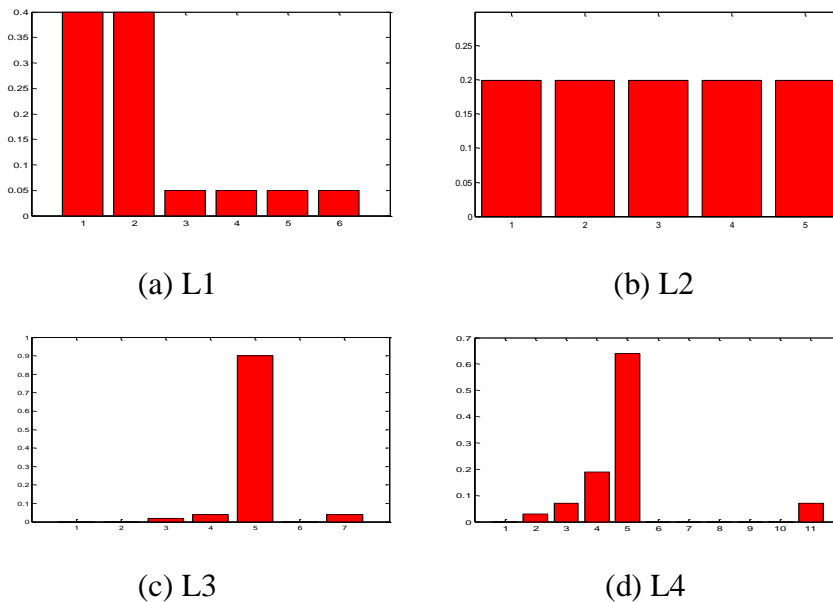


Fig. 5. Lead-time distributions used in computational experiments

For each lead-time distribution, the demand distribution follows one of three gamma distribution with parameters (k, θ) and CV values as shown in Figure 6.

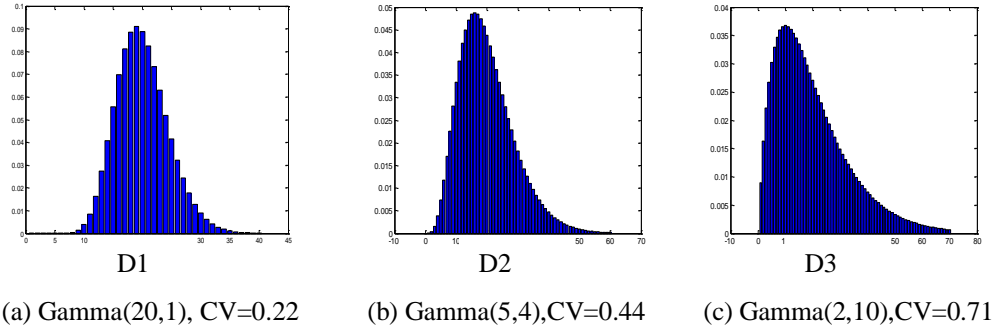


Fig. 6. Gamma demand distribution used in computational experiments

We use simulation to generate the LTD distribution sampled jointly from the period-demand and lead-time distributions, and use that to find the benchmark base-stock solution to compare with the approximation methods. We use 100,000 replications to generate the LTD distribution, use a quantile approach to determine the base-stock level, and compute the total system cost from this base-stock level. The CV values of the LTD distributions from the twelve combinations of lead-time and demand distributions in our experiments are shown in Table 2.

Table 2. CV of LTD for all combinations of demand and lead time distributions

	L1	L2	L3	L4
D1	0.67	0.49	0.15	0.37
D2	0.72	0.54	0.28	0.41
D3	0.82	0.62	0.33	0.48

4.1.2. LTD distribution parameters

The following notation is used to describe our model of the LTD distribution:

L = Random variable denoting lead time (in periods)

X = Random variable denoting lead-time demand (LTD)

μ_D = Mean of per-period demand, D

σ_D = Standard deviation of per-period demand, D

μ_L = Mean of lead time, L

σ_L = Standard deviation of lead time, L

μ_X = First moment (mean) of lead-time demand = μ_X

$\mu_2(X)$ = Second moment of lead-time demand distribution = $E(X - \mu_X)^2$

σ_X = Standard deviation of lead-time demand distribution = $\sqrt{\mu_2(X)}$

$\mu_3(X)$ = Third moment of lead-time demand distribution = $E(X - \mu_X)^3$

When the demand and lead time distributions are known, the first three moments for the LTD are as follows (Lau, 1989):

$$\mu(X) = \mu_X = \mu_D \mu_L, \quad (6)$$

$$\mu_2(X) = \mu_D^2 \mu_2(L) + \mu_L \mu_2(D), \quad (7)$$

$$\mu_3(X) = \mu_D^3 \mu_3(L) + \mu_L \mu_3(D) + 3\mu_D \mu_2(D) \mu_L. \quad (8)$$

The following eight methods are used to compute the parameters of the distribution used to estimate the LTD distribution. An inverse cdf of this approximating distribution is, in turn, used to determine the base-stock level required to meet the target customer service level. Since the actual LTD distribution resulting from the convolution of the period-demand distribution and the lead-time distribution cannot, in general, be specified analytically, simulation is used empirically to find the approximated LTD distribution, and the

corresponding base-stock level and total system cost. Ultimately we compare the resulting base-stock and cost outcomes to observe the accuracy of each approximation method.

Method 1—Normal approximation: The first two-moments of the DTMP-generated LTD distribution are set equal to the Normal approximation parameters, μ and σ^2 , as follows:

$$\mu = \mu_X, \sigma^2 = \mu_2(X).$$

Method 2—Gamma approximation: The first two-moments of the DTMP-generated LTD distribution are used to compute the gamma distribution parameters, k and θ , where k is the shape parameter and θ is a scale parameter, as follows;

$$\theta = \frac{\mu_2(X)}{\mu(X)}, k = \frac{\mu(X)}{\theta}.$$

Method 3—Poisson-geometric distribution: This combination results in the DTMP model, with Poisson demand with parameter λ and geometric lead time with parameter α . The algebraic details of the moment-matching computations in this case are provided in Appendix B, resulting in

$$\alpha = 1 - \frac{\mu_2(X) - 1}{\mu_X}, \lambda = \mu_X \alpha.$$

Method 4—Uniform-geometric distribution: This combination results in the DTMP model when demand has a discrete Uniform distribution with parameters $(0, u)$ and lead time has a geometric distribution with parameter α . The algebraic details of the moment-matching computations in this case are provided in Appendix C, resulting in

$$\alpha = \frac{12\mu_X^2 + 4\mu_X + 12\mu_2(X)}{8\mu_X^2}, u = 2\mu_X \alpha.$$

Method 5—Normal-geometric distribution: This combination results in the DTMP model when demand has a discretized Normal distribution with parameters (μ, σ^2) and lead time has a geometric distribution with parameter α . In this case, we match the first three moments of the distributions. The α of the DTMP model in this case is obtained by solving a polynomial equation. The algebraic details of the corresponding computations are provided in Appendix D. The polynomial equation and the other moment-matching equations are

$$2\mu_x^3\alpha^2 + (3\mu_x\mu_2(X) - 3\mu_x^3)\alpha + (\mu_x^3 - 3\mu_x\mu_2(X) + \mu_3(X)) = 0,$$

$$\mu = \mu_x\alpha, \text{ and } \sigma^2 = \mu_2(X)\alpha - \mu_x^2\alpha(1-\alpha).$$

Method 6—Gamma-geometric distribution: This combination results in the DTMP model when demand has a discretized gamma distribution with parameters (k, θ) and lead time has a geometric distribution with parameter α . In this case, we match the first three moments of the distributions. The algebraic details of the moment-matching computations in this case are provided in Appendix E, resulting in

$$\alpha = \frac{\mu_3(X)\mu_x - \mu_x^4 - 2\mu_2^2(X) + 4\mu_2(X)\mu_x^2 - 3\mu_x^2\mu_2(X)}{\mu_x^2\mu_2(X) - \mu_x^4}, \theta = \frac{\mu_2(X) - \mu_x^2(1-\alpha)}{\mu_x}$$

and $k = \frac{\mu_x\alpha}{\theta}$.

Method 7—Bootstrap procedure: The details are as described in the previous section.

Method 8—Simulation: This is as described in Section 4.1.1 above.

4.1.3. Experimental results

In this section, we present the results from carrying out the comparisons across the approximation methods discussed in the previous section across the twelve experiments for period-demand and lead-time distributions presented in Section 4.1.1. As an example of the results, Figure 7 illustrates the percentage cost deviation for each of the six different

approximation methods where the demand follows D2 and the lead time is L4 taken from automobile industry data (Tyworth and O’Neill, 1997). In this case, the graph demonstrates that the Normal approximation has the largest cost deviation among the methods considered, whereas the Normal-Geo and Gamma-Geo perform the best for most values of b .

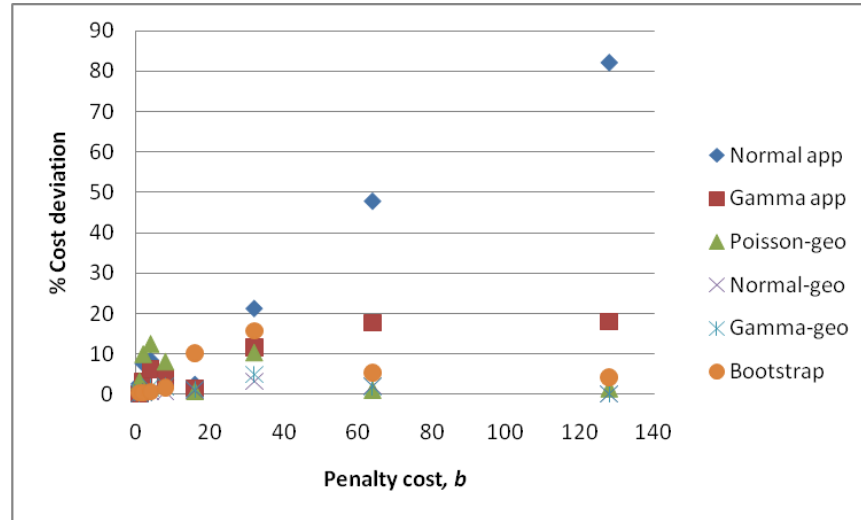


Fig. 7. Comparative results across six different approximation models for the D2-L4 case

The general observations obtained from the results for the computational experiments (Table 2) are as follows:

1. The approximation methods perform better when the CV of LTD is high than when it is low.
2. When demand has low CV, the LTD tends to have a more pronounced multimodal distribution.
3. In general, the approximation methods perform best for $b = 4$ and $b = 8$ —i.e, for lower levels of γ_{lgt} , 80% and 90%, respectively.
4. The effects of b and demand type on the cost deviations are consistent across all lead-time distributions.
5. The number of matching moments of the lead-time distribution affects the quality of the solution. Specifically, the Normal-Geo and Gamma-Geo perform better than the

Poisson-Geo and Uniform-Geo since they take into account the skewness of LTD by matching the third moment. Moreover, in some cases, the Uniform-Geo approximation fails to match the moments with the LTD and the approximation cannot be generated.

