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PRESENT VALUE OF A RENEWAL PROCESS

by

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0. Summary.

This paper studies the present cost C of a renewal process, defined as the sum of the values of the costs of the replacements considered at the starting time of the renewal process, with a compound interest.

The characteristic function of C is found when the inter-arrival times are negatively exponentially distributed; the asymptotic properties of C as the rate of interest tends to zero are studied in the general case.

1. Introduction.

Let us consider a renewal process with inter-arrival times $X_1, X_2, \ldots$ identically and independently distributed with distribution function $F(x)$ ($F(0) = 0$). Starting at time $T_o = 0$ we will have a replacement at each of the instants $T_1 = X_1, T_2 = X_1 + X_2, \ldots$.

We suppose that each renewal has a constant cost, which we assume equal to 1. So one will have to pay one (dollar, say) at time T_1, one at time T_2, and so on. It is of interest to study the present value, at time 0, of these payments, assuming compound interest.

The present value A_o, at time 0, of a sum A at time T, is given by:

(1) $$A_o = e^{-\rho T} A$$

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where ρ is the force of interest (see, for instance, [3]). So the value C_i, at time 0, of a replacement which will take place at time T_i, is given by

$$(2) \qquad C_i = e^{-\rho T_i} .$$

The present value, at time 0, of the total cost of the renewal process, which will be denoted by C, is

$$(3) \qquad C = \sum_{i=1}^{\infty} C_i = \sum_{i=1}^{\infty} e^{-\rho \sum_{j=1}^{i} X_j}$$

Let us consider the random variables Y_i defined by

$$(4) \qquad Y_i = e^{-\rho X_i} .$$

They are independent and identically distributed; we will denote by α_r their moments. If X is a random variable with distribution function F(x), and $\phi_X(t)$ is the characteristic function of X, we have

$$(5) \qquad \alpha_r = E\, Y_j^r = E\, e^{-\rho r X} = \phi_X(i\, r\, \rho) .$$

With the exception of the trivial case $P(X = 0) = 1$, we shall have:

$$(6) \qquad 0 < \alpha_r < 1 .$$

With the introduction of the Y_i, we can write:

$$C = \sum_{i=1}^{\infty} \prod_{j=1}^{i} Y_j \ , \tag{7}$$

and the moments of C are easily expressed in terms of the α_r; putting

$$\gamma_r = E\,(C^r) \ , \tag{8}$$

we have

$$\gamma_1 = E \sum_{i=1}^{\infty} \prod_{j=1}^{i} Y_j$$

$$= \sum_{i=1}^{\infty} \alpha_1^i$$

$$= \frac{\alpha_1}{1 - \alpha_1} \tag{9}$$

$$\gamma_2 = E\Big(\sum_{i=1}^{\infty} \prod_{j=1}^{i} Y_j\Big)^2$$

$$= 2\ E \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \prod_{j=1}^{i} Y_j \prod_{s=1}^{k} Y_s + E \sum_{i=1}^{\infty} \Big(\prod_{j=1}^{i} Y_j\Big)^2$$

$$= 2\ E \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \prod_{j=1}^{i} Y_j^2 \prod_{j=i+1}^{k} Y_j + E \sum_{i=1}^{\infty} \prod_{j=1}^{i} Y_j^2$$

$$= 2 \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \alpha_2^i\, \alpha_1^{k-i} + \sum_{i=1}^{\infty} \alpha_2^i$$

$$= 2\,\frac{\alpha_2}{1 - \alpha_2}\ \frac{\alpha_1}{1 - \alpha_1} + \frac{\alpha_2}{1 - \alpha_2}$$

$$= \frac{\alpha_2(1 + \alpha_1)}{(1-\alpha_2)(1-\alpha_1)} \ .$$

(10) $$\sigma^2(C) = \frac{\alpha_2 - \alpha_1^2}{(1-\alpha_1)^2\,(1-\alpha_2)} .$$

The first two moments are finite, because of (6). It can be easily seen that all the moments of C are finite; in fact, as for γ_2, each moment can be expressed as the sum of a finite number of terms which are all finite. The same result could be obtained by an argument similar to the one used to prove that N_T (i.e. the number of renewals between 0 and T) has finite moments of all orders (see [2], p. 245).

