Some Mathematical Aspects of Certain Wide-Sense Stationary Processes Relevant to Biology

by

Vincent Frank Gallucci

Biomathematics Program
Department of Statistics

Institute of Statistics Mimeo Series No. 749

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1. INTRODUCTION TO THE BIOMATHEMATICAL PROBLEMS

1.1 Biological Background

Some of the theory and methodology of random processes is often applied in the analysis of certain bioelectric data. In the context of applications, i.e., the analysis of data, a random process is a mathematical model, not a physical or biological reality. If a set of data can be usefully considered for some purpose, as if it came from a certain random process, it is not the same as saying that the physical or biological object involved is that random process. In fact, the physical or biological object may not even be statistical or random, it may be that the specification or analysis of the non-random system is not feasible (for example, a gas in a closed box).

The concept of the spectrum arises quite naturally in the study of a certain class of random processes and this concept will be central to this dissertation. Since there is more than one way to obtain a spectrum it is necessary to say that we are particularly concerned with spectra that arise from a Fourier series analysis, i.e., a Fourier spectrum.

Formulation of at least part of the biological problem in terms of a random process model has been useful in the study of electrical signals from the cortex, called an electroencephalogram or EEG and in the study of electrical signals from a muscle or group of muscles, called an electromyogram or EMG (Person and Libkind, 1970). The study of signals from the heart muscle has the special designation, electrocardiogram or ECG. The ECG data is especially attractive for the application of Fourier series techniques because there is a high degree of periodicity in even an abnormal signal. The issue of periodicity will be discussed later with respect to EEG signals. In fact, the remaining discussion will refer only to EEG signals and to signals related to the EEG. However, it is worthwhile to mention the paper by Attinger et al. (1966), in which the validity of the Fourier series as a mathematical technique in biological systems, other than the brain, is investigated.

Scientists first became aware that the brain emitted electricity in the 19th Century. Under the assumption that the signals contained useful information, various approaches have been employed to "decipher" the information. Visual analysis was a first and is a contemporary approach that has widespread clinical use. A visual analysis is an unaided search by eye, for certain EEG patterns that are known to indicate some pathologic condition of the central nervous system; for example, epilepsy or brain tumor.

Also in the 19th Century Joseph Fourier used the technique which bears his name; it was not too long before scientists and mathematicians were trying to use the Fourier series to quantify and analyze the EEG. The introduction of increasingly more efficient, automatic frequency analyzers from 1938 on until today, has led to an extensive literature. Prominent among the early work is that of Dietsch (1932), Koopman (1938) and Knott et al. (1942).

It soon became clear to many that the analysis of the EEG needed to take into consideration its apparent statistical properties. In any given record there are always irregularities that appear to be random and usually there are characteristics which tend to stabilize with increasing numbers of samples. So the EEG began to be regarded as a statistical thing proceeding in time, a time series. It was a short jump to do as N. Wiener suggested and to consider the EEG as a stationary random process (Wiener, 1958). The next logical step was to bring to bear the powerful mathematical tool of generalized harmonic analysis (Wiener, 1930).

There are however many objections to the use of Fourier series, spectra, and stationary random functions for the analysis of EEG data. Some of these objections are eloquently put forth by W. G. Walter (1960). The early workers thought of the EEG as a voltage-time function to which a Fourier series analysis could be applied. Implicit in the application should have been one of two assumptions, either that the EEG record is a periodic function with a certain "fundamental" period or, that beyone a certain point in the positive and negative directions of the time axis the function is zero. At best, an EEG record is partially repetitive and unless something is drastically wrong, it does not decrease to zero. The choice of a "fundamental" period is not generally justifiable and, in fact, two different choices of periods lead to two different Fourier spectra. Furthermore, until recently the use of automatic frequency analyzers implied that information about the phase of harmonics or frequency components would be discarded (Brazier, 1960; Bickford, 1960). Recent advances have remedied this problem (Kiss, 1970). As will be discussed later, the words stationary random process have a number of precise mathematical meanings depending upon the kind of stationarity of interest; but it has yet to be demonstrated that the EEG fits any of the criteria. This dissertation, however, is written with the assumption that an EEG record is describable by some wide-sense stationary random process. A considerable amount of attention will be paid to the phase of harmonics and the role of phase in the conditions for wide-sense stationarity; the question of the choice of a "fundamental" period will also be considered.

There is a type of electrical recording from the cortex for which the objections to Fourier series analysis are at least partially mitigated. Evoked response records are the result of recording the electrical responses to what are generally periodic stimuli, like successive clicks or flashes of light. The response pattern, subject to some apparent random fluctuation, is generally periodic in nature. Usually the responses are averaged to remove the randomness and as a consequence such records are called average evoked potentials (responses) or AEP (AER). The advisability of averaging is of course subject to discussion (Rosenblith, 1962; Donchin and Lindsley, 1969).

Having outlined the development of EEG analysis it would be interesting to see what the concerns of the contemporary researchers are and what new uses of spectra have been developed or are contemplated. The electroencephalographer is still concerned with voltage-time changes recorded from a single electrode on the scalp but in a more sophisticated way. Now he is studying the interaction of many partially correlated voltage-time changes recorded from electrodes dispersed not only over the surface (scalp), but in three dimensions. That is, he is putting his probes deep into the cortex where the generators of the electrical processes are.

Some progress seems to be in prospect for locating the cortical generators of certain frequency bands of the usual EEG spectrum. Larsen (1969) has subdivided the narrow band spectrum from one to twenty-five Hz into twenty-five, one Hz increments and, using a factor analysis technique, has investigated the interdependencies among the spectral amplitudes. D. O. Walter (1968) has described a method called Complex Demodulation which is an attempt to sort out some consistent uninteresting frequency components. And Bickford (1960) has suggested that the frequency-analysis system be used as part of a control system. Many applications of this kind exist, such as the automatic regulation of anesthesia to a patient.

1.2 Mathematical Aspects

Common mathematical representations of wide-sense stationary stochastic processes are by finite or infinite series or integrals of the form

$$\Sigma\, \xi_k e^{i\lambda_k t} \text{ or } \int e^{i\lambda t}\, d\xi(\lambda)$$

where the ξ_k are random variables and $\xi(\lambda)$ is a random function.

Subject to certain conditions given herein, the series is guaranteed to be wide-sense stationary; and subject to further conditions, the series is guaranteed to be a stochastic (Bohr-) almost periodic function.

It is well-known that if the series is defined with non-random coefficients ξ_k which are Fourier coefficients of some non-random function then, in some reasonable sense, the function can usually be constructed from the series. The situation is more complicated if the function is a non-trivial stochastic process, if for no other reason than that the set of all coefficients are described by a joint distribution function. The relation between the probabilistic properties of the coefficients and those of the stochastic process are studied.

Since the coefficients ξ_k are usually complex-valued random variables it is possible that the amplitude and phase of ξ_k are also random variables. The phase properties of a process are considered with respect to the criteria that need to be satisfied for the process to be wide-sense stationary.

2. GENERALITIES

2.1 Précis

Some of the mathematical aspects of this research were outlined in Chapter 1. We shall be more specific in this chapter. The theoretical framework is that of the Hilbert space.

The two realizations of the abstract Hilbert space that concern us are related to each other. One realization is a certain collection of random variables on a probability space; our specific interest being in the class of wide-sense stationary stochastic processes. The other realization is $L_2(a,b)$ space, where a and b may be finite or infinite.

The theorems in this work usually concern Fourier series-type expressions for stochastic processes, or realizations of processes or deterministic functions. Of particular interest will be the consequences of the wide-sense stationarity property, phase properties of the Fourier coefficients, uniqueness of convergence and mode of convergence of stochastic processes. The properties of the Fourier coefficients will be examined with the intention of answering questions about what the coefficients can tell us about the statistical properties of the function.

Essentially, Chapter 2 contains statements of known theorems and definitions which are fundamental to understanding the sequel. The accompanying discussion of these definitions and theorems usually is for the purpose(s) of pointing out how they fit into this work, to clarify something or to emphasize something. To include such information in the later chapters would tend to make the development somewhat disjoint.

The literature upon which this work is based is diverse and large. The Fourier series is a traditional tool in the hands of communications (and control) engineers, statisticians interested in time series, and mathematical analysts (and probabilists). To provide some perspective for the sequel, a brief review of some of the mathematical developments that influenced the contemporary view of Fourier-type series is presented as part of Section 2.2.

2.2 Mathematical Development

The Fourier series came into use prior to the twentieth century. For the moment, define the Fourier series of the non-stochastic function f(t) formally as,

$$(2.1) \qquad f(t) \sim \sum_{k=-\infty}^{\infty} c_k e^{i\lambda_k t} .$$

The function f(t) may be complex-valued and an element of $L_2(a,b)$; the c_k are a certain set of complex-valued coefficients which are called Fourier coefficients; and the set of λ_k, called Fourier exponents, are real-valued numbers. For real-valued f(t) (2.1) may be written in terms of sines and/or cosines.

It would be reasonable to inquire what it is about a series of the form of (2.1) that has made it of interest to mathematical theorists, engineers and statisticians. By attributing certain properties to f(t) and to the set $[\lambda_k]$ engineers found that the series had a natural relationship with certain fundamental concepts basic to their field, e.g. frequency bandwidth, filters, signal to noise ratio, white noise, etc. Because time series are often of a cyclic nature, statisticians developed analytical techniques such as the periodogram, which is closely tied to Fourier-type series analysis. Theorists find ample numbers of problems in pathologic, i.e., apparently physically uninteresting functions. It is interesting that N. Wiener, a competent mathematician, wrote a book (Wiener, 1949) devoted to establishing a dialogue, based upon the common foundation underlying their areas, between time series statisticians and communications engineers.

Coincidental with the development of the Fourier series was the increasing acceptance of the techniques of Oliver Heaviside which are now incorporated into what is called operational calculus (Churchill, 1958). A large part of operational calculus is based upon the theory of the Fourier integral and is denoted as Fourier transform theory. The Fourier transform of the non-stochastic function g(t) is defined formally as,

$$(2.2) \qquad \mathcal{F}\{g(t)\} \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt$$

The function g(t) may be a complex-valued element of $L_2(-\infty, \infty)$.

Equations (2.1) and (2.2) are stated as formal expressions to reflect the possibilities that the infinite series in (2.1) may not converge for any t, it may converge but not necessarily to f(t) (Hardy and Rogosinski, 1968, p. 5) or that the integral in (2.2) may not exist. Quite often (2.1) will converge and (2.2) for the same function will not exist and vice-versa.

Proceeding formally, it is possible to develop the equation in (2.2) from the Fourier series (Titchmarsh, 1948). The union of the methodologies of Fourier series and Fourier transforms is often termed spectral analysis or generalized harmonic analysis. The theory of generalized harmonic analysis is in large part the work of Wiener (1930); the objective being to achieve a unified theory, incorporating the desirable features of each constituent theory. Other attempts at synthesis are based upon extensive use of the Dirac delta function.

For our purposes it will be sufficient to note that the use of a Fourier transform implies a continuous spectrum; the existence of the integral in (2.2) often is tied to the finiteness of the integral of $|g(t)|^2$ over all t; i.e., g(t) is said to be a finite-energy function. The use of the Fourier series implies a discrete or line spectrum, the nature of which is part of this dissertation. It was stated earlier that certain properties are often attributed to f(t); one such property is periodicity. Clearly, the integral of $|f(t)|^2$ over all time will not be finite if f(t) is non-zero and periodic but the same integral over one period could be finite. The Fourier series represents f(t) over one period and by induction over all time. Therefore, it is often assumed that $f(t) \in L_2(a, a+d)$ where d is the period of f(t).

Because of the nature of Fourier series and periodic functions it is common practice to construct an artificial function $f_D(t)$, from a non-periodic f(t); i.e.,

$$\begin{aligned} f_D(t) &= f(t) & |t| &< D \\ &= 0 & |t| &\geq D \,. \end{aligned}$$

This is appealing because often one's interest in f(t) is only over some finite interval D. There are some difficulties for some uses but, frequently, the Fourier series analysis of $f_D(t)$ instead of f(t) is quite acceptable.

At first glance, the introduction of a wide-sense stationary (see Definition 2.8, below) stochastic process Z(t,ω) instead of f(t) or g(t) would not seem to cause much difficulty if one considers ω as a fixed element of an ensemble and treats Z(t,ω), now a realization of the process, as an ordinary function of time. This is what early workers thought but it is not quite correct. One might also question many operations done with respect to Z(t,ω) such as integration over ω or t, convergence of infinite sums, ergodic assumptions, etc., but many questions are resolved in a way consistent with the non-stochastic theory.

It is intuitively clear that a wide-sense stationary stochastic process cannot display the necessary trend to decrease to zero and therefore be termed a finite-energy process; hence, it is not useful to look for the Fourier transform of Z(t,ω). This dilemma is to some extent circumvented by the use of the well-known Fourier transform pair: the covariance (or autocorrelation) function and the spectral function. This pair of functions plays a central role in the analysis of wide-sense stationary processes and in the more encompassing, generalized harmonic analysis theory.

Assume for the moment that Z(t,ω) is both wide-sense stationary and periodic with period d (see Definitions 2.8 and 2.9). Then it can be demonstrated that the Fourier coefficients, found by a Fourier analysis of $Z(t,\omega)$, are random variables and that they are pairwise uncorrelated -- a desirable statistical property. Let us assume now that $Z(t,\omega)$ is a wide-sense stationary process, but not periodic, and that we are interested in the interval D which is a subset of the parameter set, T of the process. It is generally possible to express $Z(t,\omega)$ over D in a Fourier series just as $f_D(t)$ was expressed. However, Root and Pitcher (1955) demonstrate that the Fourier coefficients are then only asymptotically uncorrelated. This fact should be noted when reading the early literature.

The above change in the statistical properties of the coefficients is not an unimportant point. The primary reason for using a stochastic process as a mathematical model is because the properties of the statistical parameters can be defined and studied. Properties of these random Fourier coefficients can be investigated to develop an understanding of the stochastic process. The expectation of the square of the absolute value of a Fourier coefficient corresponds to its variance. So the problem of estimating the spectrum of the process becomes a problem of estimating variance components. If the process is assumed to be Gaussian, the Fourier coefficients are pairwise independent random variables and the process is then represented by a sum of independent random variables, a highly developed area of probability theory (Gnedenko and Kolmogorov, 1968).

We asked earlier why it was that the Fourier series find uses in several disciplines. The answers given were, at one level, satisfactory.

However, now that we have developed some insight into the nature of Fourier coefficients, we can carry the answers a bit further. It was noted that the Fourier series of a function implies a discrete spectrum; denote the number λ_k as a point in the spectrum and the square of the absolute value of the $k^{\underline{th}}$ Fourier coefficient as the spectral value corresponding to λ_k. The set of numbers $[\lambda_j]$, $j = 1, 2, \ldots$ are associated with the frequency domain. It may be said then, that the Fourier series lends itself to a frequency-domain interpretation; why estimate the spectral density instead of the covariance function? One answer is statistical simplicity: two frequencies in the spectrum, represented by two Fourier coefficients may be related in an uncorrelated or independent sense (if the appropriate conditions are satisfied); whereas two times, represented by the values of the realization are in general dependent or correlated. Another benefit of the frequency domain is in the area of filters and linear operators where the "frequency" λ_k is associated with the characteristic "frequency response" function for linear systems.

In theory an even wider class of processes than the class of wide-sense stationary processes can be expressed in terms of a series of uncorrelated random variables. Such series are called Karhunen-Loève expansions. The problem with using these expansions is that evaluation of the coefficients usually involves the solution of sometimes difficult integral equations and that all relation to the frequency domain is lost (Davenport and Root, 1958, p. 99).

We have pointed out that the frequency approach is securely built into the tradition but this does not imply that it is always the best approach. An alternate approach stated with speech analysis in mind, but with wider implications, is suggested in a paper by Bremermann (1968).

2.3 Mathematical Compendium

In order that the presentation of the new work in this dissertation be more efficient, this section is devoted to rather well-known definitions and theorems. The interrelationship between the established literature and the present study is discussed.

The analysis is all done in the context of an abstract Hilbert space H, say. Two realizations of this space are of interest to us, the space $L_2(a,b)$ and a certain class of random variables defined on the probability space $(\Omega, \mathcal{G}, P)$. These spaces are now defined:

Definition 2.1: Let (a,b) denote a finite or infinite interval on the real axis. Denote by $\underline{L_2(a,b)}$ the set of all complex-valued Lebesgue measurable functions $f(\cdot)$ defined on (a,b) such that $|f(\cdot)|^2$ is Lebesgue integrable on (a,b).

Definition 2.2: The space $\underline{(\Omega, \mathcal{G}, P)}$ is a probability space. That is, Ω is what we shall call the basic set[1], where $\omega \in \Omega$ is a possible

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[1] The basic set is often called the sample space.

outcome or event, so Ω is the collection of these; $\mathcal{A}$ is a sigma algebra of sets of ω. $P(A)$ is the probability measure of the set A, where $A \in \mathcal{A}$, $A \subset \Omega$.

<u>Definition 2.3</u>: An <u>abstract Hilbert space</u> is an infinite-dimensional inner product space. H is a complete metric space with respect to the metric generated by the inner product.

There is an extensive literature on Hilbert spaces so we refer only to a few properties and implications that are of special concern to us.

The inner product, defined on $L_2(a,b)$, of functions f and g is

$$(2.3) \qquad (f,g) = \int_a^b f(t)\overline{g(t)}dt \ .$$

The corresponding norm, defined on $L_2(a,b)$, of $f(t)$ is

$$\|f\| = \left(\int_a^b |f|^2 dt\right)^{\frac{1}{2}} \geq 0.$$

Note the following: if f and g are orthogonal, $(f,g) = 0$; the domain of definition need not be a one dimensional interval, but could be any higher dimensional interval; if f and g differ only on a set of Lebesgue measure zero they are not regarded as distinct.

The inner product of complex-valued random variables $\xi_1 = \xi_1(\omega)$ and $\xi_2 = \xi_2(\omega)$ defined on $(\Omega, \mathcal{A}, P)$ is

$$(2.4) \qquad (\xi_1, \xi_2) = \mathcal{E}\, \xi_1 \overline{\xi}_2,$$

where $\mathcal{E}(\cdot)$ is the expectation operator and $\mathcal{E}\,\xi_1 = \mathcal{E}\,\xi_2 = 0$. The norm of a random variable ξ_1 is

$$(2.5) \qquad \|\xi_1\| = (\mathcal{E}|\xi_1|^2)^{\frac{1}{2}} \geq 0.$$

Note the following: if ξ_1 and ξ_2 are uncorrelated random variables, $(\xi_1, \xi_2) = 0$; if random variables ξ_1 and ξ_2 differ only a set of probability measure zero they are said to be equivalent or non-distinct random variables and satisfy,

$$(2.6) \qquad P(\xi_1 \neq \xi_2) = 0 \quad \text{and/or} \quad \mathcal{E}|\xi_1 - \xi_2|^2 = 0.$$

The inner product, defined on H, of elements X and Y is denoted by (X,Y). Note that the space H is not in general separable. Since there is a norm topology on H there are notions of convergence in $L_2(a,b)$ and in the class of random variables defined on $(\Omega, \mathcal{G}, P)$. We shall return to questions of convergence and separability in the sequel.

A geometric conception of a stochastic process $Z(t,\omega)$ is as a family of random variables with parameter t ranging over some index set T and where $\omega \in \Omega$. For every fixed t, the random variable $Z(t,\omega)$ corresponds to definite point of the Hilbert space, H; when t varies over T, the result is a one-parameter family of points, or a curve in H. Of course, the word "curve" implies some notion of continuity (and therefore convergence).

Another way to conceive of a real-valued stochastic process is as a function $Z(t,\omega)$ defined on $T \times \Omega \to R$ such that for each $t \in T$, $Z(t,\omega)$

is a measurable map of $\Omega \to R$; i.e., for each t, $Z(t,\omega)$ is a random variable.[1] Let $\mathfrak{X}$ be the space of all real-valued functions on T and β be a suitably chosen sigma algebra of subsets of $\mathfrak{X}$, then $Z(t,\omega)$ defines a measurable map of $(\Omega, \mathcal{A})$ into $(\mathfrak{X}, \beta)$. The space $\mathfrak{X}$ is called the sample function or realization space.

Without going into the details, it is clear that there is an induced probability distribution on $(\mathfrak{X}, \beta)$. By choosing n arbitrary elements of T, $t_1, t_2, \ldots, t_n$ say, n random variables $Z(t_1,\omega)$, $Z(t_2,\omega), \ldots, Z(t_n,\omega)$, with a joint probability distribution, are defined. Thus, a family of finite-dimensional distributions might be constructed with the intention of deducing as much as possible about the induced probability distribution. Or, a family of finite-dimensional distributions may be known *a priori*; under what conditions does there exist a stochastic process associated with these distributions?

Such questions are answered by what is called the Kolmogorov theorem. Henceforth, the symbols FF-DD will frequently be used to denote the family of finite-dimensional distributions.

Theorem 2.1: (Kolmogorov) The FF-DD of any given stochastic process uniquely defines the probability distribution over the sample space $\mathfrak{X}$ for all sets of the sigma algebra β over the intervals in $\mathfrak{X}$. And, a necessary and sufficient condition for the existence of any stochastic

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[1] The usual abuse of notation which does not distinguish between the name and the value of a map will be followed.

process is that the given FF-DD satisfies the conditions of symmetry and consistency (see Definitions 2.4 and 2.5).

Proof: not given; see Cramér and Leadbetter (1967).

We shall now proceed to examine this crucial theorem. We have said that $\mathfrak{X}$ is the space of all real-valued functions $x(t)$ on T. Corresponding to any arbitrary finite set of t-values, t_1, t_2, ..., t_n, the random variables $Z(t_1,\omega)$, ..., $Z(t_n,\omega)$ are defined. These n random variables will have a joint n-dimensional distribution with distribution function

$$(2.7) \qquad F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P(Z(t_1,\omega) \le x_1, \ldots, Z(t_n,\omega) \le x_n) = P(Z(t_k,\omega) \le x_k; k = 1, 2, \ldots, n)$$

where $x_j = x(t_j)$.

Definition 2.4: The family of all joint probability distributions of the form (2.7) for all finite n and all possible values of t_k constitutes the family of finite-dimensional distributions (FF-DD) associated with the stochastic process $Z(t,\omega)$.

Definition 2.5: The symmetry condition requires that the n-dimensional distribution function (2.7) be symmetric in all pairs (x_j, t_j), so that $F(\cdot)$ remains invariant when the x_j and t_j are subjected to the same permutation.

