G21/4

Boundary Element Solution of Cylinder with Surface Crack

LEE Chun-yu
Shijiazhuang Railway Institute, Shijiazhuang, China

1 INTRODUCTION

Cylinder of circular cross-section are widely used in engineering structures. Surface cracks often exist on the cylinders, because of the reason of manufacturing or utilizing. A lot of disastrous failures were caused by the growth of surface cracks on cylinders. Therefore, accurate stress analyses of such surface-cracked cylinders are needed for reliable prediction of their fracture strengths and controlling their crack size. Eficient analytical method to this complicated 3-D crack problems are not available, therefore, numerical methods must be used. Considering the advantage of Boundary Element Method in reducing dimensions of problem and work load, this paper studied the semi-elliptical crack on cylinder of circular cross-section by using BEM.

2 CRACK ELEMENTS

For a linear elastic material, the displacement field and the stess field near the crack front can be described through a power series $r^{1/2}, r, r^{2/3}, \ldots$ and $r^{-1/2}, r^0, r^{1/2}, \ldots$ respectively, i.e.,

$$\left.\begin{aligned} u_i &= a_{1i} r^{1/2} + a_{2i} r + a_{3i} r^{3/2} \\ t_i &= b_{1i} r^{-1/2} + b_{2i} + b_{3i} r^{1/2} \end{aligned}\right] \quad (1)$$

In order to simulate the behavior shown in (1), this paper used the traction singular mid-side-node crack elements near the crack front(Fig. 1). Three independent sets of shape funtions N, N_d and N_t, for interpolating geometry, displacement and traction quantities respectively are chosen for this kind of crack element. Only were eight-noded quadrilateral crack elements used in this paper, their geometrical shape functions N are exactly the same as usual quadrilateral quadratic

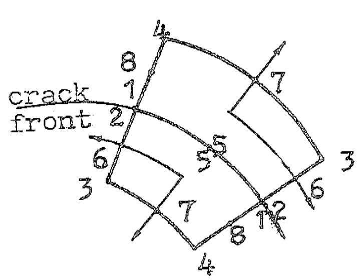

Fig.1 Crack Elements

isoparametric elements, the displacement shape functions N_d are derived by a simple change of coordinates:

$$N_{di}(\xi) = N_i(\xi') \tag{2}$$

where

$$\xi_1' = \xi_1, \quad \xi_2' = -1 + \sqrt{2(1+\xi_2)} \tag{3}$$

The traction shape functions N_t can be obtained by multiplying the N_d by a factor $T(\xi) = \sqrt{2}/\sqrt{1+\xi_2}$, that is

$$N_{ti}(\xi) = N_{di}(\xi) \cdot T(\xi) \tag{4}$$

By using above interpolating functions N_d and N_t, the displacement and traction components are expressed as

$$\left.\begin{aligned} u_i &= a_{1i} r^{1/2} + a_{2i} r \\ t_i &= b_{1i} r^{-1/2} + b_{2i} + b_{3i} r^{1/2} \end{aligned}\right] \tag{5}$$

Equation (5) guarantees that u_i and t_i show the expected behavior as $r \rightarrow 0$, althrough they are somewhat different from equation (1).

Because of the introduction of the singular factor of $1/\sqrt{1+\xi_2}$ in N_t, following coordinate transformation are necessary whenever an integration is performed over such a special crack element:

(1) The source point is not on the crack element. In this case let

$$\xi_1 = \xi_1^*, \quad 1+\xi_2 = (1+\xi_2^*)^2/2 \tag{6}$$

(2) The source point is on the crack element. In this case, above mentioned coordinate transformatin should be made in the basis of solving the singular integration over usual quadrilateral eight-noded isoparametric elements (Li et al. 1985). Expressed as:

$$\eta_1 = \eta_1^*, \quad 1+\eta_2 = (1+\eta_2^*)^2/2 \tag{7}$$

3 STRESS INTENSITY FACTOR COMPUTATION

According to LEFM, following relations between displacements near the crack front and stress intensity factor(SIF) can be obtained, when we only consider the displacements on the crack surface, i.e.

$$\begin{bmatrix} u_n \\ u_z \\ u_t \end{bmatrix} = \frac{\sqrt{2r}}{G\pi} \begin{bmatrix} K_{II}(1-\nu) \\ K_I(1-\nu) \\ K_{III} \end{bmatrix} \tag{8}$$

where u_n, u_z, u_t are displacement components with respect to intrinsic coordinates at the crack front. The above formulas show that SIF K_I, K_{II}, K_{III} are only related to u_z, u_n, u_t respectively. Take account of the value of r (in the vicinity of the crack front), the displacement of any point on the crack surface must be expressed by the nodal displacements of the crack element with its shape functions, i.e.