From (3) we obtain moreover

(11) $$C = e^{-\rho X_1}(1 + C')$$

where C' has the same distribution as C, and is independent of X_1.

2. The case of the negative exponential distribution.

A particular case of great importance in applications is given by the negative exponential distribution, which will be studied in this section. In this case, we have

$$d\,F(x) = \lambda\, e^{-\lambda x}\, dx \qquad (x > 0)$$

with $\lambda > 0$. We have also

$$\phi_X(t) = \frac{\lambda}{\lambda - i\,t} ,$$

$$\alpha_r = \phi_X(i\,\rho\,r)$$

$$= \frac{\lambda}{\lambda + r\rho} ,$$

$$EC = \frac{\lambda}{\rho} ,$$

$$\sigma^2(C) = \frac{\lambda}{2\rho} .$$

In order to study the distribution of C, we can use (11). For the characteristic function of C we have

$$\phi_C(t) = E\, e^{i\,t\,C}$$

$$= E\, e^{ite^{-\rho X_1} + ite^{-\rho X_1} C'} = E_{X_1} E[e^{ite^{-\rho X} + ite^{-\rho X} C'} \mid X = X_1]$$

$$= E\, e^{ite^{-\rho X}} \phi_C(t\, e^{-\rho X}) ,$$

therefore

$$\phi_C(t) = \int_0^{+\infty} e^{ite^{-\rho x}} \phi_C(t\, e^{-\rho x})\, d\, F(x) \quad . \tag{12}$$

Introducing the density function of the negative exponential distribution, we obtain, by a change of variables:

$$\phi_C(t) = \int_0^{+\infty} e^{ite^{-\rho x}} \phi_C(t\, e^{-\rho x})\ \lambda\, e^{-\lambda x}\ d\, x$$

$$= \frac{\lambda}{\rho} \int_0^t e^{iy}\ \phi_C(y)\ y^{\frac{\lambda}{\rho} - 1}\ t^{-\frac{\lambda}{\rho}}\ d\, y \quad .$$

In order to solve this integral equation, we transform it into a differential equation, by differentiating on both sides after multiplying by $t^{\frac{\lambda}{\rho}}$. From

$$t^{\frac{\lambda}{\rho}} \phi_C(t) = \frac{\lambda}{\rho} \int_0^t e^{iy}\, \phi_C(y)\ y^{\frac{\lambda}{\rho} - 1}\, d\, y$$

we have

$$\frac{\lambda}{\rho}\ t^{\frac{\lambda}{\rho} - 1}\ \phi_C(t) + t^{\frac{\lambda}{\rho}}\, \phi_C'(t) = \frac{\lambda}{\rho}\ e^{it}\, \phi_C(t)\ t^{\frac{\lambda}{\rho} - 1}$$

and hence

$$\phi_C'(t) = \frac{\lambda}{\rho} \quad \frac{e^{it} - 1}{t} \quad \phi_C(t) \quad ,$$

with the condition

$$\phi_C(0) = 1 .$$

We obtain finally

$$(13) \qquad \phi_C(t) = \exp \left[\frac{\lambda}{\rho} \int_0^t \frac{e^{ix} - 1}{x} \, d\, x \right] .$$

From this expression we can derive the moments of C; expanding the exponentials in series, we have

$$\begin{aligned} \phi_C(t) &= \exp \left[\frac{\lambda}{\rho} \int_0^t \left(i - \frac{x}{2!} - i \, \frac{x^2}{3!} + \ldots \right) d\, x \right] \\ &= \exp \left[\frac{\lambda}{\rho} \left(i\, t - \frac{t^2}{4} - \frac{i}{3} \frac{t^3}{3!} + \ldots \right) \right] \\ &= 1 + \frac{\lambda}{\rho} \left(i\, t - \frac{t^2}{4} - \frac{i}{3} \frac{t^3}{3!} + \ldots . \right) \\ &\quad + \frac{1}{2} \frac{\lambda^2}{\rho^2} \left(i\, t - \frac{t^2}{4} - \frac{i}{3} \frac{t^3}{3!} + \ldots \right)^2 + \ldots \quad . \end{aligned}$$