Definition 2.6: The consistency condition is expressed by the relation

$$\lim_{x_n \to +\infty} F(x_1, \ldots, x_n; t_1, \ldots, t_n) = F(x_1, \ldots, x_{n-1}; t_1, \ldots, t_{n-1})$$

Definitions 2.5 and 2.6 are taken from Cramér and Leadbetter (1967). Any FF-DD satisfying these conditions can be associated with some stochastic process.[1] The notation used in Definitions 2.5 and 2.6 is one of a number of acceptable notations. It is used in these definitions because it closely resembles the distribution function definition occurring in ordinary probability theory. However, in Section 5.3 a different notation is suggested, advantages and disadvantages are discussed and an example is given. The essential point we want to make here is that for different selections of discrete subsets of T, the domain of the corresponding distribution function, as a function of $t_1, t_2, \ldots, t_n$, is different, and therefore the functions are themselves different.

There are many ways to categorize stochastic processes. We will distinguish between those which are stationary and those which are not. Of those that are stationary we will distinguish between two types of stationarity: wide-sense stationary, henceforth also denoted by the symbols WSS, and strict-sense stationary. In order to categorize we identify some standard characteristics that will be

[1]The usual abuse of notation is adhered to, where the same symbol is used to denote the n-dimensional distribution function and its marginal distributions.

useful. The FF-DD is certainly one method of identification. Others are the mean value function $m(t)$,

$$m(t) = \mathcal{E}\, Z(t,\omega), \tag{2.8}$$

where $\mathcal{E}(\cdot)$ denotes the expectation operator; the covariance function $\mathcal{B}(t,s)$, where $m(t) = 0$ is

$$\mathcal{B}(t,s) = \mathcal{E}\, Z(t,\omega)\, \overline{Z(s,\omega)} \tag{2.9}$$

for all $t \in T$, and $s \in T$; and the characteristic function of a random vector $\vec{X}_n = \vec{X}_n(\omega) = [X(t_1,\omega),\ X(t_2,\omega),\ \ldots,\ X(t_n,\omega)]^T$, where the components of the vectors $\vec{X}_n$ and $\vec{q}$ are real-valued, and where $[\ldots\ldots]^T$ is the transpose of $[\ldots\ldots]$, is

$$\varphi(\vec{q}) = \mathcal{E}\, e^{i\vec{q}^T\vec{X}_n}. \tag{2.10}$$

It has been pointed out that for a fixed $t \in T$, the stochastic process corresponds to a random variable; for a set of t-values a random vector is formed. Characteristic functions are used extensively in Chapter 5 because, among other reasons, they are another way to express the symmetry and consistency conditions.

Nothing explicit has been said about the index set T. Another way to categorize stochastic processes is by the properties of the set T. In this dissertation we will deal with continuous parameter processes (T is a continuous set) and discrete parameter processes (T is a set of discrete elements). In particular, if $Z(t,\omega)$ is a continuous parameter process, $T = (0, \infty) \subset R^1$, if $Z(t,\omega)$ is a discrete parameter process, $T = T^* = [1, 2, \ldots, n]$ (n will often be infinite, in which case T^* is a left closed, right open interval).

Definition 2.7: A stochastic process is strict-sense stationary if all its finite-dimensional distribution functions remain the same if the whole set of points, $t_1, t_2, \ldots, t_n$, is shifted by some finite distance τ. That is, if

$$F(x_1, x_2, \ldots, x_n; t_1+\tau, t_2+\tau, \ldots, t_n+\tau) = F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$$

for any $n, t_1, t_2, \ldots, t_n$ and τ.

Definition 2.8: A stochastic process is wide-sense stationary[†] (WSS) if $\mathcal{E}\, Z(t,\omega) = m(t) = m$, a constant and if $\mathcal{B}(t,s) = \mathcal{B}(t-s) = \mathcal{B}(\tau)$ where $\tau = t-s$, $t \in T$, $s \in T$ (it will be discussed later that, without loss of generality, we will usually assume $m(t) = 0$ in this work).

In general, the mean value function and the covariance function do not uniquely specify the process but if the process if Gaussian, then $m(t)$ and $\mathcal{B}(\tau)$ completely specify the process. That is, they completely define all the distribution functions.

In this study we are generally concerned with processes that are WSS. A subclass of the class of WSS processes consists of those which are periodic.

[†]The following terms are synonymous: second-order stationary, Khinchin-sense stationary, wide-sense stationary, covariance stationary, and weakly stationary.

Definition 2.9: (a) A purely periodic stochastic process with period d is one whose covariance function is purely periodic with period d.

(b) Another definition, perhaps easier to visualize, is, $Z(t,\omega)$ is purely periodic with period d if

$$Z(t,\omega) = Z(t+d,\omega) \ \forall \ t \in T \qquad \text{a.s.}^1$$

Parts (a) and (b) are stated as definitions but it can be shown (Dubes, 1968, p. 426) that they are related by an "if, and only if" relation.

Henceforth, such processes and functions will be denoted by p.p.(d) meaning purely periodic with period d.

The discussions in Sections 1.2 and 2.2 perhaps indicate that periodicity, or the absence of it, plays a major role in the type of analysis which can be validly done. It will be useful now to clarify something which has occurred in the literature and might be misleading.

In the book edited by Rosenblith (1962), p. 84, W. M. Siebert asserts that a p.p.(d) stochastic process is not stationary and suggests useful ways to evaluate the first two moments. The assertion is correct if strict-sense stationarity is meant but not true if WSS is meant. The former condition requires that the distribution function be independent of arbitrary shifts τ, whereas it is clear that the distribution function of a p.p.(d) process is independent only of

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[1] The symbol a.s. denotes almost surely. This is the probabilistic counterpart of a.e.

shifts that are multiples of the period. In contrast, suppose $Z(t,\omega)$ is the p.p.(d) process and that $\mathcal{E}\, Z(t,\omega) = 0$. Then

$$\mathcal{B}(\tau) = \mathcal{E}\,(Z(t+\tau,\omega)\ \overline{Z(t,\omega)}) = \mathcal{E}\,(Z(t+d+\tau,\omega)\ \overline{Z(t,\omega)})$$

$$\mathcal{B}(\tau) = \mathcal{B}\,(\tau+d)$$

where Definition 2.9 (b) is used. The result is gratifying. One, it clearly shows that the covariance function is periodic; two, $\mathcal{B}(\cdot)$ does not depend on the choice of $t \in T$, confirming that $Z(t,\omega)$ is a WSS process.

A wider class of WSS processes than that of the class of purely periodic processes is the class of almost periodic WSS processes. Henceforth, such processes (and functions) will also be denoted by a.p. An extensive literature concerned with non-stochastic a.p. functions exists (Besicovitch, 1954; Bohr, 1947; Corduneanu, 1968; Maak, 1967; Wiener, 1930). The literature for stochastic a.p. functions is less extensive (Bochner, 1956; Slutsky, 1938).

"Delta-epsilon"-type definitions of almost periodic functions and processes are given by the authors listed above and need not concern us here. It is through the following theorems that the a.p. concept will be applied.

<u>Theorem 2.2</u>: (Bohr's Fundamental Theorem) The closure $\overline{s(t)}$ of the class of all finite sums $s(t) = \sum_{1}^{N} a_n e^{i\lambda_n t}$ is identical with the class of almost periodic functions.

Proof: not given; see Bohr (1947), p. 80.

Theorem 2.3: (Approximation Theorem of Bohr) Every almost periodic function $f(t)$ can be approximated by finite sums $s(t) = \sum_1^N a_n e^{i\lambda_n t}$ uniformly for $t \in (-\infty, \infty)$, where the exponents λ_n are chosen to be Fourier coefficients of $f(t)$.

Proof: not given; see Bohr (1947), p. 81.

Theorem 2.4: A random function $Z(t)$ satisfies the definition of being an almost periodic function if, and only if, it is the limit, as $p \to \infty$, uniformly in $t \in (-\infty, \infty)$ of finite exponential sums $\sum_{n=1}^{p} a_n e^{i\lambda_n t}$ for the set of real λ_n.

Proof: not given; see Bochner (1956), p. 23.

An a.p. function or process is a generalization of a p.p.(d) function or process; i.e. all p.p.(d) functions or processes are a.p. but the converse is not true.

A related theorem (Loève, 1963; Rosenblatt, 1962; Wiener, 1930) shows that certain stochastic processes $Z(t,\omega)$ can be represented in the form of a Fourier-Stieltjes integral

$$(2.11) \qquad Z(t,\omega) = \int_{-\infty}^{\infty} e^{i\lambda t} \, d\xi(\lambda,\omega)$$

where $\xi(\lambda,\omega)$ is a process of orthogonal increments.

With the intention of making a point about the numbers λ_1, λ_2, ..., λ_n, and recalling the central role that the covariance function plays in the theory of spectra, note that Bochner (1933) and Khinchin (1934) showed that the covariance function $B(\tau)$ of any WSS process can be represented in the form of an integral,

$$(2.12) \qquad \mathcal{B}(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} dF(\lambda)$$

where $F(\lambda)$ is a real, non-decreasing, bounded function called the spectral distribution function of the stochastic process. This is referred to as the Bochner-Khinchin Theorem; a proof of which is in Rosenblatt, 1962. Based on the representation theorems of probability theory, (2.12) may also be expressed by a suitable integral over a measure (Feller, 1966).

A special case of equation (2.11), which is clearly of interest in the present work, involves the representation of certain WSS processes $Z(t,\omega)$ in the form

$$(2.13) \qquad Z(t,\omega) = \sum_{k=1}^{n} \xi_k(\omega) e^{i\lambda_k t},$$

where the set of $\xi_k(\omega)$ are complex-valued random variables, the set of λ_k are (discrete) real numbers, $t \in T$, and n may be finite or infinite (Cramér and Leadbetter, 1967; Slutski, 1938), with a covariance function of the form

$$(2.14) \qquad \mathcal{B}(\tau) = \sum_{1}^{n} b_k e^{i\lambda_n t}$$

where

$$(2.15) \qquad b_k = \mathcal{E}|\xi_k(\omega)|^2 \text{ and } \mathcal{E}\ \xi_k(\omega)\ \overline{\xi_j(\omega)} = 0$$

The expression $\sum_{k=1}^{n} \mathcal{E}|\xi_k(\omega)|^2$ is often referred to as the spectral measure of $Z(t,\omega)$. If the sum is finite and the λ_k values discrete we say that $Z(t,\omega)$ is a process with a finite total spectral measure.

In Section 3.1 we shall show the known proof of a theorem that states that the necessary and sufficient conditions for a process of the form (2.13) to be WSS is that the coefficients, $\xi_k(\omega)$, be pairwise uncorrelated (orthogonal) (also see equation (2.15)). We have pointed out previously the desirability of this property.

Reference has been made to general representations of a.p. and p.p.(d) functions both in the deterministic and stochastic contexts. The present work is basically concerned with Fourier representations so we shall summarize below the Fourier series for the four cases of interest. Certain of these will be considered in detail in the sequel.

A version of equation (2.1) where f(t) is a deterministic function is rewritten as

$$(2.16) \qquad f(t) \sim \sum_{k=-n}^{n} c_k e^{i\lambda_k t} \qquad (?)$$

We distinguish between the a.p. and p.p.(d) cases as follows:

(i) If $\lambda_k = k\lambda_1$, where $\lambda_1 = \frac{2\pi}{d}$, the λ_k are said to be harmonically related; if f(t) is a purely periodic function of period d, the series represents f(t) $\forall t$; if f(t) is not p.p.(d) but d is chosen in some way, the series may still represent f(t) but over a subinterval of t; these matters have been discussed in Section 2.2. In either case, the Fourier coefficient[1], c_k, is defined by

$$(2.17) \qquad c_k = \frac{1}{D} \int_t^{t+d} f(t) e^{i\lambda_k t} dt \qquad k = 0, \pm 1, \pm 2, \ldots;$$

(ii) If λ_k are not related as in (i) but are chosen in some other suitable way, then (2.16) is said to be an almost periodic representation of f(t) and the Fourier coefficient[1], c_k, is defined by

$$(2.18) \qquad c_k = \lim_{D\to\infty} \frac{1}{d} \int_0^D f(t) e^{-i\lambda_k t} dt \qquad k = 0, +1, +2, \ldots;$$

where λ_k are non-zero real numbers. The number n in (2.16) may be finite or infinite. Integration in (2.17) and (2.18) is in the sense of Lebesgue.

The stochastic analog of (2.16) which is used to represent certain WSS stochastic processes, $Z(t,\omega)$ is

$$(2.19) \qquad Z(t,\omega) \sim \sum_{k=-n}^{n} \xi_k(\omega) e^{i\lambda_k t} \qquad (?)$$

Again, we distinguish two cases:

(iii) If the λ_k are harmonically related and $Z(t,\omega)$ is periodic with a period d, or if d is chosen in some other suitable way, the series may still represent $Z(t,\omega)$ over a subset of T (see previous discussions in this section and in Section 2.2); then the stochastic Fourier coefficient $\xi_k(\omega) = \xi_k$ is

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[1]It would be more accurate to write Fourier-Lebesgue coefficient because of the Lebesgue integrals but in the interest of brevity we do not.

$$(2.20) \qquad \xi_k = \frac{1}{d} \int_t^{t+d} Z(t,\omega)e^{-i\lambda_k t}\,dt \qquad (?) \quad k = 0, \pm 1, \pm 2, \ldots .$$

(iv) If the λ_k are not harmonically related but are chosen in some other suitable way, then (2.19) is said to be an almost periodic representation of $Z(t,\omega)$ and the corresponding stochastic Fourier coefficient $\xi_k(\omega) = \xi_k$ is

$$(2.21) \qquad \xi_k = \lim_{D\to\infty} \frac{1}{D} \int_0^{D} Z(t,\omega)e^{-i\lambda_k t}\,dt \qquad (?) \quad k = 0, +1, +2, \ldots$$

where the λ_k are non-zero real numbers. Again n may be finite or infinite (subject to convergence criteria).

The (?) that appears in four of the last six equations reflects that the modes of convergence of the possibly infinite series and of the stochastic integrals must be carefully considered. In this work (2.20) and (2.21) will be Riemann integrals in the quadratic mean sense.

Clearly, we are using the expressions "Fourier series" and "Fourier coefficients" in a number of situations and will specify the situation when it is not clear.

Much of the preceding has revolved about the set of numbers λ_k. The same λ_k occur in the representation of the process and in its covariance function. It should be noted that all Fourier series representations have discrete λ_k values which is what is meant by a discrete spectrum. When a pair of λ_k terms are not harmonically related they are said to be incommensurate.

In the sequel the stochastic process $Z(t,\omega)$ is considered to be complex-valued and may be written

(2.22) $Z(t,\omega) = X(t,\omega) + iY(t,\omega)$.

Miller (1969) discusses the properties of the three processes, $Z(t,\omega)$, $X(t,\omega)$ and $Y(t,\omega)$. Discussions up to now have merely specified (when necessary) whether $Z(t,\omega)$ was complex-valued or real-valued ($Y(t,\omega) = 0$ a.s.).

The stochastic process, say $Z(t,\omega)$, under consideration will be assumed to be of zero mean value without any loss of generality, i.e.

$$m(t) = \mathcal{E}\, Z(t,\omega) \equiv 0 \qquad \forall\ t \in T,\ \omega \in \Omega$$

This is a standard assumption in the literature. The book by Dubes (1968) does not make the assumption; the book by Loève (1963) discusses the assumption in some detail.

Some abbreviations and symbols have been introduced as this chapter developed. These are now collected for reference in the sequel. A complete list of abbreviations and symbols with the page number of first occurence precedes Chapter 1.

WSS	wide-sense stationary
FF-DD	family of finite-dimensional distributions
p.p.(d)	purely periodic of period d
a.p.	almost periodic
T	continuous parameter set
T*	discrete parameter set
s.p.(s)	stochastic process(es)
r.v.(s)	random variable(s)
$\mathfrak{X}$	space of all real-valued functions on T or T*

$\bar{x}$	complex conjugate of x
(x_n)	sequence of terms, x_n
$\{x_n\}$	discrete parameter s.p.; x_n is a r.v. for each $n \in T^*$
$[x_n]$	set of terms x_n
A^T	transpose of matrix A

Some of the footnotes which appear in this work occur because in the discussion a name was used, whereas there are other names equally acceptable and well known, which have the same meaning. To some extent this situation reflects the discussion of Section 2.2, where it was pointed out that engineers, statisticians and mathematicians have contributed, more or less in conjunction, to the development of stochastic processes. A partial list of such terms which appear frequently follows:

1. Ω: sample space; basic set; basic probability space
2. wide-sense stationary (WSS); covariance stationary; second-order stationary; Khinchin stationary; weakly stationary
3. T: index set; parameter set
4. Stochastic process; random function
5. quadratic mean; mean square
6. almost surely (a.s.); almost certainly
7. sample function; realization

3. A GENERAL SERIES REPRESENTATION

3.1 WSS and the Properties of the Coefficients

Some of the properties of finite and infinite sum representations of stochastic processes will be presented in this chapter. In preceding chapters the motivation for representing stochastic processes by a sum of terms was largely historical; i.e., it was a logical extension of the work done on deterministic problems, albeit capable of being expressed in the firm mathematical structure of Hilbert space theory.

It is possible to motivate the series representation of a WSS process expressed in terms of the orthogonal system, $e^{i\lambda_k t}$, $k \varepsilon T^*$ in a different way. Consider the s.p. as a function of two variables, $t \varepsilon T$ and $\omega \varepsilon \Omega$, and try as a reasonable analytical expression of their interrelation, the product of a random and a deterministic function, say

$$Z(t,\omega) = \xi(\omega) f(t).$$

By imposing upon this model the conditions necessary for $Z(t,\omega)$ to be WSS ($\mathcal{E}\, Z(t,\omega) = 0$, $\mathcal{B}(t,s) = \mathcal{B}(\tau)$, t and s both $\varepsilon\, T$), it can be shown (Yaglom, 1962, p. 31) that $Z(t,\omega)$ is WSS if, and only if, it is of the form

$$Z(t,\omega) = \xi(\omega) e^{i\lambda_1 t}.$$

Then, apply the principle of superposition as a possibly valid and meaningful way to represent a more general WSS s.p., say,

$$Z(t,\omega) = \sum_{k=1}^{n} \xi_k(\omega) e^{i\lambda_k t}.$$

With this type of approach one eludes the need to specify how the set of coefficients, $[\xi_k]$, and the set of exponents, $[\lambda_k]$ are determined. We have, in Chapter 2, specified some ways to construct both sets.

Theorem 3.1 given below is from Yaglom (1962). We state the theorem because it is fundamental to many theorems that follow and we give the proof (a paraphrasing of that in Yaglom, 1962) because the method employed will also be used later in this work.

Theorem 3.1: The stochastic process $Z(t,\omega) = \sum_{k=1}^{n} \xi_k(\omega)e^{i\lambda_k t}$, where n is finite, is zero mean WSS if, and only if, the coefficients $\xi_k(\omega)$ are pairwise uncorrelated random variables with the mean value zero.

Proof: (Yaglom, 1962, p. 38) By Definition 2.8, if $\mathcal{E}\, Z(t) = 0$ and if $\mathcal{B}(t,s) = \mathcal{E}\, Z(s)\overline{Z(t)}$ is a function of $\tau = t-s$ only, then Z(t) is a WSS s.p. Let us calculate $\mathcal{B}(t,s)$ but substituting $(t+\tau)$ for s and let us consider the case n = 2. The remark at the end of the proof will show the result for a general n.

$$\mathcal{E}\, Z(t+\tau)\overline{Z(t)} = \mathcal{E}[(\xi_1 e^{i\lambda_1(t+\tau)} + \xi_2 e^{i\lambda_2(t+\tau)})(\overline{\xi}_1 e^{-i\lambda_1 t} + \overline{\xi}_2 e^{-i\lambda_2 t})]$$

$$\mathcal{E}\, Z(t+\tau)\overline{Z(t)} = \mathcal{E}[\xi_1\overline{\xi}_1 e^{i\lambda_1\tau} + \xi_2\overline{\xi}_2 e^{i\lambda_2\tau} + \xi_1\overline{\xi}_2 e^{it(\lambda_1-\lambda_2)} e^{i\lambda_1\tau} + \xi_2\overline{\xi}_1 e^{it(\lambda_2-\lambda_1)} e^{i\lambda_2\tau}]$$

Then, by the linearity of the expectation operator,

$$(*) \qquad \mathcal{E}\, Z(t+\tau)\overline{Z(t)} = \mathcal{E}|\xi_1|^2 e^{i\lambda_1\tau} + \mathcal{E}|\xi_2|^2 e^{i\lambda_2\tau} + \mathcal{E}\,\xi_1\overline{\xi}_2 e^{it(\lambda_1-\lambda_2) + i\lambda_1\tau} + \mathcal{E}\,\xi_2\overline{\xi}_1 e^{it(\lambda_2-\lambda_1)+i\lambda_2\tau}$$

Suppose that $Z(t,\omega)$ is zero mean and WSS. In order for expression (*)

to be a function of τ only, say $\mathcal{B}(\tau)$, those terms of the sum which are functions of t must either cancel or be zero. Since $e^{i\lambda_k t} \neq 0$ for any values of λ_k or t and since in general, these terms do not cancel, (Remark 6 on Theorem 3.2 contains the essential relation to establish this from a linear independence argument), it follows that $\mathcal{E}\, \xi_2 \overline{\xi}_1 = \mathcal{E}\, \xi_1 \overline{\xi}_2 = 0$. That is, ξ_1 and ξ_2 are pairwise uncorrelated random variables. Furthermore, if $\mathcal{E}\, Z(t,\omega) = 0$, then

$$\mathcal{E}\, Z(t,\omega) = e^{i\lambda_1 t}\, \mathcal{E}\, \xi_1(\omega) + e^{i\lambda_2 t}\, \mathcal{E}\, \xi_2(\omega) = 0$$

implies that $\mathcal{E}\, \xi_k(\omega) = 0$, $k = 1, 2, \ldots$; that is, $\xi_k(\omega)$ are zero mean r.v.s.

The converse, if ξ_1 and ξ_2 are zero mean, (pairwise) uncorrelated r.v.s, then $Z(t,\omega)$ is WSS and zero mean follows immediately. The extension from n = 2 to a general n is straightforward (see Remark 1 on Theorem 3.1).

Q.E.D.

<u>Remark 1 on Theorem 3.1</u>: If $Z(t,\omega)$ is a zero mean WSS s.p. and is represented by a sum of n terms of the form $\xi_k e^{i\lambda_k t}$, $k = 1, 2, \ldots, n$, the expression (*) in Theorem 3.1 is replaced by

$$\mathcal{E}\, Z(t+\tau)\, \overline{Z(t)} = \sum_{j=1}^{n} \sum_{k=1}^{n} \mathcal{E}\, \xi_k \overline{\xi}_j\, e^{-i\lambda_j t + i\lambda_k (t+\tau)}$$

This sum may be written as

$$\sum_{k=1}^{n} |\xi_k|^2 e^{i\lambda_k \tau}$$

by the same arguments used in the theorem. For <u>any</u> finite n, Theorem 3.1 is established.