4.2. Specified LTD distribution

In the previous section, we have no control over the shape of the LTD distribution since it is the outcome of the convolution of the given demand and lead-time distributions. In this section, we assume that the LTD has a known distribution. We use a discretized beta distribution, by changing the shape parameters, η and φ , since it allows us to study LTD distributions that are symmetric, left-skewed or right-skewed. Thus, using the discretized

beta distribution, lead-time demand (X) has $E[X] = \frac{\eta}{\eta + \varphi}$, $\text{Var}[X] = \frac{\eta\varphi}{(\eta + \varphi)^2(\eta + \varphi + 1)}$ and

$$\text{CV}[X] = \sqrt{\frac{\varphi}{\eta \cdot (\eta + \varphi + 1)}}.$$

To represent the period demand in a finite range, we scale the discretized beta distribution to an upper bound of 60 from the standard (0,1) beta distribution. Experiments for each type of skewness—left-skewed, right-skewed, or symmetric—are conducted via 10 different combinations of the demand parameters, η and φ . Since our earlier results indicate that the CV has a major impact on the performance of each method, the increments of the beta parameters are based on the CV ranging from 0.3 to 0.8. We also specify a bimodal LTD consisting of a mix of two Poisson distributions with $\lambda=20$ and $\lambda=40$. We range the probability of each pair in the mixed distribution for a total of nine cases overall, from (0.1,0.9),(0.2,0.8),..., (0.9,0.1), as shown in Figure 8.

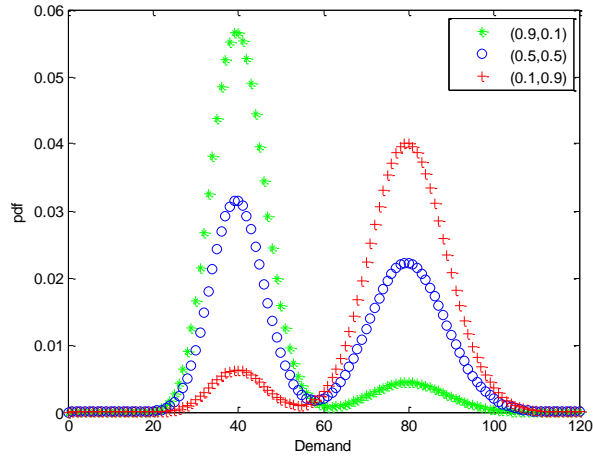


Fig. 8. Bimodal distributions generated from Poisson distributions with $\lambda=20$ and $\lambda=40$

The approximation methods used in this experiment are Normal and Gamma distributions, two approximations from our DTMP model (Poisson-Geo and Normal-Geo), and the nonparametric bootstrap procedure. The performance of each method is observed by the mean cost deviation.

4.2.1. Symmetric beta distribution

In this case, beta parameters $\eta = \varphi$ resulting in zero skewness. The average deviation from 10 different beta parameters for 8 different values of b are presented in Table 3.

Table 3. Average deviation (%) from optimal cost, symmetric LTD

b	1	2	4	8	16	32	64	128
Normal	0.00	0.98	2.13	0.98	2.87	11.18	20.78	31.17
Gamma	0.86	3.88	4.85	0.53	9.59	30.26	52.56	75.81
Poisson-Geo	3.27	11.72	17.28	5.56	16.74	47.17	80.82	116.68
Normal-Geo	0.00	0.98	2.13	0.98	2.87	11.18	20.78	31.17
Bootstrap	2.43	2.84	4.97	4.36	2.44	3.20	7.96	2.65

Across all experiments for symmetric beta demand, we observe the following:

1. The bootstrap approach outperforms all other approximation methods.
2. As expected, the Normal-Geo approximation provides results equivalent to the Normal approximation when $\alpha=1$ (i.e., perfect supply with a deterministic one-period lead time).
3. The Normal approximation and the DTMP model consistently outperform the gamma approximation.

4.2.2. Right-skewed beta distribution

The LTD is represented by a beta distribution with positive skewness coefficient. The parameters for the discretized beta distribution are $\eta, \varphi \in 2, 4, 8, 16, 32, 64, 128, \dots$ with the corresponding CVs for these parameters of 0.53, 0.6, 0.63, 0.65, 0.37, 0.41, 0.43, 0.44, 0.77, and 0.71, respectively.

Table 4. Average deviation (%) from optimal cost, right-skewed LTD

b	1	2	4	8	16	32	64	128
Normal	1.19	0.53	0.07	0.26	1.55	3.23	6.35	10.19
Gamma	0.07	0.21	0.24	0.13	0.03	0.52	2.24	5.46
Poisson-Geo	1.94	5.05	6.65	1.52	1.46	4.83	13.48	24.68
Normal-Geo	0.41	0.00	0.29	0.96	1.03	0.52	0.18	2.81
Bootstrap	1.15	1.33	1.18	2.44	1.62	3.18	2.25	7.78

Across all experiments for right-skewed beta distributions as shown in Table 4, we observe the following:

1. The Normal-Geo provides the best overall results, average deviations from the simulated cost

2. All approximation methods except for the Poisson-Geo have an average % cost deviation $< 2\%$ when $b \leq 16$ $\gamma_{igt} \leq 94.12\%$.
3. Generally, the approximations have better results for lower CV of LTD.

4.2.3. Left-skewed beta distribution

For the left-skewed LTD, it is not possible to obtain a high CV since the mean of the LTD is high relative to its standard deviation. The parameters for the beta distribution are

$\eta, \varphi \in 4, 2, 8, 2, (12, 2), (16, 2), (6, 4), (10, 4), (14, 4), (18, 4), (3, 1), (8, 1.5)$, with the range of CV from 0.08 to 0.27.

Table 5. Average deviation (%) from optimal cost, left-skewed LTD

b	1	2	4	8	16	32	64	128
Normal	1.78	1.91	0.13	0.17	11.99	21.84	33.82	45.47
Gamma	2.43	2.39	0.57	0.88	13.03	28.53	44.88	61.33
Poisson-Geo	1.33	6.19	19.65	35.03	53.43	74.14	104.71	122.02
Normal-Geo	1.78	1.12	0.04	2.41	13.03	23.21	36.70	45.87
Bootstrap	0.99	4.55	1.68	3.85	2.18	2.35	3.86	1.03

Across all experiments for left-skewed beta distributions as shown in Table 5, we observe the following:

1. Overall, the bootstrap approach performs best in approximating left-skewed LTD distributions.
2. For moderate service levels (i.e., $50 < \gamma_{igt} < 89$), the DTMP model, Normal and gamma approximations perform well with the average % deviation less than 3%.

4.2.4. Bimodal lead-time demand distribution

The probability of each pair of Poisson distributions for the bimodal distribution directly impacts the shape of the bimodal distribution. The average deviation from optimal cost is observed for different values of b as shown in Table 6.

Table 6. Average deviation (%) from optimal cost, bimodal LTD

b	1	2	4	8	16	32	64	128
Normal	2.92	2.07	1.00	1.43	3.40	5.93	10.68	14.39
Gamma	2.54	2.35	1.45	1.73	3.76	9.40	15.46	21.19
Poisson-Geo	2.58	7.91	16.73	9.07	8.77	25.51	45.66	66.12
Normal-Geo	1.81	1.14	1.12	2.66	4.41	3.31	5.07	10.49
Bootstrap	1.17	1.00	1.77	2.42	2.77	4.89	6.19	5.30

In general, we find that the Normal-Geo model works well when the peak of the left mode of the LTD distribution is higher than the right one, especially when the mixed distribution is

$$D \sim \begin{cases} \text{Poisson } \lambda = 20 & \text{w.p. } 0.9 \\ \text{Poisson } \lambda = 40 & \text{w.p. } 0.1. \end{cases} \quad \text{The Normal-Geo outperforms the Normal and}$$

gamma approximations including the bootstrap approach. However, the Normal-Geo starts to break down when b is high and provides poorer approximation. The bootstrap approach works well for most cases. However, the results from the bootstrap approach are more variable than other approximation methods with higher standard error. From the computational stand point, more intensive computational efforts are also required.

5. Conclusions

A discrete time Markov process (DTMP) model (Wangwatcharakul et al., 2009) is used to obtain the LTD distribution and find the optimal base-stock level under supply uncertainty. We investigate the robustness of various solution methods under two scenarios,

when the lead time has a geometric distribution and when the lead time has any distribution. The solution methods considered are as follows: Normal approximation, geometric approximation, Normal-Geo, Gamma-Geo, Poisson-Geo and bootstrap procedure. The DTMP-based models (Normal-Geo, Gamma-Geo, Uniform-Geo and Poisson-Geo) provide an exact base-stock level when the lead time has a geometric distribution and become approximations when the lead time has a general distribution. In the first scenario, the DTMP model represents the reality, and is used to specify the LTD distribution and find the optimal base-stock level. We specify the parameters of the LTD distribution approximations by matching the first two moments of the DTMP-generated LTD distribution with those of the approximation distribution. The Normal and gamma approximations for LTD perform well in the DTMP model setting when penalty cost, b , is low relative to inventory holding cost h , specifically in our experiments when $h = 1$ and $4 \leq b \leq 16$, corresponding to in-stock probability levels between 80% and 94%. The gamma approximation has lower deviation from the optimal cost than the Normal approximation in most experiment cases, especially for higher CV in the LTD distribution.

In the general lead-time case, the DTMP model becomes an alternative approximation to the optimal base-stock level, and the essence of the problem becomes accurately specifying the LTD distribution. The parameters for the approximation methods are determined by matching the moments with those of the specified LTD distribution, which is either sampled from independent period-demand and lead-time distributions or specified explicitly from a range of beta distribution or bimodal shapes. Among approximation methods that treat the demand and the lead time distributions independently (as opposed to seeking to generate the convolved distribution), the Normal-Geo and the Gamma-Geo perform better than the Normal and gamma approximations in most demand and lead-time scenarios used in the experiments, especially for higher LTD coefficients of variation. When the LTD is assumed to follow a discretized beta distribution, the Normal-Geo outperforms the Normal and the gamma approximations in most cases and it performs better than the bootstrap procedure when LTD has a right-skewed distribution or a bimodal distribution with the higher peak on the left.

REFERENCES

Bookbinder, J. H., Lordahl, A. E. (1989). Estimation of inventory reorder level using the bootstrap statistical procedure. *IIE Trans.* 21, 302–312.

Chopra S, Reinhardt G, Dada M (2004). The effect of lead time uncertainty on safety stocks. *Decision Science*, 35(1), 1-24.

Dullaert, W., Vernimmen, B., Aghezzaf, E., & Raa, B. (2007). Revisiting service-level measurement for an inventory system with different transport modes. *Transport Reviews*, 27(3), 273-283.

Eppen, G. D., & Martin, R. K. (1988). Determining safety stock in the presence of stochastic lead time and demand. *Management Science*, 34(11), 1380-1390.