Thus the first moments are

$$\gamma_1 = \frac{\lambda}{\rho} \ , \ \gamma_2 = \frac{\lambda}{2\rho} + \frac{\lambda^2}{\rho^2} \ , \ \text{i.e.} \ \sigma^2(C) = \frac{\lambda}{2\rho} \quad .$$

The same results could have been obtained from (9), (10) .

An interesting question concerning the random variable C is its asymptotic behaviour as ρ tends to zero. For the negative exponential distribution the study of this problem is made very easy by formula (13). Let us consider the new variable Z defined as

$$Z = \frac{C - EC}{\sigma(C)} = \frac{C - \gamma_1}{\sigma} .$$

Using (13) we have

$$\phi_Z(t) = \exp\left[\frac{-\gamma_1}{\sigma} it + \frac{\lambda}{\rho} \int_0^{\frac{t}{\sigma}} \frac{e^{ix} - 1}{x} dx \right].$$

Thus

$$\log \phi_Z(t) = \frac{-\gamma_1}{\sigma} it + \frac{\lambda}{\rho} \left(i \frac{t}{\sigma} - \frac{t^2}{4 \sigma^2} - \frac{1}{3} i \frac{t^3}{3! \sigma^3} + \dots \right)$$

$$= - \frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{1}{2}} t i + \frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{1}{2}} ti - \frac{\lambda}{\rho} \frac{2\rho}{\lambda} \frac{t^2}{4} - \frac{1}{3} \frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{3}{2}} \frac{t^3}{3!} + \dots$$

$$= - \frac{t^2}{2} - i \frac{2^{\frac{3}{2}}}{3} \lambda^{-\frac{1}{2}} \rho^{\frac{1}{2}} \frac{t^3}{3!} + \dots$$

All the terms in the last expression, except the first one, contain ρ with a positive exponent; the series is uniformly convergent for ρ in any interval of the form $0 \leq \rho \leq \Delta$; hence we have:

$$\lim_{\rho \to 0} \log \phi_Z(t) = - \frac{t^2}{2} .$$

Thus we have proved:

Theorem 1. If X has a negative exponential distribution, C is asymptotically normally distributed as ρ tends to zero.

3. The general case.

It can be easily seen that the negative exponential distribution is the only one which permits a simple solution of the integral equation (12) by a transformation into a differential equation. In the general case, since the integral equation

(12) is a homogeneous Volterra equation of second kind, the solution must be searched for among the singular solutions. Thus it appears rather difficult to find by this method the distribution of C for distributions of X other than the negative exponential one.

On the other hand, if we want to study the asymptotic properties of the distribution of C, other means are available; for instance, the investigation of the behaviour of the moments of C.

Let us first establish some lemmas.

Lemma 1. If, for an integer k, $\beta_k = E\, X^k < \infty$, then

$$(14) \qquad \alpha_r = 1 - r\,\beta_1\,\rho + \frac{r^2}{2}\,\beta_2\,\rho^2 + \dots + (-1)^k\,\frac{r^k}{k!}\,\beta_k\,\rho^k\,\theta(\rho)$$

where $0 \leq \theta(\rho) \leq 1$, and $\lim_{\rho \to 0} \theta(\rho) = 1$.

