DYNAMIC RESPONSE OF ELASTIC PLATES SUBJECT TO NONSTATIONARY RANDOM EXCITATION

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SUMMARY

Flat plate-like structural components of many types of reactors are often subjected to forces that vary in a random manner with respect to time. For simplicity, these random processes (pressures and forces) are often represented as stationary, i.e. statistical characteristics are invariant with time. However, a more realistic representation is to regard them as nonstationary. A few studies have previously been carried out by other investigators in an effort to determine response of linear mechanical systems to various types of nonstationary excitation, the linearity implying relatively small exciting forces. However, for plates subject to intensive dynamic forces, the plate motions are described by a pair of coupled nonlinear partial differential equations involving the lateral deflection and a stress (potential) function as unknowns. The objective of the present investigation is to present a technique applicable to prediction of structural response of nonlinear systems to nonstationary excitation. The nonlinearity considered is a geometric one due to finite amplitude responses of the plate to the exciting forces.

A moderately damped single-degree-of-freedom mechanical system subject to random excitation is considered and is described by a nonlinear second order differential equation involving system damping, natural frequency of the corresponding linear system, and a geometrically nonlinear restoring force (corresponding to large amplitudes of oscillation). The nonhomogeneous term of the equation represents the random excitation $f(t)$ which is expressed in the form $f(t)=A(t)\cdot n(t)$ where $A(t)$ is an envelope function and $n(t)$ is a Gaussian stationary random process with zero mean and auto-correlation $R_n(\tau)$. For any specified envelope function, the autocorrelation is readily determined. An approximate solution to the equation governing the system response can be obtained as the solution of a linearized equation involving an equivalent linear damping β_e and equivalent linear stiffness ω_e. This minimization leads to a nonlinear algebraic equation which is solved by an iteration technique. The case of a cubic-type nonlinearity (representing middle surface stretching of a plate oscillating with large amplitudes) is treated in detail. It is determined that for white noise excitation modulated by a unit step function, the transient mean-square response never exceeds the stationary response. However, the mean-square response to correlated noise modulated by a unit step function may exceed its stationary value if the power spectral density of $n(t)$ has a sharp peak. It is also demonstrated that the mean-square response of the nonlinear system may be significantly greater than the corresponding linear system response. Another significant result is that the equivalent linear damping coefficient and the equivalent linear stiffness for the nonstationary random excitation are found to be identical to those found by previous investigators for the stationary process. The present investigation represents the first rigorous application of equivalent linearization to nonstationary nonlinear systems.

In summary, a technique has been developed for prediction of statistical characteristics of nonlinear response of flat plate-like structural elements subject to nonstationary random excitation. (Sponsored by AFOSR Grant 72-2340).

1. INTRODUCTION

The transient mean-square response of a linear single-degree-of-freedom mechanical system to certain types of nonstationary random excitation has been studied by many authors [1,2,3,4]. The nonstationary input was taken in the form of a product of a well-defined envelope function, A(t) and a stationary Gaussian noise with zero mean, n(t).

Caughey and Stumpf [1] have examined the case in which the envelope function A(t) is a unit step function and n(t) is assumed to be either white noise or broad-band noise whose power spectral density has no sharp peaks. Results of their analysis were applied to the determination of the structural response to earthquake ground motion. Bolotin [2] has determined the mean-square response of a structure represented by a second order differential equation to earthquake excitation. In his analysis, he considered the ground acceleration to be characterized by the product of an exponentially decaying harmonic correlation function and an envelope function, $A(t) = Ae^{-ct}$.

In a recent paper [3], Barnoshi and Maurer have formulated the time varying mean-square response of a linear single-degree-of-freedom system in terms of the system frequency response function and the generalized spectral density function of the input excitation. They considered the envelope function to be either the unit step function or a rectangular step function. Bucciarelli and Kuo [4] have recently obtained an approximate expression for the mean-square response to excitation characterized by a general envelope function subject only to the restriction that the envelope function is slowly varying. Their work also gave an estimated maximum value of the mean-square response.