Fricker, R. D., & Goodhart, C. A. (2000). Applying a bootstrap approach for setting reorder points in military supply systems. *Naval Research Logistics*, 47(6), 459-478.

Heuts, R., Lieshout, J., Baken, K. (1986) *Operations Research*, 30, B1-B14.

Keaton, M. (1995). Using the gamma distribution to model demand when lead time is random. *Journal of Business Logistics*, 16, 107–131.

Lau, H.S. (1989). Toward an inventory control system under non-normal demand and lead-time uncertainty. *Journal of Business Logistics*, 10, 89–103.

Lau, H. S., & Lau, A. H. L. (2003). Nonrobustness of the normal approximation of lead-time demand in a (Q, R) system. *Naval Research Logistics*, 50(2), 149-166.

Tyworth, J.E., Guo, Y., and Ganeshan, R. (1996). Inventory control under gamma demand and random lead time. *Journal of Business Logistics*, 17, 291–304.

Tyworth, J. E., & O'Neill, L. (1997). Robustness of the normal approximation of lead-time demand in a distribution setting. *Naval Research Logistics*, 44(2), 165-186.

Vernimmen, B., Dullaert, W., Willeme, P., & Witlox, F. (2008). Using the inventory-theoretic framework to determine cost-minimizing supply strategies in a stochastic setting. *International Journal of Production Economics*, 115(1), 248-259.

Wang, M. C., & Rao, S. S. (1992). Estimating reorder points and other management science applications by bootstrap procedure. *European Journal of Operational Research*, 56(3), 332-342.

Wangwacharakul, W., King, R. E., Warsing, D. P.(2009). Optimal base-stock levels for an inventory system with imperfect supply. Working paper, North Carolina State University.

APPENDICES

APPENDIX A

Percent deviation from the optimal cost for different demand distributions

1. Percent deviations from the optimal cost of the approximation methods when demand follows uniform distribution, $D \sim U(0, 40)$.

<i>b</i>	$\alpha=0.9, CV=0.644$			$\alpha=0.7, CV=0.738$			$\alpha=0.5, CV=0.822$		
	N	G	B	N	G	B	N	G	B
1	0.35	0.51	0.06	1.27	0.50	1.27	4.60	0.01	0.00
2	0.00	1.69	1.11	2.71	0.06	0.23	5.25	0.49	0.11
4	0.08	0.52	3.88	4.05	1.78	0.05	1.48	0.04	2.38
8	1.62	3.73	1.62	0.13	0.13	1.42	0.11	0.11	6.56
16	2.90	5.96	9.94	0.94	0.02	2.08	1.05	0.00	7.01
32	0.12	1.41	5.81	3.79	0.06	2.15	5.86	0.00	4.30
64	4.82	0.53	5.24	11.18	0.03	9.68	16.21	0.06	1.21
128	16.53	0.43	5.14	27.08	0.13	17.95	32.81	0.18	0.01

2. Percent deviations from the optimal cost of the approximation methods when demand follows Normal distribution, $D \sim N(\mu = 20, \sigma = 5)$.

<i>b</i>	$\alpha=0.9, CV=0.396$			$\alpha=0.7, CV=0.586$			$\alpha=0.5, CV=0.729$		
	N	G	B	N	G	B	N	G	B
1	6.21	1.96	0.48	12.69	3.95	0.27	7.56	1.15	2.34
2	11.81	2.91	1.49	6.97	3.12	0.56	5.47	0.62	0.09
4	7.73	7.73	3.21	1.30	0.21	0.23	1.68	0.32	0.82
8	1.56	2.68	0.40	0.30	0.30	1.71	0.03	0.11	4.26
16	1.72	0.25	11.19	1.42	0.05	6.26	1.30	0.00	12.44
32	9.30	2.35	25.94	6.19	0.42	9.32	6.82	0.18	27.60
64	18.03	3.98	18.03	18.28	2.06	11.65	18.31	0.78	50.46
128	35.70	9.11	21.99	35.38	3.82	20.05	36.64	1.45	88.15

3. Percent deviations from the optimal cost of the approximation methods when demand follows gamma distribution, $D \sim \text{Gamma } k = 16, \theta = 1.25$.

b	$\alpha=0.9, CV=0.396$			$\alpha=0.7, CV=0.586$			$\alpha=0.5, CV=0.729$		
	N	G	B	N	G	B	N	G	B
1	0.35	0.51	0.06	1.27	0.50	1.27	4.60	0.01	0.00
2	0.00	1.69	1.11	2.71	0.06	0.23	5.25	0.49	0.11
4	0.08	0.52	3.88	4.05	1.78	0.05	1.48	0.04	2.38
8	1.62	3.73	1.62	0.13	0.13	1.42	0.11	0.11	6.56
16	2.90	5.96	9.94	0.94	0.02	2.08	1.05	0.00	7.01
32	0.12	1.41	5.81	3.79	0.06	2.15	5.86	0.00	4.30
64	4.82	0.53	5.24	11.18	0.03	9.68	16.21	0.06	1.21
128	16.53	0.43	5.14	27.08	0.13	17.95	32.81	0.18	0.01

APPENDIX B

Derivation for Poisson-geometric parameters for a Poisson period demand and a geometric lead time

$$\mu_D = \lambda, \mu_2(D) = \lambda, \mu_L = \frac{1}{\alpha}, \text{ and } \mu_2(L) = \frac{1-\alpha}{\alpha^2}$$

Substituting the above parameters from expressions(6) and (7),

$$\begin{aligned}\mu_X &= \frac{\lambda}{\alpha} \\ \mu_2(X) &= \lambda^2 \left[\frac{1-\alpha}{\alpha^2} \right] + \frac{\lambda}{\alpha} \\ \mu_2(X) &= \frac{\lambda}{\alpha} \left[\frac{\lambda}{\alpha} (1-\alpha) \right] + 1 \\ \mu_2(X) &= \mu_X \mu_X (1-\alpha) + 1\end{aligned}$$

$$\begin{aligned}\alpha &= 1 - \frac{\frac{\mu_2(X)}{\mu_X} - 1}{\mu_X} \\ \lambda &= \mu_X \alpha\end{aligned}$$

APPENDIX C

Derivation for Uniform-geometric parameters for a uniform period demand and a geometric lead time

$$\mu_D = \frac{u}{2}, \mu_2(D) = \frac{(u+1)^2 - 1}{12}, \mu_L = \frac{1}{\alpha}, \text{ and } \mu_2(L) = \frac{1-\alpha}{\alpha^2}$$

Substituting the above parameters from expressions(6) and (7),

$$\mu_X = \frac{u}{2\alpha}$$
$$\mu_2(X) = \frac{u^2}{4} \left[\frac{1-\alpha}{\alpha^2} \right] + \frac{(u+1)^2 - 1}{12\alpha}$$

$$\alpha = \frac{12\mu_X^2 + 4\mu_X + 12\mu_2(X)}{8\mu_X^2}$$

$$u = 2\mu_X\alpha$$

APPENDIX D

Derivation for Normal-geometric parameters for a Normal period demand and a geometric lead time

$$\mu_D = \mu, \mu_2(D) = \sigma^2, \mu_3(D) = 0, \mu_L = \frac{1}{\alpha}, \mu_2(L) = \frac{1-\alpha}{\alpha^2}, \text{ and } \mu_3(L) = \frac{(1-\alpha)(2-\alpha)}{\alpha^3}$$

Substituting the above parameters from expressions(6), (7), and (8),

$$\begin{aligned} \mu_X &= \frac{\mu}{\alpha} \\ \mu_2(X) &= \mu^2 \left[\frac{1-\alpha}{\alpha^2} \right] + \frac{\sigma^2}{\alpha} \\ \mu_3(X) &= \mu^3 \left[\frac{(1-\alpha)(2-\alpha)}{\alpha^3} \right] + 3\mu\sigma^2 \left[\frac{1-\alpha}{\alpha^2} \right] \end{aligned}$$

After solving the above equations, we obtain the α , μ , and σ^2 for the Normal-Geo from the following equations,

$$\begin{aligned} 2\mu_X^3\alpha^2 + (3\mu_X\mu_2(X) - 3\mu_X^3)\alpha + (\mu_X^3 - 3\mu_X\mu_2(X) + \mu_3(X)) &= 0, \\ \mu &= \mu_X\alpha, \\ \sigma^2 &= \mu_2(X)\alpha - \mu_X^2\alpha(1-\alpha). \end{aligned}$$

APPENDIX E

Derivation for the Gamma-geometric parameters for a gamma period demand and a geometric lead time

$$\mu_D = k\theta, \mu_2(D) = k\theta^2, \mu_3(D) = 2k\theta^3,$$

$$\mu_L = \frac{1}{\alpha}, \mu_2(L) = \frac{1-\alpha}{\alpha^2}, \text{ and } \mu_3(L) = \frac{(1-\alpha)(2-\alpha)}{\alpha^3}$$

Substituting the above parameters from expressions(6), (7), and (8),

$$\mu_x = \frac{k\theta}{p}$$

$$\mu_2(X) = (k\theta)^2 \left[\frac{1-\alpha}{\alpha^2} \right] + \frac{k\theta^2}{\alpha}$$

$$\mu_3(X) = (k\theta)^3 \left[\frac{(1-\alpha)(2-\alpha)}{\alpha^3} \right] + \frac{2k\theta^3}{\alpha} + 3k^2\theta^3 \left[\frac{1-\alpha}{\alpha^2} \right]$$

After solving the above equations, we obtain the α , θ , and k for the Gamma-Geo as follows:

$$\alpha = \frac{\mu_3(X)\mu_x - \mu_x^4 - 2\mu_2^2(X) + 4\mu_2(X)\mu_x^2 - 3\mu_x^2\mu_2(X)}{\mu_x^2\mu_2(X) - \mu_x^4}, \theta = \frac{\mu_2(X) - \mu_x^2(1-\alpha)}{\mu_x}$$

$$\text{and } k = \frac{\mu_x\alpha}{\theta}.$$

CHAPTER 4

Computing Base-stock Levels for a Serial Inventory System with Imperfect Supply

Abstract

We propose a computational method to determine optimal base-stock levels in a two-location inventory system consisting of one distributor (upstream) and one retailer (downstream), with a fixed one-period nominal lead time from distributor to retailer but an unreliable supplier to the distributor. The distributor and retailer both operate under a periodic review base-stock control policy. The customer demand is assumed to follow any discrete probability distribution. Supply uncertainty is modeled as a Bernoulli trial, with a specified probability that the total quantity on-order is filled in one period. The objective is to minimize the joint system cost. Since a closed-form solution appears to be unattainable, direct numerical computation is the only viable approach to finding an optimal solution. As the state space of this model increases, however, computational complexity becomes an issue. In order to reduce the search space and computational time, we develop two approximation methods to find a good starting solution. The approximation methods are based on a decomposition method to solve the problem sequentially by optimizing the separate single-site inventory problems (distributor, retailer) under supply risk with a modified input parameter at the upstream node, namely an implied backorder penalty. The implied backorder penalty is required in this decentralized system to force the upstream node to carry some inventory. This penalty cost reflects the cost impact of the distribution delays to the retailer. The first approximation method provides a functional relationship between the input parameters and the implied backorder penalty. The second method numerically searches for the implied backorder penalty to minimize the total system cost for the decentralized system. The approximate solution could serve as an initial solution for a steepest descent search algorithm to obtain the optimal solution. Numerical experiments are conducted to evaluate the effectiveness of the approximation methods. The cost deviations from the optimal solutions are presented. We also observe the impact of changing input parameters, such as backorder penalty and demand shape, on the base-stock levels and the total system cost.

