A METHOD OF SOLUTION OF THE EIGENPROBLEMS OF LARGE STRUCTURAL SYSTEMS IN AN ARBITRARILY SPECIFIED RANGE

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Presented is a method of calculation of all eigenvalues and corresponding eigenvectors of the eigenproblem $\mathbf{A}\,\mathbf{U} = \mathbf{B}\,\mathbf{U}\,\Lambda$ in an arbitrarily specified range of the eigenspectrum for real symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of high rank. A version of the method of subspace iterations was elaborated, two new concepts including :

1. an extension of the symmetric version of the well known simultaneous vector iteration (SVI) method to an arbitrary domain of the eigenspectrum,
2. application of the Chebychev acceleration in a non-classical way, that is in the direct form of the product of subsequent linear terms instead of the recurrence formula being in common use.

A computer program in the FORTRAN IV code for was worked out and various engineering examples (vibrations of structures) were solved.

1. Introduction

Analysis of free vibrations as well as bifurcational stability of large discrete structures yield general eigenproblem

$$\mathbf{A}\,\mathbf{U} = \mathbf{B}\,\mathbf{U}\,\Lambda \qquad (1)$$

for real symmetric matrices **A** and **B**. The present paper concerns calculation of a group of eigenvalues and corresponding eigenvectors in an arbitrarily specified range of the eigenspectrum of the problem mentioned above. We assume matrices **A** and **B** as being too large to be stored incore. In order to solve this problem a version of the subspace iteration method was elaborated. Due to its numerous advantages this method was intensively developed in last years [2,11]. The method is based on simultaneous iteration of a group vectors with direct access only to part of matrices **A** and **B**.

To subspace iteration methods one may count

- the simultaneous vector iteration method for symmetric [3,8,12] and non-symmetric [3,5] eigenproblem if converted into the standard form $\bar{\mathbf{A}}\,\mathbf{u} = \lambda\,\mathbf{u}$;
- a method of minimization of a subspace of Rayleigh quotients presented by Schwarz [9];
- the Lanczos method generalized into subspace iteration [4];
- the Block-Stodola method in subspace [10].

All methods mentioned above however, are restriced to determining a group of the extremal eigenvalues (eigenvectors). Only the SVI method for nonsymmetric standard problem might be applied to solve the problem in an arbitrary inner part of the eigenspectrum. Such an approach would be very laborous becouse as much as two groups of vectors (right and left) have to be iterated (biiteration).

There are also other methods, not mentioned here, that might be used to solution of the eigenproblem in question. One may count here methods useful for matrices of small or medium size only as well as methods designed for subsequent determining of single eigenvalues like the inverse iteration method combined with a shift or the bisection method [1,7].

The present paper is concerned a generalization of 'symmetric' version of the classical SVI method [1] into an arbitrarily specified range of eigenspectrum, despite the fact that matrix $\bar{\mathbf{A}}$ of the corresponding standard eigenproblem is non-symmetric then.

Main advantages of the 'symmetric' version of SVI, when comparing with non-symmetric' one, like banded structure of matrices, various techniques of acceleration of convergence process and relative economy of calculations are preserved [13].

2. Formulation of the problem

The eigenvalue problem (1) is considered where **A** and **B** are real symmetric matrices of large size $n \times n$ and banded structure. Matrix **A** is positive-definite. Both matrices are stored in a mass story with direct access. Required are m ($m \ll n$) eigenvectors and corresponding eigenvalues λ_{i+k}, $k = 1,2,\ldots,m$ closet to certain number l regarding the norm $|\lambda_{i+k} - l|$. The following order takes place

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_i \leqslant \lambda_{i+1} \leqslant \lambda_{i+2} \leqslant \ldots \leqslant \lambda_{i+m} \ldots \leqslant \lambda_n . \quad (2)$$

The eigenvalue problem (1) might be analysed by the general, but very laborous and not always convergent approach for nonsymmetric matrices. An attempt was made, therefore, to find an effective procedure, similar to the classical 'symmetrical version' of the SVI method (preserving its advantages when compared with 'nonsymmetric' version) but general enough to solve the eigenproblem in question. This modified SVI procedure is based on the following three interrelated eigenproblems :

1° 'given' presented in the general form

$$\mathbf{A}\,\mathbf{U} = \mathbf{B}\,\mathbf{U}\,\Lambda_1 \quad , \qquad (3)$$

where $\Lambda_1 = \mathrm{diag}[\lambda_{i+1}, \lambda_{i+2}, \ldots \lambda_{i+m}]$ is a diagonal matrix $m \times m$ consisting of real eigenvalues $\lambda_{i+1}, \ldots, \lambda_{i+m}$ written in increasing order, $\mathbf{U} = [u_{i+1} \ldots u_{i+m}]$ is a rectangular matrix $n \times m$, m columns of which are corresponding eigenvectors in the specified range $[\check{a}, \check{b}]$ of the eigenspectrum K.

