Error Computation and Optimal Mesh in Elasticity and Elastoplasticity

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Abstract

The proposed method is based on the computation of the error on the constitutive law associated with a finite element mesh. The computation of this error is made using a statically admissible stress field which is obtained explicitly from the finite element solution (displacement approach). From the knowledge of the error, an optimal mesh can be derived. This approach is different from the one developed by BABUSKA and RHEINBOLT which only leads to error indicators and, more recently from the one developed by ZIENKIEWICZ et al.

This work is based on the method of P. LADEVEZE which has been previously applied to thermal problems. This method allows to construct a statically admissible stress field using the following procedure :

1) a force density is derived from the "kinematic" finite element stress field, solving explicitly local problems associated with each vertix,
2) given the force density on the edges of elements, the "static" stress field is then derived explicity for each element of the mesh.

This numerical procedure only needs a small amount of new calculations in addition to the finite element analysis. Since the two stress field are avalaible, it is then possible to compute, element by element, the local contribution to the error on the constitutive law. This allows a local mesure of the mesh quality and permits a means of obtaining, at any location, the new optimal mesh size which will lead to a uniform error of given value. The mesh size being avalaible at any location, a semi-automatic or automatic mesh generation procedure can be applied.

In plasticity, the definition of the error on the constitutive law has been extended in order to take into account the incremental and non-reversible nature of the stress-strain relationship.

1. Introduction

The use of finite element programs based on the displacement approach is now generalized in most fields of structural and mechanical engineering. Among the parameters that govern the accuracy and the cost of an analysis, the mesh holds a place of prime importance, which is not well understood by many users, as it defines the range of displacement fields in which the approximate solution will be found.

Of course the experience of users yields meshes which give satisfactory results for given types of calculations, but each new problem requires preliminary tests in order to obtain a reliable discretization. It must also be noticed that the large choice of preprocessors offered to engineers now renders the use of finite elements very easy for unskilled users who are not always in a position to check the results they obtain from programs. In addition to these remarks, a good mesh for making calculations which involve materials nonlinearities (elastoplasticity - damage), under loading conditions having arbitrary variations is not easy to be found a priori, because very important changes may occur in the stress field during the analysis. Thus measures of the quality of a mesh are obviously needed.

The proposed method is based on the computation of the error on the constitutive law associated to a finite element mesh. It is based on the following mechanical remark : if the kinematic and equilibrium equations and boundary conditions of a well defined problem are exactly satisfied, then the constitutive relation still need not be satisfied. This deficiency in satisfying the constitutive relations may provide a means of identifying the error which has a clear physical meaning.

The computation of this error is carried using the finite element displacement solution which satisfies the kinematic constraints and a statically admissible stress field. This stress field is obtained explicitely from the finite element solution, and the numerical procedure only needs a small amount of new calculations in addition to the first analysis.

The contribution of each element to the total error allows a local measure of the mesh quality and this permits a means of obtaining, at any location, the new optimal mesh size which will lead to a uniform given value of the error. As the procedure used to get the new mesh size involves a linearization, in some cases, more than one iteration can be necessary to get a uniform value of the error. The mesh size being available at any location, a semi-automatic or automatic mesh generation procedure can be applied.

In addition to this automatic mesh refinement procedure, the error is used to control the iterative schemes.

This approach is different of the one proposed by BABUSKA and RHEINBOLT [1] which only leads to error indicators and more recently from the one developed by ZIENKIEWICZ et al [4]. The method has been implemented using a constant strain triangle. First numerical results are shown.

2. Basic notations

At the present time, the method is developed for small displacements plane problems. Hereafter, we consider a polygonal domain Ω which has been discretized in constant strain triangular elements Ω_i, $i \in \{1, 2, \ldots N\}$ and Ω is submitted to a given set of boundary conditions.

Let $\bar{u}$ and $\bar{\sigma}$ be, respectively the displacement and stress fields associated to the exact solution of the problem. σ and $\varepsilon(u)$ are, respectively, the stress and strain fields associated to the kinematically admissible displacement field u, solution of the finite element problem.

3. Error on the constitutive law

If we assume that a kinematically admissible displacement field u and a statically admissible stress field $\hat{\sigma}$ are known, the only relationship which is not verified by the couple $(u, \hat{\sigma})$ is the constitutive law.