Proof: We have

$$e^{-r\rho x} = \sum_{i=0}^{k-1} (-1)^i\,\frac{r^i}{i!}\,x^i\,\rho^i + (-1)^k\,\frac{r^k}{k!}\,x^k\,\rho^k\,\theta_1(x, \rho) \quad ,$$

where

$$e^{-\rho r x} \leq \theta_1(x, \rho) \leq 1 .$$

If we integrate this equation, and use the mean value theorem, then we find

$$\int_0^{+\infty} e^{-r\rho x}\, d\,F(x) = \sum_{i=1}^{k-1} (-1)^i\,\frac{r^i}{i!}\,\beta_i\,\rho^i + (-1)^k\,\frac{r^k}{k!}\,\rho^k \int_0^{+\infty} x^k\,\theta_1(x,\rho)\, d\,F(x)$$

$$= \sum_{i=1}^{k-1} (-1)^i\,\frac{r^i}{i!}\,\beta_i\,\rho^i + (-1)^k\,\frac{r^k}{k!}\,\rho^k\,\beta_k\,\theta(\rho) \quad ,$$

where $0 \leq \theta(\rho) \leq 1$. Moreover

$$\theta(\rho) = \frac{1}{\beta_k} \int_0^{+\infty} x^k \, \theta_1 (x,\rho) \, d\, F(x) \quad .$$

Hence it follows from the hypotheses and the properties of $\theta_1(x,\rho)$:

$$\lim_{\rho \to 0} \theta (\rho) = 1 \, .$$

The lemma is thus proved.

<u>Lemma 2.</u> If $E\, X^k < \infty$ $(k \geq 1)$ then, for every $t \geq 0$,

$$\lim_{\rho \to 0} \rho^{-k} \, E \, e^{-\rho t X} (e^{-\rho X} - \alpha_1)^h = \begin{cases} E(X - \beta_1)^k & \text{if } h = k \\ 0 & \text{if } h > k \end{cases}$$

<u>Proof:</u> We can write, as in lemma 1,

$$e^{-\rho x} = 1 - \rho \, x \, \theta_1 \, (x, \rho) \quad ,$$

where $e^{-r\rho x} \leq \theta_1(x,\rho) \leq 1$; and

$$\alpha_1 = 1 - \rho \, \beta_1 \, \theta \, (\rho) \quad ,$$

where $0 \leq \theta \, (\rho) \leq 1$ and $\lim_{\rho \to 0} \theta \, (\rho) = 1$.

Then

$$\rho^{-k} E \, e^{-\rho t X} (e^{-\rho X} - \alpha_1)^h = E \, e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left(\frac{e^{-\rho X} - \alpha_1}{\rho} \right)^k$$

$$= E \, e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left[\frac{1 - \rho X \, \theta_1(x,\rho) - 1 + \rho \, \beta_1 \theta(\rho)}{\rho} \right]^k$$

$$= E \, e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left[\beta_1 \theta(\rho) - X \, \theta_1(X,\rho) \right]^k .$$

Moreover, for $h\text{-}k \geq 0$,

$$|e^{-\rho tX}(e^{-\rho X} - \alpha_1)^{h-k}[\beta_1\theta(\rho) - X\theta_1(X,\rho)]^k| \leq |\beta_1\theta(\rho) - X\theta_1(X,\rho)|^k$$

$$\leq 2^k|\beta_1^k\theta^k(\rho) + X^k\theta_1^k(X,\rho)|$$

$$\leq 2^k(\beta_1^k + X^k) .$$

Since the expectation of $2^k(\beta_1^k + X^k)$ is finite, we can interchange the expectation and lim signs, and the lemma is proved.

Let us consider now the central moments $\bar{\gamma}_r$ of C. We have

$$\bar{\gamma}_r = E(C - EC)^r = E(C - \gamma_1)^r$$

From (11) we obtain a recurrence formula for the moments γ_r:

$$\begin{aligned}\gamma_r &= E\,C^r \\ &= E\,e^{-r\,\rho\,X_1}(1 + C')^r \\ &= E\,e^{-r\,\rho\,X_1}\,E(1 + C')^r \\ &= \alpha_r\,E\sum_{j=o}^{r}\binom{r}{j}C'^j \\ &= \alpha_r\sum_{j=o}^{r}\binom{r}{j}\gamma_j\end{aligned}$$