1. Introduction

There are two classes of base-stock policy: local and echelon (Galleo and Zipkin 1999). The local base-stock policy operates under decentralized control. Each stage controls its inventory independently as a one-node inventory system. In contrast, the echelon base-stock policy employs a centralized control policy. Chen and Zheng (1994) show that the echelon base-stock policy is optimal for both continuous and discrete review settings.

Policy evaluation and optimization of serial inventory systems have been studied extensively in the past. However, the study of models with supply risk is limited. The common assumptions are that the demand process is continuous compound Poisson and there are constant lead times between stages (Mohebbi, 2003). Chao and Zhou (2007) develop bounds and heuristics for this optimal echelon base-stock policy. The bounds are obtained from the decomposition of the derivative of the cost function. The explicit equations they obtained represent the bounds and the optimal echelon base-stock policies for both average and discounted cost criteria.

We consider a two-node serial system with one retailer, who fulfills customer demand, and one distributor, whose supplier is not fully reliable (i.e., exhibits imperfect supply). Each node operates under a periodic-review, base-stock replenishment policy in an infinite horizon. For simplicity, we assume a common review period for the retailer and distributor. We assume that the nominal replenishment lead time is a constant one period for both nodes. However, supply to the upstream node (distributor) is subject to uncertainty, modeled as a Bernoulli trial with a specified probability, α , that the total quantity on-order from the distributor's supplier is filled in one period. The discrete, stochastic demand fulfilled by the retailer is independent and identically distributed (i.i.d) in each period. Unfulfilled orders are fully backlogged for both nodes. However, a linear penalty cost is incurred at both nodes each time the shortage occurs. The customer demand and cost factors are stationary.

This model serves as an extension of the one-node system with supply risk in Wangwatcharakul et al. (2009). The objective function consists of two cost elements, holding

and backordering costs. We employ a discrete-time Markov process (DTMP) to evaluate the cost of the solution. The two-node system leads to a two-dimensional state space of on-hand inventory at nodes 1 and 2, denoted by Ω . The ultimate goal of the study is to find the optimal base-stock level for each node to minimize the total cost of the system.

In Section 2, we describe the model and present a computational method based on a two-dimensional discrete-time Markov process model. Since a closed-form solution appears to be unattainable, direct numerical computation is used to find an optimal solution. We also observe the impact of changing input parameters, such as backorder penalty and demand shape, on the base-stock levels and the total system cost. When the state space of this model increases, however, computational complexity becomes an issue. In Section 3, we develop two approximation methods to find a good starting solution in order to reduce the search space and computational time. The approximation methods are based on a decomposition method to solve the problem sequentially by optimizing the separate single-site inventory problems (distributor, retailer) under supply risk with a modified input parameter at the upstream node, namely an implied backorder penalty. The implied backorder penalty is required in this decentralized system to force the upstream node to carry inventory and reflect the cost impact of the distribution delays to the retailer. This approximate solution could serve as an initial solution for a steepest descent search algorithm to obtain the optimal solution. In Section 4, numerical experiments are conducted to evaluate the effectiveness of the approximation methods. The cost deviations from the optimal solutions are presented.

2. Model

2.1 Model specification

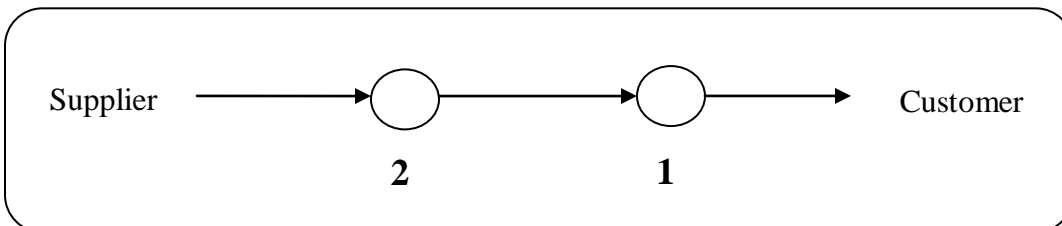


Fig.1. System configuration

Node 1 represents a retailer. Node 2 is a distributor node connected with an outside supplier as shown in Figure 1. The inventory management is under periodic review base-stock control policy. The customer demand is i.i.d. and arrives at node 1. At the end of each period, node 1 places an order to node 2 to bring its inventory position (comprised of the sum of on-hand inventory and all outstanding orders) to the base-stock level S_1 . There is no ordering lag time from node 1 to node 2. Node 2 replenishes the order from node 1 in full if possible or partially if the on-hand inventory is less than the order size. A penalty cost is incurred each period a stockout occurs. Node 2 places an order to the outside supplier to bring its inventory position back up to the base-stock level S_2 . The supplier has an infinite supply or no capacity constraint, but the availability of the supply has a factor of α , which is the probability that the total quantity on-order arrives in the current period. The following notation is used to describe the model.

S_1 = base-stock level of node 1

S_2 = base-stock level at node 2

i = on-hand inventory level at node 1 physically available to supply the customer demand

j = on-hand inventory level at node 2 physically available to supply order from node 1,
including the backorders about to ship to node 1

G_1 = maximum backorder allowed at node 1

G_2 = maximum backorder allowed at node 2

Ω = state space of the system = $(i, j) : -G_1 \leq i \leq S_1, -G_2 \leq j \leq S_2$

α = probability that the total quantity on-order arrives at the distributor in the current period

D = the customer demand in a period

$p_d = P\{D=d\}, d=0,1,2,3,\dots$

h_1 = holding cost per unit per period for node 1

h_2 = holding cost per unit per period for node 2

b_1 = backorder penalty cost per unit per period for node 1

b_2 = backorder penalty cost per unit per period for node 2

$$[x]^+ = \max(x, 0)$$

$$[x]^- = \max(-x, 0)$$

The objective function is as follows

$$TC(S_1, S_2) = \sum_{i=-G_1}^{S_1} \sum_{j=-G_2}^{S_2} h_1[i]^+ + h_2[j]^+ + b_1[i]^- \pi_{ij} \quad (9)$$

where π_{ij} denotes the steady state probability of having on-hand inventory i and j units at node 1 and 2, respectively. Note that $TC(S_1, S_2)$ contains a backorder penalty cost term for only node 1 since the node 2 backorder penalty cost paid by the distributor to the retailer results in an equivalent *negative* cost (i.e., a cash inflow) to node 1, with no net effect on $TC(S_1, S_2)$ in total. Later in the paper, when we decompose $TC(S_1, S_2)$ into its component costs, terms containing the node 2 backorder penalty cost, b_2 , will appear in the independent-node cost functions.

2.2. System dynamics

During some general period t , six events determine the state transition from (i, j) to (i', j') . The sequence of these events is as follows and displayed in Figure 2.

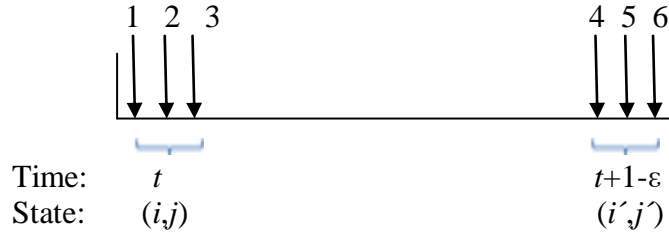


Fig.2. Sequence of events during a period

1. Node 1 places an order with node 2
2. Node 2 ships to node 1 (Deduction from node 2 inventory)

3. Order at node 2 to supplier
4. Receipt at node 1 from node 2
5. Receipt at node 2 from supplier
6. Demand arrives and is fulfilled (to the extent supported by on-hand inventory) at node 1 (Deduction from node 1 inventory)

We include some assumptions to simplify the model as follows.

1. The demand at node 1 occurs anytime during the period, but it is accumulated and fulfilled at the end of period, specifically at the time of event 6.
2. The depletions of the inventories of nodes 1 and 2 happen at different points in time.
3. Node 2 incurs inventory cost during the shipping at event 2 until the order arrives at node 1 at the time of event 4.
4. The on-hand inventory is evaluated at the time of event 6, but each node places an order to its supplier at the beginning of the next period.

The dynamics of the system can be represented according to following notation.

d_t = realized customer demand at time t

I_t = on-hand inventory of node 1 at time t

J_t = on-hand inventory of node 2 at time t

O_t = order quantity placed from node 1 to node 2 at time t

Z_t = amount shipped from node 2 to node 1 at time t

In order to have a finite state space for on-hand inventories, the maximum backorders allowed in node 1 and node 2 are set to be G_1 and G_2 , respectively. We assume the system begins operation in state (S_1, S_2) . Under base stock control, node 2 places an order with its supplier equal to the order from node 1 (O_t) to bring its inventory back up to the base-stock level. In each period, the on-hand inventory and on-order quantity are updated as follows.

1. $I_{t+1} = \max(I_t + Z_t - d_t, -G_1)$
2. $J_{t+1} = (1-y) J_t - \max(O_t, Z_t) + y J_t^+ + S_2 - J_t^+ + O_t - Z_t$,

where $y = \begin{cases} 0 & \text{w.p. } 1-\alpha \\ 1 & \text{w.p. } \alpha \end{cases}$

- If the supplier is unable to ship, $(1-y) = 1$ and $J_{t+1} = J_t - \max(O_t, Z_t)$.
 - If $J_t \leq S_2$, O_t is no less than Z_t . The term $\max(O_t, Z_t)$ implies that we would let J_t go down to a negative value if we don't have enough on-hand inventory to supply the order from i . Also, J_{t+1} would not go beyond $-G_1$ since we have a constraint on $\max(O_t)$ (within $J_t + G_2$)
 - If $J_t > S_2$, this implies that the backorders just arrived at node 2 from the supplier. We would ship the previous backorder at the same time that we replenish O_t . In this scenario, Z_t could exceed O_t . So, we take $\max(O_t, Z_t)$

- If the supplier is able to ship, $y=1$ and $J_{t+1} = J_t^+ + S_2 - J_t^+ + O_t - Z_t$

Node 1 places an order of O_t to node 2 at time 3 and will receive it at time 5

- In case of $J_t < 0$, $J_t^+ = 0$ and $S_2 - J_t^+ = S_2 - J_t$
- In case of backorder just arrived, $J_t > S_2$
 $J_t^+ = J_t$ and $S_2 - J_t^+ = 0$

3. $O_t = \min(S_1 - I_t + \min(J_t, 0) - J_t - S_2^+, J_t + G_2)$

The term $\min(J_t, 0)$ is used to prevent double ordering stock already on order. In the case when node 2 receives the backorder from the supplier, it automatically ships the backorder to node 1, but still incurs the inventory cost until the receipt at node 1. The term $J_t - S_2^+$ represents in-transit inventory from the previous backorder. Instead of

placing a constraint $\max(O_t, -G_2)$ on node 2, the term, $J_t + G_2$, represents the upper bound of the allowable order from node 1 to node 2.

$$4. \quad Z_t = J_t - S_2^+ + \max \left[\min \left[J_t - J_t - S_2^+, O_t \right], 0 \right]$$

$$= J_t - S_2^+ + \left[\min \left[J_t - J_t - S_2^+, O_t \right] \right]^+$$

In the case when $J_t > S_2$, the term $J_t - S_2^+$ adjusts the on-hand inventory after deducting the backorder ready to ship to node 1 and this amount of $J_t - S_2^+$ would be shipped from node 2 to node 1 at time 2 from the diagram at the same time of the replenishment of new order from node 1 to node 2.