In all the above studies, the systems treated were linear. The present study presents an approximate solution to the problem of transient mean-square response of a simple nonlinear system to a nonstationary random excitation. Only systems with geometric nonlinearities (rather than materials nonlinearities) are considered and the nonlinear differential equation is linearized by an equivalent linearization technique. The results are directly applicable to determination of response of an elastic flat rectangular or circular plate subject to random lateral loading when the plate is approximated as a single-degree-of-freedom system characterized by its central deflection. Obviously the results also apply to other one-degree-of-freedom mechanical systems excited by nonstationary random forces.

2. ANALYSIS

Consider a lightly damped single-degree-of-freedom mechanical system subjected to a random excitation and governed by the equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2(y(t) + g(y)) = f(t) \tag{2.1}$$

where

ζ = fraction of critical damping

ω_n = natural frequency of the corresponding linear system

$$g(y) = \sum_{k=1}^{N} \mu_k y^{2k+1} \qquad \mu_k \geq 0 \tag{2.2}$$

The nonstationary random excitation f(t) is expressed by

$$f(t) = A(t)n(t) \tag{2.3}$$

where A(t) is a well-defined envelope function and n(t) is a Gaussian stationary random process with zero mean and autocorrelation function $R_n(\tau)$.

We are to determine the mean-sqaure response $E[y^2(t)]$ to an input f(t) when the envelope function is a unit step function:

$$A(t) = u(t) \tag{2.4}$$

and n(t) has the autocorrelation functions

$$R_n(\tau) = 2\pi K_o \delta(\tau) \tag{2.5}$$

Although various methods can be applied to determine the response of nonlinear systems, the equivalent linearization technique will be used here. This technique was developed by Krylov and Bogoliuvov for the treatment of nonlinear systems under deterministic excitations, and then Booton [5] and Caughey [6] applied this technique to problems of random vibrations.

We assume that an approximate solution of (2.1) can be obtained from the linearized equation

$$\ddot{y} + 2\beta_e \dot{y} + \omega_e^2 y = f(t) \tag{2.6}$$

where β_e is the equivalent linear damping coefficient and ω_e^2 is the equivalent linear stiffness. The error "e" due to linearization is given by the difference between (2.1) and (2.6), i.e.,

$$e = 2(\zeta\omega_n - \beta_e)\dot{y} + (\omega_n^2 - \omega_e^2)y + g(y)\omega_n^2 \tag{2.7}$$

The variables β_e and ω_e^2 are chosen so as to minimize the mean-square error $E[e^2]$. The resulting values involve $E[y^2]$, $[\dot{y}^2]$ and $E[y\dot{y}]$, hence it is necessary to know the probability density function $p(y,\dot{y})$. In general, however, $p(y,y)$ is not known. If the input is Gaussian and the nonlinearities of the system are small, then the response of the linearized equation (2.6) is also Gaussian. Therefore, the assumption is made that the probability density function $p(y,\dot{y})$ is Gaussian with covariances to be determined. Before constructing the probability density function, however, it is necessary to find ensemble averages of y and $\dot{y}$ and these are readily found by Duhanel's integral to be

$$E[y] = \int_0^t h(t-\tau)E[f(\tau)]d\tau \tag{2.8}$$

and if we assume that $E[f(t)] = 0$, then

$$E[y] = 0 \tag{2.9}$$

Similarly, the ensemble average of y is obtained as

$$\begin{aligned} E[\dot{y}\,] &= \int_0^t \frac{\partial}{\partial t} h(t-\tau)E[f(\tau)]d\tau \\ &= 0 \end{aligned} \tag{2.10}$$