By the well-known relations between moments and central moments

$$\bar{\gamma}_r = \sum_{j=o}^{r}(-1)^{r-j}\binom{r}{j}\gamma_1^{r-j}\gamma_j \quad ,$$

$$\gamma_r = \sum_{j=o}^{r}\binom{r}{j}\bar{\gamma}_j\gamma_1^{r-j} \quad ,$$

we may derive a recurrence formula for the central moments of C:

$$\bar{\gamma}_r = \sum_{j=o}^{r} (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \gamma_j$$

$$= \sum_{j=o}^{r} (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{t=o}^{j} \binom{j}{t} \gamma_t$$

$$= \sum_{j=o}^{r} (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{t=o}^{j} \binom{j}{t} \sum_{i=o}^{t} \binom{t}{i} \bar{\gamma}_i \gamma_1^{t-i}$$

$$= \sum_{j=o}^{r} (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{i=o}^{j} \bar{\gamma}_i \sum_{t=i}^{j} \frac{j!}{(j-t)!\, i!\, (t-i)!} \gamma_1^{t-i}$$

$$= \sum_{j=o}^{r} (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{i=o}^{j} \bar{\gamma}_i \binom{j}{i} (1+\gamma_1)^{j-i} .$$

Substituting γ_1 by (9), we have:

$$\bar{\gamma}_r = \sum_{i=o}^{r} \sum_{j=i}^{r} (-1)^{r-j} \binom{r}{j} \frac{\alpha_1^{r-j}}{(1-\alpha_1)^{r-j}} \alpha_j \bar{\gamma}_i \binom{j}{i} \frac{1}{(1-\alpha_1)^{j-i}}$$

$$= \sum_{i=o}^{r} \bar{\gamma}_i \frac{1}{(1-\alpha_1)^{r-i}} \sum_{j=i}^{r} (-1)^{r-j} \binom{r}{j} \binom{j}{i} \alpha_1^{r-j} \alpha_j$$

$$= \sum_{i=o}^{r} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{\gamma}_i \sum_{j=i}^{r} \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j}$$

$$= \alpha_r \bar{\gamma}_r + \sum_{i=o}^{r-1} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{\gamma}_i \sum_{j=i}^{r} \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} .$$

We obtain finally

$$(15) \quad \bar{\gamma}_r = \frac{1}{1-\alpha_r} \sum_{i=o}^{r-1} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{\gamma}_i \sum_{j=i}^{r} \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} .$$

Since

$$\sum_{j=i}^{r} \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} = \sum_{j=i}^{r} \binom{r-i}{j-i} \; E\, e^{-\rho j X} (-\alpha_1)^{r-j}$$

$$= E \sum_{j=i}^{r} \binom{r-i}{j-i} \; e^{-\rho j X} (-\alpha_1)^{r-j}$$

$$= E\, e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i} \quad ,$$

(15) can also be written as

$$(15') \qquad \bar{\gamma}_r = \frac{1}{1-\alpha_r} \sum_{i=o}^{r-1} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{\gamma}_i \; E\, e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i} .$$

We can now proceed to study the convergence of the central moments $\bar{\gamma}_r$.

Theorem 2. If, for $r > 0$, $E\, X^{\frac{r}{2}+1} < \infty$, then

$$(16) \qquad \lim_{\rho \to o} \rho^{\frac{r}{2}} \bar{\gamma}_r = K_r$$

where

$$K_r = \begin{cases} \dfrac{(\beta_2 - \beta_1^2)^{\frac{r}{2}} \; r!}{2^r (\frac{r}{2})! \; \beta_1^{\frac{3}{2} r}} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases}$$

Proof: We will prove this theorem by induction. Clearly (16) holds for $r = 0, 1$, since $\gamma_o = 1$, $\gamma_1 = 0$. Now let us assume that it holds for $0, 1, \ldots r-1$, with $r \geq 2$, and prove it must consequently hold for r.