When $J_t > 0$ and $O_t > 0$, node 2 ships $\min [J_t, O_t]$. If $J_t < 0$, there is no shipment from node 2 at that period. The universal term to cover all the scenarios is

$$\left[\min \left[J_t - J_t - S_2^+, O_t \right] \right]^+$$

To better understand the dynamics of the system, the following statements clarify some mechanics of the system:

1. The total on-order amounts (from node 2 to supplier) could be accumulated from multiple periods which can be represented as $\max [S_2 - J_t, 0]$. If node 2 were backordered with negative J_t after the receipt of its on-order from supplier, its on-hand inventory is higher than S_2 . When this happens, on-order from node 2 to supplier is zero.
2. Node 2 holds in-transit inventory cost while shipping to node 1.

2.3. Computing transition probabilities

The state transition probability matrix of the discrete-time Markov process (DTMP), P , has dimensions $(m \times n) \times (m \times n)$, where $m = G_1 + S_1 + 1$ (since $i \in [-G_1, S_1]$, an interval that includes zero, provided $G_1, S_1 > 0$) and $n = G_1 + G_2 + S_1 + S_2 + 1$. The system is

stationary-i.e., the transition probabilities and state space do not change over time. The system dynamics above are used to construct the transition probability matrix. The state transition probabilities are computed based on the input parameters such as the demand distribution and the supplier uncertainty factor (α). Since backordering is allowed when stockouts occur, node 2 could have a maximum backorder of G_1+G_2 . For large-valued customer demand distributions, P can be quite large, which can result in a time-consuming search for the optimal solution, or in exhausting the memory capacity of the computational system.

Instead of using exhaustive search through the entire state space, we can apply a steepest descent search to accelerate the search process. A good starting solution is the key to reduce the search space. There are two approximation methods proposed here to obtain a good starting solution. Each method determines the optimal base-stock for each node based on the one-stage Markov model developed in Wangwatcharakul et al. (2009). The system solutions are calculated sequentially from node 2 to node 1. In this case, the optimal fill rate of node 2 is used as an input parameter to the node 1 problem and provides the supply uncertainty factor necessary to solve for the node 1 base-stock level using the approach of Wangwatcharakul et al. (2009). Thus, the approximations to S_2^* and S_1^* are calculated sequentially. The approximation methods are described in detail in Sections 3.1 and 3.2.

2.4. Numerical experiments

In this section, we present numerical examples to observe the impact of the input parameters, α , b_1 , and h_2 , on the optimal base-stock levels (S_1^* and S_2^*) one parameter at a time. The two-node model is completely specified by α , b_1 , h_1 , h_2 , and the demand distribution. The demand is assumed to follow a discretized beta distribution due to its flexibility, allowing us to model a wide variety of demand shapes—including symmetric, right-skewed, left-skewed, and U-shape—by specifying different shape parameters. In this experiment, the discretized beta demand is parameterized to have a constant mean of 10 for

each case but different standard deviation (SD) and skewness as shown in Table 1. The resulting mass functions of demand are shown in Figure 3.

Table 1. Summary of demand parameters used in this experiment

Beta(a,b)	1.5,1.5	2.62,2.62	5,5	2,3	2,7.8	3,12	6,2.4	4,3.2	7,2.1
SD	5	4	3	0.51	3.81	1.47	0.55	0.18	0.67
Skewness	0	0	0	0.10	0.53	0.29	-0.23	-0.03	-0.26

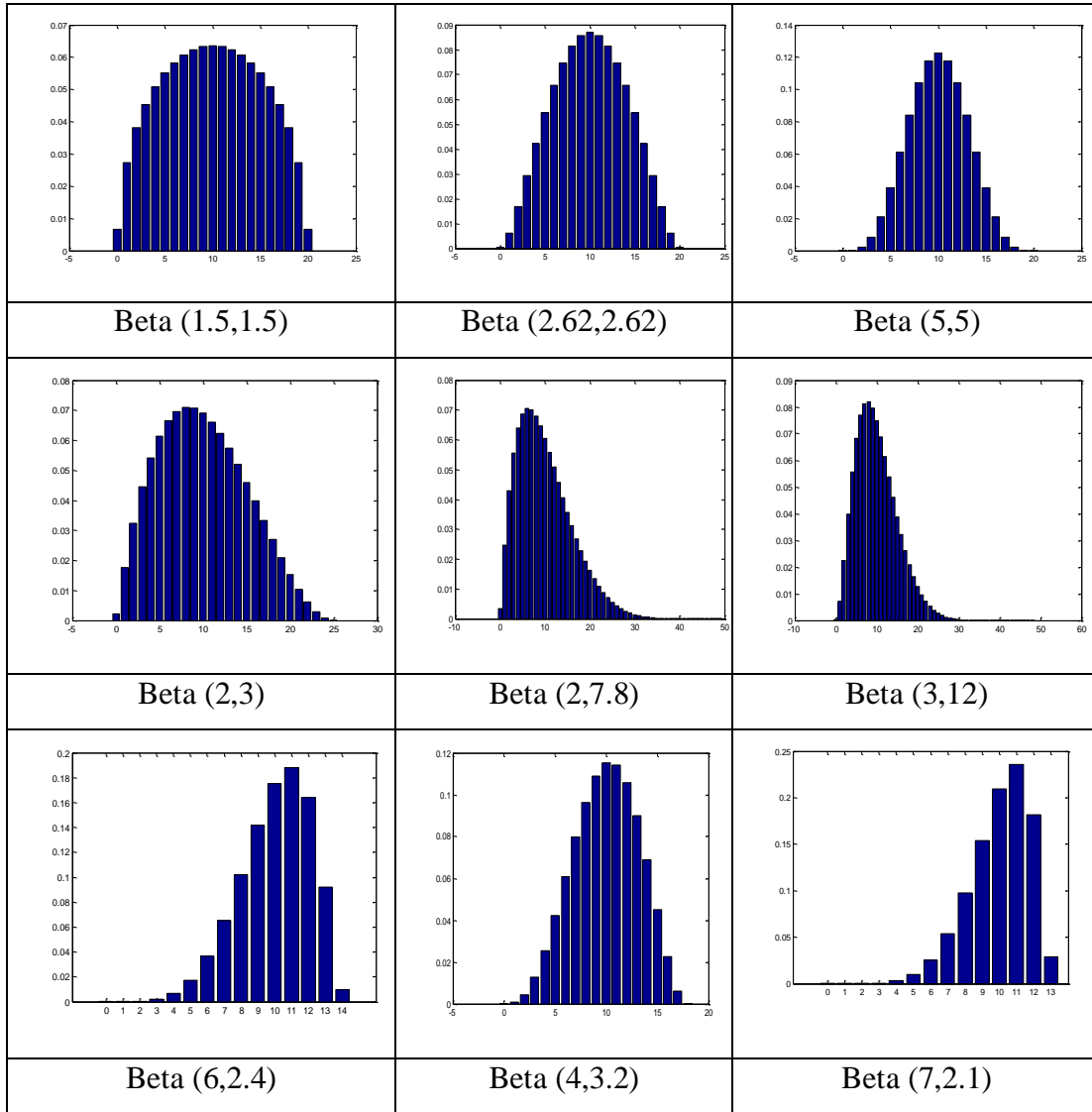


Fig.3. Different beta demand distributions

In our computational experiments, the value of h_1 is set to 1, and a variety of problem instances are generated by selecting $h_2 \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$, $b_1 \in \{10, 20, 40\}$, and $\alpha \in \{0.7, 0.75, 0.8, 0.85, 0.9, 0.95\}$. This results in a total of 810 problem instances. Note that the results are consistent across demand distributions used in the experiment. Hence, in the following sections we arbitrarily shown the results for the beta(2.62,2.62) case.

2.4.1. Impact of h_2 on the optimal base-stock levels

It is reasonable to assume that $h_2 \leq h_1$. As h_2 increases relative to h_1 , the system holds less safety stock at node 2 and holds more safety stock at node 1 as shown in Figure 4. When α or h_1 increases, the system holds more inventory and obviously results in higher total cost. When h_2 increases, S_2 decreases while S_1 increases. This results in higher total system cost even with lower S_2 .

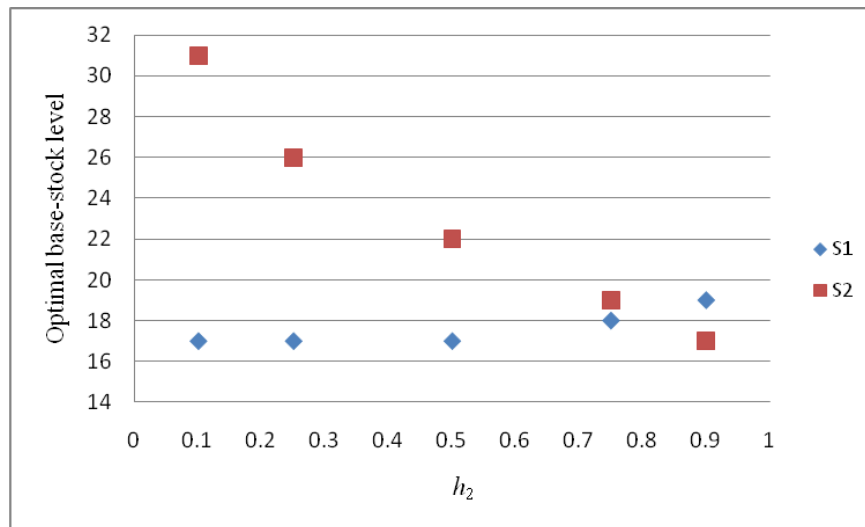
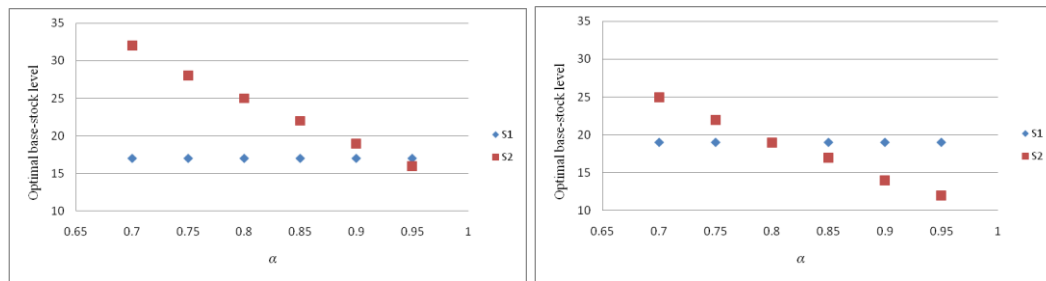


Fig.4. Optimal base-stock levels vs. h_2 , when $b_1 = 20$, $\alpha = 0.85$

2.4.2. Impact of α on the optimal base-stock levels

The supply uncertainty has an impact on the optimal base-stock level of the upstream node and has no impact on the optimal base-stock of the downstream node as shown in Figure 5. The higher levels of supply uncertainty (lower α) result in a higher optimal base-stock level of node 2 to protect the system against the uncertainty. The total system cost increases when the system has higher supply uncertainty. The difference in the system cost can be used as a benchmark to consider alternative suppliers that are more reliable, but whose ancillary costs might be higher (but not modeled here).