$$\begin{bmatrix} u_n \\ u_z \\ u_t \end{bmatrix} = \sum_{i=1}^{8} \begin{bmatrix} N_{di} u_n^{(i)} \\ N_{di} u_z^{(i)} \\ N_{di} u_t^{(i)} \end{bmatrix} \tag{9}$$

First, suppose crack surface is the symmetric plane of a crack problem, thus the displacements of the point at the crack front must be zero, i.e., $u_n^{(i)} = u_z^{(i)} = u_t^{(i)} = 0$, $(i=1,5,2)$. For simplicity, only u_z is considered,

$$u_z = \sum_{i=1}^{8} N_{di} u_z^{(i)} = \sqrt{1+\xi_2} \cdot u_z' \tag{10}$$

where

$$\begin{aligned} u_z' = {} & \frac{1}{4}(1+\xi_1)\sqrt{2}(\xi_1 - 2 - \sqrt{2}) u_z^{(3)} + \frac{1}{4}(1-\xi_1)[-\sqrt{2}(\xi_1 + 2 + \sqrt{2})] u_z^{(4)} \\ & + \frac{1}{2}(1+\xi_1)(2+\sqrt{2}) u_z^{(6)} + \frac{\sqrt{2}}{2}(1-\xi_1^2) u_z^{(7)} + \frac{1}{2}(1-\xi_1)(2+\sqrt{2}) u_z^{(8)} \end{aligned} \tag{11}$$

In the above formula, the second order $(1+\xi_2)$ has been neglected. If crack surface is not the symmetric plane of the problem, the displacement $u_z^{(i)}$ in formula (11) should be replaced by the half of the difference between displacements on upper crack surface and counterpart on the lower crack surface.

Second, it is supposed that the contour of crack is an ellipse (Fig.2, special case: circle) which can be described by parametric equations $x=c\cos\theta$, $y=a\sin\theta$. Then introduce polar coordinates on the crack(the center of elliptic crack as the origin, elliptic crack surface as $\rho-\theta$ plane), so any point on the crack can be written as

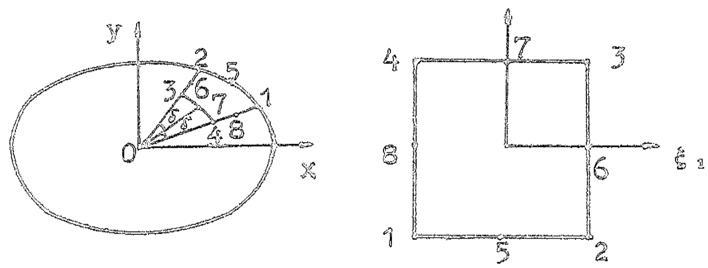

Fig. 2 Crack Surface And Arbitrarily Chosen Crack Element

$$\left.\begin{array}{l} x=\rho a\cos\theta,\ 0\leqslant\rho\leqslant1 \\ y=\rho b\sin\theta,\ 0\leqslant\theta\leqslant2\pi \end{array}\right] \tag{12}$$

we arbitrarily choose a crack element in crack surface (Fig. 2), those eight-nodal coordinates are

$$\left.\begin{array}{ll} \rho_1=1 & \theta_1=\theta_0-\delta \\ \rho_2=1 & \theta_2=\theta_0+\delta \\ \rho_3=1-L & \theta_3=\theta_0+\delta \\ \rho_4=1-L & \theta_4=\theta_0-\delta \\ \rho_5=1 & \theta_5=\theta_0 \\ \rho_6=1-L/2 & \theta_6=\theta_0+\delta \\ \rho_7=1-L & \theta_7=\theta_0 \\ \rho_8=1-L/2 & \theta_8=\theta_0-\delta \end{array}\right] \tag{13}$$

where L is the size of the crack element along ρ-coordinate, $0<L<1$. From equation (13) and shape function N, we can obtain the coordinates of any point in the crack element:

$$\left.\begin{array}{l} \rho=\sum_{i=1}^{8}\rho_i N_i=1-\dfrac{L}{2}(1+\xi_2) \\ \theta=\sum_{i=1}^{8}\theta_i N_i=\theta_0-\xi_1\delta \end{array}\right] \tag{14}$$

Let $A^2=c^2\cos^2\theta+a^2\sin^2\theta$, $r_1^2=\rho^2(c^2\cos^2\theta+a^2\sin^2\theta)$, hence

$$1+\xi_2=\frac{2(A-r_1)}{AL} \tag{15}$$

Substitute expression (15) and expression (10) into expression (8), we obtain

$$K_I=\frac{E\pi}{2(1-\nu^2)\sqrt{AL}}\cdot\frac{\sqrt{A-r_1}}{\sqrt{r}}\cdot u_z' \tag{16}$$

In the above formula term $\dfrac{\sqrt{A-r_1}}{\sqrt{r}}$ can be determined by taking into account geometric relation (Luo et al. 1988), see Fig. 3.