From (15') we have

$$(17) \qquad \rho^{\frac{r}{2}} \, \bar{\gamma}_r = \sum_{i=1}^{r-1} \rho^{\frac{r}{2}} \, T_i$$

where

$$\rho^{\frac{r}{2}} T_i = \rho^{\frac{r}{2}} \frac{1}{1-\alpha_r} \frac{1}{(1-\alpha_1)^{r-i}} \binom{r}{i} \bar{\gamma}_i \; E \, e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}$$

$$= \binom{r}{i} \frac{\rho}{1-\alpha_r} \left(\frac{\rho}{1-\alpha_1}\right)^{r-i} \rho^{\frac{i}{2}} \bar{\gamma}_i \; \rho^{-(\frac{r}{2}+1-\frac{i}{2})} E \, e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i} .$$

By lemma 1 we obtain:

$$\lim_{\rho \to 0} \frac{\rho}{1-\alpha_r} = \frac{1}{r \, \beta_1} \quad ,$$

so that

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i = \binom{r}{i} \frac{1}{r \, \beta_1} \frac{1}{\beta_1^{r-i}} K_i \lim_{\rho \to 0} \rho^{-(\frac{r}{2}+1-\frac{i}{2})} E \, e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i} .$$

Since $\frac{r}{2} + 1 - \frac{i}{2} \leq \frac{r}{2} + 1$, we have

$$E \, X^{\frac{r}{2}+1-\frac{i}{2}} < \infty$$

and we can use lemma 2. So, if $i < r-2$, we have $\frac{r}{2} + 1 - \frac{i}{2} < r-i$, and then

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i = 0 .$$

If on the contrary $i = r - 2$, i.e. $\frac{r}{2} + 1 - \frac{i}{2} = r-i$, we have:

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_{r-2} = \binom{r}{2} \frac{1}{r} \frac{1}{\beta_1^3} K_{r-2} \, E(X - \beta_1)^2$$

$$= \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2} \quad .$$

Finally, for $i = r - 1$, we can write

$$E\, e^{-(r-1)\rho X} \left(e^{-\rho X} - \alpha_1\right) = \alpha_r - \alpha_1 \alpha_{r-1}$$

and, by lemma 1, we have

$$\lim_{\rho \to 0} \rho^{-\frac{3}{2}} (\alpha_r - \alpha_1 \alpha_{r-1}) =$$

$$= \lim_{\rho \to 0} \rho^{-\frac{3}{2}} \left\{1 - r\beta_1\rho + O_1(\rho^2) - [1-(r-1)\beta_1\rho + O_2(\rho^2)][1-\beta_1\rho + O_3(\rho^2)]\right\}$$

$$= \lim_{\rho \to 0} \rho^{-\frac{3}{2}} [1 - r\beta_1\rho - 1 + \beta_1\rho + (r-1)\beta_1\rho + O(\rho^2)]$$

$$= 0$$

Hence

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_{r-1} = 0 \ .$$

We thus have:

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} \bar{\gamma}_r = \sum_{i=0}^{r-1} \lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i$$

$$= \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2} \ .$$

If r is odd, so is $r-2$, and then $K_r = K_{r-2} = 0$. If r is even, we have

$$\binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2} = \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} \frac{(\beta_2 - \beta_1^2)^{\frac{r}{2}-1} (r-2)!}{2^{r-2} (\frac{r}{2} - 1)! \, \beta_1^{\frac{3}{2}(r-2)}}$$

$$= \frac{1}{2^r} \frac{r!}{(\frac{r}{2})!} \frac{(\beta_2 - \beta_1^2)^{\frac{r}{2}}}{\beta_1^{\frac{3}{2} r}} = K_r \ .$$

The theorem is thus proved.

Theorem 2 enables us to establish a sufficient condition for the asymptotic normality of C.

<u>Theorem 3</u>. If all the moments of X are finite, then the distribution of $(C - EC)/\sigma(C)$ tends to the standard normal distribution as ρ decreases to zero.