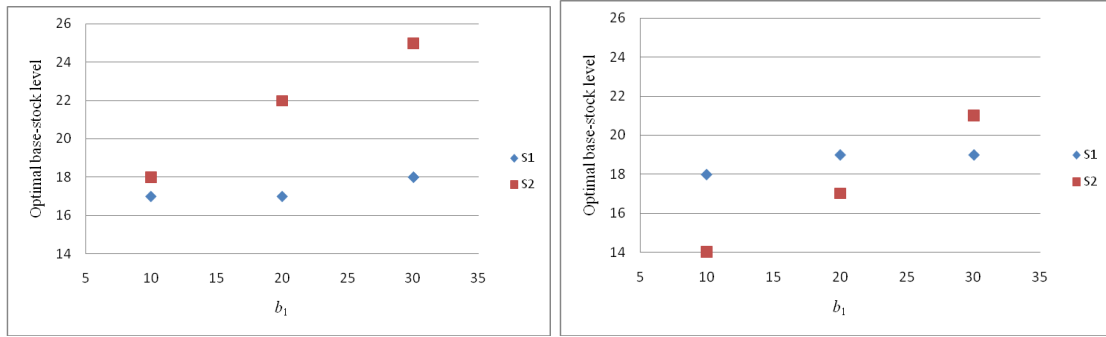
(a) $h_2 = 0.5$

(b) $h_2 = 0.9$

Fig.5. Optimal base-stock levels vs. α , when $b_1 = 20$

2.4.3. Impact of b_1 on the optimal base-stock levels

The higher backorder penalty from the customer drives the system to carry more safety stock and results in higher optimal base-stock levels for both nodes even though the impact is more pronounced on the upstream node as shown in Figure 6. The higher safety stock to protect against higher customer penalty cost results in higher total system cost.



(a) $h_2 = 0.5$

(b) $h_2 = 0.9$

Fig.6. Optimal base-stock levels vs. b_1 , when $\alpha = 0.85$

When the state space of the system increases, computational time becomes a significant issue. In order to reduce the search space and computational time, we develop two approximation methods to find a good starting solution. From the approximate solution, we can employ a steepest descent search algorithm to obtain the optimal solution instead of using exhaustive search.

3. Decentralized approximation methods

Since solving for the optimal solution for the two-node system can require a considerable amount of time, especially when the problem size grows, we present two approximation methods to finding a good starting solution and thereby reducing the search space. The approximation methods are based on a decomposition method to solve the problem sequentially by optimizing the separate single-site inventory problems (distributor, retailer) under supply risk with a modified input parameter at the upstream node, namely an implied backorder penalty (\hat{b}_2). The implied backorder penalty is needed under the decentralization scheme to force the upstream node to carry inventory to supply the downstream node demand and prevent stockouts at the downstream node. The base-stock level of the upstream node is determined based on its holding cost and its implied backorder

penalty. This base-stock level impacts the demand fulfillment capability and the replenishment lead time of the downstream node. The objective function in this decentralized system uses a transfer payment of node 2 to node 1 in terms of the stockout penalty at node 2. The following sections show how to determine the implied backorder penalty and the base-stock levels of the system under this decentralized system.

3.1. Method1: Functional form of implied backorder penalty

There are several methods to determine the implied backorder penalty when a multi-echelon problem is broken into a series of single-echelon problems (Axsäter, 2005). In this section, we seek a functional form of the implied backorder penalty (\hat{b}_2) as a function of input parameters α , h_2 , b_1 , and the CV and skewness of the demand distribution. We examine the functional form across three beta demand shapes: symmetric, right-skewed and left skewed. A total of 810 problem instances, as presented in section 2.4, were used to study the functional relationship. Those instances were solved, and the optimal base-stock levels were determined.

In this section, we first demonstrate how to obtain the functional relationship and then how to use that relationship to obtain the approximated base-stock levels. The optimal base-stock level of node 2 (S_2^*) is the key to finding the functional relationship of the implied backorder penalty. Hence, the procedure starts from optimally solving the two-node system to obtain S_2^* by minimizing the objective function in equation (9). For a given value of S_2^* in each instance, we back-solve the one-node system (Wangwatcharakul et al., 2009) to find an associated range of the backorder penalty that results in the same base-stock level S_2^* where the other input parameters are held the same. The mean value of the backorder penalty range is taken to represent the implied backorder penalty of node 2 for a given set of input parameters. Linear regression is used to capture the relationship between the input parameters and the implied backorder penalty (\hat{b}_2) to obtain the functional form in the decentralized setting.

The linear regression model shows a high degree of relationship with the coefficient of determination $R_{adj}^2 = 0.89$. This shows that 89% of the variability can be explained by the regression model. The linear regression equation generated from the 810 problem instances is

$$\sqrt{\hat{b}_2} = 5.3317 + .05413b_1 - 3.5728 \alpha - .62088 h_2 - 1.72821 \nu - .76865\Psi,$$

where ν *see DW comment about this** is the coefficient of variation and Ψ is the skewness of the demand distribution.

Figure 7 shows the relationship between the implied backorder penalty from the decentralized model and the fitted values from the linear regression model.

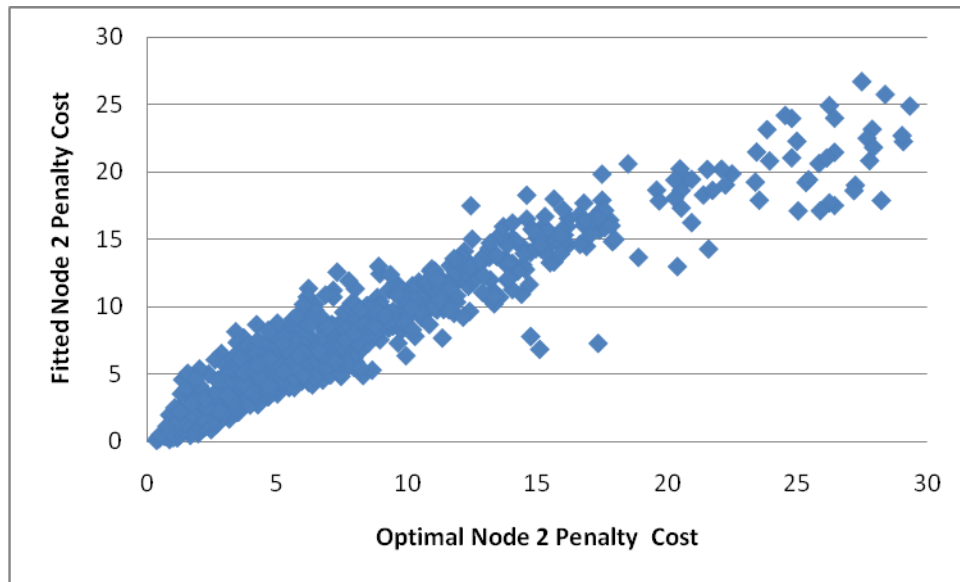


Fig.7. Optimal node 2 penalty cost vs. Fitted node 2 penalty cost

For a given set of input parameters, the procedure to determine the approximated base-stock levels are as follows.

- Determine the implied backorder penalty (\hat{b}_2) from the linear regression equation.

- The approximated base-stock level of node 2 (S_2) is determined using a single-node system (Wangwatcharakul et al., 2009) to minimize sum of holding and backorder costs of node 2.
- Since S_2 implies the demand fulfillment capability for node 1, we use the fill rate of node 2 to characterize the supply uncertainty factor (α) of the downstream node 1 which results in a stochastic replenishment lead time.
- The approximated base-stock level for node 1 (S_1) is determined to minimize the total system cost for the downstream node using the single-node system.
- The total system cost, $TC(\tilde{S}_1, \tilde{S}_2)$, is evaluated using expression (1)

We implement this method on the same set of training data with 810 instances to assess the effectiveness of this approximation method. We measure the cost deviation from the optimal solution across three shapes of the beta demand. Specifically, we compute

$$\Delta_{\%} = \frac{TC(S_1, S_2) - TC(S_1^*, S_2^*)}{TC(S_1^*, S_2^*)} \times 100\% .$$

The performance of this method is empirically evaluated through nine different sets of beta demand cases. In each beta demand case, the value of h_1 is set to 1, and a variety of problem instances are generated by selecting $h_2 \in 0.1, 0.25, 0.5, 0.75, 0.9$, $b_1 \in 10, 20, 40$, and $\alpha \in 0.7, 0.75, 0.8, 0.85, 0.9, 0.95$. Table 2 summarizes the mean, median, maximum and minimum cost deviation for a total the 90 problem instances in each of the nine demand cases.

Table 2. Summary of cost deviation from optimal (expressed as %) for method 1 for different demand parameters

Beta(a,b)	1.5,1.5	2.62,2.62	5,5	2,3	2,7.8	3,12	6,2.4	4,3.2	7,2.1
MEAN	0.34	0.15	0.17	0.51	3.81	1.47	0.55	0.18	0.67
MEDIAN	0.11	0.04	0.06	0.10	0.53	0.29	0.23	0.03	0.26
MAX	7.73	1.78	0.98	12.20	77.96	34.55	7.23	1.14	8.32
MIN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The approximation method performs well for this set of data overall. The results indicate small deviations from the optimal cost when using the median performance metric, with the largest median deviation equal to 0.53% across all demand distribution shapes. From all 810 instances, our results start to break down when α and h_2 are high for some demand parameters.

3.2. Method 2: Decentralized objective function

The approximated base-stock levels are solved sequentially from the upstream node to the downstream node for both approximation methods. In the first method, the implied backorder penalty of node 2 is determined by a functional relationship. The approximated base-stock level for node 1 (S_1) depends on S_2 and the associated fill rate. With the second method, the approximated base-stock levels for node 2 (\bar{S}_2) is determined by using the implied backorder penalty determined by finding a value such that the decentralized system cost is minimized.