Remark 2 on Theorem 3.1: It will not be proven here but intuitively it is reasonable that Theorem 3.1 will be valid for a non-finite n and a suitable mode of convergence. In fact, some of the Fourier series expressions in Section 2.3 are examples of special cases of infinite series.

3.2 Real-valued, WSS Processes and the Properties of the Coefficients

In Theorem 3.1 the coefficient $\xi_k(\omega)$ is a complex-valued r.v. It is always possible then to express ξ_k as,

$$(3.1) \qquad \xi_k(\omega) = \tfrac{1}{2}\,(\alpha_k(\omega) - i\beta_k(\omega)) \qquad k = 1, 2, \ldots, n,$$

where $\alpha_k = \alpha_k(\omega)$ and $\beta_k = \beta_k(\omega)$ are real-valued r.v.s.

In analysis the complex-valued s.p. $Z(t,\omega)$ is often used but in practice, one observes only real-valued processes, $X(t,\omega)$. The next theorem states the conditions that the elements of the sum,

$$(3.2) \qquad Z(t,\omega) = \sum_{j=1}^{n} \xi_j e^{i\lambda_j t}$$

must satisfy in order that $Z(t,\omega)$ be real-valued (n, finite or infinite).

The method of proof and the statement of the conditions as being necessary and sufficient are perhaps original but certainly the conditions are not. For the non-stochastic case, the conditions are usually given in the context of expressing a complex-valued Fourier series in terms of a real-valued Fourier series involving sine and cosine functions.

Theorem 3.3 will consider the second order properties that the coefficients α_k and β_k possess.

Theorem 3.2: Let $Z(t,\omega)$ be a zero mean, WSS stochastic process[1] which can be represented for all $t \in T$ in the form of (3.2) where the λ_j are distinct, non-zero, real-valued numbers, the $\xi_j(\omega)$ are distinct, non-zero, complex-valued random variables expressable as in (3.1), and n is a finite, even positive integer. A necessary and sufficient condition that $Z(t,\omega)$ be real-valued is that the indices be assignable to ξ_k and λ_k in such a way that $\lambda_{2k} = -\lambda_{2k-1}$ and $\xi_{2k}(\omega) = \bar{\xi}_{2k-1}(\omega)$ for $k = 1, 2, \ldots, n/2$.

Proof: For convenience, let $Z(t,\omega) = Z(t)$, $\xi_k(\omega) = \xi_k$, $\alpha_k(\omega) = \alpha_k$, and $\beta_k(\omega) = \beta_k$. Let $n = 4$ for computational ease; we shall see that this does not affect the generality of the proof. Substitution of ξ_k in $Z(t)$ yields

$$
\begin{aligned}
Z(t) = \tfrac{1}{2} [&(\alpha_1 - i\beta_1)(\cos \lambda_1 t + i \sin \lambda_1 t) + \\
&(\alpha_2 - i\beta_2)(\cos \lambda_2 t + i \sin \lambda_2 t) + \\
&(\alpha_3 - i\beta_3)(\cos \lambda_3 t + i \sin \lambda_3 t) + \\
&(\alpha_4 - i\beta_4)(\cos \lambda_4 t + i \sin \lambda_4 t)] \quad \forall\, t \in T \text{ and a.s.}
\end{aligned}
$$

[1]The possibility that $Z(t,\omega) = 0 \ \forall\, t$, a.s. is excluded.

or

$$(*) \quad Z(t) = \tfrac{1}{2}\,[(\alpha_1 \cos\lambda_1 t + \beta_1 \sin\lambda_1 t + \alpha_2 \cos\lambda_2 t + \beta_2 \sin\lambda_2 t) + (\alpha_3 \cos\lambda_3 t + \beta_3 \sin\lambda_3 t + \alpha_4 \cos\lambda_4 t + \beta_4 \sin\lambda_4 t)] + \tfrac{i}{2}\,[(\alpha_1 \sin\lambda_1 t - \beta_1 \cos\lambda_1 t + \alpha_2 \sin\lambda_2 t - \beta_2 \cos\lambda_2 t) + (\alpha_3 \sin\lambda_3 t - \beta_3 \cos\lambda_3 t + \alpha_4 \sin\lambda_4 t - \beta_4 \cos\lambda_4 t)] \quad \forall\ t \text{ and a.s.}$$

For sufficiency, show that if $\xi_1 = \bar{\xi}_2$ and $\xi_3 = \bar{\xi}_4$, $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$, then Z(t) is real-valued. From $\xi_k = \frac{1}{2}(\alpha_k - i\beta_k)$, the following identification scheme is constructed for k = 1, 2, 3, 4:

$\alpha_1 = \alpha_2 = \eta_1$, $\alpha_3 = \alpha_4 = \eta_2$, $\beta_1 = -\beta_2 = \zeta_1$, and $\beta_3 = -\beta_4 = \zeta_2$, $\lambda_1 = -\lambda_2 = \kappa_1$, and $\lambda_3 = -\lambda_4 = \kappa_2$. Substitution of η_j, ζ_j, κ_j, j = 1, 2 into (*) yields the real-valued expression (the imaginary terms cancel),

$$Z(t) = [\eta_1 \cos\kappa_1 t + \zeta_1 \sin\kappa_1 t + \eta_2 \cos\kappa_2 t + \zeta_2 \sin\kappa_2 t] \quad \forall\ t \in T \text{ and a.s.}$$

which demonstrates that the condition is sufficient for Z(t) to be real-valued.

For necessity, show that if the imaginary part of expression (*) is zero, then $\xi_1 = \bar{\xi}_2$, $\xi_3 = \bar{\xi}_4$, $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$. The imaginary part of expression (*) set equal to zero yields

$$(**) \quad \tfrac{1}{2}\,[(\alpha_1 \sin\lambda_1 t - \beta_1 \cos\lambda_1 t + \alpha_2 \sin\lambda_2 t - \beta_2 \cos\lambda_2 t) + (\alpha_3 \sin\lambda_3 t - \beta_3 \cos\lambda_3 t + \alpha_4 \sin\lambda_4 t - \beta_4 \cos\lambda_4 t)] = 0 \quad \forall\ t \in T \text{ and a.s.}$$

For any $1 \leq k \leq n$ $(=4)$, $\alpha_k = |\xi_k| \cos \Psi_k$ and $\beta_k = -|\xi_k| \sin \Psi_k$, where $\Psi_k = \Psi_k(\omega)$ is the angle, arc tan $\left(\frac{-\beta_k}{\alpha_k}\right)$. Substituting into (**),

$$[(|\xi_1| \cos \Psi_1 \sin \lambda_1 t + |\xi_1| \sin \Psi_1 \cos \lambda_1 t +$$

$$|\xi_2| \cos \Psi_2 \sin \lambda_2 t + |\xi_2| \sin \Psi_2 \cos \lambda_2 t) +$$

$$(|\xi_3| \cos \Psi_3 \sin \lambda_3 t + |\xi_3| \sin \Psi_3 \cos \lambda_3 t +$$

$$|\xi_4| \cos \Psi_4 \sin \lambda_4 t + |\xi_4| \sin \Psi_4 \cos \lambda_4 t)] =$$

$$[|\xi_1| (\cos \Psi_1 \sin \lambda_1 t + \sin \Psi_1 \cos \lambda_1 t) +$$

$$|\xi_2| (\cos \Psi_2 \sin \lambda_2 t + \sin \Psi_2 \cos \lambda_2 t) +$$

$$|\xi_3| (\cos \Psi_3 \sin \lambda_3 t + \sin \Psi_3 \cos \lambda_3 t) +$$

$$|\xi_4| (\cos \Psi_4 \sin \lambda_4 t + \sin \Psi_4 \cos \lambda_4 t)] =$$

$$(\dagger) \qquad [|\xi_1| \sin (\lambda_1 t + \Psi_1) + |\xi_2| \sin (\lambda_2 t + \Psi_2) +$$

$$|\xi_3| \sin (\lambda_3 t + \Psi_3) + |\xi_4| \sin (\lambda_4 t + \Psi_4)] = 0 \quad \forall\ t \in T \text{ and a.s.}$$

By taking successive derivatives of (†) the numbers λ_k, to increasing powers, will appear as coefficients. The derivatives are taken with respect to t and since the functions are random functions, we specify that the derivatives are defined in the quadratic mean sense. Some factors relevant to taking the q.m. derivatives are given in Remark 5. For computational purposes we will take six derivatives of (†); if the zeroth derivative (the functions themselves) are counted, seven derivatives are taken. In general, then, (2n-1) derivatives must be taken. Rewriting (†),

$$\sum_{k=1}^{4} |\xi_k| \sin (\lambda_k t + \Psi_k) = 0 \quad \forall\ t \in T \text{ and a.s.}$$

The seven derivatives may be expressed by the set of equations,

$$\sum_{k=1}^{4} \lambda_k^{\,j} \, |\xi_k| \, \{\sin (\lambda_k t+\Psi_k)\}^{(j)} \qquad j = 0, 1, 2, 3, \ldots, 6 \quad \forall\, t \in T$$

and a.s. Consider the four equations in which the λ_k have exponents 0, 2, 4, 6 and note that the set of equations is valid for all t. Therefore it is no essential specialization to consider the set of equations for t = 0. Now, let $x_k = -\beta_k = |\xi_k| \sin \Psi_k$ so that the set of equations is

$$\sum_{k=1}^{4} \lambda_k^{\,j} x_k = 0 \quad \text{a.s.}, \; j = 0, 2, 4, 6$$

These four equations correspond to four hyperplanes in x-space. It follows that the random vector $\vec{X} = (x_1, x_2, \ldots, x_4)$ is almost surely in the intersection of the four planes. If the coefficient matrix of the four equations is non-singular the intersection consists of one point $\vec{X} = (0, 0, 0, 0)$ which is excluded by the hypothesis. Therefore, the determinant of the coefficients is zero. That is,

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^{2} & \lambda_2^{2} & \lambda_3^{2} & \lambda_4^{2} \\ \lambda_1^{4} & \lambda_2^{4} & \lambda_3^{4} & \lambda_4^{4} \\ \lambda_1^{6} & \lambda_2^{6} & \lambda_3^{6} & \lambda_4^{6} \end{bmatrix} = 0$$

This is in the form of the well-known Vandermonde determinant (Smirnov, 1964). Therefore,

$$(\dagger *) \quad (\lambda_1^{2} - \lambda_2^{2})\,(\lambda_1^{2} - \lambda_3^{2})\,(\lambda_1^{2} - \lambda_4^{2})\,(\lambda_2^{2} - \lambda_3^{2})$$
$$(\lambda_2^{2} - \lambda_4^{2})\,(\lambda_3^{2} - \lambda_4^{2}) = 0$$

There are many combinations of λ_k values that will satisfy (†*).

Five combinations are listed below as representative of the different possible cases. We then indicate that three of these should be excluded because they contradict the hypothesized condition that the λ_k be distinct.

(1) $\lambda_1 = +\lambda_2$ and $\lambda_3 = +\lambda_4$ $\ni$ $\lambda_1 \neq \lambda_3$ .....excluded

(2) $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$ $\ni$ $\lambda_1 \neq \lambda_3$

(3) $\lambda_1 = +\lambda_2$ and $\lambda_3 = -\lambda_4$ $\ni$ $\lambda_1 \neq \lambda_3$ .....excluded

(4) $\lambda_1 = \pm\lambda_2$ and λ_3, λ_4 arbitrary

(5) $\lambda_1 = \lambda_2 = \pm\lambda_3$ and λ_4 arbitrary .....excluded

Since no special conditions are associated with any k-value, it is acceptable to say that showing the consequences of e.g., case (4), is equivalent to showing the consequences of $\lambda_3 = -\lambda_4$, λ_1, λ_2 arbitrary. We call this a symmetry argument.

A brief discussion of the excluded cases may be found in Remark 2. In what follows we will use the result: given a sum of sinusoids, e.g., $a_1 \sin \varphi_1 t + a_2 \sin \varphi_2 t = 0$ a.s. and $\forall$ t, where $0 \neq \varphi_1 \neq \varphi_2 \neq 0$, it follows that $a_1 = a_2 = 0$. Remark 6 is devoted to a discussion of this result.

Case (2) ($\lambda_1 = -\lambda_2$, $\lambda_3 = -\lambda_4$):

Substituting into (**),

$$[\alpha_1 \sin \lambda_1 t - \beta_1 \cos \lambda_1 t - \alpha_2 \sin \lambda_1 t - \beta_2 \cos \lambda_1 t +$$

$$\alpha_3 \sin \lambda_3 t - \beta_3 \cos \lambda_3 t - \alpha_4 \sin \lambda_3 t - \beta_4 \cos \lambda_3 t] =$$

$$[|\delta_1| (\cos \theta_1 \sin \lambda_1 t + \sin \theta_1 \cos \lambda_1 t) +$$

$$|\delta_3| (\cos \theta_3 \sin \lambda_3 t + \sin \theta_3 \cos \lambda_3 t)] =$$

$$[|\delta_1| \sin (\lambda_1 t + \theta_1) + |\delta_3| \sin (\lambda_3 t + \theta_3)] = 0 \quad \forall\ t, \text{ a.s.}$$

where, $\delta_1 = |\delta_1(\omega)| = +[(\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2]^{\frac{1}{2}}$ and $\delta_3 = |\delta_3(\omega)| = [(\alpha_3 - \alpha_4)^2 + (\beta_3 + \beta_4)^2]^{\frac{1}{2}}$ and θ_1 and θ_2 are the angles corresponding to δ_1 and δ_3, respectively.

The implication is that $\delta_1 = \delta_3 = 0$ a.s. (Remark 6). It follows that $\alpha_1 = +\alpha_2$, $\beta_1 = -\beta_2$ and $\alpha_3 = +\alpha_4$, $\beta_3 = -\beta_4$ a.s. When substituted into $\xi_k = \frac{1}{2}(\alpha_k - i\beta_k)$, the result is $\xi_1 = \bar{\xi}_2$ and $\xi_3 = \bar{\xi}_4$. Substitution of these values of the complex coefficients in equation (3.2) where n is an even integer now, yields the real-valued process

$$Z(t) = (\alpha_1 \cos \lambda_1 t + \beta_1 \sin \lambda_1 t + \alpha_3 \cos \lambda_3 t + \beta_3 \sin \lambda_3 t) \quad \forall\ t \text{ and a.s.}$$

<u>Case (4) ($\lambda_1 = \pm\lambda_2$ and λ_3, λ_4 arbitrary):</u>

$\lambda_1 = +\lambda_2$ is excluded by hypothesis. For $\lambda_1 = -\lambda_2$ substitution into (**) yields,

$$[(\alpha_1 \sin \lambda_1 t - \beta_1 \cos \lambda_1 t - \alpha_2 \sin \lambda_1 t - \beta_2 \cos \lambda_1 t) +$$

$$(\alpha_3 \sin \lambda_3 t - \beta_3 \cos \lambda_3 t + \alpha_4 \sin \lambda_4 t - \beta_4 \cos \lambda_4 t)] =$$

$$[|\delta_1| \sin (\lambda_1 t + \theta_1) + \alpha_3 \sin \lambda_3 t - \beta_3 \cos \lambda_3 t +$$

$$\alpha_4 \sin \lambda_4 t - \beta_4 \cos \lambda_4 t] = 0 \quad \forall\ t \text{ and a.s.}$$

where $\delta_1 = +[(\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2]^{\frac{1}{2}}$ and θ_1 is the appropriate angle associated with δ_1.

Again, the implication is that $\delta_1 = 0$, but now, $\alpha_3 = \beta_3 = \alpha_4 = \beta_4 = 0$ also. It follows that $\xi_3 = \xi_4 = 0$ which contradicts the non-zero condition of the hypothesis. Hence, case (4) is not possible.

Of the five cases considered that satisfy the determinant, only case (2) is neither excluded by the conditions of the hypothesis on the exponents nor leads to excluded conditions in the hypothesis on the coefficients. By the symmetry argument, equally valid are the remaining relations, $\lambda_\ell = -\lambda_m$, $\ell \neq m$, etc.

Case (4) contained arbitrary exponents λ_3 and λ_4. By the symmetry argument, any arbitrary exponents will lead to excluded results.

This completes the necessity part of the proof for n = 4. It has been indicated throughout that n can be any even number. More than five cases will result if n > 4. These cases are always selected by examination of how the appropriate Vandermonde determinant can be made equal to zero, subject to the condition that expressions of the form of (**) are also satisfied. Therefore, the selection of n = 4 was made with no essential loss of generality.

Q.E.D.

Remarks on Theorem 3.2:

1. It is repeated again that the theorem is proven for the class of WSS process that can be expressed in the form of (3.2), where the only conditions imposed on the series are those required for the sum to be a WSS process.

2. It is possible to argue that the hypothesis of the theorem is too rigid in requiring that n be an even number and that the λ_k and ξ_k be a.s. non-zero and distinct. Of course, some of these conditions are interrelated: if $\xi_k = 0$ a.s. or $\xi_k = \xi_j$ a.s., $k \neq j$ then (3.2) with an even n could be written with an odd n. It can be shown that using the techniques of the proof any of the forementioned conditions could be relaxed with no detriment to the conclusions.

3. In analogy with Remark 2 on Theorem 3.1, Theorem 3.2 could be shown for a non-finite n and a suitable mode of convergence.

4. In the standard Fourier series treatment of the $\left(e^{i\lambda_k t}\right)$ base set the coefficients and exponents are numbered: $k = \ldots -n, \ldots -1, 0, +1, \ldots, +n, \ldots$. Then, if the process (or function) is real-valued, $\xi_k = \overline{\xi}_{-k}$ and $\lambda_k = -\lambda_{-k}$ $\forall\ k \in T^*$. The concept of negative frequencies is not a problem, but a discussion may be found in Stuart (1961), p. 48.

5. In the necessity part of the proof repeated q.m. derivatives with respect to t are taken of the expression,

(i) $$\sum_{i=1}^{4} \xi_i \sin(\lambda_i t + \Psi_i) = 0 \quad \forall\ t \text{ and a.s.}$$

Clearly, if the first order q.m. derivative with respect to t of

(ii) $$\xi_1 \sin(\lambda_1 t + \Psi_1)$$

exists, the first order q.m. derivative of (i) exists. The derivative will, as one might expect, be

(iii) $$\lambda_1 \xi_1 \cos(\lambda_1 t + \Psi_1),$$

which also has a q.m. derivative (a sine function), and so on.

Therefore, that the derivative of (ii) is (iii) is sufficient to demonstrate that $n^{\underline{th}}$ order q.m. derivatives of (i) exist.

The second order s.p. $A(t,\omega)$ is said to be differentiable in the q.m. sense if $\underset{h\to 0}{\text{l.i.m.}}\ \frac{1}{h}\,[A(t+h,\omega)-A(t,\omega)]$ exists $\forall (t,t+h) \subset T$. We write then,

$$A'(t) = \underset{h\to 0}{\text{l.i.m.}}\ \frac{1}{h}\,[A(t+h) - A(t)] = \underset{h\to 0}{\text{l.i.m.}}\ Q(t,h)$$

which is equivalent to writing $\lim_{h\to 0} \mathcal{E}|A'(t) - Q(t,h)|^2 = 0$.

The derivative may be evaluated directly or by the limit of the generalized second derivative of the covariance function of $A(t,\omega)$ (Loève, 1963, p. 470). We do not do the algebra here. The derivative of $\xi_1(\omega) \sin (\lambda_1 t+\Psi_1)$ is as one might expect, $\lambda_1 \xi_1(\omega) \cos (\lambda_1 t+\Psi_1)$.

It follows from the preceding discussion that the derivative of (i) exists to arbitrary order.

That the derivative exists for all $t \in T$ and a.s. is based upon the general theory of q.m. calculus. Lukacs (1968) in the proof of the corollary to theorem 5.3.1 explains that $\forall\, t \in T$ is required essentially because the increment $(t,t+h)$ is a subset of T. The a.s. statement follows from a theorem (Lukacs, 1968, p. 37) that if a sequence of r.v.s converges in q.m. to two r.v.s, then the two r.v.s are equivalent (equal a.s.).

6. Consider the following equation:

(i) $\quad a_1 \sin \theta_1 t+a_2 \sin \theta_2 t = 0 \quad \forall\, t \in T$ and a.s.

where θ_1 and θ_2 are distinct non-zero real numbers and a_1 and a_2 are random variables. We assert that $a_1 = a_2 = 0$, a.s. Rather than actually proving the assertion three related procedures are indicated. This is useful because these techniques are a good demonstration of the inner consistency of this dissertation and of the nice properties of the Hilbert space structure.

Consider the space $L_2(0, 2\pi)$, the element $0 \in L_2(0, 2\pi)$, and the orthogonal base $(\sin \theta_k t)$ in $L_2(0, 2\pi)$. It is known that in the Euclidian n-space R^n, every vector $x \in R^n$ may be written in the form $x = \sum_1^n (x, e_k) e_k$, where $(e_k)_{k=1, n}$ is a complete orthonormal base. By letting $x = 0$, (i) is clearly established. The Hilbert space representation is an extension where the sequence of partial sums of terms $\sum_1^n (x, e_k) e_k$ can be shown (Kolmogorov and Formin, 1961) to deviate least in the L_2-norm, for specified n, from x. In either case, (x, e_k) will correspond to the a_k in (i) and must be zero.

It is instructive to consider (i) from the almost periodic function theory discussed earlier. The uniqueness theorem of H. Bohr states that if the Fourier coefficients are zero the corresponding a.p. function is identically zero (Bohr, 1947).

Finally, using the well-known theorem that a system of homogeneous linear equations will have non-trivial solution if, and only if, the appropriate determinant is zero, in this case, for all $t \in T$. Pick two elements of T, say t_1 and t_2; and construct the determinant

$$\det \begin{bmatrix} \sin \theta_1 t_1 & \sin \theta_2 t_1 \\ \sin \theta_1 t_2 & \sin \theta_2 t_2 \end{bmatrix}$$

which can be shown to be non-zero for certain t_1 and t_2, proving that $a_1 = a_2 = 0$ a.s. by contradiction.

We note that similar arguments may be made if (i) is replaced by

$a_1 \sin \theta_1 t + a_2 \cos \theta_2 t = 0 \quad \forall\, t \in T$ and a.s.

or if there are more terms in the sum.

Equation (2.22) shows that any complex-valued s.p. can be written as the sum of a real-valued and an imaginary-valued process. In the next theorem we assume that $Z(t,\omega)$ is real-valued and denote it by $X(t,\omega)$.

Theorems 3.1 and 3.2 specified the conditions that the r.v.s ξ_k have to satisfy for $Z(t,\omega)$ to be WSS and real-valued, respectively. Theorem 3.3 which follows, specifies the conditions that random variables α_k and β_k in (3.1) have to satisfy when $Z(t,\omega)$ is WSS as well as real-valued.