<u>Proof</u>: Theorem 2 gives, for $r = 2$:

$$\lim_{\rho \to 0} \rho \, \sigma^2(C) = \frac{\beta_2 - \beta_1^2}{2 \beta_1^3}$$

Thus we have:

$$\lim_{\rho \to 0} E \left[\frac{C - EC}{\sigma(C)} \right]^r = \lim_{\rho \to 0} \frac{\bar{\gamma}_r}{\sigma^r(C)}$$

$$= \lim_{\rho \to 0} \frac{\rho^{\frac{r}{2}} \bar{\gamma}_r}{\rho^{\frac{r}{2}} \sigma^r(C)} = \left(\frac{2 \beta_1^3}{\beta_2 - \beta_1^2} \right)^{\frac{r}{2}} K_r$$

$$= \frac{1}{2^{\frac{r}{2}}} \frac{r!}{(\frac{r}{2})!}$$

which is equal to the r-th moment of the normal variate $N(0,1)$. Thus the moments of $(C - EC)/\sigma(C)$ converge to the moments of the normal distribution, and this is sufficient to ensure that the distribution of the variate converge to the normal distribution (see, for instance, [1] p. 110). The theorem is thus proved.

The asymptotic normality is of course an important feature of C; if permits us to approximate C with a normal variate when ρ is small. It would be interesting to have some necessary condition in order that the asymptotic distribution be normal; unfortunately a condition of this kind appears rather difficult to establish.

However, the proof given above requires as an essential condition the existence of all the moments of X; so that one should expect that, if this condition is dropped, the moments of $(C - EC)/\sigma(C)$ do not converge to the moments of the normal variate. That this is actually true will be seen later.

Lemma 3. If $E X < \infty$ and

$$\lim_{\rho \to 0} \rho^{-t} E (e^{-\rho X} - \alpha_1)^k = 0 \qquad (0 < t \leq k)$$

then, for every $q < t$:

$$E X^q < \infty$$

Proof: We will show first that, under the hypotheses above, we have:

$$(18) \qquad \lim_{\rho \to 0} \rho^{-t} E |e^{-\rho X} - \alpha_1|^k = A$$

where $0 \leq A < \infty$. In fact, we can write:

$$(19) \qquad \rho^{-t} E(e^{-\rho X} - \alpha_1)^k = \rho^{-t} \int_0^{-\frac{1}{\rho}\log \alpha_1} (e^{-\rho x} - \alpha_1)^k dF(x) + \rho^{-t} \int_{-\frac{1}{\rho}\log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k dF(x)$$

Now, since $\beta_1 = E\,X < \infty$,

$$\lim_{\rho \to 0} \left(- \frac{1}{\rho} \log \alpha_1\right) = - \lim_{\rho \to 0} \frac{1}{\alpha_1} \frac{d\,\alpha_1}{d\,\rho}$$

$$= \lim_{\rho \to 0} \frac{\int_0^{+\infty} x\, e^{-\rho x}\, dF(x)}{\int_0^{+\infty} e^{-\rho x}\, dF(x)}$$

$$= \beta_1$$

Moreover

$$\rho^{-t} |e^{-\rho x} - \alpha_1|^k = |e^{-\rho x} - \alpha_1|^{k-t} \, |\beta_1\, \theta(\rho) - x\, \theta_1(x, \rho)|^t$$

$$\leq 2^t \, (\beta_1^t + x^t) .$$

Hence

$$\lim_{\rho \to 0} \rho^{-t} \int_0^{-\frac{1}{\rho} \log \alpha_1} (e^{-\rho x} - \alpha_1)^k \, d\,F(x) = \frac{A}{2} ,$$

where

$$\frac{A}{2} = \begin{cases} \int_0^{\beta_1} (\beta_1 - x)^t \, d\,F(x) & \text{if } k = t \\ 0 & \text{if } k > t . \end{cases}$$

Since, by hypothesis, the left side of (19) tends to zero, we have also

$$\lim_{\rho \to 0} \rho^{-t} \int_{-\frac{1}{\rho} \log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k \, dF(x) = - \frac{A}{2} .$$

We have finally

$$\rho^{-t}\, E|e^{-\rho x} - \alpha_1|^k = \rho^{-t} \left| \int_0^{\frac{1}{\rho}\log \alpha_1} (e^{-\rho x} - \alpha_1)^k \, d\, F(x) \right| + \rho^{-t} \left| \int_{-\frac{1}{\rho}\log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k dF(x) \right|,$$

and (18) follows.