2° 'auxiliary' defined in the standard form

$$\mathbf{H}\,\mathbf{Y} = \mathbf{Y}\,\Lambda_2 \qquad (4)$$

obtained due to the shift

$$(\mathbf{A} - l\mathbf{B}) = \mathbf{B}\,\mathbf{U}\,(\Lambda_1 - l\mathbf{I}) \qquad (5)$$

of the eigenspectrum K by the scalar value $l \cong (\check{a} + \check{b})/2$;

$$\mathbf{H} \equiv \mathbf{L}^{-1}\,\mathbf{B}\,\mathbf{L}^{-T}\,\mathbf{D} \quad . \qquad (6)$$

is a nonsymmetric matrix $n \times n$ obtained due to a decomposition

$$\mathbf{A} - l\mathbf{B} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^{T} \qquad (7)$$

into triangular matrices $\mathbf{L}$, $\mathbf{L}^T$ and diagonal matrix **D**. We have also then

$$\Lambda_2 \equiv (\Lambda_1 - l\mathbf{I})^{-1} = \left[\frac{1}{\lambda_{i+1} - l}, \ldots, \frac{1}{\lambda_{i+m} - l}\right] , \qquad (8)$$

$$\mathbf{Y} \equiv \mathbf{D}\,\mathbf{L}^{T}\mathbf{U} = [y_{i+1}, \ldots, y_{i+m}] . \qquad (9)$$

Eigenvalues $(\lambda_{i+k} - l)^{-1}$, $k = 1,2,\ldots,m$ are show in the Fig. 1.

3° 'related' defined in the standard form

$$\mathfrak{H}\,\mathbf{V} = \mathbf{V}\,\Lambda_3 . \qquad (10)$$

Matrix $\mathcal{H}$ n x n is then symmetric like in classical SVI method [8], therefore eigenvectors $\mathbf{V}$ are orthogonal. Due to the decomposition

$$\mathbf{A} = \mathcal{L}\,\mathcal{L}^T \tag{11}$$

into Cholesky foctors $\mathcal{L}$, $\mathcal{L}^T$ we obtain

$$\mathcal{H} = \mathcal{L}^{-1}\,\mathbf{B}\,\mathcal{L}^{-T}, \tag{12}$$

$$\Lambda_3 = \Lambda_1^{-1} = \mathrm{diag}\,[\,\lambda_{i+1}^{-1}, \ldots, \lambda_{i+m}^{-1}\,], \tag{13}$$

$$\mathbf{V} = \mathcal{L}^T\mathbf{U} = [\,v_{i+1}, \ldots, v_{i+m}\,] \tag{14}$$

defining eigenvalues λ_{i+k}^{-1} and eigenvectors v_{i+k}, k = 1,2,...,m of the problem 3° .

All eigenvalues/vectors closet to 1 might be obtained by biiteration method [3,5]. Such an approach, however, would not be effective. Autors' proposal presented in the paper, is based on a combination of the problem 2° used to make a power step (an increase of required eigenvectors) and problem 3° , where an orthonormalization process of trial vectors is performed. In the practical realization of the method, instead of permanent transformations between trial vectors of the problems 2° and 3° , one 'resultant' eigenproblem $\mathbf{P}\,\mathbf{V} = \mathbf{V}\,\Lambda_2$ is solved for symmetric matrix $\mathbf{P} = \mathcal{L}^{-1}\,\mathbf{L}^T\mathbf{D}\,\mathbf{L}^{-1}\,\mathbf{B}\,\mathcal{L}^{-T}$. This prcedure may be completed by other one improving the convergence like projection in the subspace of Rayleigh quotients and Chebychev acceleration [1,2,8,13] .

3. Convergence acceleration

To improve the convergence of the iteration process a modyfied Chebychev acceleration procedure was introduced . Instead of the reccurence one [1] , the direct formula of the matrix Chebychev polynomial in the form of a product of subsequent linear terms was applied. Such approach is similar to that one which is used in the iterative Richardson method of solution of simultaneous linear algebraic equations. It is different, however, from the common approach based on the reccurence formula [8,12].