The results of Theorem 3.3 are known and are generally associated with Fourier series expressions like (2.19) and involve the evaluation of the integrals which define the Fourier coefficients. The method of proof given here is not associated with any particular definition of the coefficients and is perhaps original.

<u>Theorem 3.3</u>: If $X(t,\omega)$ is a real-valued, zero mean, WSS stochastic process, if $X(t,\omega)$ is written in a series of the form

$$(3.3) \qquad X(t,\omega) = \sum_{k=1}^{m} \xi_k e^{i\lambda_k t} \quad \forall\, t \in T,\ \text{a.s.},$$

and if each coefficient $\xi_k = \xi_k(\omega)$ can be written in the form of equation (3.1); then, the coefficients α_k and β_k have the following covariance properties:

$$\mathcal{E}\,\alpha_k\,\alpha_j = \mathcal{E}\,\beta_k\,\beta_j = 0 \quad \forall\, k \neq j$$

$$\mathcal{E}\,\alpha_k\,\beta_j = 0 \quad \forall\, k, j$$

where k = 1, 2, ..., m/2 and j = 1, 2, ..., m/2.

Proof: The proof of this theorem is based upon Theorem 3.2, which showed that a necessary and sufficient condition for $X(t,\omega)$ to be real-valued is that

$$\lambda_{2k-1} = -\lambda_{2k} \text{ and } \xi_{2k-1} = \overline{\xi}_{2k} \text{ for } k = 1, 2, \ldots, m/2$$

and upon the technique employed in the proof of Theorem 3.1.

Without loss of essential generality, let m = 4 so that $X(t,\omega)$ is written

$$X(t,\omega) = \xi_1 e^{i\lambda_1 t} + \xi_2 e^{i\lambda_2 t} + \xi_3 e^{i\lambda_3 t} + \xi_4 e^{i\lambda_4 t}$$

$$X(t,\omega) = \xi_1 e^{i\lambda_1 t} + \overline{\xi}_1 e^{-i\lambda_1 t} + \xi_3 e^{i\lambda_3 t} + \overline{\xi}_3 e^{-i\lambda_3 t}$$

The correlation function of $X(t,\omega)$ ($\mathcal{E}\,X(t,\omega) = 0$) is

$$\mathcal{B}(\tau) = \mathcal{E}\,\{X(t+\tau)\overline{X(t)}\}$$

$$\mathcal{B}(\tau) = \mathcal{E}\left\{\left(\xi_1 e^{i\lambda_1 (t+\tau)} + \overline{\xi}_1 e^{-i\lambda_1 (t+\tau)} + \xi_3 e^{i\lambda_3 (t+\tau)} + \overline{\xi}_3 e^{-i\lambda_3 (t+\tau)}\right)\left(\overline{\xi}_1 e^{-i\lambda_1 t} + \overline{\overline{\xi}}_1 e^{+i\lambda_1 t} + \overline{\xi}_3 e^{-i\lambda_3 t} + \overline{\overline{\xi}}_3 e^{+i\lambda_3 t}\right)\right\}$$

$$
\text{(i)}\quad \mathcal{B}(\tau) = \mathcal{E}\Big\{ \xi_1\bar{\xi}_1 e^{i\lambda_1\tau} + \xi_1\xi_1 e^{i\lambda_1(2t+\tau)} + \xi_1\bar{\xi}_3 e^{i\lambda_1(t+\tau)-i\lambda_3 t} +
$$

$$
\xi_1\xi_3 e^{i\lambda_1(t+\tau)+i\lambda_3 t} + \bar{\xi}_1\bar{\xi}_1 e^{-i\lambda_1(2t+\tau)} + \xi_1\bar{\xi}_1 e^{-i\lambda_1\tau} +
$$

$$
\bar{\xi}_1\bar{\xi}_3 e^{-i\lambda_1(t+\tau)-i\lambda_3 t} + \bar{\xi}_1\xi_3 e^{-i\lambda_1(t+\tau)+i\lambda_3 t} +
$$

$$
\xi_3\bar{\xi}_1 e^{i\lambda_3(t+\tau)-i\lambda_1 t} + \xi_1\xi_3 e^{i\lambda_3(t+\tau)+i\lambda_1 t} +
$$

$$
\xi_3\bar{\xi}_3 e^{i\lambda_3\tau} + \xi_3\xi_3 e^{i\lambda_3(2t+\tau)} + \bar{\xi}_3\bar{\xi}_1 e^{-i\lambda_1 t-i\lambda_3(t+\tau)} +
$$

$$
\bar{\xi}_3\xi_1 e^{i\lambda_1 t-i\lambda_3(t+\tau)} + \bar{\xi}_3\bar{\xi}_3 e^{-i\lambda_3(2t+\tau)} + \xi_3\bar{\xi}_3 e^{-i\lambda_3\tau}\Big\}
$$

As in the proof of Theorem 3.1 all terms of the form $\mathcal{E}\,\square\,\cdot$, which are multiplied by an exponential which is a function of t, must be zero. Of these, choose $\mathcal{E}\,\xi_1\,\xi_3 = \mathcal{E}\,\xi_1\,\bar{\xi}_3 = 0$ which we may expand in terms of α_k and β_k,

$$
\mathcal{E}[\tfrac{1}{2}(\alpha_1 - i\beta_1)\,\tfrac{1}{2}(\alpha_3 - i\beta_3)] = \mathcal{E}[\tfrac{1}{2}(\alpha_1 - i\beta_1)\,\tfrac{1}{2}(\alpha_3 + i\beta_3)] = 0
$$

$$
\mathcal{E}[(\alpha_1\alpha_3 - \beta_1\beta_3) - i(\alpha_3\beta_1 + \alpha_1\beta_3)] = \mathcal{E}[(\alpha_1\alpha_3 + \beta_1\beta_3) + i(\alpha_1\beta_3 - \alpha_3\beta_1)] = 0
$$

Therefore,

$$
\mathcal{E}(\alpha_1\alpha_3 - \beta_1\beta_3) = \mathcal{E}(\alpha_1\alpha_3 + \beta_1\beta_3) = 0
$$

and

$$
-\mathcal{E}(\alpha_3\beta_1 + \alpha_1\beta_3) = +\mathcal{E}(\alpha_1\beta_3 - \alpha_3\beta_1) = 0
$$

Furthermore,

$$\mathcal{E}\,\alpha_1\alpha_3 = +\mathcal{E}\,\beta_1\beta_3 \text{ and } \mathcal{E}\,\alpha_1\alpha_3 = -\mathcal{E}\,\beta_1\beta_3$$

(ii) and

$$\mathcal{E}\,\alpha_3\beta_1 = -\mathcal{E}\,\alpha_1\beta_3 \text{ and } \mathcal{E}\,\alpha_1\beta_3 = +\mathcal{E}\,\alpha_3\beta_1$$

The four equations in (ii) can only be satisfied if $\mathcal{E}\,\alpha_1\alpha_3 = \mathcal{E}\,\beta_1\beta_3 = \mathcal{E}\,\alpha_1\beta_3 = \mathcal{E}\,\alpha_3\beta_1 = 0$. Similar results follow for other pairs of cross-products in (i). Generalizing (see Remark 1, following this theorem), the result is

$$\left.\begin{aligned} &\mathcal{E}\,\alpha_k\,\alpha_j = \mathcal{E}\,\beta_k\,\beta_j = 0 \\ &\mathcal{E}\,\alpha_k\,\beta_j = 0 \end{aligned}\right\} \begin{aligned} &\forall\, k \neq j, \text{ where } k = 1, 2, \ldots, m/2 \\ &\text{and } j = 1, 2, \ldots, m/2. \end{aligned}$$

Consider the pair, $\mathcal{E}\,\xi_1\,\xi_1$ and $\mathcal{E}\,\overline{\xi}_1\,\overline{\xi}_1$; from (i) we know that these also must be zero: $\mathcal{E}\,\xi_1\,\xi_1 = \mathcal{E}\,\overline{\xi}_1\,\overline{\xi}_1 = 0$

Expanding,

$$\mathcal{E}[(\alpha_1 - i\beta_1)\,(\alpha_1 - i\beta_1)] = \mathcal{E}[(\alpha_1 + i\beta_1)\,(\alpha_1 + i\beta_1)] = 0$$

$$\mathcal{E}[(\alpha_1{}^2 - \beta_1{}^2) - 2i\alpha_1\beta_1] = \mathcal{E}[(\alpha_1{}^2 - \beta_1{}^2) + 2i\alpha_1\beta_1] = 0$$

Hence,

$$\mathcal{E}\,\alpha_1\,\beta_1 = 0$$

By consideration of the pairs $\mathcal{E}\,\xi_3\,\xi_3$ and $\mathcal{E}\,\overline{\xi}_3\,\overline{\xi}_3$, $\mathcal{E}\,\alpha_3\,\beta_3 = 0$ follows. Generalizing, the result is

$$\mathcal{E}\,\alpha_k\,\beta_k = 0 \quad \forall\, k, \text{ where } k = 1, 2, \ldots, m/2.$$

Hence,

$$\left.\begin{aligned} &\mathcal{E}\,\alpha_k\,\alpha_j = \mathcal{E}\,\beta_k\,\beta_j = 0 \qquad \forall\, k \neq j \\ &\mathcal{E}\,\alpha_k\,\beta_j = 0 \qquad\qquad\qquad \forall\, k, j \end{aligned}\right\} \begin{aligned} &k = 1, 2, \ldots, m/2 \\ &\text{and } j = 1, 2, \ldots, m/2 \end{aligned}$$

Q.E.D.

Remark 1 on Theorem 3.3: This theorem was essentially proven through re-indexing equation (3.3) and using the odd integers from 1 to m. At the point of first generalization however, the equally acceptable technique of indexing consecutively from 1 to m/2 was introduced. This slight "incongruity" is intended to be a simplification. The statement of the theorem is in terms of the latter system.

A related situation typically occurs in the indexing change when going from an exponential to a sine-cosine form of the Fourier series.

Remark 2 on Theorem 3.3: From the theorem, nothing of interest can be said about the terms $\mathcal{E}\,\alpha_k\,\alpha_j$ and $\mathcal{E}\,\beta_k\,\beta_j$, $k = j$, $k = 1, 2, \ldots, m/2$, where α_k and β_k are real-valued random variables. We show below that these are not generally zero.

Recall from the proof of Theorem 3.1 that the coefficient,

$$\xi_k = \frac{1}{2}\,(\alpha_k - i\beta_k)$$

is not in general zero valued. It is not difficult to show that

$$\mathcal{E}|\xi_k|^2 = \frac{1}{4}\,[\mathcal{E}|\alpha_k|^2 + \mathcal{E}|\beta_k|^2],$$

the left hand side of which appears in the covariance function as a b_k, and is not generally zero or the covariance function would be zero.

Hence the variances $\mathcal{E}|\alpha_k|^2$ and $\mathcal{E}|\beta_k|^2$ are related to the term $\mathcal{E}|\xi_k|^2$, at least one of which is clearly greater than zero.

3.3 Periodicity and Coefficients

We have discussed in previous Sections (2.2 and 2.3) the role that periodicity has in this work. There is, however, much more that can be said and many unanswered questions.

It was pointed out that if a stochastic process is WSS and purely periodic with period d (p.p.(d)) and if the coefficients, ξ_k, in a series of the form of (3.2) are found using (2.20), then they are pairwise uncorrelated; if the process is not p.p.(d) then the coefficients are asymptotically uncorrelated (pairwise). Let us consider the latter situation now.

The number d must be chosen in order to define a fundamental frequency $\lambda_1 = \frac{2\pi}{d}$, say, such that $\lambda_k = k\lambda_1$, for k = 1, 2, ..., are harmonically related and where the coefficients ξ_k are normalized by $\frac{1}{d}$ (see Section 2.2 and equations (2.20) and (2.21)). For a given choice of d, a set of λ, $[\lambda_k]$, and a set of coefficients, $[\xi_k]$, are defined. For a different choice of d, different sets, $[\lambda_k]$ and $[\xi_k]$, are defined.

If the process were p.p.(d), it could be shown (Bohr, 1947, p. 9; Thomas, 1969, p. 151) that the partial sums of the Fourier series defined as above are approximations of the process in the mean square sense[1]. Since we are assuming that the process is not p.p.(d) several reasonable questions can be asked. For example, how does one choose d to optimize the rate of convergence for a given mode of convergence.

[1]For mean-square convergence is the same as quadratic mean convergence.

Some of the earliest uses of stochastic Fourier series was for the analysis of noise problems. In these analyses the workers often made the assumption which we have said is incorrect: *i.e.* they assumed that Fourier coefficients resulting from non-periodic processes would be statistically uncorrelated or independent (see Section 2.2). While we know differently now, it does not mean that for every problem this assumption was incorrect. Essentially, the signal was assumed to extend from $t = 0$ on; it was cut into strips of length d and a Fourier analysis done on each strip. This resulted in a large set of Fourier coefficients (of the form described in equation (2.20)) which differed somewhat from strip to strip. That is, one looked for statistical properties of the process by examining the statistical properties of the Fourier coefficients.

Since it is assumed that the process being studied is not periodic, how did they decide how large d should be? If the process was a noise process it was common to perform the analysis after the transients introduced by long time-delay elements had dissipated and to choose lengths d that were long compared to all the periods occurring which were of interest. The works of Lawson and Ulhenbeck (1950) p. 34, Rice (1954) p. 48, and Usov and Orlov, (1968) describe the techniques in greater detail.

We discussed earlier that the Fourier coefficients from non-periodic processes are uncorrelated in the limit as $d \to \infty$. It should be noted that recent work (Thomas, 1969, p. 151) gives an empirical guide for determining when coefficients are approximately uncorrelated (*i.e.*, how large d should be).

4. CONVERGENCE AND EQUIVALENCE RELATIONS

4.1 Précis

In this chapter we begin to make extensive use of the fact that a finite or infinite series which purports to represent a random function or stochastic process must do so with respect to both $\omega \in \Omega$ and $t \in T$. Up to now these questions were hedged (e.g., the (?) in Section 2.3) primarily so as not to restrict the generality of Chapter 3. Now we shall be explicit and say that our interest is in limits in quadratic mean with respect to $\omega \in \Omega$ and, for infinite series, uniform convergence in $t \in T$.

In Section 3.2 finite series representations of WSS s.p.s were discussed. We specified that the representation be valid $\forall\ t \in T$, although a.e. in $t \in T$ might also be acceptable.

Our study has been limited to stochastic processes in which both the parameter set and the state space form a continuum. Based on advanced calculus considerations, a function is guaranteed to be continuous if its infinite series representations converge uniformly in $t \in T$. In this respect we refer back to the Approximation Theorem of Bohr, given as Theorem 2.3.

In the next chapters we will extend our study to include discrete parameter processes. The intention will be to see if it is valid to view the set of Fourier coefficients as a discrete parameter s.p. and then, taking advantage of the statistical properties of s.p.s, to learn more about the process $Z(t,\omega)$, say, under investigation.

4.2 Quadratic Mean Convergence and Equivalence Relations

We will now formalize the notion of equivalent random variables first mentioned in Section 2.3. Let $\mathfrak{X}$ be a set of r.v.s which has Hilbert space structure as defined in Definition 2.3.

Definition 4.1: Two r.v.s U and V which are both elements of $\mathfrak{X}$ are said to be equivalent random variables if $P(U \neq V) = 0$.

Consider the r.v.s in $\mathfrak{X}$ and identify all those which are equivalent to any one other member. In some way we wish to omit from consideration random variables which have this property. We say therefore, that $\mathfrak{X}$ is a set of r.v.s which contains exactly one representative for each equivalence class and redefine $\mathfrak{X}$ to be a set of non-equivalent r.v.s over $(\Omega, \mathcal{G}, P)$.

We repeat (equation (2.5)) that the norm of a r.v. $W \in \mathfrak{X}$ is,

$$\|W\| = (\mathcal{E}|W|^2)^{\frac{1}{2}} .$$

Convergence with respect to this norm is called quadratic mean convergence.

Definition 4.2: A sequence of r.v.s $W_n \in \mathfrak{X}$ for all n, is said to converge in quadratic mean[1] to a r.v. W if

[1]We have noted elsewhere that quadratic mean convergence is synonomous with mean square convergence, each being abbreviated by q.m. and m.s., respectively; the limit operation in such cases being denoted by $\underset{n\to\infty}{\text{l.i.m.}}$, where $n\to\infty$ may be omitted if no confusion can result.

(i) all W_n and W have finite second moments

(ii) $$\lim_{n\to\infty} \mathcal{E}|W_n - W|^2 = 0$$

The following theorem occurs in Lukacs (1968), p. 47 and is written below with minor changes.

<u>Theorem 4.1</u>: A sequence of r.v.s (W_n) converges in q.m. to a r.v., W, where W_n and W are elements of $\mathfrak{X}$, if and only if, it is possible to find for any $\varepsilon > 0$ a number N_ε such that

$$\mathcal{E}(|W_m - W_n|^2) \leq \epsilon \text{ for } n, m \geq N_\varepsilon .$$

Proof: -not given

It is apparent that this theorem states a form of Cauchy criterion for q.m. convergence (in ω). It is with the use of such a theorem that one establishes the completeness of a Hilbert space of r.v.s. A complete argument establishing this fact may be found in Lukacs (1968), p. 62.

In Section 2.3 we mentioned the papers of Slutsky (1938) and Bochner (1956). Both papers deal, at least in part, with a.p. stochastic processes. We will refer quite often to the Bochner paper because of its greater mathematical detail. None of the results obtained here occur in the Bochner paper; however, some results we obtain do occur in the Slutsky paper as either unproven statements or as theorems proven, in what today, might be called a heuristic way.

Bochner (1956) discusses the a.p. random function $Z(t,\omega)$ and its representation by a series. Let us write this representation as

(4.1) $\quad Z(t,\omega) \sim \Sigma\, \xi_k(\omega) e^{i\lambda_k t}$ for some $t \in T$ and $\omega \in \Omega$

where the coefficients $\xi_k = \xi_k(\omega)$ are called Fourier coefficients. He also states what we shall henceforth refer to as the Bochner conditions; $Z(t,\omega)$ is WSS and a.p. if:

$$
(4.2) \quad \begin{cases} \text{(i)} & \text{its spectrum is purely discontinuous} \\ & (\cdot\ni\cdot\ \lambda_1 < \lambda_2 < \lambda_3 < \ldots) \\ \text{(ii)} & \mathcal{E}\, \xi_k \overline{\xi}_j = 0 \quad k \neq j \\ \text{(iii)} & \sum_k \mathcal{E}|\xi_k|^2 < +\infty \end{cases}
$$

It should be noted that condition (ii) corresponds to part of the results of Theorem 3.1.

Definition 4.3: Let $S_n(t,\omega)$ be the $n^{\underline{th}}$ partial sum associated with the series in (4.1),

$$S_n(t,\omega) = \sum_1^n \xi_k(\omega) e^{i\lambda_k t}\ .$$

For a fixed t, a sequence of $S_n(t,\omega)$ is a sequence of r.v.s. In analogy with Definition 4.2 we now state criteria for a sequence of $S_n(t,\omega)$ to converge uniformly in $t \in T$ and in q.m. in $\omega \in \Omega$ to a s.p. $Z(t,\omega)$. We shall refer to such convergence as "quadratic mean convergence uniform in t".

Definition 4.4: The sequence $(S_n(t,\omega))$ is said to converge in quadratic mean uniform in t to $Z(t,\omega)$ for arbitrary fixed t, if given an $\varepsilon > 0$, $\exists$ a number $N_{\varepsilon,t}$ $\cdot\ni\cdot$ $\forall\ n > N_{\varepsilon,t}$

$$\mathcal{E}|Z(t,\omega) - S_n(t,\omega)|^2 \leq \epsilon$$

and where the number $N_{\varepsilon,t}$ can be chosen independently of the choice of t (and if, furthermore, $Z(t,\omega)$ and $S_n(t,\omega)$ have finite second moments for each n).

Definition 4.4 clearly contains the features of both quadratic mean convergence and uniform convergence.

Bochner's paper (1956, Theorem 5.1.1) states that $Z(t,\omega)$ is an a.p. stochastic process if, and only if, $(S_n(t,\omega))$ converges in q.m. uniform in t. In the sequel, we will refer to WSS stochastic processes $Z(t)$ which can be represented by

$$(4.4) \qquad Z(t) = \Sigma\, \xi_k e^{i\lambda_k t} \text{ uniformly in } t;\ \text{q.m. in } \omega,$$

as series "in the sense of Bochner".

Definitions 4.2 and 4.4 have been concerned with q.m. convergence in ω. We will generalize the notion of q.m. convergence in the following way. In analogy with the concept of q.m. convergence of sequences of r.v.s $W_k(\omega)$, $k \varepsilon T^*$, a generalization (Loève, 1963, p. 469; Lukacs, 1968, p. 100) is found by replacing T^* by T (bounded or unbounded) and examining the q.m. convergence of $W_r = W_r(\omega)$ as $r \varepsilon T$ tends to r_o, a limit point of T. That is,

$$(4.5) \qquad \underset{r \to r_o}{\text{l.i.m.}}\ W_r = W(\omega)$$

is defined by the condition,

$$(4.6) \qquad \lim_{r \to r_o} \mathcal{E}(|W_r - W(\omega)|^2) = 0$$

where W_r and $W(\omega)$ in (4.5) and (4.6) have finite second moments. We will make use of this generalization in the sequel.

Bochner does not specify how the coefficients, ξ_k, in (4.4) are to be found other than to require that they satisfy the conditions in (4.2). We will adapt the definition used in the non-random a.p. theory (and in some sense that used by Slutsky (1938)). A discussion of these coefficients and related matters is given in Section 2.3.

<u>Definition 4.5</u>: The <u>Fourier coefficient</u>, $\xi_k = \xi_k(\omega)$, associated with the WSS s.p. $Z(t,\omega)$ is,

$$(4.7) \qquad \xi_k(\omega) = \underset{D\to\infty}{\text{l.i.m.}} \frac{1}{D} \int_0^D e^{-i\lambda_k t} Z(t,\omega)dt \neq 0,$$

where $t \varepsilon T$, $\omega \varepsilon \Omega$, $k \varepsilon T^*$, $\lambda_k \varepsilon R^1$ and where the integral is a Riemann integral in q.m.

Since "l.i.m." refers to q.m. convergence, we define the following partial "sum",

<u>Definition 4.6</u>: The <u>partial "sum"</u>, $\xi_{k,D} = \xi_{k,D}(\omega)$, associated with (4.7) is,

$$\xi_{k,D} = \frac{1}{D} \int_0^D e^{-i\lambda_k t} Z(t,\omega)dt,$$

where the same qualifications apply as in Definition 4.5.