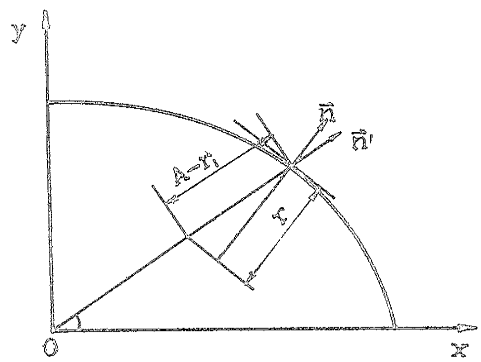

Fig. 3 Geomitric Relation

$$\frac{A-r_1}{r}=\frac{1}{\cos(n,\ n')}=\frac{a\ b}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)(a^2\cos^2\theta+b^2\sin^2\theta)}} \tag{17}$$

Substitute expression (17) into expression (16), then

$$K_{I}=f(\theta)\cdot u_z' \tag{18}$$

where

$$f(\theta)=\frac{E\pi\ \sqrt[4]{b^2\cos^2\theta+a^2\sin^2\theta}}{2(1-\nu^2)\sqrt{abL}},\ \theta=\theta_0-\xi_1\delta \tag{19}$$

Similarly, following formulas can be obtained:

$$K_{II}=f(\theta)\cdot u_n' \tag{20}$$

$$K_{III}=f(\theta)\cdot u_t' \tag{21}$$

where

$$u_n'=\frac{1}{4}(1+\xi_1)\sqrt{2}(\xi_1-2-\sqrt{2})u_n^{(3)}+\frac{1}{4}(1-\xi_1)[-\sqrt{2}(\xi_1+2+\sqrt{2})]u_n^{(4)}$$
$$+\frac{1}{2}(1+\xi_1)(2+\sqrt{2})u_n^{(6)}+\frac{\sqrt{2}}{2}(1-\xi_1^2)u_n^{(7)}+\frac{1}{2}(1-\xi_1)(2+\sqrt{2})u_n^{(8)} \tag{22}$$

$$u_t'=\frac{1}{4}(1+\xi_1)\sqrt{2}(\xi_1-2-\sqrt{2})u_t^{(3)}+\frac{1}{4}(1-\xi_1)[-\sqrt{2}(\xi_1+2+\sqrt{2})]u_t^{(4)}$$
$$+\frac{1}{2}(1+\xi_1)(2+\sqrt{2})u_t^{(6)}+\frac{\sqrt{2}}{2}(1-\xi_1^2)u_t^{(7)}+\frac{1}{2}(1-\xi_1)(2+\sqrt{2})u_t^{(8)} \tag{23}$$

4 RESULT AND DISCUSSION

To verified the feasibility of above mentioned method and the reliability of self-made computer program, two examples are computed, which are a uniformly loaded cylinder of circular cross-section with a central penny-shaped crack and a finite-thickness plate subjected to tension with a semi-elliptical surface crack. Through error analysis, we obtained a optimized size of crack elements along its radiation from crack front, its value is L=0.16a (a is the crack depth). It have been proved that good agreement with related references (Benthem et al. 1973 and McGowan 1980) can be chieved when the optimized size was used.

Considering the supposition of plane strain is not proper to the region near the intersection of the crack and the free surface, we made a modified formula for SIF in this region:

$$K_I' = \frac{K_I}{\theta^*}[(1-\nu^2)(\theta^*-\theta)+\theta] \tag{24}$$

where $\theta^*=12.5°$ is the bound angle of the transition region.

Using this self-made progrom, semi-elliptical surface crack of cylinder was studied. Two load conditions (axial tension and pure bending); four relative crack-depth (a/R=0.05, 0.1, 0.2, 0.4) and five elliptical shapes (a/c=0.2, 0.4, 0.6, 0.8, 0.98) were computed. Obtained stress intensity magnification factor (SIMF) as a function of a/R shown in Fig. 4, where SIMF defined as

$$M = \frac{K_I(\theta)}{(p_0\sqrt{\pi a \Phi})[\sin^2\theta+(a/c)^2\cos^2\theta]^{1/4}} \tag{25}$$

where $\Phi=1+1.464(a/c)^{1.65}$ for $a/c\leqslant 1$.

Distribution of SIF along crack front shown at Fig.5.

In pure bending case, the conditions considered was the same as tension and the numerical results shown at Fig. 6 and Fig. 7.