3.1 Elasticity

For any pair$(u, \hat{\sigma})$, the quadratic error on the constitutive law is defined by :

$$e = \|\hat{\sigma} - \sigma\| = \|\hat{\sigma} - C.\varepsilon(u)\| \tag{1}$$

and the associated relative error :

$$e_r = \frac{e}{\|\sigma\|} \tag{2}$$

where $\|\sigma\|^2 = \int_{\Omega} \sigma^T . C^{-1} . \sigma \, d\Omega$

the PRAGER-SYNGE theorem may be written :

$$4\mathcal{E}^{*2} = \frac{\|\bar{\sigma} - \sigma\|^2 + \|\bar{\sigma} - \hat{\sigma}\|^2}{\|\sigma\|^2} = e_r^2 \tag{3}$$

$$\mathcal{E}^* = \frac{\|\bar{\sigma} - \sigma^*\|}{\|\sigma\|}$$

where $\sigma^* = \frac{1}{2}(\hat{\sigma} + \sigma)$

The contributions of Ω_i to $\mathcal{E}^*$ are local measures of the error on the constitutive law.

3.2 Plasticity

For the sake of simplicity, we consider the PRANDLT-REUSS' model. The strain rate $\dot{\varepsilon}$ is related to the stress rate $\dot{\sigma}$ by :

$$\dot{\varepsilon} = C^{-1}.\dot{\sigma} + \gamma . h(s_{II}) \langle s^T . \dot{s} \rangle_+ . s \tag{4}$$

where h is a function characterizing the material behaviour,
s is the stress deviator,
s_{II} is the second invariant of s
$\langle \ \rangle_+$ denotes the "positive part of".

γ takes the values 1 or 0. $\gamma = 1$ if the stress is on the boundary of the convex elastic domain, $\gamma = 0$ otherwise.

Using the theory of convex functions [7], [8], it can be shown that the constitutive relation may be defined by two dual functions φ and φ^*.

$$\begin{aligned} \varphi &: \quad \dot{\varepsilon} \longmapsto \varphi(\dot{\varepsilon}) = \frac{1}{2}\left(\dot{\varepsilon}^T.C.\dot{\varepsilon} - \frac{\gamma h}{1 + h.s^T.C.s} \langle s^T.C.\dot{\varepsilon} \rangle_+^2\right) \\ \varphi^* &: \quad \dot{\sigma} \longmapsto \varphi^*(\dot{\sigma}) = \frac{1}{2}\left(\dot{\sigma}^T.C^{-1}.\dot{\sigma} + \gamma h . \langle s^T.\dot{s} \rangle_+^2\right) \end{aligned} \tag{5}$$

such that, $\forall \dot{\varepsilon}, \dot{\sigma}$

$$\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\sigma}) - \dot{\sigma}^T.\dot{\varepsilon} \geq 0 \tag{6}$$

and

$$\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\sigma}) - \dot{\sigma}^T.\dot{\varepsilon} = 0 \tag{7}$$

if, and only if, the constitutive law is verified.

The quantity

$$e^2(t) = \int_{\Omega} \left(\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\sigma}) - \dot{\sigma}^T . \dot{\varepsilon} \right) d\Omega \qquad (8)$$

is therefore the error on the constitutive law, where t is a kinematic time.

The following quantities are associated to the couple $(u,\hat{\sigma})$ to measure the error which is made on the constitutive relations :

$$\varepsilon^{*2} = \frac{1}{2} \left(\int_{\Omega} \left(\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\hat{\sigma}}) - \dot{\hat{\sigma}}^T \dot{\varepsilon} \right) d\Omega \right) \Big/ \left(\int_{\Omega} \left(\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\hat{\sigma}}) \right) d\Omega \right) \qquad (9)$$

for an interval of time $[0,T]$ E^* and E^{**} are introduced :

$$E^{*2} = \frac{1}{T} \int_0^T \varepsilon^{*2} dt \qquad (10)$$

and

$$E^{**2} = \frac{1}{2} \left(\int_0^T dt \int_{\Omega} \left(\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\hat{\sigma}}) - \dot{\hat{\sigma}}^T \dot{\varepsilon} \right) d\Omega \right) \Big/ \left(\int_0^T dt \int_{\Omega} \left(\varphi(\dot{\varepsilon}) + \varphi^*(\dot{\hat{\sigma}}) \right) d\Omega \right) \qquad (11)$$

Obviously, the couple $(u,\hat{\sigma})$ is the exact solution on the interval $[0,T]$ if, and only if,

- ε^* is zero on $[0,T]$ or
- E^* or E^{**} is zero.

Hence, these quantities are global measures of the quality of $(u,\hat{\sigma})$. The contributions of the elements to these expressions are local measures of the error.

4. Explicit derivation of the field $\hat{\sigma}$ from the finite element solution

The statically admissible field is built up applying the method of P. LADEVEZE [2] and [3] to the two-dimensional continuum problems. This method [5], [6] takes advantage of the fact that the finite element stress field σ, despite its imperfections, gives a good approximation of the mean value of the exact stress field $\overline{\sigma}$ within the elements. Therefore, the knowledge of the stress field σ are used to build up $\hat{\sigma}$ without any requirement for a new global calculation.