Thus, the assumed Gaussian probability density function $p(y,\dot{y})$ takes the form

$$p(y,\dot{y}) = \frac{1}{2\pi(\det(K))^{\frac{1}{2}}} \exp(-ay^2 + 2by\dot{y} - c\dot{y}^2) \tag{2.11}$$

where

$$\begin{aligned} a &= E[\dot{y}^2]/(2\det(K)) \\ b &= E[y\dot{y}]/(2\det(K)) \\ c &= E[y^2]/(2\det(K)) \\ \det(K) &= E[y^2]E[\dot{y}^2] - (E[y\dot{y}])^2 \end{aligned} \tag{2.12}$$

This leads to

$$2\beta_e = 2\zeta\omega_n \tag{2.13}$$

$$\omega_e^2 = \omega_n^2 \{1 + \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!}(E[y^2])^k \tag{2.14}$$

It is interesting to observe that the above equivalent linear damping $2\beta_e$ and stiffness ω_e^2 are identical to those found for a stationary process in which case $E[y\dot{y}]$ is equal to zero.

If the nonlinearity is involved only in the velocity term such as $g(\dot{y})$ it is possible to demonstrate that the equivalent linear damping and stiffness for a nonstationary process are identical to those for a stationary process.

In all cases, the mean-square response $E[y^2]$ at any time t is obtained from the expected value of $(y(t))^2$ over the ensemble response. Use of Duhamel's integral leads to

$$E[y^2] = \frac{1}{\omega_d^2} \int_0^t \int_0^t \exp\{-\zeta\omega_n(2t-\tau-\tau')\}\sin\omega_d(t-\tau')\sin\omega_d(t-\tau)A(\tau)A(\tau')R_n(\tau-\tau')d\tau d\tau' \tag{2.15}$$

where

$$\omega_d^2 = \omega_n^2 \{1 + \sum_{k=1}^{N} \mu_k \frac{(2k+1)!}{2^k k!} (E[y^2])^k - \zeta^2\} \tag{2.16}$$

If the input is white noise, then (2.15) becomes

$$E[y^2] = \frac{\pi K_o}{\omega_d^2} \int_0^t \exp\{-2\zeta\omega_n(t-\tau)\} A^2(\tau)\sin^2\omega_d(t-\tau)d\tau \tag{2.17}$$

For the unit step envelope function defined by (2.4), we have

$$\begin{aligned} E[y^2] &= \frac{\pi K_o}{\omega_d^2} \int_0^t \exp\{-2\zeta\omega_n(t-\tau)\}\sin^2\omega_d(t-\tau)d\tau \\ &= \frac{\pi K_o}{4\zeta\omega_n(\zeta^2\omega_n^2+\omega_d^2)} [1 - e^{-2\zeta\omega_n t}(1 + \frac{2\zeta^2\omega_n^2}{\omega_d^2}\sin^2\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin^2\omega_d t)] \end{aligned} \tag{2.18}$$

Employing (2.16), (2.18) becomes a nonlinear algebraic equation for $E[y^2]$ since ω_d^2 is a function of $E[y^2]$ in (2.16). This type of equation generally has more than one solution. However, from physical considerations the desired solution will be that one close to the solution of the corresponding linear system because only a weakly nonlinear system is being considered. It is convenient to solve this system through use of Newton's method of tangents together with iteration. As an example, let us consider the case

$$g(y) = \mu y^3 \tag{2.19}$$

For various values of μ and damping coefficient ζ, $E[y^2]$ is computed and the normalized plots such as shown in Figure 1 result. The normalization factor is determined by the stationary mean-square response of the linear system

$$E[y_o^2] = \pi K_o/4\zeta\omega_n^3 \tag{2.20}$$

The parameter μ is chosen in such a manner that given μ the stationary mean-square response reaches 40 percent, 60 percent and 80 percent of $E[y_o^2]_s$.