Therefore, as in (4.5) one could rewrite (4.7) as the convergent limit of r.v.s,

$$\xi_k = \underset{D\to\infty}{\text{l.i.m.}} \xi_{k,D}$$

or, as in (4.6)

$$\lim_{D\to\infty} \mathcal{E}(|\xi_k - \xi_{k,D}|^2) = 0$$

where ξ_k and $\xi_{k,D}$ have finite second moments for arbitrary $k \varepsilon T^*$.

Up to this point, we have said that the λ_k values are real-valued numbers (elements of R^1) and purely discontinuous in nature (Bochner conditions (4.2)); i.e., they form a discrete spectrum. We have also said that a special case of interest is when the λ_k are chosen such that they are harmonically related (see the discussion between (2.11) and (2.22)). The series in (4.4) is a Fourier series if the ξ_k are defined as Fourier coefficients ((4.7) or (2.20)); i.e., if the ξ_k are defined, in a certain way, in terms of Z(t). Is there any way to define the set of λ_k, in a corresponding way, in terms of Z(t)?

The answer is yes and no. Certainly, if the λ_k are to be harmonically related then that is a condition. The rest of the answer is that in any case, if ξ_k are given by (4.7) then at least one property of the corresponding λ_k must be that the expression in (4.7) is non-zero. In fact, in the non-random theory of almost periodic functions the exponents are those real numbers for which the coefficients are non-zero.

In practice, it would be helpful to have a condition on the choice of λ_k that is dependent upon the covariance function rather than upon the coefficients. Such a condition is known (Yaglom, 1962, p. 35) and is clearly related to (4.7). We place no further conditions upon the choice of λ_k.

The following theorem[1] is given to guarantee that the definition we have given of Fourier coefficients does in fact satisfy the

[1] Slutsky (1938) works with real-valued Fourier coefficients and, where referring to the corresponding statement of the Bochner conditions, writes that they are shown to be satisfied, "sans difficulté".

Bochner conditions (4.2). The enumeration, (i), (ii), ..., (vi), refers to justification for the indicated operation involved in the transition from one line of the proof to the next; these are listed immediately following the proof.

Theorem 4.2: If $Z(t,\omega)$ is a zero mean, WSS s.p. with finite total spectral measure, then the coefficients ξ_k, as defined in Definition 4.5, satisfy the Bochner conditions (4.2) as used in (4.4).

Proof: From Definition 4.5, $\xi_k(\omega) = \underset{D\to\infty}{\text{l.i.m.}} \frac{1}{D}\int_0^D Z(t,\omega)e^{-i\lambda_k t}dt.$ Consider the product,

$$\mathcal{E}\,\xi_k\,\overline{\xi}_j = \mathcal{E}\left\{\underset{D_1\to\infty}{\text{l.i.m.}} \frac{1}{D_1}\int_0^{D_1} Z(t_1,\omega)e^{-i\lambda_k t_1}dt)\right.$$

$$\left.(\underset{D_2\to\infty}{\text{l.i.m.}} \frac{1}{D_2}\int_0^{D_2} \overline{Z(t_2,\omega)}e^{+i\lambda_j t_2}dt_2)\right\}$$

(i)

$$= \lim_{D_1}\lim_{D_2}\frac{1}{D_1D_2}\,\mathcal{E}\left\{\int_0^{D_1} Z(t_1)e^{-i\lambda_k t_1}dt_1)\right.$$

$$\left.(\int_0^{D_2}\overline{Z(t_2)}e^{+i\lambda_j t_2}dt_2)\right\}$$

(ii)

$$= \lim_{D_1}\lim_{D_2}\frac{1}{D_1D_2}\int_0^{D_1}\int_0^{D_2}\mathcal{E}\,Z(t_1)\overline{Z(t_2)}e^{-i\lambda_k t_1+i\lambda_j t_2}dt_2\,dt_1$$

$$= \lim_{D_1}\lim_{D_2}\frac{1}{D_1D_2}\int_0^{D_1}\int_0^{D_2}\mathcal{B}(t_1-t_2)e^{-i\lambda_k t_1+i\lambda_j t_2}dt_2\,dt_1$$

(iii)

$$= \lim_{D_1} \lim_{D_2} \frac{1}{D_1 D_2} \int_0^{D_1} \int_0^{D_2} \int_{-\infty}^{\infty} e^{-i\lambda_k t_1 + i\lambda_j t_2} \cdot$$

$$e^{+i\lambda(t_1 - t_2)} \mu(d\lambda) dt_2 dt_1$$

(iv)
$$\mathcal{E}\, \xi_k \overline{\xi}_j = \lim_{D_1} \lim_{D_2} \frac{1}{D_1 D_2} \int_{-\infty}^{\infty} \int_0^{D_1} \int_0^{D_2} e^{it_1(\lambda - \lambda_k)} \cdot$$

$$e^{-it_2(\lambda - \lambda_j)} dt_2 dt_1 \; \mu(d\lambda)$$

(v)
$$= \lim_{D_1} \lim_{D_2} \frac{1}{D_1 D_2} \sum_{\ell} \mu_\ell \int_0^{D_1} \int_0^{D_2} e^{it_1(\lambda_\ell - \lambda_k)} \cdot$$

$$e^{-it_2(\lambda_\ell - \lambda_j)} dt_2 dt_1$$

(vi)
(*) $$\mathcal{E}\, \xi_k \overline{\xi}_j = \sum_{\ell} \lim_{D_1} \lim_{D_2} \frac{1}{D_1 D_2} \mu_\ell \int_0^{D_1} \int_0^{D_2} e^{it_1(\lambda_\ell - \lambda_k)} \cdot$$

$$e^{-it_2(\lambda_\ell - \lambda_j)} dt_2 dt_1$$

It is important to recall that the values of λ_k from the discontinuous spectrum (4.2) correspond to the values from the discontinuous spectral measure in a one-to-one way. Hence, the $n^{\underline{th}}$ non-zero value of λ_ℓ corresponds to the spectral measure value where $\ell = n$. We will now evaluate the double integral in (*).

Consider the case where $k = j$. Then for some value of ℓ in the sum in equation (*), $\lambda_\ell = \lambda_k = \lambda_j$. The double integral reduces to the product $(D_1 D_2)$ and for that ℓ, the expression is equal to μ_k. For other values of ℓ, the exponents are non-zero and the absolute value of the integral is less than a real number independent of D_1 and D_2. In the limit with respect to D_1 and D_2 the expression goes to zero. Hence,

$$\mathcal{E}\ \xi_k(\omega)\ \overline{\xi_j(\omega)} = \mu_k \qquad k = j$$

By hypothesis, the sum of all μ_k is finite so it follows that condition (iii) of (4.2) is satisfied.

Consider the case where $k \neq j$. Then there will occur for each $k \neq j$ the two incidences, $\lambda_\ell = \lambda_k \neq \lambda_j$ and $\lambda_\ell = \lambda_j \neq \lambda_k$. For other values of ℓ, the exponents are both non-zero, the integrals are as above; and in the limit on D_1 and D_2 the expression goes to zero. When $\lambda_\ell = \lambda_k \neq \lambda_j$ equation (*) is

$$\mathcal{E}\ \xi_k(\omega)\overline{\xi_j(\omega)} = \sum_\ell \lim_{D_1} \lim_{D_2} \frac{1}{D_1 D_2}\, \mu_\ell \int_0^{D_1} dt_1 \int_0^{D_2} e^{it_2(\lambda_\ell - \lambda_j)}\, dt_2$$

Integration over t_1 removes the limit over D_1, but the limit over D_2 of the bounded integrand (function of t_2) again is zero. When $\lambda_\ell = \lambda_j \neq \lambda_k$, a similar situation occurs. Hence,

$$\mathcal{E}\ \xi_k(\omega)\overline{\xi_j(\omega)} = 0 \qquad k \neq j$$

Q.E.D.

The following items numbered (i) to (vi) correspond to the numbers in the proof of Theorem 4.2 and are given for purposes of rigour.

(i) $\xi_{k,D_\ell} = \underset{D_\ell \to \infty}{\text{l.i.m.}} \frac{1}{D_\ell} \int_0^{D_\ell} Z(t_\ell) e^{-i\lambda_k t_\ell} dt_\ell$

and let $\xi_k = \underset{D_\ell}{\text{l.i.m.}}\ \xi_{k,D_\ell}$, where $D_\ell \in R^1$

and $\ell = 1, 2$. Then,

$$\mathcal{E}[(\underset{D_1}{\text{l.i.m.}}\ \xi_{k,D_1})\ (\underset{D_2}{\text{l.i.m.}}\ \bar{\xi}_{j,D_2})] = \lim_{D_1} \lim_{D_2} \mathcal{E}\ \xi_{k,D_1}\ \bar{\xi}_{j,D_2}$$

follows directly from Loève (1963), p. 469.

(ii) The operation is valid if the Riemann integrals:

$\int_0^{D_\ell} Z(t_\ell) e^{i\lambda_k t_\ell} dt_\ell$, $\ell = 1, 2$ exist in the q.m. sense, then it is known that

$$\mathcal{E}[(\int_a^b g(t_1) Z(t_1) dt_1)\ (\int_a^b h(t_2) \overline{Z(t_2)} dt_2)] =$$

$$\int_a^b \int_a^b g(t_1) \overline{h(t_2)}\ \mathcal{E}\ Z(t_1) \overline{Z(t_2)} dt_2\, dt_1$$

(Cramér and Leadbetter, 1967, p. 88).

(iii) We have alluded earlier to the fact that a covariance function $\mathcal{B}(\tau)$ (equation (2.12)) of a WSS process can be written as an integral over a measure. Therefore we may write, if $\mathcal{B}(\tau) = \mathcal{E}\ Z(t+\tau)\overline{Z(t)}$ is a continuous function independent of t (and since $\mathcal{B}(\tau)$ is positive definite),

$$\mathcal{B}(\tau) = \int_{-\infty}^{\infty} e^{+i\lambda\tau} \mu(d\lambda)$$

$\lambda \in R^1$, $\tau \in T$ (Feller, 1966, p. 586).

(iv) The integrand is non-negative and integrable on $[0, D_1] \times [0, D_2] \times (-\infty, \infty)$ so the operation's validity is based on application of Fubini's Theorem (Royden, 1963, p. 233).

(v) The set of λ_k are purely discontinuous so the integral is appropriately replaced by a summation.

(vi) The sum and limits may be interchanged because the expression satisfies the conditions of Lebesgue's Dominated Convergence Theorem (Hewitt and Stromberg, 1965) for a discrete measure.

The hypothesis of Theorem 4.2 includes the condition that $Z(t,\omega)$ be a WSS and zero mean s.p. We show below that the mean value of the Fourier coefficients is almost always zero.

<u>Lemma 4.1</u>: If the first moment of the s.p. $Z(t,\omega)$ is constant, then the mean value of all coefficients, $\xi_k = \xi_k(\omega)$, defined as in Definition 4.5 is zero.

Proof: From Definition 4.5,

$$\mathcal{E}\,\xi_k = \mathcal{E}\ \underset{D\to\infty}{\text{l.i.m.}}\ \frac{1}{D}\int_0^D Z(t)e^{-i\lambda_k t}\,dt.$$

From Definition 4.6, $\mathcal{E}\, \xi_k = \mathcal{E}\, \underset{D\to\infty}{\text{l.i.m.}}\, \xi_{k,D}$

Since $\xi_k = \underset{D\to\infty}{\text{l.i.m.}}\, \xi_{k,D}$ means $\lim_{D\to\infty} \mathcal{E}(\xi_k - \xi_{k,D})\,(\overline{\xi_k - \xi_{k,D}}) = 0$ and since it can be shown that convergence in q.m. implies convergence in the first mean, it follows that $\lim_{D} \mathcal{E}(\xi_k - \xi_{k,D}) = 0$.

That is, $\mathcal{E}\, \xi_k = \lim_{D} \mathcal{E}\, \xi_{k,D}$.

The integral is a function of λ_k and ω and is therefore an ordinary r.v. Then,

$$\mathcal{E}\, \xi_k = \lim_{D} \frac{1}{D} \int_0^D e^{-i\lambda_k t}\, \mathcal{E}\, Z(t)dt$$

$$= \lim_{D} \frac{m}{D} \int_0^D e^{-i\lambda_k t} dt$$

$$\mathcal{E}\, \xi_k = 0,\ k = 1, 2, \ldots$$

The limit goes to zero because the integral is bounded $\forall$ k and is multiplied by the $\frac{1}{D}$ factor. The number m is the constant, $\mathcal{E}\, Z(t,\omega)$.

Q.E.D.

We note that if the exponents are harmonically related and if k = 0 is permissible, then $\lambda_k = \frac{2\pi}{D} k$ becomes $\lambda_o = 0$. Denoting the corresponding coefficient as ξ_o, and by the use of (2.20), $\mathcal{E}\, \xi_o$ would be

$$\mathcal{E}\, \xi_o = \mathcal{E}\, Z(t)\, \frac{1}{D} \int_0^D dt = \mathcal{E}\, Z(t,\omega)$$

Therefore, if the process is not a zero mean, $\mathcal{E}\, \xi_o$ reflects that fact by a non-zero value. The term $\mathcal{E}\, \xi_o$ is commonly called the average value. A similar result also applies to (4.7) if Z(t) is not a zero mean process.

It should be noted that Lemma 4.1 places no conditions on the second moments of $Z(t,\omega)$; therefore, the conclusion follows for processes which are not WSS.

Since we have just digressed from zero mean processes we note again that in the sequel, all processes are zero mean.

<u>Theorem 4.3</u>: If $\sum_{1}^{\infty} \xi_k e^{i\lambda_k t}$ is a zero mean, WSS stochastic process as in (4.1) with finite total spectral measure then the sequence of partial sums, $S_n(t,\omega) = \sum_{1}^{n} \xi_k e^{i\lambda_k t}$, converges in quadratic mean uniform in $t \in T$.

Proof: Formally apply the Cauchy criterion specified in Theorem 4.1 to the sequence of partial sums; i.e., if the inequality,

$$(*) \qquad \mathcal{E}(|S_n(t,\omega) - S_m(t,\omega)|^2) < \epsilon$$

is satisfied for any $\epsilon > 0$ and for all $n, m > N_\varepsilon$, then the sequence of sums $S_n(t,\omega)$ converges in q.m. in ω to a s.p. $Z(t,\omega)$. That is, $\lim_{n\to\infty} \mathcal{E}(|Z(t,\omega)-S_n(t,\omega)|^2) = 0$. It will be demonstrated now that for a fixed t, equation (*) is an expression concerning r.v.s only (i.e., independent of t).

$$\mathcal{E}(|S_m(t,\omega)-S_n(t,\omega)|^2) = \mathcal{E}\left(\left|\sum_{k=1}^{m} \xi_k e^{i\lambda_k t} - \sum_{k=1}^{n} \xi_k e^{i\lambda_k t}\right|^2\right) =$$

$$\mathcal{E}\left(\left|\sum_{n+1}^{m} \xi_k e^{i\lambda_k t}\right|^2\right) = \mathcal{E}\left[\sum_{n+1}^{m} \xi_k e^{i\lambda_k t}\right]\left[\sum_{n+1}^{m} \bar{\xi}_j e^{-i\lambda_j t}\right] =$$

$$\mathcal{E}\left(\sum_{k=n+1}^{m} |\xi_k|^2 + \sum_{k<j} \xi_k \bar{\xi}_j f_1(t,\lambda_k,\lambda_j) + \sum_{j<k} \bar{\xi}_k \xi_j f_2(t,\lambda_k,\lambda_j)\right) .$$

The functions $f_1(t,\lambda_k,\lambda_j)$ and $f_2(t,\lambda_k,\lambda_j)$ contain imaginary exponentials which have a conjugate relationship to each other and the last two summations are with respect to k and j between n + 1 and m. Continuing,

$$\mathcal{E}(|S_m(t,\omega)-S_n(t,\omega)|^2) = \sum_{n+1}^{m} \mathcal{E}|\xi_k|^2 +$$

$$\Sigma\, \mathcal{E}\, \xi_k \bar{\xi}_j f_1(t,\lambda_k,\lambda_j) + \Sigma\, \mathcal{E}\, \bar{\xi}_k \xi_j f_2(t,\lambda_k,\lambda_j) =$$

$$\sum_{n+1}^{m} \mathcal{E}|\xi_k|^2 ;$$

where, $\mathcal{E}\, \xi_k\, \bar{\xi}_j = \mathcal{E}\, \bar{\xi}_k\, \xi_j = 0 \quad \forall\, k \neq j$, follows from the WSS nature of Z(t).

The right hand side of the last equation can be made arbitrarily small because of the finiteness of the total spectral measure and is clearly time-independent.

Q.E.D.

This theorem demonstrates that q.m. convergence in ω of the series in (4.1) is uniform in $t \in T$. Incidentally, this result is to be expected intuitively from the WSS condition on $Z(t,\omega)$.

Corollary to Theorem 4.3: If the coefficients, ξ_k, are Fourier coefficients as defined in Definition 4.5, then the series in (4.1) converges in quadratic mean uniform in t.

Proof: -trivial.

The concept of equivalent r.v.s in Definition 4.1 has a counterpart in the theory of q.m. convergence (Lukacs, 1968, p. 37).

Theorem 4.4: If $\text{l.i.m.}\ W_j(\omega) = W(\omega)$ and $\text{l.i.m.}\ W_j(\omega) = V(\omega)$, then W and V are equivalent random variables.

Proof: -not given.

There is also a counterpart of the concept of equivalent r.v.s for stochastic processes (Cramér and Leadbetter, 1967),

Definition 4.7: Two s.p.s $W(t,\omega)$ and $V(t,\omega)$ are equivalent[1], if, for every $t \in T$, $W(t,\omega)$ and $V(t,\omega)$ are equivalent r.v.s.

The following theorem is a consequence of Theorem 4.4 and Definition 4.7.

Theorem 4.5: If any sequence of partial sums, $(S_n(t,\omega))$, defined as in Definition 4.1 is such that $\underset{n\to\infty}{\text{l.i.m.}}\ S_n(t,\omega) = Z'(t,\omega)$ and $\underset{n\to\infty}{\text{l.i.m.}}\ S_n(t,\omega) = Z''(t,\omega)$ both limits being in quadratic mean and uniform in t, then $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent s.p.s.

Proof: For an arbitrary $t \in T$, the following are limits in q.m. of r.v.s

$$\underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z'(t,\omega) \text{ and } \underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z''(t,\omega).$$

By Theorem 4.4, $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent r.v.s. By Definition 4.7, $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent s.p.s if $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent for all $t \in T$; i.e., if for all t,

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[1]Equivalent stochastic processes will be discussed further in Chapter 7 in connection with separable stochastic processes.

(*) $\underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z'(t,\omega)$ and $\underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z''(t,\omega)$

But of course the equations in (*) hold for all t because $S_n(t,\omega)$ converges uniformly in t by hypothesis.

Q.E.D.

It is of interest to note that assuming the convergence of the sequence of r.v.s. $(W_j(\omega))$ in mean square in ω to $W(\omega)$ and $V(\omega)$ and then concluding that W and V are equivalent r.v.s. is quite similar to the well known uniqueness theorem in metric space analysis that if $x_n \to x$ and $x_n \to y$ then $x = y$. Therefore, Theorem 4.4 is a uniqueness theorem. And, by extension, Theorem 4.5 is also sort of a uniqueness theorem.

Looking back we note that we have established the following: if $\text{l.i.m.}\ W_j(\omega) = W(\omega)$ and $\text{l.i.m.}\ W_j(\omega) = V(\omega)$, then $W(\omega)$ and $V(\omega)$ are equivalent r.v.s. (Theorem 4.4); and, if $\underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z'(t,\omega)$ and $\underset{n}{\text{l.i.m.}}\ S_n(t,\omega) = Z''(t,\omega)$ in q.m. uniformly in t, then $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent s.p.s (Theorem 4.5).

5. THE FOURIER COEFFICIENTS AS A STOCHASTIC PROCESS

5.1 Précis

Preceding chapters have demonstrated that it is an advantage to know that a certain phenomena can be described by a stochastic process because it imposes certain conditions upon the probabilistic nature of the realizations. If the s.p. has a series representation of the type discussed other conditions are imposed and knowing the parameter set, T, imposes further conditions.

It was noted in Section 4.1 that one objective of this research is to ascertain what further knowledge about $Z(t,\omega)$ can be inferred from the statistical properties of the set of coefficients which occur in a series representation.

Since the knowledge of the criteria that $Z(t,\omega)$ has to fulfill to be a WSS s.p. was so useful, it is reasonable to inquire if perhaps the set of coefficients form a certain type s.p. Ignoring the underlying family of distributions and the associated conditions, it is clear that the set of coefficients $[\xi_k]$, $k = 1, 2, \ldots, n$, where n may be finite or not, could quite possibly be a discrete parameter s.p. with T^* as a parameter set[1]. This idea and its implications are pursued in this chapter.

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[1]This is what R. C. Dubes (1968) has done on page 427 where he notes that each coefficient is a r.v. for every integer k and then asserts that the coefficients are a discrete parameter s.p.

5.2 The Class of Associated Processes and the Family of Finite-Dimensional Matrices

To motivate Definition 5.1, we begin this section with an example. The definition assumes that a set of Fourier coefficients can be validly termed a stochastic process. The validity of the assumption will be scrutinized in Section 5.3.

Suppose that a s.p. is represented by a series, $\Sigma \xi_k e^{i\lambda_k t}$, whose exponents are harmonically related (*i.e.*, a period, d, has been selected) and whose coefficients are thereby found from (2.20). Suppose further, that another selection of a period is made, d*, say. Finally, suppose that the series of terms found by using d as a period and the series found by using d* as a period, each converge in q.m. uniform in t over d and d*, respectively. This is one possibility among several, that Definition 5.1 deals with.

The definition specifies how to construct a class of discrete parameter s.p.s by stating the three properties all elements of the class must have in common; one property being that all elements of the class be associated in the same way with the same continuous parameter s.p.

Definition 5.1: Discrete parameter s.p.s $\{\eta_k(\omega)\}$ and $\{\eta'_k(\omega)\}$, $k \in T^*$, are said to be elements of the class of processes associated with $Z(t,\omega)$ if:

(i) $\{\eta_k\}$ and $\{\eta'_k\}$ occur as coefficient processes in two series of the form of (4.1).

(ii) these series both sum (converge, if T^* is unbounded) in some sense, in t and ω; both series in the same sense.

(iii) elements of both coefficient processes satisfy the Bochner conditions.

We do not distinguish between elements of this class of s.p.s but rather we denote any member of the class by the process $\{\xi_k\}$, say. Obviously, the class of s.p.s equivalent to $\{\xi_k\}$ in the sense of Definition 4.7 is a special case of a class of associated processes. Indeed, the class of s.p.s constructed from r.v.s equivalent to the r.v. $\xi_k(\omega)$, for every k, are also associated with $Z(t,\omega)$. We have, however, ruled out consideration of these cases by agreeing (Section 4.2) to consider only one element of the class of equivalent s.p.s.