Now, since $\lim_{\rho \to 0} \alpha_1 = 1$, given a number K with $0 < K < 1$, by making ρ small enough, we can make $\alpha_1 > K$. Then we have:

$$E|e^{-\rho X} - \alpha_1|^k \geq \int_{\frac{1}{\rho}\log\frac{1}{K}}^{+\infty} (\alpha_1 - e^{-\rho x})^k \, d\, F(x)$$

$$\geq (\alpha_1 - K)^k \int_{\frac{1}{\rho}\log\frac{1}{K}}^{+\infty} d\, F(x)$$

$$= (\alpha_1 - K)^k \left[1 - F\left(\frac{1}{\rho}\log\frac{1}{K}\right)\right].$$

If we put $\frac{1}{\rho}\log\frac{1}{K} = x$, we obtain

$$x^t\left[1 - F(x)\right] = \left(\log\frac{1}{K}\right)^t \rho^{-t}\left[1 - F\left(\frac{1}{\rho}\log\frac{1}{K}\right)\right]$$

$$\leq \frac{\left(\log\frac{1}{K}\right)^t}{(\alpha_1 - K)^k}\, \rho^{-t}\, E|e^{-\rho X} - \alpha_1|^k .$$

Then from (18) it follows that

$$\overline{\lim_{x \to +\infty}}\, x^t\left[1 - F(x)\right] \leq \left(\log\frac{1}{K}\right)^t (1 - K)^{-k} A .$$

This establishes the lemma.

<u>Theorem 4.</u> If $E X^2 < \infty$, then in order that all the moments of $(C - EC)/\sigma(C)$ converge to the moments of the normal distribution $N(0,1)$, it is necessary that all the moments of X be finite.

<u>Proof:</u> The proof will be by induction. Since $E X^2 < \infty$, the convergence of $E[(C - EC)/\sigma(C)]^k$ to the k-th central moment of the normal variate $N(0,1)$ is equivalent to

$$(20) \qquad \lim_{\rho \to 0} \rho^{\frac{k}{2}} \bar{\gamma}_k = K_k$$

We will assume that $E X^{\frac{r}{2}} < \infty$ for an integer $r \geq 4$, and that (20) holds for every integer k, and we will prove that $E X^{\frac{r+1}{2}} < \infty$. The theorem will thus be proved.

With the notation of theorem 2, (17) holds; since (20) holds for $k < r$, we have again:

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i = \binom{r}{i} \frac{1}{r \beta_1} \frac{1}{\beta_1^{r-i}} K_i \lim_{\rho \to 0} \rho^{-(\frac{r}{2} + 1 - \frac{i}{2})} E\, e^{-\rho i X}(e^{-\rho X} - \alpha_1)^{r-i} .$$

For $i \geq 2$, that is $\frac{r}{2} + 1 - \frac{i}{2} \leq \frac{r}{2}$, we have $E X^{\frac{r}{2} + 1 - \frac{i}{2}} < \infty$.

Hence, for $i \geq 2$, the conclusions of theorem 2 hold, that is

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i = \begin{cases} K_r & \text{if } i = r-2 \\ 0 & \text{if } i \neq r-2 \end{cases}$$

Moreover, for $i = 1$, $\gamma_1 = 0$; thus

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} \bar{\gamma}_r = \sum_{i=0}^{r-1} \lim_{\rho \to 0} \rho^{\frac{r}{2}} T_i = K_r + \lim_{\rho \to 0} \rho^{\frac{r}{2}} T_o .$$

Since (20) holds for $k = r$, the last limit above must be zero, that is:

$$\lim_{\rho \to 0} \rho^{\frac{r}{2}} \frac{1}{1 - \alpha_r} \frac{1}{(1-\alpha_1)^r} E(e^{-\rho X} - \alpha_1)^r = 0 .$$

$$\lim_{\rho \to 0} \rho^{-(\frac{r}{2} + 1)} E(e^{-\rho X} - \alpha_1)^r = 0 .$$

By lemma 3, this implies that $E X^q < \infty$ for every $q < \frac{r}{2} + 1$, in particular for $q = \frac{r+1}{2}$. The theorem is thus proved.

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