If the damping is small, (2.18) can be approximated by

$$E[y^2] \approx E[y_o^2]_s \frac{(1 - e^{-2\zeta\omega_n t})}{1 + 3\mu E[y^2]} \tag{2.21}$$

from which the following approximate solution is obtained:

$$E[y^2] \approx \frac{1}{6\mu}\{[1 + 12\mu E[y_o^2]_s(1 - e^{-2\zeta\omega_n t})]^{\frac{1}{2}} - 1\} \tag{2.22}$$

From the above results it is easy to demonstrate that the transient mean-square response for both linear and nonlinear systems does not exceed the stationary mean-square response to white noise.

If instead of (2.5) the function n(t) has the autocorrelation

$$R_n(\tau) = K_o \exp(-\alpha|\tau|)\cos\beta\tau \tag{2.23}$$

(termed correlated noise) then (2.15) becomes

$$E[y^2] = \frac{K_o e^{-2\zeta\omega_n t}}{\omega_d^2} \int_0^t\int_0^t \exp[\zeta\omega_n(\tau+\tau') - \alpha|\tau-\tau'|]A(\tau)A(\tau')\sin\omega_d(t-\tau)$$

$$\sin\omega_d(t-\tau')\cos\beta(\tau-\tau')d\tau d\tau' \tag{2.24}$$

Use of (2.4) in (2.24) leads to the desired expression for $E[y^2]$. In the interest of brevity this lengthy expression is not given. However, inspection of it reveals that the mean-square response depends upon interrelationships between damping ζ, the corresponding linear system natural frequency ω_n, the decay constant and the frequency β of the correlation function.

For a white noise input, only the value of damping of the system influences the time required to attain stationarity. However, for the correlated noise input, the time required for the response to reach a stationary value is influenced not only by the system damping coefficient ζ but also by the decay constant α of the input noise. Inspection of results indicates that as α decreases, i.e. the power spectral density has a sharp peak at some frequency, then the transient response tends to exceed the stationary value. Another interesting result is that the nonlinear response becomes greater than the corresponding linear response under certain conditions even if the system has hardening spring-type nonlinearity. An example of this is shown in Figure 2.

3. CONCLUSIONS

The time varying mean-square response of a nonlinear single-degree-of-freedom mechanical system to nonstationary random excitation characterized by the product of an envelope function and a stationary Gaussian random process has been considered. A unit step envelope function was considered in conjunction with both correlated and white noise with zero mean. The nonlinear governing equation was linearized by the method of equivalent linearization.

For the nonstationary process it has been shown that the equivalent linear damping coefficient and the equivalent linear stiffness for the system with nonlinearities involved only in displacements or only in velocities are the same as those for the stationary process.

The mean-square response depends upon the coefficients of the system equation, the shape of the envelope function, and the parameters of the autocorrelation of the process n(t). It was proved that for white noise modulated by a unit step function, the transient mean-square response never exceeds the stationary response. However, the mean-square response to correlated noise modulated by a unit step function may exceed its stationary value, especially when the power spectral density of the process n(t) has a sharp peak, and its maximum value becomes several times the stationary value.

It has also been shown that the mean-square response of the system with cubic hardening spring-type nonlinearity may be greater than the corresponding linear system response under certain conditions.

The results presented are directly applicable to determination of response of a single-degree-of-freedom mechanical system to nonstationary random excitation. In the case of a flat rectangular plate subject to random lateral loading the response at the center of the plate is frequently the one of greatest interest and in this case the vibrating plate may be represented as a one-degree-of-freedom system.

REFERENCES

1. CAUGHEY, T. K., and STUMPF, H. J., "Transient Response of a Dynamic System Under Random Excitation," Journal of Applied Mechanics, 28, 4, 1961.
2. BOLOTIN, V. V. "Statistical Methods in Structural Mechanics," Translated by S. Aroni, Holden-Day, Inc., San Francisco, 1969.
3. BARNOSKI, R. L., and MAURER, J. R., "Mean-Square Response of Simple Mechanical Systems to Nonstationary Random Excitation," Journal of Applied Mechanics, 36, 2, 1969.
4. BUCCIARELLI, L. L. and KUO, C., "Mean Square Response of a Second-Order System to Nonstationary Random Excitation," Journal of Applied Mechanics, 37, 3, 1970.
5. BOOTON, R. C. "The Analysis of Nonlinear Control Systems with Random Inputs, Proceedings of the Symposium on Nonlinear Circuit Analysis, Vol. II, 1953.
6. CAUGHEY, T. K. "Equivalent Linearization Techniques," Journal of the Acoustical Society of America, 35, 11, 1963.