Other possibilities of interest to us are processes $\{\xi_k(\omega)\}$ and $\{\xi'_k(\omega)\}$ say, which satisfy Definition 5.1 where the elements:

(a) are calculated in different ways and where the set of λ_k is not equal to the set of λ'_k for every $k \varepsilon T^*$

(b) are calculated in different ways and where the set of λ_k are equal to the set of λ'_k for every $k \varepsilon T^*$.

Of particular interest is (b) when the elements of $\{\xi_k(\omega)\}$ are calculated from (4.7); where the elements of $\{\xi'_k(\omega)\}$ are found in any other fashion; and where the set of λ_k and λ'_k are equal for every $k \varepsilon T^*$. We return in Theorem 5.1 to this possibility.

We will now construct and study a transformation matrix that maps the space of random Fourier coefficients to the space of r.v.s. constructed from time samples of the s.p. $Z(t,\omega)$.

We shall, as usual, assume that $Z(t,\omega)$ is a WSS, continuous parameter s.p. We choose an arbitrary, finite set of elements from T, say $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$, and evaluate the process at these points. The result is a k-dimensional random vector, $\vec{Z}_k(\omega) = [Z(t_{n_1}, \omega), \ldots, Z(t_{n_k}, \omega]^T$; the k elements are r.v.s. and have a corresponding distribution function, $F(x_1, \ldots, x_k; t_{n_1}, \ldots, t_{n_k}) = P(Z(t_j, \omega) \leq x_j;\ j = n_1, n_2, \ldots, n_k)$ where $x_j = x(t_j)$. A complete description of what is involved is found in Section 2.3.

Clearly, a family of such finite dimensional distributions can be constructed. We will assume that the family of finite-dimensional distributions (FF-DD) satisfies the Kolmogorov theorem conditions of symmetry and consistency.

Denoting $Z(t_{n_j}, \omega)$ by $Z(t_{n_j})$, the vector $\vec{Z}_k(\omega)$ may be written as

$$(5.1) \qquad Z(t_{n_j}) = \sum_{\ell=1}^{m} \xi_\ell(\omega) \exp[i\lambda_\ell t_{n_j}] \quad j = 1, 2, \ldots, k; \text{ or,}$$

alternatively as,

$$(5.2) \qquad \begin{bmatrix} Z(t_n) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Z(t_{n_k}) \end{bmatrix} = \begin{bmatrix} \exp[i\lambda_1 t_{n_1}] \ldots \exp[i\lambda_m t_{n_1}] \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \exp[i\lambda_1 t_{n_k}] \ldots \exp[i\lambda_m t_{n_k}] \end{bmatrix} \begin{bmatrix} \xi_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \xi_m(\omega) \end{bmatrix} .$$

Subject to convergence conditions, m in (5.1), may be infinite; k is always a finite integer. We denote the vector of coefficients in (5.2) by $\vec{\xi}_m(\omega)$ and the matrix of imaginary exponentials by E_{km}. Then (5.2) may be written

$$(5.3) \qquad \vec{Z}_k(\omega) = E_{km}\, \vec{\xi}_m(\omega)$$

where the matrix E.. is in general, rectangular and non-random.

We will now consider the elements of E_{km} when $Z(t,\omega)$ is evaluated at a different subset of T, $[t_{\ell_1}, t_{\ell_2}, \ldots, t_{\ell_k}]$. For each t_{ℓ_i} there exists a $\tau_i \ni t_{\ell_i} = t_{n_i} + \tau_i$ for $i = 1, 2, \ldots, k$. The resulting $E_{\cdot\cdot}$ matrix is of the same form as that occurring for the set $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ and the elements are related by the fact that the exponents are translated by appropriate τ_i values. Unless $\tau_i = 0, \forall\, i$, we say that the matrices are different. If it is not possible given a certain subset of T, $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ say, to find any other subset of T $\ni$ the two corresponding matrices are the same, then we say that $E_{\cdot\cdot}$ is uniquely related to $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$. We wish to pursue this question of uniqueness further.

The following lemma provides a condition upon the set $[\lambda_k]$ such that only non-unique relations may occur.

<u>Lemma 5.1</u>: If the matrix E_{km} corresponds to the subset of T, $[t_1, t_2, \ldots, t_k]$ and E'_{km} corresponds to a different subset of T, $[s_1, s_2, \ldots, s_k]$ where both E_{km} and E'_{km} refer to the same set $[\lambda_1, \lambda_2, \ldots, \lambda_m]$, then $E_{km} \neq E'_{km}$ if at least one ratio λ_p/λ_q is irrational $(1 < p < m,\ 1 < q < m)$, $\lambda_1 < \lambda_2 < \ldots < \lambda_m$.

Proof: We say that two matrices are equal if corresponding elements of each are equal. Therefore, if the conclusion of the lemma is valid for the two element set $[\lambda_1, \lambda_2]$, $0 < \lambda_1 \neq \lambda_2 > 0$, and the singleton subsets of T, [t] and [s], $0 < s \neq t > 0$, ($k = 1$, $m = 2$) it will be valid for larger k and m values. The matrices are

$$E_{12} = \begin{bmatrix} e^{i\lambda_1 t} & e^{i\lambda_2 t} \end{bmatrix} \qquad E'_{12} = \begin{bmatrix} e^{i\lambda_1 s} & e^{i\lambda_2 s} \end{bmatrix}$$

where $s = t+\tau$. In order that $E_{12} = E'_{12}$ the following set of simultaneous equations must be satisfied

$$e^{i\lambda_1 t} = e^{i\lambda_1 s} = e^{i\lambda_1 t} e^{i\lambda_1 \tau}$$

$$e^{i\lambda_2 t} = e^{i\lambda_2 s} = e^{i\lambda_2 t} e^{i\lambda_2 \tau}$$

Therefore,

(*) $$1 = e^{i\lambda_1 \tau} \text{ and } 1 = e^{i\lambda_2 \tau}$$

or, τ must satisfy

$$\tau = 2\pi\, n/\lambda_1 \text{ and } \tau = 2\pi\, \ell/\lambda_2$$

where n and ℓ are integers, which implies

$$\lambda_1/\lambda_2 = n/\ell$$

That is, λ_1/λ_2 must be rational. Therefore, if at least one λ_p/λ_q is irrational,

$$E_{km} \neq E'_{km}.$$

Q.E.D.

Note the following observations:

Remark 1: If the number of elements in the subsets of T are different then the corresponding matrices $E_{\cdot\cdot}$ and $E_{\cdot\cdot}$ cannot be equal. This possibility was purposely omitted from the statement of the lemma.

Remark 2: If the elements of the set $[\lambda_j]$ are harmonically related then no conclusion can be drawn from Lemma 5.1 other than it is possible that $E_{\cdot\cdot}$ and $E_{\cdot\cdot}$ are equal.

Remark 3: As k and m assume larger values the possibility of at least one element of E_{km} and E'_{km} being different is enhanced (see Theorem 6.3 for further discussion of this point).

Remark 4: We note that if τ in $s = t+\tau$ is a multiple of 2π or is 0, equations (*) are trivially satisfied. Non-trivial cases result if at least one τ, corresponding to the same location in the "t and s" matrices, respectively, is not a multiple of 2π, or 0.

Henceforth, when a fixed set of exponents $[\lambda_1, \lambda_2, \ldots, \lambda_n]$, n finite or not, is such that at least one ratio λ_p/λ_q, say, is irrational, we will refer to the set as an "incommensurate set".

Theorem 5.1: If $Z(t,\omega)$ is a s.p. expressible in the form of (4.4) with parameter set T, if the exponents form an incommensurate set, $[\lambda_i]$, and if the set of random coefficients is an element of the class of discrete parameter s.p.s associated with $Z(t,\omega)$; then, there exists a one-one correspondence between the family of non-random matrices of the form of E_{km} in (5.3) and the FF-DD which uniquely defines $Z(t,\omega)$.

Proof: From Theorem 4.5 the series in (4.4) is guaranteed to converge (or sum) to a process $Z(t,\omega)$, say, which is an element of a certain equivalence class of processes. Let $\{\xi_k\}$ be the set of coefficients in (4.4) regarded as a s.p.; by hypothesis, this s.p. represents all possible sets of coefficients that satisfy the convergence criteria.

$Z(t,\omega)$ is a s.p. and therefore is uniquely specified by a FF-DD (Kolmogorov Theorem). For every choice of finite k and set $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ there is only one k-dimensional distribution function corresponding to the random vector $\vec{Z}_k(\omega) = [Z(t_{n_1})\ Z(t_{n_2})\ \ldots\ Z(t_{n_k})]^T$.

To every $\vec{Z}_k(\omega)$ there corresponds a product $E_{km}\vec{\xi}_m(\omega)$ as given in (5.3). From Lemma 5.1 and the incommensurate exponent set, E_{km} is uniquely defined for every choice $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$, $\forall$ finite k. And since $\vec{\xi}_m(\omega)$ in (5.3) is common to all the random vectors $\vec{Z}_k(\omega)$ independent of what set $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ is selected, it follows that the matrix E_{km} solely reflects that selection. That is, to every $Z_k(\omega)$ there corresponds an E_{km}, and only one.

Therefore, to every element of the FF-DD there corresponds one matrix E_{km}; and to the FF-DD there corresponds a family of E_{km} matrices.

Q.E.D.

The next theorem is similar to the preceding one. It considers the case where the set of exponents are harmonically related.

Theorem 5.2: If $Z(t,\omega)$ is expressible as in (4.4) with parameter set T, if the set of exponents are harmonically related, if the set of coefficients is an element of the class of discrete parameter s.p.s associated with $Z(t,\omega)$, if the set $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ for all finite k is such that $t_{n_j} \varepsilon (t, t+d)$, $\forall j$, and arbitrary $t \varepsilon T$, and if d is the period for the harmonic relationship; then, there exists a one-one correspondence between the family of non-random matrices of the form E_{km} in (5.3) and the FF-DD which uniquely defines $Z(t,\omega)$.

Proof: Essentially by contradiction. Consider an arbitrary element of E_{km}, say $e^{i\lambda_\ell t}$, where $\lambda_\ell = 2\pi\ell/d$ and ℓ is an integer between 1 and m. Without loss of generality let t = 0. Choose two different values of T, $t_1 < d$ and $s_1 < d$, such that $e^{i\lambda_\ell t_1} = e^{i\lambda_\ell s_1}$. That is, $i2\pi\ell t_1/d = i2\pi\ell s_1/d$ or $t_1 = s_1$, a contradiction.

Therefore, by selecting values of t and s in (0, d) the only way elements of corresponding E_{km} matrices could be equal is if the respective values of t = s. It follows then, that to every $\vec{Z}_k(\omega)$ there corresponds only one E_{km} and of course only one k-dimensional distribution function. The argument is now as clearly as in Theorem 5.1.

Q.E.D.

5.3 Stochastic Process of Fourier Coefficients And a Sampling Relation

In this section we continue to pursue an answer to the questions, what can be said about the random vector $\vec{\xi}_m(\omega)$ in terms of $Z(t,\omega)$? Is there any assurance that a FF-DD for the coefficients $\xi_k(\omega)$ can be meaningfully constructed; i.e., will the FF-DD satisfy the Kolmogorov conditions of symmetry and consistency? Such questions are fundamental to the use of Fourier coefficients as elements of a stochastic process.

One way to approach such questions is to try to use (5.3) and to demonstrate that $\vec{Z}_k(\omega)$ and $\vec{\xi}_k(\omega)$ are both elements of ℓ_2-space,

$$\overline{\overrightarrow{Z_k(\omega)}}^T = (Z_1, Z_2, \ldots, Z_k, 0, 0, \ldots)^T \in \ell_2(1, k)$$

and

$$\overline{\overrightarrow{\xi_m(\omega)}}^T = (\xi_1, \xi_2, \ldots, \xi_m, 0, 0, \ldots)^T \in \ell_2(1, m).$$

Then, if $E^{-1}{}_{km}$ exists, it might be possible to write $\vec{\xi} = E^{-1}{}_{km}\vec{Z}_k$, where $E^{-1}{}_{km}$ is the inverse of matrix E_{km}. The difficulty is that $E^{-1}{}_{km}$ can be shown to exist only for certain restrictive values of k and m.

Another way to try to gain insight about $\vec{Z}_k(\omega)$ is to take advantage of the uniqueness relation between probability distribution functions and characteristic functions. Consider two statements of the uniqueness theorem:

(i) A distribution function is uniquely determined by its characteristic function (Gnedenko and Kolmogorov, 1968, p. 50).

(ii) Two distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ are identical if, and only if, their characteristic functions $\varphi_1(\square)$ and $\varphi_2(\square)$ are identical (Lukacs, 1960, p. 35).

Recall that for a real-valued random vector $\vec{W}$, the characteristic function was given by

(2.10) $$\varphi_{\vec{W}}(\vec{s}) = \mathcal{E}\, e^{+i\vec{s}^T\vec{W}}$$

where $\vec{s}$ is a real-valued, non-random vector.

Suppose that vector $\vec{W}$ is related to vector $\vec{V}$ by $\vec{W} = A\vec{V}$, where A is a non-random, real, rectangular matrix. Then the following lemma is known.

Lemma 5.2: The characteristic function of $\vec{W}$ may be expressed as

$$\varphi_{\vec{W}}(\vec{s}) = \varphi_{\vec{V}}(A^T\vec{s})$$

Proof: Let $\varphi_{\vec{V}}(\vec{T}) = \mathcal{E}\, e^{i\vec{r}^T\vec{V}}$ and $\varphi_{\vec{W}}(\vec{s}) = \mathcal{E}\, e^{i\vec{s}^T\vec{W}} = \mathcal{E}\, e^{i\vec{s}^T A\vec{V}}$,

where

$\vec{W} = A\vec{V}$. If $\vec{r}^T = \vec{s}^T A$ or $\vec{r} = A^T\vec{s}$, we can write

$$\varphi_{\vec{W}}(\vec{s}) = \varphi_{\vec{V}}(A^T\vec{s}) .$$

Q.E.D.

If matrix A is A_{km}, say (therefore, $\vec{W}$ and $\vec{s}$ are k x 1; $\vec{V}$ and $\vec{r}$ are m x 1) the equation just proved is

(5.4) $$\varphi_{\vec{W}}(\vec{s}) = \varphi_{\vec{V}}(A_{mk}\vec{s}) \text{ where } \vec{W} = A_{km}\vec{V}$$

Application of the Kolmogorov theorem requires that the FF-DD satisfy the conditions of symmetry and consistency (see Section 2.3). We will repeat these conditions here in terms of distribution functions and then state corresponding conditions in terms of characteristic functions. The uniqueness relationship between a distribution function and a characteristic function guarantees that to every member of the FF-DD there corresponds one characteristic function. Hence, there is a family of finite-dimensional characteristic functions which corresponds in a one-one manner to the FF-DD.

As has been our custom, let $x_j = X(t_j)$, $j = 1, 2, 3, \ldots, n$.

(i) symmetry: for every finite n, the n-dimensional distribution function $F_{x_1; x_2, \ldots, x_n}(v_1, v_2, \ldots, v_n) =$ $F(v_1, v_2, \ldots, v_n; t_1, t_2, \ldots, t_n)$ is symmetric in all pairs $(v_j, x_j) \Leftrightarrow (v_j, t_j)$, such that when v_j and x_j (or t_j) are subjected to the same permutation the distribution function is invariant; e.g., if n = 2,

$$F_{x_1, x_2}(v_1, v_2) = F_{x_2, x_1}(v_2, v_1)$$

$$= F(v_2, v_1; t_2, t_1) .$$

(ii) consistency: for every finite n, a typical marginal (n - 1)-dimensional distribution function is

$$\lim_{v_n \to \infty} F_{x_1, x_2, \ldots, x_n}(v_1, v_2, \ldots, v_n) =$$

$$F_{x_1, x_2, \ldots, x_{n-1}}(v_1, v_2, \ldots, v_{n-1}) =$$

$$F(v_1, v_2, \ldots, v_{n-1}, t_1, \ldots, t_{n-1}) .$$

In terms of characteristic functions,

(i') symmetry: for every finite n, the n-dimensional characteristic function

$$\varphi_{x_1, x_2, \ldots, x_n}(\vec{s}) = \mathcal{E} \exp\,[i(x_1 s_1 + x_2 s_2 + \ldots + x_n s_n)]$$

is symmetric in all pairs (x_j, s_j), such that when v_j and s_j are subjected to the same permutation the characteristic function is invariant; e.g., if n = 2,

$$\varphi_{x_1, x_2}(s_1, s_2) = \varphi_{x_2, x_1}(s_2, s_1)$$

(ii') consistency: for every finite n, a typical (n - 1)-dimensional characteristic function is

$$\varphi_{x_1, x_2, \ldots, x_n}(s_1, s_2, \ldots, s_n)\Big|_{s_n=0} =$$

$$\varphi_{x_1, x_2, \ldots, x_{n-1}}(s_1, s_2, \ldots, s_{n-1})\ .$$

Devising appropriate notation has been a problem in (i), (ii), (i') and (ii'). The difficulty basically stems from the fact that it would be more accurate to display the actual elements, $[t_1, t_2, \ldots, t_n]$, selected in specifying a distribution function or a characteristic function, but later on in this section it will become notationally clumsy to do so. Therefore, to some large extent, the notation is selected for ease of usage.

Consider the following, for n = 2: The function $F_{x_1, x_2}(v_1, v_2)$ is also written as $F(v_1, v_2; t_1, t_2)$; the former displaying the fact that realizations are involved, the latter displaying the distinct elements of T. The function $\varphi_{x_1, x_2}(s_1, s_2)$ could have been written as $\varphi(s_1, s_2; t_1, t_2)$ but was not because it will be very convenient later, to write $\varphi_{x_1, x_2}(s_1, s_2)$ as $\varphi_{\vec{X}}(s_1, s_2)$, where $\vec{X} = \vec{X}_n = [x_1 \; x_2]^T$. In addition to problems such as these, we have noted before the usual abuse of notation whereby an n-dimensional and an (n - 1)-dimensional function have the same name.

By choosing all finite values of n, a unique family of finite-dimensional characteristic functions can be constructed to correspond in a one-to-one fashion with a FF-DD. In Lemma 5.3 we show that if the latter family satisfies the Kolmogorov conditions the former family is guaranteed to.

We return now to the specific topic of investigation, the relationship between $Z(t,\omega)$ and $\vec{\xi}(\omega)$. Recall that the random vector $\vec{Z}_k(\omega)$ is constructed by the choice of $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$, all elements of the parameter set T. The right hand side of

$$(5.3) \qquad \vec{Z}_k(\omega) = E_{km}\vec{\xi}_m(\omega)$$

displays the choice of t_{n_i} only in the matrix E_{km}. Therefore, for another choice of elements of T, $[t_{q_1}, t_{q_2}, \ldots, t_{q_k}]$ say, the random vector $\vec{\xi}_m(\omega)$ is unchanged but E_{km} is different. The E_{km} matrix is as shown in equation (5.2) and consists of elements of the form $e^{i\lambda_j t_\ell}$.

It would be attractive to use (2.10) to write the characteristic function of $\vec{Z}_k(\omega)$ as "$\varphi_{\vec{Z}}(\vec{s})$" and the characteristic function of $\vec{\xi}_m(\omega)$ as "$\varphi_{\vec{\xi}}(\vec{r})$" and then to use Lemma 5.2 and equations (5.3) and (5.4) to obtain

$$\text{"}\varphi_{\vec{Z}}(\vec{s}) \overset{?}{=} \varphi_{\vec{\xi}}(\vec{r}) \overset{?}{=} \varphi_{\vec{\xi}}(E_{mk}\vec{s})\text{"}$$

where

$$\vec{r} = E_{mk}\vec{s}$$

However, in (2.10) the vectors corresponding to $\vec{r}$, $\vec{s}$, $\vec{Z}$ and $\vec{\xi}$ must be real-valued. Hence, if $Z(t,\omega)$ is complex-valued, "$\varphi_{\vec{Z}}(\cdot)$" and "$\varphi_{\vec{\xi}}(\cdot)$" will not be defined.

The characteristic function approach can still be useful in the attempt to learn about the coefficient vector $\vec{\xi}_j(\omega)$ if we are willing to restrict consideration to processes $X(t,\omega)$, which are real-valued or if we define the vector process $\vec{U}(t,\omega) = [X(t,\omega) \quad Y(t,\omega)]^T$ where $Z(t,\omega) = X(t,\omega) + iY(t,\omega)$. In either case the characteristic functions corresponding to vectors $\vec{X}_k(\omega)$ or $\vec{U}_{2k}(\omega)$ could be defined. We will restrict our attention to the process $X(t,\omega)$ and the vector $\vec{X}_k(\omega)$, thereby returning to matters considered in Chapter 3.

In accordance with Theorem 3.2, $Z(t,\omega)$ is a zero-mean, real-valued s.p. $X(t,\omega)$ [1] if, and only if, $\xi_{2k-1}(\omega) = \bar{\xi}_{2k}(\omega)$ and $\lambda_{2k-1} = -\lambda_{2k}$ for k = 1, 2, ..., m/2 where m is the number of points in the spectrum. From the proof of sufficiency in Theorem 3.2, it is possible to write $X(t,\omega)$ as

$$(5.5) \qquad X(t,\omega) = \sum_{j=1}^{m/2} (\eta_j \cos \kappa_j t + \zeta_j \sin \kappa_j t)$$

where $\xi_\ell = \frac{1}{2}(\alpha_\ell - i\beta_\ell)$, $\ell = 1, 2, \ldots, m$; $\alpha_{2n-1} = \alpha_{2n} = \eta_n$, $\beta_{2n-1} = -\beta_{2n} = \zeta_n$, $\lambda_{2n-1} = -\lambda_{2n} = \kappa_n$ for n = 1, 2, ..., m/2. Clearly ξ_j, η_j, ζ_j, α_j, β_j are real-valued random variables.

In equation (5.3) the E_{km} matrix consists of complex-valued elements. We now define a new matrix, D_{km}, which consists of real-valued

[1] For the remainder of this chapter the process $Z(t,\omega)$ being real-valued, will be denoted by $X(t,\omega)$.

elements, and is related to equation (5.5) in the same sense as E_{km} is related to equation (4.4). As in Theorem 3.2, the number m is a finite, even integer.

