The Creep Compliance, The Relaxation Modulus and the Complex Compliance of Linear Viscoelastic, Homogeneous, Isotropic Materials

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OBJECTIVE : To obtain the creep compliance, the relaxation modulus and the complex compliance derived from the concept of mechanical resistance for the constitutive equation of a class of linear viscoelastic, homogeneous , isotropic materials.

METHODS : Based on (1) The concept of mechanical resistance (Wong, 1970 & 1986) ;(2) The definition of creep compliance J(t) as the strain function in time (t) resulted from the application of a unit step stress,and the definition of relaxation modulus Y(t) as the stress in time (t) caused by the application of a unit step strain (Flugge,1967; Bland, 1960; Freudenthal, 1950; Lee, 1956); (3) The convolution theorem in Laplace Transform(Wylie, 1960) and (4) The information and the references in an article entitled " The Formulation and Solution of The Governing Equations of Motion for Three-Dimensional Linear Viscoelastodynamics " (Wong, 1986), the OBJECTIVE can be fulfilled.

RESULTS : (1) The creep compliance, the relaxation modulus and the complex compliance obtained by means of mechanical resistance concept in the one dimensional constitutive equations are compatible with those as obtained by means of balance of forces as shown by (Flugge, 1967) ; (2) The constitutive law for a class of linear viscoelastic, homogeneous, isotropic materials can thus be proposed and obtained ; (3) The wave speed for the linear viscoelastic materials can thus be systematically defined by means of electrical circuit analysis.

CONCLUSIONS : The concept of mechanical resistance by means of the electrical circuit analogy offers several features : (1) Extremely complicated material models can be dealt with a simple manner by utilizing electrical net work analysis; (2) Without suppressing the time variable the governing equations are maintained in their general form such that a wider class of boundary value problems can be accommodated ; (3) The solutions of many physical problems in linear viscoelastodynamics are thus greatly simplified because they can be expressed in the same form of mathematical functions but with different wave speeds that represent a type of linear viscoelastic materials.

Denoting the mechanical resistance or impedance $Z(\frac{\partial}{\partial t})$, it can be related to the creep compliance J(t) and the relaxation modulus Y(t) in the following:

$$\epsilon(t) = Z(\frac{\partial}{\partial t})\ \sigma(t) \tag{1}$$

The Laplace Transform of equation (1) is

$$\epsilon(s) = Z(s)\,\sigma(s) \tag{2}$$

The inverse Laplace Transform of equation (2) should be the strain function $\epsilon(t)$ itself and thus

$$\epsilon(t) = \mathcal{L}^{-1}\left[Z(s)\,\sigma(s)\right] = \mathcal{L}^{-1}\left[\frac{Z(s)}{s}\, s\,\sigma(s)\right] = \left[J(t) * \frac{d}{dt}\sigma(t)\right] \tag{3}$$

$$J(t) = \mathcal{L}^{-1}\frac{Z(s)}{s} \tag{4}$$

$$Y(t) = \mathcal{L}^{-1}\frac{1}{s\,Z(s)} \tag{5}$$

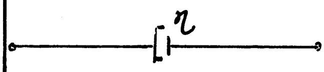
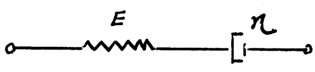
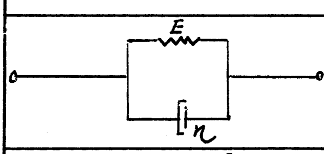
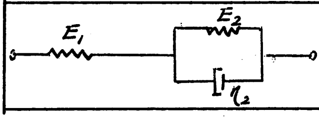
$$\epsilon(t) = Z\left(\frac{\partial}{\partial t}\right)\sigma(t) = \sigma_0 J(t) \tag{6}$$

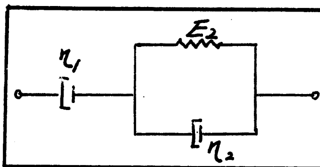
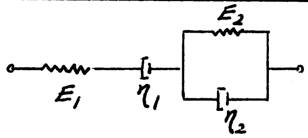
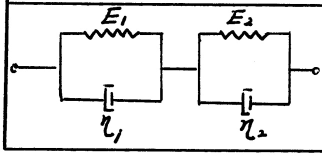
$$\sigma(t) = \frac{\epsilon(t)}{Z\left(\frac{\partial}{\partial t}\right)} = \epsilon_0 Y(t) \tag{7}$$

The complex compliance is $Z(i\omega) = G_1(\omega) + i\,G_2(\omega)$ (8)

As indicated by (Wong, 1970 & 1986) , the linear viscoelastic materials represented by equation (1) can be summarized in the following table of eight examples in order to compare the results with (Flugge,1967)