$\hat{\sigma}$ is chosen as an extension of the finite element stress field σ and satisfies, by definition, eq. (12) on each element

$$\int_{\Omega_i} (\hat{\sigma} - \sigma) \, d\Omega = 0 \qquad (12)$$

$\hat{\sigma}$ is obtained in two steps :

1. determination of a force density $\hat{F}$, defined on each interelement boundary in order to ensure the continuity of the stress vector on the associated boundary,
2. $\hat{F}$ being known on each interelement boundary, deviation of $\hat{\sigma}$ on each element.

Determination of $\hat{F}$

The method described in [3] for thermal problems is generalized to the two-dimensional plane stress problem : the force density $\hat{F}$ is chosen to be linear on each element edge. It is obtained as the solution of a very simple linear system equations for each node of the mesh. In most cases, this linear system admits an infinity of solutions, which is the case for each internal node. In this case, it is possible to choose a solution which minimizes a convenient measure of the difference between $\hat{\sigma}$ and σ on the edges of elements.

Determination of $\hat{\sigma}$

Each triangle is subdivided into 3 sub-triangles and $\hat{\sigma}$ is derived explicitely [5], [6] on the element once the force density is known on its edges.

5. Optimal mesh

In elasticity, several criteria may be selected to define an optimal mesh. The most suitable, in our opinion, is related to the error contribution of elements. These contributions are defined by (13)

$$\varepsilon_i^{*2} = \frac{1}{4} \int_{\Omega_i} (\hat{\sigma} - \sigma)^T C^{-1} (\hat{\sigma} - \sigma)\, d\Omega \,/\, \|\sigma\|^2 \qquad (13)$$

and we have, from [2] and [3]

$$\varepsilon^{*2} = \sum_i \varepsilon_i^{*2} \qquad (14)$$

A mesh is said to be $\bar{\theta}$ - satisfactory if

$$\varepsilon^* \leq \bar{\theta}$$

It is said to be $\bar{\theta}$ - optimal if the element contributions ε_i^* to ε^* are uniform, in other words if

$$\varepsilon_i^{*2} = \frac{1}{N} \bar{\theta}^2$$

N being the number of elements in the mesh.

Once the error associated to a first mesh is avalaible, the optimal mesh is defined using the behaviour of the error as a function of the element size. For instance, the factor of reduction R_i of an element Ω_i is given by (15).

$$R_i = \frac{\bar{\theta}^2}{\varepsilon_i^* \sum_{j=1}^{N} \varepsilon_j^*}$$

In plasticity, this procedure can be used at the beginning of increments to initiate the mesh, replacing the elastic rigidity C by the one used for iterations.

6. Examples

A computer program has been developed to carry out two dimensional calculations.

The method was tested on a pure bending example for which the exact solution is known. The error ε^* was found to be 26 % over the error calculated from the exact solution.

In thermal problems, the construction of the optimal mesh was made to be automatic [3]. The results are very satisfactory and usually obtained in one iteration.

In elasticity, a semi-automatic procedure has been used, so the mesh presented below could be improved. Nevertheless it is much better than a regular mesh which would have the same number of elements. The boundary conditions are shown Fig. 1. A quasi optimal mesh leading to a value of ε^* near 5 % is shown Fig. 2.

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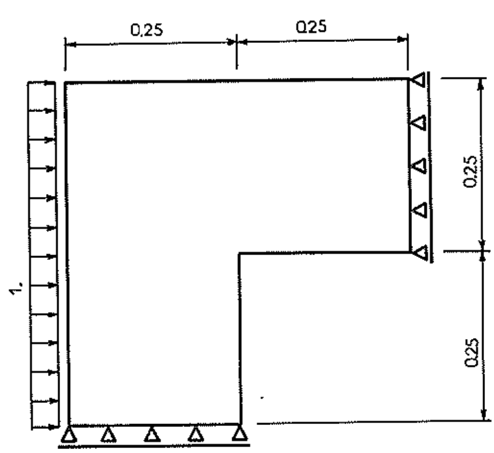

Figure 1 : L-shaped domain boundary conditions

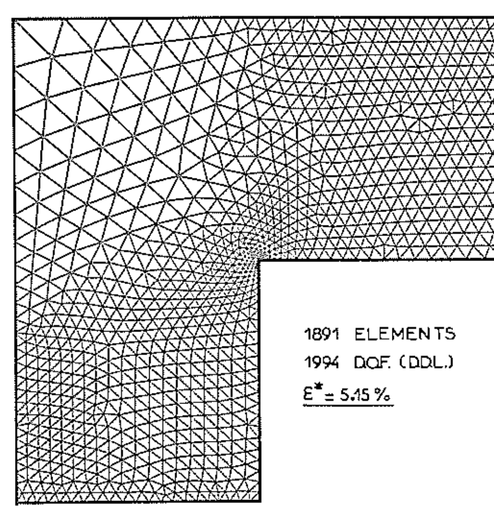

Figure 2 : Quasi-optimal mesh.