Definition 5.2: The D_{km} matrix consists of cosine and sine elements,

$$D_{km} = \begin{bmatrix} \cos \kappa_1 t_1 & \cos \kappa_2 t_1 \ldots\ldots \cos \kappa_{\frac{m}{2}} t_1 & \sin \kappa_1 t_1 \ldots\ldots \sin \kappa_{\frac{m}{2}} t_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cos \kappa_1 t_k & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \sin \kappa_{\frac{m}{2}} t_k \end{bmatrix}$$

In analogy to equation (5.3), we write the vector equation,

$$(5.6) \qquad \vec{X}_k(\omega) = D_{km}\, \vec{\delta}_m(\omega)$$

where $\vec{X}_k(\omega)$ corresponds to the real-valued s.p. $\vec{X}(t,\omega)$ evaluated at $[t_1, t_2, \ldots, t_k]$; $\vec{X}_k(\omega) = [x_1, x_2, \ldots, x_k]^T$, $\vec{\delta}_m(\omega) = [\eta_1, \eta_2, \ldots, \eta_{m/2}, \zeta_1, \ldots, \zeta_{\frac{m}{2}}]^T$, and $x_j = X(t_j)$.

From (2.10) the characteristic function of the real-valued random vector $\vec{X}_k = \vec{X}_k(\omega)$ is

$$(5.7) \qquad \varphi_{\vec{X}}(\vec{s}) = \mathcal{E}\, e^{i\vec{s}^T\vec{X}_k}$$

and the characteristic function of $\vec{\delta}_m = \vec{\delta}_m(\omega)$ is

$$(5.8) \qquad \varphi_{\vec{\delta}}(\vec{r}) = \mathcal{E}\, e^{i\vec{r}^T\vec{\delta}_m}$$

By use of Lemma 5.2 and equations (5.4) and (5.6) $\varphi_{\vec{X}}(\vec{s})$ may be written

(5.9) $\varphi_{\vec{X}}(\vec{s}) = \varphi_{\vec{\delta}}(D_{mk}\vec{s}) = \varphi_{\vec{\delta}}(\vec{r})$

provided $\vec{r}$ and $\vec{s}$ are connected by

(5.10) $\vec{r} = D_{mk}\vec{s}$,

where $D_{mk} = D_{km}^{T}$.

For every finite subset of T which is selected to determine an $\vec{X}_k = \vec{X}_k(\omega)$ there is a characteristic function $\varphi_{\vec{X}}(\vec{s})$ and, from (5.9), there is a corresponding characteristic function $\varphi_{\vec{\delta}}(\vec{r})$. We have discussed how each $\varphi_{\vec{X}}(\vec{s})$ corresponds to a finite-dimensional cistribution function and how a family of finite-dimensional characteristic functions, $\varphi_{\vec{X}}(\vec{s})$, can be constructed to correspond to the FF-DD. Therefore, in some sense, there is a family of finite-dimensional characteristic functions, $\varphi_{\vec{\delta}}(\vec{r})$, which also correspond to the FF-DD.

It is necessary to carefully distinguish between two uses of the word "family" when discussing a collection of characteristic functions of the random vectors, $\vec{\delta}_{\ell}(\omega)$. On the one hand is the family of characteristic functions each of the form $\varphi_{\vec{\delta}}(\vec{q})$ corresponding to the vector $\vec{\delta}_m(\omega)$, by Lemma 5.2; where, for any choice $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$, $\vec{\delta}_m(\omega)$, is unchanged and the "family" is generated by different choices of subsets of T. On the other hand is the family of characteristic functions each of the form $\Phi_{\vec{\delta}}(\vec{r})$, each corresponding to a random vector $\vec{\delta}_{\ell}(\omega)$, for any finite, positive integer $\ell \ni 0 \le \ell \le m$.

It is also worth noting here that the respective values of k and m could also influence the meaning of "family". For example if $k < m$ (the set of elements of T is less than the number of points in the spectrum) then the domain of the corresponding function $\varphi_{\vec{\delta}}(\vec{r})$ will be a subset of the domain of the same function, if s = r. The question is, in constructing the family will the two functions be counted as different? And, if k > m, what then? We shall, in constructing the family count each case as a separate element of the family.

Using the word family as in the case of $\varphi_{\vec{\delta}}(\vec{q})$, and recalling the statements of the consistency and symmetry conditions in terms of characteristic functions, we want to know whether the assumption that the family of characteristic functions, each of the form $\varphi_{\vec{X}}(\vec{s})$, satisfies these conditions necessarily implies that the family of characteristic functions, each of the form $\varphi_{\vec{\delta}}(\vec{q})$, satisfies parallel conditions.

<u>Lemma 5.3</u>: Let $\varphi_{\vec{\delta}}(\vec{r})$ be a member of the family of characteristic functions corresponding to the random vector $\vec{\delta}_m(\omega)$ and to the set $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$. If the FF-DD constructed from all possible sets of t_{n_j}, for all finite k, satisfies the consistency and symmetry conditions, then the whole family of characteristic functions also satisfies these conditions.

Proof: The characteristic function corresponding to one member of the FF-DD is $\varphi_{\vec{X}}(\vec{s})$, say. From (5.9) $\varphi_{\vec{X}}(\vec{s}) = \varphi_{\vec{\delta}}(D_{mk}\vec{s}) = \varphi_{\vec{\delta}}(\vec{r})$. As usual, there are m coefficients which constitute $\vec{\delta}_m(\omega)$; without loss of essential generality let k = 2, so that $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}] = [t_1, t_2]$.

Let the elements of $D_{mk} = D_{m2}$ be denoted by a_{jn}, where n = 1, 2; j = 1, 2, ..., m. Let $x_j = X(t_j)$, and define the following vectors:

$$\vec{X}_k = X_2(\omega) = [x_1, x_2]^T, \ \vec{s} = [s_1, s_2]^T, \ \vec{\delta}_m = \vec{\delta}_m(\omega) =$$

$$[\eta_1, \eta_2, \ldots, \eta_{\frac{m}{2}}, \zeta_1, \zeta_2, \ldots, \zeta_{\frac{m}{2}}]^T,$$

and $\vec{r} = [r_1, r_2, \ldots, r_m]^T$. For k = 2, and using (5.10); $\vec{r} = D_{m2}\vec{s}$ is written,

$$\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ r_m \end{bmatrix} = \begin{bmatrix} a_{11} \ldots a_{12} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ a_{m1} \ldots a_{m2} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} a_{11} s_1 + a_{12} s_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} s_1 + a_{m2} s_2 \end{bmatrix} .$$

The consistency conditions are:

(i) $\varphi_{\vec{X}}(\vec{s}) \big|_{s_2=0} = \varphi_{x_1, x_2}(s_1, 0) = \varphi_{x_1}(s_1)$ and

$\varphi_{\vec{X}}(\vec{s}) \big|_{s_1=0} = \varphi_{x_1, x_2}(0, s_2) = \varphi_{x_2}(s_2)$

From above,

$$\varphi_{\vec{X}}(\vec{s}) = \varphi_{\vec{\delta}}(\bar{r})$$

$$= \varphi_{\vec{\delta}}([(a_{11} s_1 + a_{12} s_2), (a_{21} s_1 + a_{22} s_2), \ldots, (a_{m1} s_1 + a_{m2} s_2)]^T).$$

From (i),

$$\text{(ii)}\begin{cases}\text{(a)} & \varphi_{\vec{X}}(s_1,\ 0) = \varphi_{\vec{\delta}}\ ([a_{11}s_1,\ a_{21}s_1,\ \ldots,\ a_{m_1}s_1]^T) \\ \text{(b)} & \varphi_{\vec{X}}(0, s_2) = \varphi_{\vec{\delta}}\ ([a_{12}s_2,\ a_{22}s_2,\ \ldots,\ a_{m_2}s_2]^T)\ .\end{cases}$$

From (i) and (5.9)

$$\text{(iii)}\begin{cases}\text{(a)} & \varphi_{x_1}(s_1) = \varphi_{\vec{\delta}}(D_{m_1}\vec{s}) = \varphi_{\vec{\delta}}([a_{11}s_1,\ a_{21}s_1,\ \ldots,\ a_{m_1}s_1]^T) \\ \text{(b)} & \varphi_{x_2}(s_2) = \varphi_{\vec{\delta}}(D_{m_2}\vec{s}) = \varphi_{\vec{\delta}}([a_{12}s_2,\ a_{22}s_2,\ \ldots,\ a_{m_2}s_2]^T)\end{cases}$$

Equations (ii) (a) and (iii) (a) are equal and equations (ii) (b) and (iii) (b) are equal; therefore, for k = 2, the consistency conditions are satisfied.

The symmetry condition, for k = 2, where $x_j = X(t_j)$ is,

(iv) $\varphi_{x_1,\ x_2}(s_1,\ s_2) = \varphi_{x_2,\ x_1}(s_2,\ s_1)$.

To express (iv) in terms of $\varphi_{\vec{\delta}}(\vec{r})$ requires that we define matrix $D'_{mk} = D'_{m_2}$,

$$D'_{m_2} = \begin{bmatrix} a_{12} & a_{11} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{m_2} & a_{m_1} \end{bmatrix} = \begin{bmatrix} \cos \kappa_1 t_2 & \cos \kappa_1 t_1 \\ \vdots & \vdots \\ \cos \kappa_{\frac{m}{2}} t_2 & \cos \kappa_{\frac{m}{2}} t_1 \\ \sin \kappa_1 t_2 & \sin \kappa_1 t_1 \\ \vdots & \vdots \\ \sin \kappa_{\frac{m}{2}} t_2 & \sin \kappa_{\frac{m}{2}} t_1 \end{bmatrix}$$

and vectors $\vec{s}'$ for k = 2,

$$\vec{s}' = [s_2 s_1]^T .$$

Then,

$$\varphi_{x_2, x_1}(\vec{s}') = \varphi_{\vec{\delta}}(D'_{m_2}\vec{s}')$$

and we may write the symmetry condition with respect to $\vec{\delta}_m$, for k = 2, as

(v) $\varphi_{\vec{\delta}}(D_{m_2}\vec{s}) = \varphi_{\vec{\delta}}(D'_{m_2}\vec{s}')$.

To show that (v) is satisfied is straightforward.

The family of finite-dimensional characteristic functions of the form $\varphi_{\vec{\zeta}}(\vec{r})$ is constructed by choosing all possible sets $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ for all finite k. By proceeding as above, for arbitrary k, the conclusion follows.

Q.E.D.

In order to be able to make use of the properties of the characteristic functions we have developed it would be useful to demonstrate that the characteristic function $\varphi_{\vec{\delta}}(\vec{r})$ corresponding to $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ will be the same function as the characteristic function $\varphi_{\vec{\delta}}*(\vec{r}*)$ corresponding to $[t*_{n_1}, t*_{n_2}, \ldots, t*_{n_k}]$ for any finite k; i.e., that

$$\varphi_{\vec{\delta}}(\vec{r}) = \varphi*_{\vec{\delta}}(\vec{r}*)$$

whenever $\vec{r} = \vec{r}*$, over the whole domain of $\vec{r}$.

From Lemma 5.2

$$\varphi_{\vec{X}}(\vec{s}) = \varphi_{\vec{\delta}}(D_{mk}\vec{s}) = \varphi_{\vec{\delta}}(\vec{r})$$

and

$$\varphi_{\vec{X}*}(\vec{q}) = \varphi*_{\vec{\delta}}(D*_{mk}\vec{q}) = \varphi*_{\vec{\delta}}(\vec{r}*)$$

where

$$\text{(i)} \quad \begin{cases} \vec{r} = D_{mk}\vec{s} \\ \vec{r}* = D*_{mk}\vec{q} \end{cases}$$

If $\vec{r} = \vec{r}*$ it follows that

$$\text{(ii)} \qquad D_{mk}\vec{s} = D*_{mk}\vec{q}$$

and from the definition of characteristic functions,

$$\text{(iii)} \qquad \varphi_{\vec{\delta}}(D_{mk}\vec{s}) = \mathcal{E}\ e^{i(D_{mk}\vec{s})^T\vec{\delta}_m}$$

and

$$\text{(iv)} \qquad \varphi*_{\vec{\delta}}(D*_{mk}\vec{q}) = \mathcal{E}\ e^{i(D*_{mk}\vec{q})^T\vec{\delta}_m}$$

Substitution of (ii) into (iii) yields $\mathcal{E}\ e^{i(D*_{mk}\vec{q})^T\vec{\delta}_m}$ which is the same as the right hand side of (iv). Since $\vec{r}$ is an element of an m-dimensional vector space and r* is an element of an m-dimensional space we conclude that for at least some points $\vec{r}$ and $\vec{r}*$ in these vector spaces,

$$\text{(v)} \qquad \varphi_{\vec{\delta}}(\vec{r}) = \varphi*_{\vec{\delta}}(\vec{r}*) \quad \text{whenever } \vec{r} = \vec{r}*.$$

We note the following facts: equation (i) is a transformation from a k-dimensional space of which $\vec{s}$ is a generic element into an m-dimensional space of which $\vec{r}$ is a generic element; m is fixed by (4.4) as the size of the spectrum or equivalently, as the number of Fourier coefficients in $\vec{\delta}_m = [\delta_1, \delta_2, \ldots, \delta_m]^T$; the Bochner conditions (4.2) specify that the δ_j are mutually uncorrelated, or orthogonal.

To be meaningful, the characteristic function must be identified on its full domain and not on a subspace, therefore $\vec{r}$ must be of full m-dimension. This is consistent, of course, with $\vec{\delta}_m$ being m-dimensional. In order for $\vec{r}$ to be of full m-dimension in equation (i) $\vec{s}$, which is of k-dimension, must be such that it is of at least dimension m $(k \geq m)$ and D_{mk} must be of rank m. A similar line of reasoning applies to $\vec{r}*$.

Matrix D_{mk} $(k \geq m)$ will be of rank m if it contains m linearly independent rows. It is straightforward to show that this is a generalization of that considered for random variables in Remark 6 to Theorem 3.2. For example, if $m = 4$, $k = 5$, the four rows are linearly independent if, for $j = 1, 2, \ldots, 5$,

$$a_1 \cos \kappa_1 t_j + a_2 \cos \kappa_2 t_j + a_3 \sin \kappa_1 t_j + a_4 \sin \kappa_2 t_j = 0$$

implies, $a_1 = a_2 = a_3 = a_4 = 0$.

It will be useful now to recall statement (i) of the uniqueness theorem for characteristic functions which states that a characteristic function uniquely specifies a distribution function for a random vector. We have thus proven the following,

Theorem 5.3: Let $\vec{\delta}_m$ be a random vector of coefficients, let $[t_{n_1}, t_{n_2}, \ldots, t_{n_k}]$ and $[t*_{n_1}, t*_{n_2}, \ldots, t*_{n_k}]$ be two arbitrary, finite subsets of T and let $\varphi_{\vec{\delta}}(\cdot)$ and $\varphi*_{\vec{\delta}}(\cdot)$ be the corresponding characteristic functions, respectively. Then, these characteristic functions are identical, *i.e.*, the distribution function of $\vec{\delta}_m$ is independent of the choice of the above mentioned subsets of T.

The implication of the proof of Theorem 5.3 is that as the spectrum increases in size, larger and larger subsets of T must be taken to specify the distribution function of the vector of Fourier coefficients $\vec{\delta}_m$.

Indeed, this conclusion is in harmony with known sampling theorems, but is not as specific as these theorems.

6. PHASE PROPERTIES OF STOCHASTIC PROCESSES

6.1 Précis

This chapter presents a different aspect of the problem to which we have been addressing ourselves, viz., what can the statistical properties of the coefficients, ξ_k, tell us about the stochastic model, $Z(t,\omega)$, of the phenomena under study? Associated with each coefficient ξ_k is a phase angle φ_k, also a random variable. The questions to be considered are, do the phase angles have any correspondence with the biological phenomena of interest (e.g. those underlying the EEG signal) and what, if any, mathematical tools can be used to display this correspondence.

Section 6.2 is a general discussion of the following: how the phase is related to the coefficients; a partial explanation, at best, of why little effort has gone into the study of phase properties whereas a great deal of effort has gone into the study of Fourier series in EEG analysis; and a brief description of some work that has been done.

Section 6.3 considers the problem in the context of the series in Chapter 3. Some rather interesting conclusions about the role that phase plays with respect to the WSS condition are displayed.

Section 6.4 considers the phase angle from a deterministic perspective; it is influenced somewhat by one of the papers discussed in Section 6.2.

6.2 General Discussion

We are concerned in this section with the phase properties of the "frequency components" of a stochastic process or of a deterministic function. Generally, in what little literature there is on the subject of phase, "frequency components" is replaced by "harmonic components". We prefer the former terminology because it is more general, thereby allowing the use of the material in Chapter 3. (The harmonic case being the special case where the λ_k are harmonically related and the stochastic coefficients are found from (2.20). In either case, if $\xi_k(\omega)$ is the complex-valued coefficient, we specify a random amplitude coefficient γ_k by

$$(6.1) \qquad \gamma_k = \gamma_k(\omega) = |\xi_k(\omega)| \quad \forall\ k\ \varepsilon\ T^*$$

and a random phase coefficient φ_k by

$$(6.2) \qquad \varphi_k = \varphi_k(\omega) = \arg(\xi_k(\omega)) \quad \forall\ k\ \varepsilon\ T^*,$$

where γ_k and φ_k are real-valued r.v.s. The three coefficients are of course related by

$$(6.3) \qquad \xi_k = \xi_k(\omega) = \gamma_k e^{-i\varphi_k} .$$

A similar situation applies to the non-random case, where the coefficient (for the harmonic case) would be found from (2.17).

The discussions in Chapters 1 and 2 pointed out that the analysis of EEG-type data has been strongly based upon techniques developed for the field of communication theory. One of the standard (perhaps the singly most popular) methods of analysis consists of looking at the spectrum of the process or function; viz., spectral analysis. That is, one examines the relative magnitudes of the terms $\mathcal{E}|\xi_k|^2$ for $k \varepsilon T^*$. Some of the relevant theory is touched on in Chapter 2. We will not discuss in this dissertation the assumptions and details of spectral analysis; however, references previously quoted, such as Thomas (1969), Dubes (1968) and Davenport and Root (1958), discuss these matters at length.

We do note however, that the spectrum destroys all phase information. That is, consider (6.3) and evaluate $\mathcal{E}|\xi_k|^2$ as follows,

$$(6.4) \qquad \mathcal{E}|\xi_k|^2 = \mathcal{E}\ \xi_k\bar{\xi}_k = \mathcal{E}(\gamma_k e^{-i\varphi_k}\bar{\gamma}_k e^{+i\varphi_k}) = \mathcal{E}|\gamma_k|^2.$$

There are then, an infinite number of processes (or functions), each with different phase properties, which have the same spectrum.

For a long time this seems to have been of little or no concern to those doing EEG analysis with spectral methods. As indicated in Section 1.2 and by the papers we will discuss next, the situation may be changing. The paper by Usov and Orlov (1968) is concerned with the fact that the Fourier series analysis (harmonic analysis, based on picking a segment of length D, etc., see Section 3.3) of a s.p. does not yield sufficient information to reconstruct the record. They seem to trace the problem to their inability to ascertain initial phase relations for the harmonics. They go on to suggest a relatively simple "criterion of the quality of congruence" of the $k^{\underline{th}}$ phase, from a comparison of the $k^{\underline{th}}$ harmonic, apparently calculated from doing a

Fourier series, as the length D is shifted about. They conclude that more statistical work must be done and that the phase parameters they measured "are not uncorrelated with the phenomena investigated".

A second paper of interest (Brugge et al., 1969) studies the time-domain responses of the discharges in single auditory nerve fibers of a monkey, in response to complicated periodic sounds. The objective was to study the responses as the initial phase angles of the component frequencies were systematically varied. A lot of data was statistically analyzed and one of their conclusions was that the neurons they were concerned with are very sensitive to a phase shift of a component frequency. They note that this seems at odds with the classical view. They too feel that much work remains to be done.

6.3 Time Dependent Phase

The foundations for this section were essentially put down in Chapter 3 where the discussion concerned the series representation of a stochastic process $Z(t) = Z(t,\omega)$,

$$(6.5) \qquad Z(t,\omega) = \sum_{1}^{n} \xi_k(\omega) e^{i\lambda_k t} \qquad \forall\ t \in T$$

where $\xi_k = \xi_k(\omega)$ are random variables, the set of λ_k is purely discontinuous, and n may be non-finite. For the moment, we do not require that Z(t) be WSS but note Theorem 3.1 which says, in part, that a necessary and sufficient condition for Z(t) to be WSS is that

$$\mathcal{E}\ \xi_k \overline{\xi}_j = 0 \qquad k \neq j \quad .$$

As in Chapter 3, we impose pointwise convergence with respect to t and do not specify a mode with respect to ω.

A central point of the proof of Theorem 3.1 was that $\mathcal{E}|\xi_k|^2$ not be a function of t ε T. Suppose we wrote ξ_k as a function of t ε T, say $\xi_k(t) = \gamma_k(t)e^{-i\varphi_k(t)}$. Since $\gamma_k(t)$ and $\varphi_k(t)$ remain real-valued, $\mathcal{E}|\xi_k|^2 = \mathcal{E}|\gamma_k(t)|^2$; i.e., the phases cancel out but the t-variable remains as part of $\gamma_k(t)$. The proof of Theorem 3.1 would tolerate at most, variation with respect to τ. However, let us agree that henceforth, we will consider γ_k to be independent of t and τ, but that $\varphi_k = \varphi_k(t)$ is at least possible. When ξ_k are Fourier coefficients, their definitions as integrals over t preclude any t-dependence.

In the sequel, the symbols $[x_k]$ will refer to the collection of x_k $\forall$ k ε T*. Then $[\varphi_k(t)]$ and $[\gamma_k]$ are the collection of all phases and amplitudes, respectively.

Definition 6.1: The r.v. $\Phi_k(t,s)$ is called the phase increment random variable, where

$$(6.6) \qquad \Phi_k(t,s) = \varphi_k(t) - \varphi_k(s) \text{ for each } k \in T^*$$

and $[\Phi_k(t,s)] = [\varphi_k(t) - \varphi_k(s)]$ is called the collection of phase increments, $\forall$ k ε T*; t, s ε T.

Theorem 6.1: Let Z(t,ω) be a zero mean s.p. of the form of (6.5), written as

$$(6.7) \qquad Z(t,\omega) = \sum_1^n \gamma_k e^{-i\varphi_k(t)} e^{+i\lambda_k t} \qquad \forall\ t \in T$$

If the amplitudes in the collection, $[\gamma_k]$, are pairwise uncorrelated, then a sufficient condition for Z(t,ω) to be WSS is that the phase increments in the collection $[\Phi_k(t,s)]$ depend on t, s ε T only through the difference (t-s).