TABLE 1

Model	Name	Mechanical Constitutive Resistance
	elastic solid	$Z_1 = \frac{1}{E}$
	viscous fluid	$Z_2 = \frac{1}{\eta\frac{\partial}{\partial t}}$
	Maxwell fluid	$Z_3 = \frac{1}{E} + \frac{1}{\eta\frac{\partial}{\partial t}}$
	Kelvin solid	$Z_4 = \frac{1}{E + \eta\frac{\partial}{\partial t}}$
	3 parameter solid	$Z_5 = \frac{1}{E_1} + \frac{1}{E_2 + \eta_2\frac{\partial}{\partial t}}$

	3 parameter fluid	$Z_6 = \dfrac{1}{\eta_1 \frac{\partial}{\partial t}} + \dfrac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$
	4 parameter fluid	$Z_7 = \dfrac{1}{E_1} + \dfrac{1}{\eta_1 \frac{\partial}{\partial t}} + \dfrac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$
	4 parameter solid	$Z_8 = \dfrac{1}{E_1 + \eta_1 \frac{\partial}{\partial t}} + \dfrac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$

From TABLE 1 denoting m as the subscfipt of $Z_m(\frac{\partial}{\partial t})$ when m = 1 (elastic soid);m = 2 (viscous fluid); etc., the same subscript m is used to denote the creep compliance $J_m(t)$; relaxation modulus $Y_m(t)$; complex compliance $Z_m(i\omega)$ for a kind of viscoelastic material m.
Noticing that the partial derivatives with respect to time (t) in $Z_m(\frac{\partial}{\partial t})$ can be replaced by (s) in Laplace Transform and that can also be replaced by $i\omega$ for the Complex compliance $Z_m(i\omega)$, the $J_m(t)$; $Y_m(t)$ and $Z_m(i\omega)$ can therefore be obtained from equations (3),(4) and (5) for a material model "m".

As a result of these systematical representations, the creep compliance for material models "m" are listed in the following:

$$J_1(t) = \mathcal{L}^{-1} \frac{1}{s\,E} = \frac{1}{E}$$

$$J_2(t) = \mathcal{L}^{-1} \frac{1}{s^2 \eta} = \frac{t}{\eta}$$

$$J_3(t) = \mathcal{L}^{-1}\left(\frac{1}{s\,E} + \frac{1}{\eta\, s^2}\right) = \frac{1}{E} + \frac{t}{\eta}$$

$$J_4(t) = \mathcal{L}^{-1}\left(\frac{1}{s} \quad \frac{1}{(E + \eta s)}\right) = \frac{1}{E} - \frac{1}{E} e^{-\frac{E}{\eta}t}$$

$$J_5(t) = \mathcal{L}^{-1}\left(\frac{1}{E_1 s} + \frac{1}{s(E_2 + \eta_2 s)}\right) = \frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{E_2} e^{\frac{E_2}{\eta_2}t}$$

$$J_6(t) = \mathcal{L}^{-1}\left(\frac{1}{\eta_1 s^2} + \frac{1}{s(E_2 + \eta_2 s)}\right) = \frac{t}{\eta_1} + \frac{1}{E_2} - \frac{1}{E_2} e^{-\frac{E_2}{\eta_2}t}$$

$$J_7(t) = \mathcal{L}^{-1}\left(\frac{1}{E_1 s} + \frac{1}{\eta_1 s^2} + \frac{1}{s(E_2+ \eta_2 s)}\right) = \frac{1}{E_1} + \frac{t}{\eta_1} + \frac{1}{E_2} - \frac{1}{E_2} e^{-\frac{E_2}{\eta_2}t}$$

$$J_8(t) = \mathcal{L}^{-1}\left(\frac{1}{s(E_1+\eta_1 s)} + \frac{1}{s(E_2+ \eta_2 s)}\right) = \frac{1}{E_1} - \frac{1}{E_1} e^{-\frac{E_1}{\eta_1}t} + \frac{1}{E_2} - \frac{1}{E_2} e^{-\frac{E_2}{\eta_2}t}$$

The relaxation modulus $Y_m(t)$ for material model "m" can also be listed:

$$Y_1(t) = \mathcal{L}^{-1}\left(\frac{1}{s\, Z_1(s)}\right) = \mathcal{L}^{-1}\left(\frac{E}{s} \right) = E$$

$$Y_2(t) = \mathcal{L}^{-1}\left(\frac{\eta s}{s} \right) = \mathcal{L}^{-1}(\eta) = \eta\delta(t)$$

$$Y_3(t) = \mathcal{L}^{-1}\left(\frac{E}{(s + E/\eta)}\right) = E\, e^{-\frac{E}{\eta}t}$$

$$Y_4(t) = \mathcal{L}^{-1}(\eta + E/s) = E + \eta\, \delta(t)$$

$$Y_5(t) = \mathcal{L}^{-1}\left(\frac{E_1(E_1 +\eta_2 s)}{s (E_1 + E_2+\eta_2 s)} \right) = \frac{E_1^2}{E_1+ E_2}\left(1 - e^{-\frac{E_1+E_2}{\eta_2}t}\right) + E_1\, e^{-\frac{E_1+E_2}{\eta_2}t}$$