Two levels of iteration are applied now. On the first one k single steps of the simultaneous vector iterations - each one consisting of : a power step for matrix $\mathbf{P}$, a projection onto a subspace of Rayleigh quotients and orthonormalization - are replaced by a series of k steps

$$p_k(\mathbf{P})\,\mathbf{V}^j = \prod_{i=1}^{k}\,[\,(1+\delta_i\mu^*)\mathbf{I} - \delta_i\,\mathbf{P}\,]\,\mathbf{V}_{i-1}^j = \mathbf{V}^{j+1} \tag{15}$$

consistent with the Chebychev polynomial of degree k. Here $\mathbf{V}^j = [\,v_1^j, \ldots, v_m^j\,]$ are trial vectors ; $\delta_1, \ldots, \delta_k$ coefficients derived directly from zeros of the Chebychev polynomial

$$p_k(\mu) = \prod_{i=1}^{k}\,[\,1 - \delta_i(\mu - \mu^*)\,], \tag{16}$$

$$\mu = \frac{1}{\lambda - l}\,, \quad \delta_i = [\,\tfrac{1}{\sigma} - a\cdot\cos(\tfrac{2i-1}{2k})\pi\,]^{-1}.$$

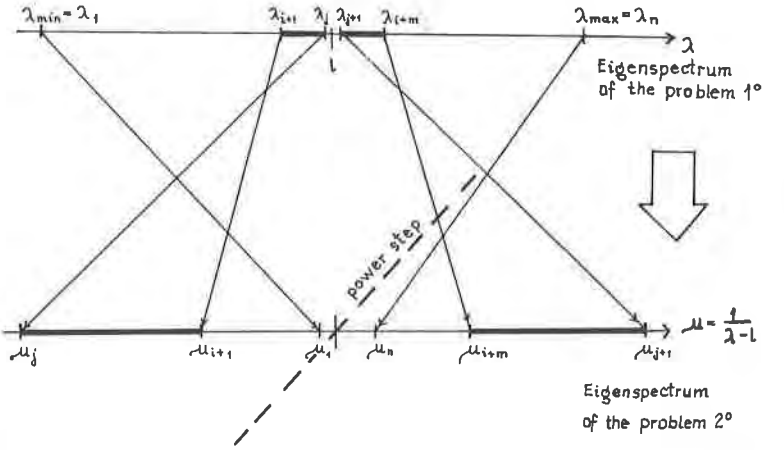


Fig. 1. Transformation of eigenvalues from the problem 1° to 2°.

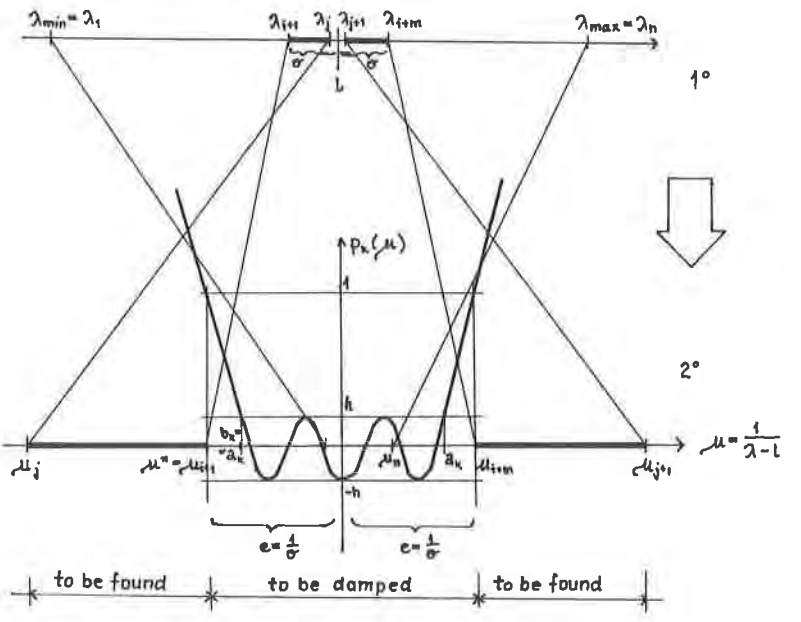


Fig. 2. The Chebychev polynomial $p_k(\mu)$ used for acceleration of calculations in the eigenproblem 2° .

Each one of these k steps consists of a projection onto the subspace $\mathfrak{A}_i = (\mathbf{V}_i^j)^T \mathbf{V}_i^j$ of Rayleigh quotients and orthonormalization of the trial vectors $\mathbf{V}_i^j$. Graphical interpretation of the polynomial $\rho_k(\mu)$ applied to the eigenproblem 2° presents Fig. 2. On the second level the same iteration process is repeated so many times until the convergence is sufficient for $r \leqslant m$ required eigenvalues/vectors.