Decentralized system cost (TC_D) is

$$TC_D = \sum_{i \in \Omega_1, j \in \Omega_2} (h_1 i^+ + b_1 i^-) \pi_i - (b_2 j^-) \pi_j + \sum_{j \in \Omega_2} (h_2 j^+ + b_2 j^-) \pi_j,$$

where $\Omega_1 =$ state space of node 1 = $-G_1, \dots, -2, -1, 0, 1, 2, \dots, S_1$,

and $\Omega_2 =$ state space of node 2 = $-G_2, \dots, -2, -1, 0, 1, 2, \dots, S_2$.

With this method, the system objective function is obtained from the summation of each single-node objective function with a transfer payment of node 2 to node 1 in terms of the stockout penalty. The supply uncertainty factor for node 1 is implied by the fill rate of node 2. Therefore, the optimal base-stock level for node 1 (\bar{S}_1) relies on \bar{S}_2 and the fill rate from node 2. Figure 8 depicts the relationship of the implied backorder penalty (\hat{b}_2) and the local system cost for node1, node 2, and the total system cost for the decentralized system when demand distribution is beta(2,3), $b_1 = 20$, $h_2 = 0.5$, and $\alpha = 0.85$. When \hat{b}_2 increases, node 2 cost increases, but node 1 cost decreases. The total system cost appears to be convex as shown in Figure 8.

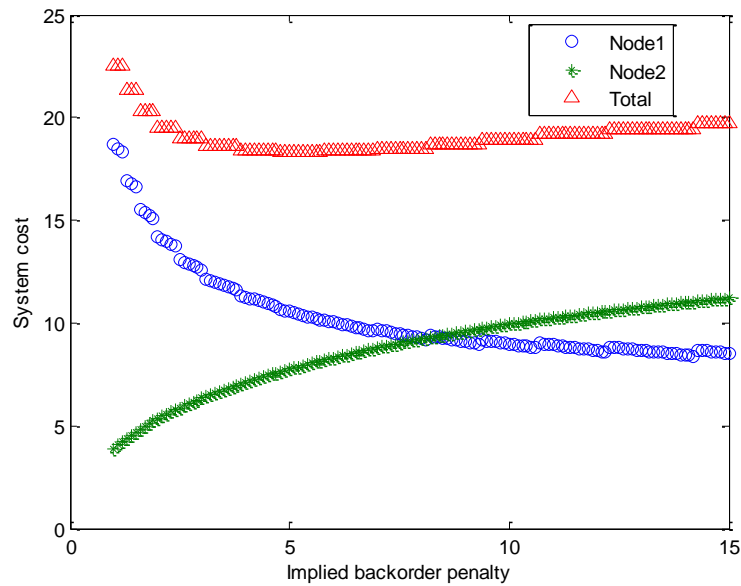


Fig.8. Total system cost for decentralized system vs. \hat{b}_2

Hence, we use an exhaustive search to find an implied backorder penalty that minimizes the total system cost. The approximated base-stock levels from the decentralized system (\bar{S}_1, \bar{S}_2) are used to evaluate the total system cost in the centralized system. This cost

represents the cost for the approximation model and then is compared with the optimal cost in terms of the % cost deviation.

To assess the effectiveness of this approximation method, we implement this method on the same set of data as in section 3.1. Table 3 summarizes the mean, median, maximum and minimum values of cost deviation for the total of 90 instances for each demand case.

Table 3. Summary of cost deviation from optimal (expressed as %) for method 2 for different demand parameters

Beta(a,b)	1.5,1.5	2.62,2.62	5,5	2,3	2,7.8	3,12	6,2.4	4,3.2	7,2.1
MEAN	0.89	0.76	0.71	1.09	1.16	1.11	1.28	0.74	1.72
MEDIAN	0.35	0.32	0.36	0.50	0.64	0.61	0.53	0.30	0.70
MAX	5.52	5.18	6.29	6.35	5.46	5.58	9.66	6.73	28.22
MIN	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

For the specified set of parameters, approximation method 2 performs worse than method 1 when we compare the mean and median cost deviation with the highest mean and median cost deviation of 1.28% and 0.7%, respectively. This could be because the testing data set is the same as the data used to obtain the functional form of implied backorder penalty in method 1. However, the worst case of method 2 is better than method 1 on average. The next section investigates the robustness of the approximation methods for some extreme demand shapes.

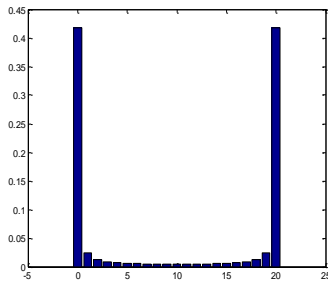
4. Experimental results

In this section, we conduct a numerical study on the effectiveness of the two approximation methods by assessing the deviation from the optimal cost through the estimation of the implied backorder penalty. We investigate the robustness of the functional form of the implied backorder penalty to a variety of extreme demand cases. We compare the results from the approximation methods with the optimal solution and present percent

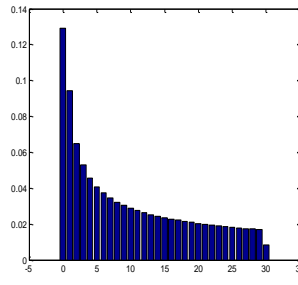
cost deviation from the optimal cost. The demand follows a discretized beta distribution with specified parameters which make the shape of the distribution close to U-shape, exponential and uniform distributions as shown in Table 4. The corresponding mass functions are plotted in Figure 9.

Table 4. Beta demand distributions used in the experiment

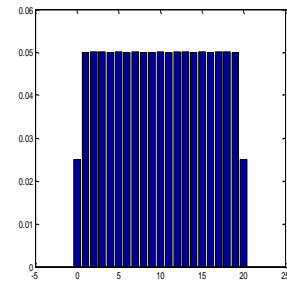
Demand	Shape	Mean	SD	CV	Skewness
Beta(0.05,0.05)	U	10	9.549	.9549	0
Beta(0.5,1)	exponential	10	8.97	.897	.635
Beta(1,1)	uniform	10	5.788	.5788	0



(a) Beta(.05,.05)



(b) Beta(.5,1)



(c) Beta(1,1)

Fig.9. Beta demand distributions used in the experiment

Other input parameters are as follows:

$$b_1=20, h_1=1, h_2 \in 0.1, 0.25, 0.5, 0.75, 0.9, \alpha \in 0.7, 0.75, 0.8, 0.85, 0.9, 0.95 .$$

The total of 30 instances were performed for each demand case. We compare mean, median, maximum, and minimum deviation from the optimal cost for each approximation method as in Table 5. The detailed results of the experiment can be found in Appendix A.

Table 5. Summary of % cost deviation from the experiment

Shape	U		exponential		uniform	
	Method1	Method2	Method1	Method2	Method1	Method2
MEAN	8.79	7.98	4.29	1.15	0.53	0.89
MEDIAN	2.61	1.21	1.85	0.85	0.21	0.69
MAX	51.7	44.61	22.06	3.19	2.11	4.77
MIN	0.00(7)	0.00(13)	0.11	0.00(1)	0.00(5)	0.00(2)

This experiment aims to demonstrate the robustness of the approximation methods for changes in the demand distribution. We use three extreme demand shapes. The median cost deviations for method 2 are low as 1.21%, .85%, and .69% for the U-shape, exponential-like and uniform-like demand distribution, respectively.

For the U-shape demand case, approximation method 2 performs better than method 1 with mean and median of 7.98% and 1.21% cost deviation, respectively. Both approximations perform best for the uniform-like distribution since the moments and shapes of the demand distribution used to generate the functional form of the implied backorder penalty in method 1 are closer to those of the uniform-like demand distribution. Approximation method 1 starts to break down when the moments of the demand distribution are significantly different from those of the data in Table 1 such as in the U-shape demand. However, the results show a small mean deviation of 2.61%. Even though there are cases of input parameters that result in high worst case percent cost deviation, the overall results observing from the mean and median are satisfied. Both methods appear to have good overall outcomes. Method 1 is, however, easier to implement in practical situations.

5. Conclusions

We develop a computational method to determine optimal base-stock levels in a two-location, periodic-review inventory system with a fixed, one-period nominal lead time to re-supply the downstream location but with an unreliable supplier to the upstream location. The

objective is to minimize the joint costs of inventory at the upstream and downstream locations and backorder penalty costs at the downstream location. The periodic nature of the inventory system and discrete distributions for customer demand and re-supply lead time allow the system to be modeled as a discrete-time Markov process (DTMP). We obtain the optimal base-stock levels for this system using an exhaustive numerical search. A numerical study is conducted to investigate the influence of supply uncertainty, holding costs, and backorder penalty costs on the optimal base-stock levels and optimal system cost.

To reduce the computational time and search space, we develop two approximation approaches to finding a good starting solution. Both approaches approximate an implied backorder penalty cost at the upstream node to force it to hold sufficient inventory to protect the downstream node from demand uncertainty. The approximation approaches are based on a decomposition method to solve separate single-site inventory problems (distributor and retailer) sequentially, but with different methods to compute the implied backorder penalty at the distributor that induces near-optimal base-stock levels at both locations. The implied backorder penalty is used to compute the approximate base-stock level at the upstream node using a single-site inventory system. The fill rate corresponding to the approximated S_2 is then used as the supply uncertainty factor in computing the downstream node base-stock level. In the first method, we use a linear regression to develop an expression for the implied backorder penalty as a function of the model input parameters. The functional form is robust with respect to changes in the shape of the demand distribution and provides a median deviation from the optimal cost of less than 1% across our computational experiments. The second method numerically searches for the implied backorder penalty that minimizes the sum of the independent costs, iteratively, at the upstream and downstream locations. Based on a numerical study, these approximation methods are found to be effective and accurate, demonstrating reasonably small deviations from the optimal cost. The solutions from the approximation methods can be used as a starting point of the search process to reduce the search space and computational time, especially when the problem size is large.

Extension of the models and decomposition-based approximation procedures to a serial system with more than two echelons is the subject of future work. The DTMP model

will likely be computationally intensive to finding the optimal base-stock levels. Hence, the approximation methods are promising to finding the approximated solution effectively. Another possible extension is to consider multiple suppliers under supply risk where one supplier is less expensive but more unreliable. In this system configuration, important operating policy issues are, for example, how to split the input stream between suppliers and under what conditions to use which supplier.