$$Y_6(t) = \mathcal{L}^{-1}\left(\frac{{}_1 s(E_2+ \eta_2 s)}{s(E_2+(\eta_1+ \eta_2)s)} \right) = \frac{\eta_1\ E_2}{\eta_1 + \eta_2}\, e^{-\frac{E_2}{\eta_1+\eta_2}t} - \frac{\eta_1 \eta_2 E_2}{(\eta_1+ \eta_2)^2} e^{-\frac{E_2}{\eta_1 +\eta_2}t}$$

$$Y_7(t) = \mathcal{L}^{-1}\left(\frac{1}{s (1/E_1 + 1/\eta_1 s + 1/(E_2+\eta_2 s))}\right)$$

$$= (E_1E_2/ \eta_2 q - E_1 a/q)\, e^{-at}\sinh qt + E_1 e^{-at}\cosh qt$$

where $a=(E_2/\eta_2 +E_1/\eta_1+E_1/\eta_2)/2$

$$q= \left(\sqrt{(E_1/\eta_2)^2 + 2(E_1/\eta_2)(E_2/\eta_2 + E_1/\eta_1) + (E_1/\eta_1 - E_2/\eta_2)^2}\right)/ 2$$

$$Y_8(t) = \mathcal{L}^{-1}\left(\frac{1}{s\ (\ 1/(\ E_1+\eta_1 s) + 1/(E_2 + \eta_2 s\ ))}\right)$$

$$= \frac{E_1E_2}{E_1+E_2}\left(1 - e^{-\frac{E_1+E_2}{\eta_1+\eta_2}t}\right) + \frac{E_1\eta_2 + E_2\eta_1}{\eta_1+\eta_2}\ e^{-\frac{E_1+E_2}{\eta_1+\eta_2}t}$$

$$- \frac{\eta_1\eta_2}{(\eta_1+\eta_2\ )^2}\ (\ E_1 + E_2)\ e^{-\frac{E_1+E_2}{\eta_1+\eta_2}t}$$

The complex compliance $Z_m(i\omega)$ are obtained and listed in the following:

$Z_1(i\omega) = 1/E$

$Z_2(i\omega) = 1/\eta i\omega = -\ i\ /\eta\omega$

$Z_3(i\omega) = 1/E + 1/\eta i\omega = 1/E + i\ (-1/\eta\omega)$

$Z_4(i\omega) = 1/(E + i\eta\omega) = E/(\ E^2 +\eta^2\omega^2) + i\ (-\ \eta\omega\ /(E^2+\eta^2\omega^2))$

$Z_5(i\omega\ ) = 1/E_1 + 1/(E_2 + \eta_2 i\omega\ ) = (1/E_1+ E_2/(E_2^2+\eta_2^2\omega^2)) + i\ (-\eta_2\omega/(E_2^2+\eta_2^2\omega^2)\ )$

$Z_6(i\omega) = 1/\eta_1 i\omega + 1/(\ E_2+ \eta_2 i\omega\ ) = E_2/(E_2^2+\eta_2^2\omega^2) + i\ (-1/\eta_1\omega -\eta_2\omega\ /(E_2^2+\eta_2^2\omega^2)$

$Z_7(i\omega\ ) = 1/E_1+1/\ \eta_1 i\omega + 1/(E_2+\eta_2 i\omega\ )=1/E_1+E_2/(E_2^2+\eta_2^2\omega^2)+i(-1/\eta_1\omega -\eta_2\omega/(E_2^2+\eta_2^2\omega^2))$

$$Z_8(i\omega) = 1/(E_1+\eta_1 i\omega) + 1/(E_2+\eta_2 i\omega\ )$$

$$= E_1/(E_1^2+\eta_1^2\omega^2) +E_2/(E_2^2+\eta_2^2\omega^2) + i(-\eta_1\omega\ /(E_1^2+\eta_1^2\omega^2)- \eta_2\omega/(E_2^2+\eta_2^2\ \omega^2\ ))$$

As can be seen from the above, in the one-dimensional case the linear operator which interrelates the stress and the strain varies with the constitute law of the material. Likewise do the corresponding two linear operators in the three-dimensional case where one of the operators governs the dilatational motions of the material, the other the shearing motions. The constitutive law for a class of isotropic, homogeneous,linearly viscelastic materials has been proposed and published by (Wong, 1970 & 1986) in which the various operators were listed and compared with all those having been obatined by (Flugge,1967) and by (Bland, 1960). All these work done before lay down the foundation for research and development not only in viscoelastodynamics but also in neurosciences for neuro-network analysis.

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APPENDIX:

Equation (3) can be extended and written in the following forms:

$$\epsilon(t) = J(t) * \frac{d}{dt}\sigma(t) = \int_0^t J(t-\lambda)\, d\sigma(\lambda) = \int_0^t \frac{d\sigma(t-\lambda)}{dt} J(\lambda)\, d\lambda$$

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