Effectiveness of the Chebychev acceleration, briefly described above, depends on values of polynomial parametrs assumed. These are : k - polynomial degree ; a , b - bounds of the interval of the eigenspectrum, where eigenvalues/vectors are damped (see Fig. 2) ; $\delta_1, \ldots, \delta_k$ polynomial zeros. In the paper [13] the optimal choice of these parameters was analysed. Thus k was found, which minimize number of iterations. Parameters a and b as well as the optimal order of $\delta_1, \ldots, \delta_k$ were also determined from the requirement of minimal round-off errors in the iteration prcedure. Numerous tests carried out [13] confirmed an essential influence of the optimal choice of parameters on the convergence rate.

4. Computer code

One computer program in the FORTRAN IV code for CDC 7216 was worked out, both modyfied presented here and the classical SVI version including. This program may be used to determine all eigenvalues/vectors in an arbitrarily specified range of the eigenspectrum of large system as well as to find a single eigenvalue/eigenvector, the extremal or the closest to a given one.

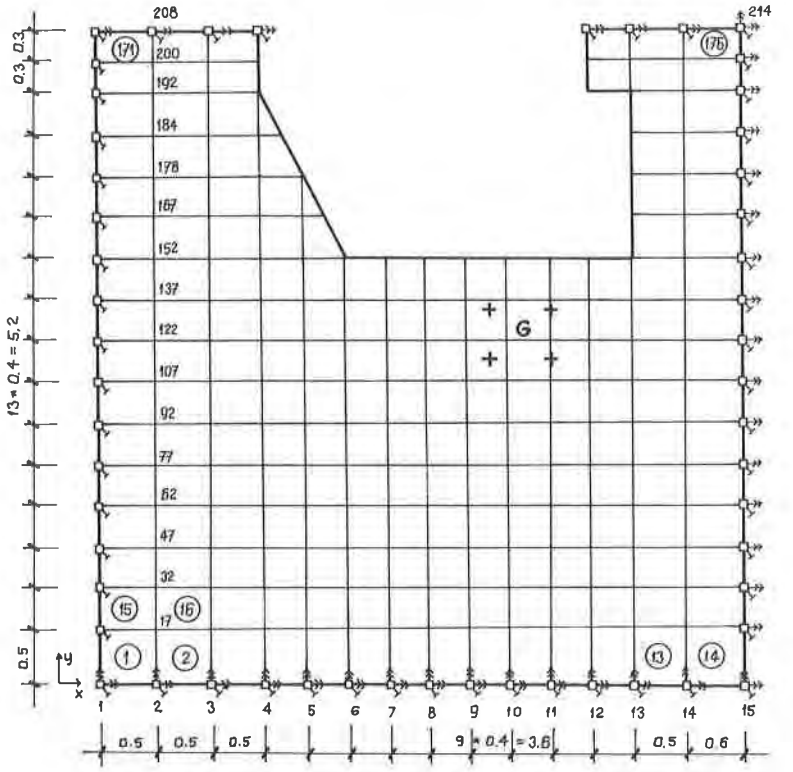
In course of realization of the program matrices **L** , **L** and **B** are stored in the form of an upper half band in a magnetic disc with direct access. All data may be read in form of blocks therefore. Blocks may be read in by columns or by rows, that depends on operation which is performed.

Initial data require four cards only provided we have in our disposal **A** and **B** matrices stored in a mass store.

5. Numerical example

A real engineering structure was considered. Calculated were eigenmodes and eigenfrequences of a vibrating ceiling of blowing room see Fig.3 in a prescribed interval of the eigenspectrum in a neighbourhood of the frequence n_f of vibrations forced by a fan. Global stiffness and mass matrices together with specified lumped masses were automatically generated. The conditioning number for the discretisation shown in the Fig. 3 $k = \lambda_{max} / \lambda_{min} \approx 500\,000$. The group of five eigenfrequences/mods the closet to the frequency of forced vibrations $n_f = 877$ 1/min as well as the group five the smallest and five the largest eigenfrequences/mods were found. Calculation time required to obtain the accuracy of the results up to six significant places is presented below.

Case	smallest		largest		closest to n_f	
	eigen-freq.	eigen-mods	eigen-freq.	eigen-mods	eigen-freq.	eigen-mods
Num.of quantities	5	5	5	5	5	5
Time in sec. CPU of the CDC 7216	4	10	4	10	6	15



number degrees of freedom half band number of elem. in matrices K, M

520 × 51 = 26 520

G - dead weight of the fan 3 000 kG
frequency of forced vibrations n_f = 877 1/min ,
t = 0.12m , E = $16 \cdot 10^4$ kG/cm² = 15690.6 MPa , ν = 0.16667,
ρ = 0.0024 kG/cm³ = 0.02354 N/m³.

Fig. 3. Plate of ceiling in blowing room.

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