REFERENCES

- Axsäter S. and Rosling K. (1993). Installation vs. echelon stock policies for multilevel inventory control. *Management Science*, 39, 1274–1280.
- Axsäter S. (2005). A simple decision rule for decentralized two-echelon inventory control. *Int. J. Production Economics*, 93-94, 53-59.
- Chao X. and Zhou S. X. (2007). Probabilistic solution and bounds for serial inventory system with discounted and average cost. *Naval Research Logistics*, 54, 623-631.
- Chen F. and Zheng Y. S. (1994). Lower bounds for multi-echelon stochastic inventory systems, *Management Science*, 40, 1426–1443.
- Federgruen A. and Zipkin P. (1984). Computational issues in an infinite horizon multi-echelon inventory problem with stochastic demand, *Operations Research*, 32, 818–836.
- Gallego G. and Ozer O. (2005). A new algorithm and a new heuristic for serial supply systems, *Operations Research Letters*, 33, 349–362.
- Gallego G. and Zipkin P. (1999). Stock positioning and performance estimation in serial production-transportation systems, *Manufacturing & Service Operations Management*, 1, 77–88.
- Mohebbi E. (2003). Supply interruptions in a lost-sales inventory system with random lead time. *Computers & Operations Research*, 30, 411–426.
- Shang K. H. and Song J. S. (2003). Newsvendor bounds and heuristics for optimal policies in serial supply chains, *Management Science*, 49, 618–638.
- Wangwacharakul, W., King, R., Warsing. D. (2009). Optimal base-stock levels for an inventory system with imperfect supply. Working paper, North Carolina State University.

APPENDICES

APPENDIX A

The experimental results: deviation from optimal cost (expressed as %)

1. Demand follows beta(.05,.05)

1.1 Approximation method 1

	$h_2 = .1$	$h_2 = .25$	$h_2 = .5$	$h_2 = .75$	$h_2 = .9$
$\alpha = .95$	3.89	0.00	0.00	51.71	7.90
$\alpha = .9$	23.52	3.19	0.00	0.00	20.79
$\alpha = .85$	1.24	17.95	0.76	0.00	0.00
$\alpha = .8$	0.00	32.05	8.74	0.50	0.00
$\alpha = .75$	2.04	1.50	21.63	5.77	1.66
$\alpha = .7$	5.96	0.01	27.55	16.52	8.99

Mean = 8.79, Median = 2.61

1.2 Approximation method 2

	$h_2 = .1$	$h_2 = .25$	$h_2 = .5$	$h_2 = .75$	$h_2 = .9$
$\alpha = .95$	3.89	0.00	0.00	0.00	0.00
$\alpha = .9$	23.52	3.19	0.00	0.00	0.00
$\alpha = .85$	0.00	17.95	0.76	0.00	0.00
$\alpha = .8$	0.00	34.85	8.74	0.50	0.00
$\alpha = .75$	2.04	44.61	21.63	5.77	1.66
$\alpha = .7$	9.66	0.01	35.14	16.52	8.99

Mean = 7.98, Median = 1.21

2. Demand follows beta(.5,1)

2.1 Approximation method 1

	$h_2=.1$	$h_2=.25$	$h_2=.5$	$h_2=.75$	$h_2=.9$
$\alpha = .95$	0.11	3.93	13.50	22.06	17.67
$\alpha = .9$	0.27	1.50	6.96	10.32	12.11
$\alpha = .85$	0.97	0.80	4.18	5.47	5.15
$\alpha = .8$	1.52	0.60	2.01	3.49	2.67
$\alpha = .75$	0.56	1.31	1.16	2.77	1.76
$\alpha = .7$	0.25	1.23	0.89	1.77	1.93

Mean = 4.29, Median = 1.85

2.2 Approximation method 2

	$h_2=.1$	$h_2=.25$	$h_2=.5$	$h_2=.75$	$h_2=.9$
$\alpha = .95$	0.41	2.02	3.02	2.81	2.32
$\alpha = .9$	0.01	1.31	1.43	2.27	1.83
$\alpha = .85$	0.65	0.22	0.87	1.73	1.34
$\alpha = .8$	1.52	0.08	0.36	0.75	0.80
$\alpha = .75$	1.61	0.84	0.07	0.53	0.11
$\alpha = .7$	1.76	3.20	0.48	0.00	0.08

Mean = 1.15, Median = .85

3. Demand follows beta(1,1)

3.1 Approximation method 1

	$h_2=.1$	$h_2=.25$	$h_2=.5$	$h_2=.75$	$h_2=.9$
$\alpha = .95$	1.57	0.00	0.11	0.00	2.07
$\alpha = .9$	0.15	2.11	0.10	0.00	0.28
$\alpha = .85$	0.00	0.39	0.64	0.29	0.27
$\alpha = .8$	0.02	0.15	0.76	1.70	0.55
$\alpha = .75$	0.02	0.06	0.24	1.55	1.43
$\alpha = .7$	0.00	0.08	0.13	0.19	0.98

Mean = .53, Median = .21

3.2 Approximation method 2

	$h_2=.1$	$h_2=.25$	$h_2=.5$	$h_2=.75$	$h_2=.9$
$\alpha = .95$	0.00	0.11	0.85	3.31	4.77
$\alpha = .9$	0.15	0.51	0.10	1.19	1.98
$\alpha = .85$	0.02	0.08	0.64	0.13	0.14
$\alpha = .8$	0.14	0.00	0.76	0.79	0.10
$\alpha = .75$	0.63	0.29	0.74	1.55	1.43
$\alpha = .7$	1.02	1.21	0.81	1.38	2.02

Mean = .89, Median = .69

CHAPTER 5

Conclusions

In this dissertation, we have presented an extensive study of inventory systems operating under infinite-horizon, periodic-review base-stock control policies to fulfill stochastic downstream demand, but subject to imperfect (i.e., less than 100% reliable) upstream supply.

In Chapter 2, we consider a single-site inventory system serving demand that follows a general discrete distribution and served by a supplier whose replenishment lead time follows a geometric distribution, resulting from a Bernoulli trial-based model of supply uncertainty. Our model of this system can be viewed as an extension of the newsvendor model with supply risk. A discrete-time Markov process (DTMP) is introduced to solve the problem analytically. The steady state distribution of the on-hand inventory can be calculated from recursive equations given the demand distribution parameters and the probability that replenishment arrives at the end of the current fulfillment period. We present two complementary scenarios. In the first scenario, when penalty cost is specified, a critical fractile approach is used to obtain the optimal base-stock level to minimize the total system cost. For specific demand distributions we are able to develop closed-form solutions for these outcomes, and we also develop expressions for upper and lower bounds on the corresponding optimal service level. In the second scenario, when the penalty cost is not known, we seek a base-stock level that satisfies a service-level target. The classical approach relies on the lead-time demand (LTD) distribution, which may be difficult to specify, especially if one must convolve generally specified lead-time and demand distributions. A popular approximate method is to assume a Normally distributed LTD distribution. We have shown that there are deviations of the Normal approximation to the optimal result from the DTMP model under our model assumptions.

In Chapter 3, we present methods to compute the based-stock levels from approximate LTD distributions. We investigate quality of various approximate solutions

under two scenarios, when the lead time has a geometric distribution and when the lead time has any distribution. The solution methods considered are as follows: Normal approximation, geometric approximation, Normal-Geo, Gamma-Geo, Poisson-Geo, Uniform-Geo and bootstrap procedure. The DTMP-based models; Normal-Geo, Gamma-Geo, Uniform-Geo and Poisson-Geo provide optimal base-stock level where lead time has a geometric distribution and become approximations where lead time has a general distribution. In the first scenario, the DTMP model represents the reality, and is used to specify the LTD distribution and find the optimal base-stock level. We specify the parameters of the solution methods by matching the first two moments of the LTD distribution with those of the approximation methods. The Normal and gamma approximations for LTD perform well in this setting when penalty cost, b , is low relative to inventory holding cost h , i.e., when $h = 1$ and $4 \leq b \leq 16$, which corresponds to in-stock probability levels between 80% and 94%. The gamma approximation has lower deviation from optimal than the Normal approximation in most experiment cases, especially for higher LTD coefficients of variation. In the general lead-time case, as indicated above, the DTMP model becomes an alternative approximation to the optimal base-stock level, and the essence of the problem becomes accurately specifying the LTD distribution. The parameters for the approximation methods are determined by matching the moments with those of the specified LTD distribution. Among approximation methods that treat the demand and the lead time distributions independently (as opposed to seeking to generate the convolved distribution), the Normal-Geo and the Gamma-Geo perform better than the Normal and gamma approximations in most demand and lead-time scenarios used in the experiments, especially for higher LTD coefficients of variation. When the LTD is assumed to follow a discretized beta distribution, the Normal-Geo outperforms the Normal and the Gamma approximations in most cases and it performs better than the bootstrap procedure when LTD has a right-skewed distribution or a bimodal distribution with the higher peak on the left.

The results from this single-node model allow us to gain insights that serve as a building block for computing base-stock levels in more complex, multi-echelon supply chains (i.e., serial systems or assembly and distribution networks). Due to the complexity of

computing optimal inventory levels in such systems, determining good policies for the management of material flows of these types of realistic systems is an active area of research. Accordingly, in Chapter 4 we develop computational methods to determine optimal base-stock levels in a two-location, periodic-review inventory system with a fixed, one-period nominal lead time to re-supply the downstream location, but with an unreliable supplier to the upstream location.

The overall objective in the models we develop and solve in Chapter 4 is to minimize the joint costs of inventory holding at the upstream and downstream locations and backorder penalty costs at the downstream location. Due to the periodic nature of the inventory system and given discrete distributions for customer demand and re-supply lead time, the system is modeled as DTMP. We obtain the optimal base-stock levels in this system using an exhaustive numerical search and conduct a computational study to investigate the influence of the supply uncertainty factor, holding cost, and backorder penalty cost on the optimal base-stock levels and optimal system cost.

To reduce the computational time and search space in finding optimal base-stock levels, we develop two approximation approaches to finding a good starting solution by computing the implied backorder penalty cost at the upstream node that forces it to hold sufficient inventory to protect the downstream node from demand uncertainty. These decomposition-based approximation approaches solve the separate single-site inventory problems (distributor, retailer) sequentially, but with different methods for computing the implied backorder penalty at the distributor that induces near-optimal base-stock levels at both locations. In the first method, we use a linear regression to develop an expression for the implied backorder penalty as a function of the model input parameters. The functional form is robust with respect to changes in the shape of the demand distribution and provides a median deviation from the optimal cost of less than 1% across our computational experiments. The second method numerically searches for the implied backorder penalty that minimizes the sum of the independent costs, iteratively, at the upstream and downstream locations. Based on our numerical study, these approximation methods are found to be effective and accurate, demonstrating reasonably small deviations from the optimal cost.

For future research, a first step would be to extend our analysis to a serial system with more than two echelons, ultimately developing procedures to apply the decomposition-based approximation methods presented in this dissertation to that more complex problem. A DTMP-based model of this system is likely to require intensive computation to find the optimal base-stock levels. Hence, our approximation methods represent an intriguing and promising approach to finding good approximate solution. Another possible extension is to consider a two-stage assembly system under a sourcing-option strategy. In such a system, multiple suppliers—not necessarily identical in terms of price or lead-time performance would be considered for some components in the assembly in order to mitigate supply risk. This system configuration, however, requires us to determine effective operating policies for such issues as splitting the input stream between suppliers, and specifying the conditions under which to invoke use of one supplier versus the other, among many other possibilities.