Proof: $Z(t,\omega)$ will be WSS if the covariance function $B(t,s) = B(\tau)$, where τ = t-s. Compute B(t,s) in the usual way and let n = 2; it has been demonstrated in Theorem 3.1 that we do this with no essential loss of generality.

$$B(t,s) = \mathcal{E}[(\xi_1 e^{i\lambda_1 t} + \xi_2 e^{i\lambda_2 t})(\bar{\xi}_1 e^{-i\lambda_1 s} + \bar{\xi}_2 e^{-i\lambda_2 s})]$$

$$= \mathcal{E}[(\gamma_1 e^{-i\varphi_1(t)} e^{+i\lambda_1 t} + \gamma_2 e^{-i\varphi_2(t)} e^{+i\lambda_2 t})(\gamma_1 e^{+i\varphi_1(s)} e^{-i\lambda_1 s} + \gamma_2 e^{+i\varphi_2(s)} e^{-i\lambda_2 s})]$$

$$\text{(i)} \quad B(t,s) = \mathcal{E}(|\gamma_1|^2 e^{i\lambda_1\cdot(t-s)} e^{-i[\varphi_1(t)-\varphi_1(s)]}) + \mathcal{E}(|\gamma_2|^2 e^{i\lambda_2\cdot(t-s)} e^{-i[\varphi_2(t)-\varphi_2(s)]}) .$$

As in Theorem 3.1, coefficients of exponentials of the form $e^{i\lambda_k t} e^{-i\lambda_j s}$ are to be set equal to zero if Z(t) is to be WSS; i.e., the amplitudes are pairwise uncorrelated, or $\mathcal{E}\, \gamma_1 \gamma_2 = \mathcal{E}\, \gamma_2 \gamma_1 = 0$ (in general, $\mathcal{E}\, \gamma_k \gamma_j = 0 \quad \forall\, k \neq j$).

Let τ = (t-s) and substitute into (i), which may be written as

$$\text{(6.8)} \qquad B(t,s) = \sum_{k=1}^{n=2} e^{i\lambda_k \tau}\, \mathcal{E}(|\gamma_k|^2 e^{-i[\varphi_k(t)-\varphi_k(s)]})$$

Equation (6.8) has been shown for n = 2 and is easily generalized for arbitrary finite n. In order for B(t,s) to be a function of τ = t-s, say $B(\tau)$, it is sufficient that $\Phi_k(t,s)$ be a function of τ = t-s, say $\Phi_k(\tau)$, for each k ε T*, $\tau \subset$ T.

Q.E.D.

Theorem 6.1 demonstrates that the phase of the coefficients in $Z(t,\omega)$ may be dependent upon $t \varepsilon T$ and yet $Z(t,\omega)$ may still be WSS. In retrospect, this result is expected when the phase dependence is in keeping with the general characteristic associated with WSS processes; viz., that the covariance function depends upon interval lengths rather than specific values of $t \varepsilon T$.

It is the stated objective of this section to emphasize that the properties of the phase should be studied more thoroughly in EEG work. With this in mind, let us examine some of the ramifications of Theorem 6.1.

We now consider some of the conditions under which $\Phi_k(t,s)$ will be a function of $\tau = (t-s)$ only. With the collection $[\Phi_k(t,s)]$ in mind, let us briefly discuss a s.p., $v(t)$, whose first and second moments exist but whose covariance function is such that $B_v(t,s) \neq B_v(\tau)$. We can define an increment process $W(t,s) = v(t)-v(s)$, say, and let us say that it has the property that $W(t_1, s_1)$ and $W(t_2, s_2)$ are stochastically independent when $[t_1, s_1]$, $[t_2, s_2]$ are non-overlapping intervale; i.e., $v(t)$ is an independent increment process. In general, the probability distribution of $W(t,s)$ depends on t and s. Suppose, however, that in this case, $W(t,s)$ has the properties that $\mathcal{E}\, W(t,s) = 0$, $\mathcal{E}(W(t,s))^2 = \sigma^2|t-s| = \sigma^2 \tau$ $\forall\, t > s > 0$, $t, s \varepsilon T$, $\sigma^2 > 0$ and $W(t,s)$ is normally distributed; then $v(t)$ is said to be a Wiener process (Wiener-Lévy or Brownian process).

It is not too difficult to intuitively associate the rapid variations of an EEG record with the rapid fluctuations one imagines in

Brownian motion and to think of the variations as being caused in part, by phase fluctuations. Therefore, one might tentatively associate $\varphi_k(t)$ with $v(t)$ and $\Phi_k(t,s)$ with $W(t,s)$.

A related, but less sophisticated possibility is to consider the s.p. $v(t)$ with the property that the corresponding increment process $v(t+\tau) - v(t)$, is such that its distribution function depends only upon the length τ and not upon the endpoints. Such a s.p., $v(t)$, is said to be a homogeneous process.

In fact, consider a phase relation

$$\varphi_k(t) = \alpha_k t + \beta_k \qquad \forall\ t \in T \text{ and } \forall\ k \in T^*,\ \alpha_k \text{ and } \beta_k \text{ r.v.s.}$$

Then,

$$\varphi_k(t+\tau) - \varphi_k(t) = \alpha_k \tau \qquad \forall\ k \in T^*,\ \forall\ t \in T,$$

which may be denoted as

$$\Phi_k(\tau) = \alpha_k \tau\ .$$

The preceding discussion has been phrased in terminology implying that the set of phases $\varphi_k(t)$ is a s.p. with parameter set $t \in T$. This should not be confused with the discussions in Chapter 4 which centered about the question of whether the set of coefficients, γ_k is a s.p. with parameter set $k \in T^*$.

6.4 Phase Differences Between Frequency Components

The paper by Brugge, et al (1969) contains a number of histograms which are the result of responses (the rate and temporal distribution of neuronal discharges) to phase varying stimuli. The consisted of a sum of two sinusoids of different frequency and phase;

responses were recorded as the initial phase of the higher frequency component was varied relative to the initial phase of the lower.

The pattern that developed in the shape of the histograms depended very much upon the phase differences. As the phase difference approached 2π, the shape of the histograms approached that of the zero phase difference histogram. It is clear then, that knowing if and when a point of zero phase difference will occur is an important aspect in designing experiments and interpreting data dealing with the phase sensitivity of neurons. With this in mind, Theorem 6.2 is stated and proved below.

In looking at an EEG record it is apparent that more than two frequency components are present. Again, the question of the occurrence of times of zero phase difference is important. Consider the question, given n frequency components, what is the liklihood of finding such times? Theorem 6.3 considers this question.

We define the following terms for use in Theorems 6.2 and 6.3. Let

$$(6.9) \qquad R(t) = \sum_{\ell=1}^{n} c_\ell e^{i(\omega_\ell t + \varphi_\ell)}$$

where the c_ℓ are complex constants called amplitudes; ω_ℓ are angular frequencies, which are not necessarily harmonically related; and φ_ℓ are initial phase angles associated with the ℓ^{th} frequency ω_ℓ. There is no probabilistic aspect to (6.9).

Theorem 6.2: Given two components of different angular frequency and different initial phase, it is always possible to find a time, $t = t_o$, such that they are instantaneously in phase. Furthermore, t_o will occur within every period of the lower frequency, independent of the values of the initial phases if, and only if, the higher frequency is at least twice as great as the lower one.

Proof: Let $R(t) = c_j e^{i(\omega_j t+\varphi_j)} + c_k e^{i(\omega_k t+\varphi_k)}$, where ω_k, ω_j, φ_k, and φ_j are defined in (6.8). The subscripts j, k are used rather than 1, 2, say, to emphasize that there may be many terms in a sum like (6.9) and that this theorem applies to any pair of them. The two components will be in-phase when the arguments are equal; i.e., when $t = t_o$,

$$\omega_j t_o + \varphi_j = \omega_k t_o + \varphi_k$$

or

$$\text{(i)} \qquad t_o = (\varphi_j - \varphi_k)/(\omega_k - \omega_j)$$

where it is given that $\omega_k \neq \omega_j$ and $\varphi_k \neq \varphi_j$. That is, t_o is the time when the frequency components are instantaneously in phase; t_o exists and is non-zero.

We denote the period of the lower frequency by T_j and that of the higher frequency by T_k, so that $T_j > T_k$. Note that for an angular frequency ω_ℓ, $\omega_\ell = 2\pi f_\ell = 2\pi/T_\ell$.

Rearranging (i) and substituting in the appropriate values of ω_k and ω_j,

$$t_o = \frac{\varphi_j-\varphi_k}{\omega_k-\omega_j} = \frac{1}{2\pi}\,(\varphi_j-\varphi_k)\left[\frac{1}{1/T_k - 1/T_j}\right] = \frac{1}{2\pi}\,(\varphi_j-\varphi_k)\left[\frac{1}{\frac{T_j-T_k}{T_j\,T_k}}\right]$$

(ii) $$t_o = \frac{1}{2\pi}(\varphi_j - \varphi_k)T_j \left[\frac{T_k}{T_j - T_k}\right] .$$

The maximum value of $(\varphi_j - \varphi_k)$ that need be considered is 2π radians. Denoting the corresponding maximum of t_o by t_{om},

(iii) $$t_{om} = \frac{1}{2\pi}(2\pi)T_j \left[\frac{T_k}{T_j - T_k}\right] = T_j \left[\frac{T_k}{T_j - T_k}\right] \geq t_o .$$

Let us arbitrarily pick $t = 0$ as the origin and measure periods from there. Suppose then, that $t_o \in [0, T_j]$; we show now that this implies that $\omega_k \geq 2\omega_j$. From the preceding, $t_o \leq t_{om} \leq T_j$ and $t_{om} = T_j \left[\frac{T_k}{T_j - T_k}\right]$; in order for t_{om} to satisfy the second half of the inequality, $\left[\frac{T_k}{T_j - T_k}\right] \leq 1$. That is, $T_k \leq T_j - T_k$, or $2T_k \leq T_j$, or $T_k \leq \frac{1}{2}T_j$, or $\omega_k \geq 2\omega_j$. The "only if" part of the theorem is established.

Suppose that $\omega_k \geq 2\omega_j$; we show now that this implies that $t_o \leq T_j$. If $\omega_k \geq 2\omega_j$ then,

$T_j \geq 2T_k$, or $T_j - T_k \geq T_k$, or $(T_k/(T_j - T_k)) \leq 1$. From (iii),

$$t_o \leq T_j \left[\frac{T_k}{(T_j - T_k)}\right] = t_{om} \leq T_j ,$$

or $t_o \leq T_j$.

Q.E.D.

Remark on Theorem 6.2: This theorem is stated in such a way that an incommensurate relationship is allowed between ω_k and ω_j. If ω_k and ω_j are harmonically related, it is a special case also covered by the theorem.

An informal statement of a generalization of the next theorem is, as the number of frequency contributions in R(t) in (6.9) increases, it gets increasingly less likely that a time will be found when all the frequencies are simultaneously and instantaneously in-phase. This is neither a surprising conclusion, nor is it trivial.

Note that the theorem can be proven for larger n than $n = 4$, but we will not do it here; the procedure is basically the same as is used in the proof of the theorem for $n = 4$ but more would have to be said about selecting which equations to analyze and so on.

We mentioned on an earlier page that knowing the prospect of finding times of simultaneous zero phase difference among all of the constituent frequencies could be of interest in designing certain experiments which are concerned with detection of phase differences. We have mentioned one such experiment.

In a slightly different vein, studies of musical-instrument tones (Risset and Mathews, 1969) have demonstrated that the frequency spectrum alone is a poor descriptor of musical tone; there is even an indication that "warmth" in piano tones is correlated with inharmonicity of the waveshape.

We note that Theorem 6.3 was referred to in Remark 3 to Lemma 5.1. This lemma seeks to demonstrate that by selecting times, $t_1, t_2, \ldots, t_k$, and times, $s_1, s_2, \ldots, s_k$, their respective matrices E_{km} will not be identical under certain conditions. The elements of the matrix are of the form $e^{i\lambda t}$; the point is that for some τ, $e^{i\lambda t} = e^{i\lambda(t+\tau)}$, and for some t and s, $e^{i\lambda(t+\tau)} = e^{i\lambda s}$, where $t = t_j$, $s = s_j$ for a fixed j. The question relevant to Theorem 6.3 is for a fixed set $[\lambda_k]$, will a set of τ-values occur, and if so how frequently, such that the corresponding matrices are equal?

Lemma 5.1 was proved for λ_k which are not commensurate, the lemma not being valid for λ_k which have a harmonic relation. Theorem 6.3 is proven for the general case, the frequencies being specified only as unequal.

While the statement and proof of Theorem 6.3 are entirely deterministic in nature, we show that the likelihood of the four frequency components being simultaneously in-phase is small by constructing a 4-dimensional hyperspace, each initial phase, φ_k, being a coordinate. Then a set of two appropriate simultaneous equations with four unknowns must be satisfied. The solution is a set of parallel 2-dimensional planes in 4-space. In general, these 2-dimensional planes do not lie in a coordinate plane of the 4-space.

Theorem 6.3: Given 4 unequal frequency components of the form used in (6.9), the set of all initial angles φ_j, $j = 1, 2, \ldots, n$, such that all of the components are simultaneously in-phase, is of dimension 2. This set is the union of a small number of parallel, 2-D planes.

Proof: Let $R(t) = \sum_{\ell=1}^{4} c_\ell e^{i(\omega_\ell t+\varphi_\ell)}$. From Theorem 6.2, there exists a time, t_o, for every pair of elements of the sum, such that they are instantaneously in-phase; for the elements k and j, denote t_o by t_{kj}, so that

$$t_{kj} = \frac{\varphi_j-\varphi_k}{\omega_k-\omega_j}$$

Because of the periodic nature of each element, there are many times t_{kj} when the $k^{\underline{th}}$ and $j^{\underline{th}}$ term are in-phase. We denote these by $t_{kj}(m)$, $m \in T^*$, which are calculated from,

$$2\pi m + \omega_j t + \varphi_j = \omega_k t + \varphi_k$$

$$t_{kj}(m) = \frac{\varphi_j - \varphi_k}{\omega_k - \omega_j} + \frac{2\pi m}{\omega_k - \omega_j} \qquad m \in T^*,$$

where, $t_{kj} = t_{kj}\ (0)$.

Note the symmetry condition, $t_{kj}(m) = t_{jk}(m)$ and that terms $t_{kk}(\cdot)$ do not occur because $\omega_k \neq \omega_j$ by hypothesis.

The question now is whether we can find <u>one</u> time for which <u>all</u> pairs of terms are in-phase. Consider the 4 terms characterized by the frequencies $\omega_p > \omega_q > \omega_r > \omega_s$. Times of zero phase difference are: $t_{pq}(m_1)$, $t_{qr}(m_2)$, $t_{rs}(m_3)$, $t_{pr}(m_4)$, $t_{ps}(m_5)$, $t_{qs}(m_6)$, where $m_i \in T^*$ and $i = 1, 2, 3, 4, 5, 6$.

Recall that

$$\text{(i)} \qquad t_{kj} = t_{kj}(0) = \frac{\varphi_j - \varphi_k}{\omega_k - \omega_j}\ , \ k > j.$$

Hence, the points of zero phase differences between elements of each pair (we know such points exist by Theorem 6.2) are,

$$t_{pq}(m_1) = t_{pq}(0) + \left(\frac{2\pi}{\omega_p - \omega_q}\right) m_1 \qquad t_{ps}(m_5) = t_{ps}(0) + \left(\frac{2\pi}{\omega_p - \omega_s}\right) m_5$$

$$t_{qr}(m_2) = t_{qr}(0) + \left(\frac{2\pi}{\omega_q - \omega_r}\right) m_2 \qquad t_{qs}(m_6) = t_{qs}(0) + \left(\frac{2\pi}{\omega_q - \omega_s}\right) m_6$$

$$t_{rs}(m_3) = t_{rs}(0) + \left(\frac{2\pi}{\omega_r - \omega_s}\right) m_3$$

$$t_{pr}(m_4) = t_{pr}(0) + \left(\frac{2\pi}{\omega_p - \omega_r}\right) m_4$$

In order that a time exist where all terms are instantaneously in-phase, it is necessary that there exist a 6-tuple $(m^*_1, m^*_2, \ldots, m^*_6)$ such that

(ii) $\quad t_{pq}(m^*_1) = t_{qr}(m^*_2) = t_{rs}(m^*_3) = t_{pr}(m^*_4) = t_{ps}(m^*_5) = t_{qs}(m^*_6).$

From (ii) consider the set of equalities,

(iii)
$$t_{pq}(m^*_1) = t_{qr}(m^*_2)$$
$$t_{qr}(m^*_2) = t_{rs}(m^*_3)$$

These may be written in terms of frequency and phase as,

(iv)
$$\frac{\varphi_q-\varphi_p}{\omega_p-\omega_q} + \frac{2\pi}{\omega_p-\omega_q}\, m^*_1 = \frac{\varphi_r-\varphi_q}{\omega_q-\omega_r} + \frac{2\pi}{\omega_q-\omega_r}\, m^*_2$$
$$\frac{\varphi_r-\varphi_q}{\omega_q-\omega_r} + \frac{2\pi}{\omega_q-\omega_r}\, m^*_2 = \frac{\varphi_s-\varphi_r}{\omega_r-\omega_s} + \frac{2\pi}{\omega_r-\omega_s}\, m^*_3$$

which may be written using the shortened notation $\omega_{kj} = \omega_k-\omega_j$ as

(v)
$$\left[\frac{m^*_1}{\omega_{pq}} - \frac{m^*_2}{\omega_{qr}}\right] = \frac{1}{2\pi}\left[\frac{\varphi_r}{\omega_{qr}} - \frac{\varphi_q}{\omega_{qr}} - \frac{\varphi_q}{\omega_{pq}} + \frac{\varphi_p}{\omega_{pq}}\right]$$
$$\left[\frac{m^*_2}{\omega_{qr}} - \frac{m^*_3}{\omega_{rs}}\right] = \frac{1}{2\pi}\left[\frac{\varphi_s}{\omega_{rs}} - \frac{\varphi_r}{\omega_{rs}} - \frac{\varphi_r}{\omega_{qr}} + \frac{\varphi_q}{\omega_{qr}}\right] \quad .$$

After some gruesome algebra (v) may be written as,

$$\omega_{qr} m^*_1 - \omega_{pq} m^*_2 = \frac{1}{2\pi} [\omega_{pq}\varphi_r - \omega_{pr}\varphi_q + \omega_{qr}\varphi_p]$$

(vi)

$$\omega_{rs} m^*_2 - \omega_{qr} m^*_3 = \frac{1}{2\pi} [\omega_{qr}\varphi_s - \omega_{qs}\varphi_r + \omega_{rs}\varphi_q].$$

The right hand side may be written as,

$$\text{(vii)} \quad \frac{1}{2\pi} \begin{bmatrix} 0 & \omega_{pq} & -\omega_{pr} & \omega_{qr} \\ \omega_{qr} & -\omega_{qs} & +\omega_{rs} & 0 \end{bmatrix} \begin{bmatrix} \varphi_s \\ \varphi_r \\ \varphi_q \\ \varphi_p \end{bmatrix}$$

The equations in (vi) and the coefficient matrix in (vii) are, of course, the result of the selection of the two equations in (iii). The coefficient matrix is of rank 2. Suppose, in order to satisfy (ii) we choose from (ii) a third equation, e.g., $t_{rs}(m^*_3) = t_{pr}(m^*_4)$, and possibly a fourth and a fifth. And then following the steps (iii) to (vii) above, the result would be larger coefficient matrices. It can be shown that these matrices are also of rank 2.

Therefore, it is sufficient to judiciously choose two equations from (ii) to find the appropriate phase solution. This solution will be the union of a number of parallel 2-dimensional hyperplanes in 4-space as the numbers m^*_1, m^*_2 and m^*_3 are varied. The remark following the end of the proof explains this more fully.

Furthermore, since only the values of φ_k, k = 1, 2, 3, 4, that lie in $[-\pi, \pi)$ are of interest, a hypercube is defined in 4-space within (and on) which solutions must lie.

The theorem is therefore established.

Q.E.D.

Remark: That the coefficient matrix is of rank 2 is interpreted to mean that there are two linearly independent equations. The equations in (vi) define 3-dimensional hyperplanes in the 4-space $(\varphi_s, \varphi_r, \varphi_q, \varphi_p)$. The intersection of these 3-dimensional hyperplanes is a 2-dimensional hyperplane corresponding to the two linearly independent equations. All this is for a fixed set, (m^*_1, m^*_2, m^*_3). As these (m^*_1, m^*_2, m^*_3) are varied, hyperplanes parallel to the first are defined.

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8. APPENDICES

8.1 Separable Stochastic Processes

This section neither introduces anything that is not in the literature, nor does it relate in any continuous fashion with the preceding material. The intent is simply to point out that some concepts occasionally referred to earlier, have considerably more depth than is readily apparent. We refer to the notion of separable stochastic processes.

Theorem 4.5 contains the result that s.p.s $Z'(t,\omega)$ and $Z''(t,\omega)$ are equivalent, where $t \in T$ (continuous parameter set). This is legitimate, but is also more complicated than simply applying the discrete parameter set notion that $Z'(t_\ell,\omega) = Z''(t_\ell,\omega)$, $\forall\ \ell \in T^*$. The reason is that T is not countable and we are therefore forced to consider limits along T and such limits need not be measurable. The ordinary framework of probability theory is not constructed to consider such situations. Essentially then, we are interested in that subclass of a specific class of equivalent (continuous T) processes which is subject to the analytical machinery available. That there always is such a choice within any equivalence class is guaranteed by a theorem of Doob which states that within every equivalent class of s.p.s there is always at least one which has the desired properties; it is called a separable process.

Such conumdrums are usually evaded in one of two ways. One way is to study only those processes which satisfy certain regularity properties. Then, one is guaranteed that the same probability is assigned by two equivalent processes for any event. The regularity conditions can be rather "loose". The books of Cramér and Leadbetter (1967) and Thomasian (1969) consider this approach in some detail. A related approach is to consider processes which are defined by some "nice" analytical expression. This was somewhat the situation in the preceding, whenever a series converged uniformly in t.

The references given so far have a slightly practical bent to them and are lighter on the conceptual issues. For discussions of the latter, the books of Loève (1963) and, of course, Doob (1953), are useful.

8.2 Abbreviations, Symbols and Page of First Occurrence

WSS	wide-sense stationary	19
FF-DD	family of finite-dimensional distributions	17
p.p.(d)	purely periodic of period d	22
a.p.	almost periodic	23
a.s.	almost surely	22
T	continuous parameter set	20
T*	discrete parameter set	21
s.p.(s)	stochastic process(es)	31
q.m.	quadratic mean	37
r.v.(s)	random variable(s)	33
Ω	basic probability space	14
$\mathfrak{X}$	space of all real-valued functions on T or T*	17
$\bar{x}$	complex conjugate of x	15
(x_n)	sequence of terms, x_n	54
$\{x_n\}$	discrete parameter s.p.; x_n is a r.v. for each $n \in T^*$	70
$[x_n]$	set of terms, x_n	21
A^T	transpose of matrix A	20

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