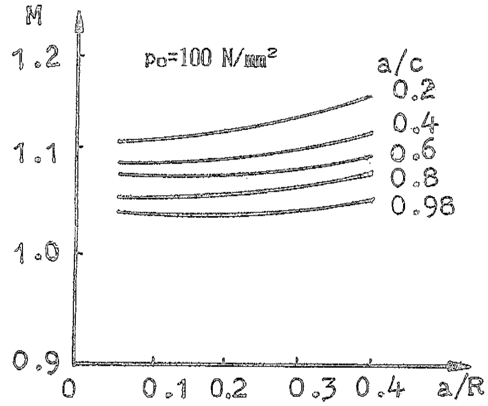

Fig. 4 SIMF as a Function of a/R
(Tension, $\phi=90°$)

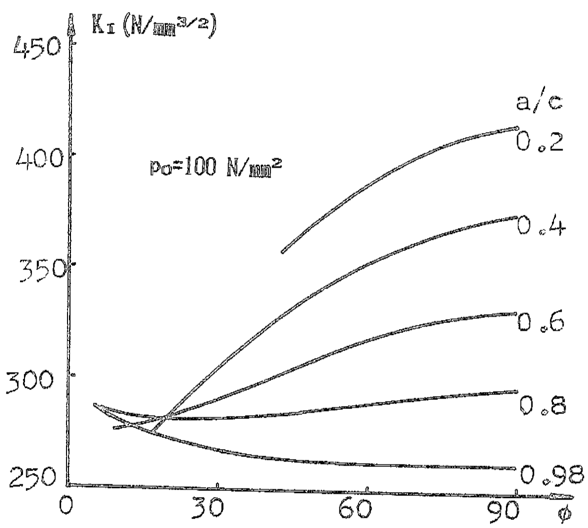

Fig. 5 Relation between K_I and ϕ
(Tension, a/R=0.1)

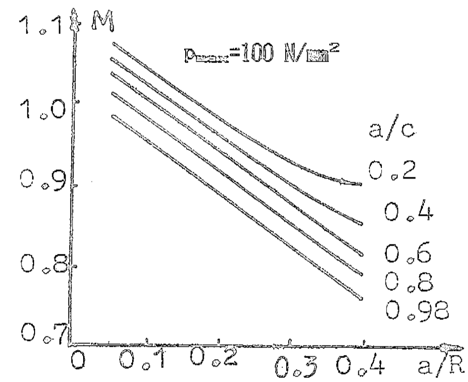

Fig. 6 SIMF as a Function of a/R
(Bending, ϕ=90°)

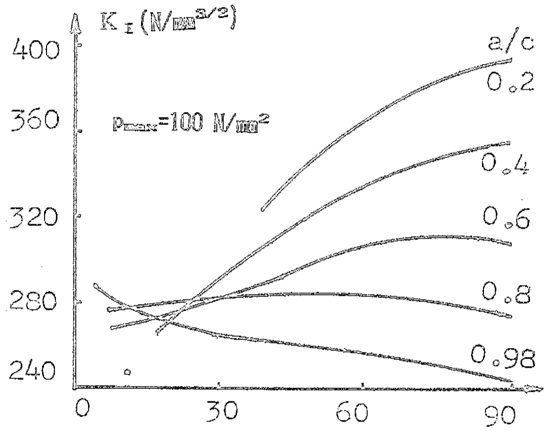

Fig. 7 Relation between K_I and ϕ
(Bending, a/R=0.1)

5 CONCLUSION

(1) The method for estimating SIF based on traction singular crack element and by introducing parametric equation of ellipse was verified to be reliable and has higher accuracy.
(2) The SIMF were mainly influenced by relative crack-depth a/R, crack depth to length ratio a/c and elliptical angle ϕ.
(3) From Fig.4 and Fig.7, it can be seen that the crack growth tend to the crack depth to length ratio a/c equal to 0.6~0.8. For shallow surface crack, the maximum SIF occurred at the deepest point, on the other hand, for close to semi-circular surface crack, the minimum SIF occurred at the deepest point. These show the same as the striped graghs of the growth of fatigue section of axle.

6 REFERENCES

Li Hong-bao and Han Guo-ming (1985). A New Method for Evaluating Singular Integrals in Stress Analysis of Solids by the Direct Boundary Element Method. Int J Num Engng, Vol. 21: 2071-2098.

Luo,G.M. and Zhang,Y.Y.(1988). Application of boundary element method with singular and isoparametric elements in three dimensional crack problems. Engng. Fracture Mech, Vol. 29: pp.97-106.

Benthem,J.P. and Koiter,W.T.(1973). Asymtotic Approxipations to Crack Problems. Mechanics of Fracture, Vol. 1: pp.131-178.

McGowan,J.J.(1980). A Critical Evaluation of Numerical Solution to the "Benchmark" Surface-flaw Problem. Experimental Mechanics, Vol. 20: pp.253-264.