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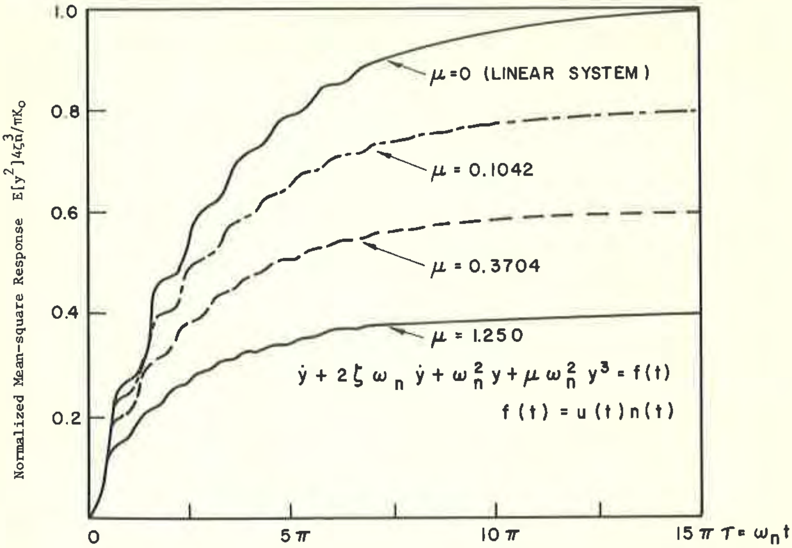

Figure 1: Mean-square Response of the Nonlinear Systems with various Nonlinearities to White Noise Modulated by a Unit Step Function. System Damping $\zeta = 0.05$

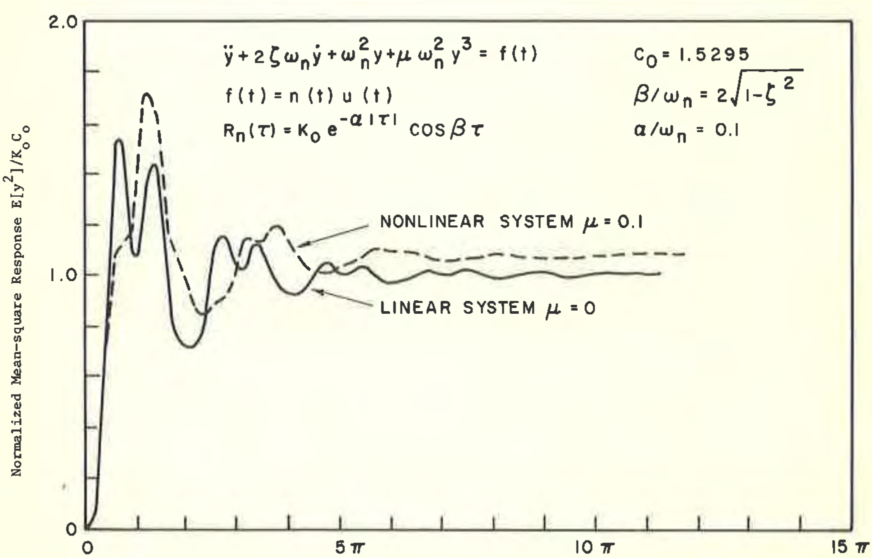

Figure 2: Mean-square Response of the Systems to Correlated Noise modulated by a Unit Step Function. System Damping $\zeta = 0.1$