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STRAIN-RATE DEPENDENT PLASTICITY IN THERMO-MECHANICAL TRANSIENT ANALYSIS

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The thermo-mechanical behavior of LWR fuel rods during reactor transients is generally governed by the strain-rate properties of the clad material. Relevant constitutive modeling requires extensive material experiments that simulate actual in-reactor thermal histories and environmental conditions. Unfortunately, however, the prevalent sources of available data are simple experiments involving internally heated pressurized tubes. The material data are generally presented in the form of average circumferential strain and nominal stress as functions of temperature and heating rates. Such data implicitly includes environmental and time history effects that are unique to the experiment and, hence, are difficult to generalize to actual in-reactor transients.

Considering the present limitations of the available data, a plausible approach for such generalization is to characterize the strain-rate response of the clad material as function of true-stress, temperature, time and environmental conditions, which can then be fitted within a theoretical framework of an inelastic constitutive model. The traditional approach of constructing such a model is to partition the strain tensor into elastic plastic and creep components. While this treatment may appear to be phenomenologically artificial, it provides a convenient analytical device for using material data that is usually available in this partitioned form. However, one of the difficulties encountered in practice, as a result of this strain decomposition, is that the resulting stress-strain relations do not adequately account for the high strain-rate sensitivity of the material. Consequently, attempts to predict high strain-rate behavior (the transitional behavior between creep and plasticity) using constitutive relations based on the combined, but uncoupled, formulation of instantaneous plasticity and creep, invariably fail. The use of the so-called unified theories, which are basically one-dimensional emperical formulas, limit the computational procedure to the highly inefficient initial-strain approach and also utilize arbitrary analogies for extension to three-dimensional states.

In this paper, we present a constitutive formulation that deals continuously with the entire strain-rate range and has the desirable advantage of utilizing existing material data. The derivation makes use of strain-rate sensitive stress-strain curve and strain-rate dependent yield surface. By postulating a strain-rate dependent von Mises yield function and a strain-rate dependent kinematic hardening rule, we are able to derive incremental stress-strain relations that describe the strain-rate behavior in the entire deformation range spanning high strain-rate plasticity and creep. The model is sufficiently general as to apply to any materials and loading histories for which data is available.

1. Introduction

In the implementation of inelastic constitutive models within the framework of the finite element computational method, the accepted procedure is to partition the strain tensor into elastic, plastic and creep components. This treatment provides a convenient analytical device for using material data that is usually available in this partitioned form. It is also consistent with well established analytical theories in creep and plasticity. In the context of these theories, the strain incremental tensor is written as follows

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij} + \dot{\varepsilon}^{c}_{ij}dt \tag{1}$$

The plastic strain increment $d\varepsilon^{p}_{ij}$ and the creep strain increment $\dot{\varepsilon}^{c}_{ij}dt$ are treated independently although they form parts of the same phenomenon. Although the classical instantaneous plasticity theory does not recognize the true time rate of change of the strains, it is used as a model for the equation of state creep theory in which the creep strain ε^{c}_{ij}, unlike the plastic strains ε^{p}_{ij}, are true time-dependent quantities. Specifically, the evolution law

$$\dot{\varepsilon}^{c}_{ij} = \frac{3\,\dot{\varepsilon}^{c}_{e}}{2\sigma_{e}}\,S_{ij} \tag{2}$$

which describes the creep strain rate-stress relations was fashioned after the plasticity theory and therefore is valid only for stationary creep in which no stress redistribution takes place [1] . In actual practice, however, this restriction on equation (2) is generally ignored. The material data $\dot{\varepsilon}^{c}_{e}$, which is the effective creep strain rate, is usually obtained in standard creep test and it includes a primary (transient) creep component and a steady state (minimum creep rate) component. Although both components are part of one continuous process, they are often treated separately in numerical analysis. For instance, the strain hardening rule is used in conjunction with equation (2) to model the transient effects of primary creep. The steady state component offers no particular numerical difficulties, and its treatment is actually more faithful to the theoretical limitations of equation (2) on which it is based.

This somewhat adhoc treatment of the creep problem in combination with the plasticity problem has worked out fairly well in analyses involving low and middle temperature regimes. However, in high-temperature applications particularly thermo-mechanical transients, some difficulties have been encountered which stem from the fundamental assumptions of the theory. For instance, if one examines the stress vs. strain-rate data generated in simple tension tests in which the strain-rate is varied over the entire range of practical interest, one observes a strain-rate transitional region that is difficult to duplicate analytically using uncoupled creep and plasticity theories. The basic difficulty lies in the fact that the classical framework of instantaneous plasticity and stationary creep theories are in effect bounding analytical theories which, when applied to this transitional region, become mere extrapolations, i.e., the plasticity formulation is extrapolated from above to higher strain rates. Furthermore, the material data that is available for quantifying these analytical theories are often ill-defined in this transitional region; hence, the resulting constitutive relations are in general incapable of correctly predicting the material behavior throughout the entire strain-rate range of interest.

The development of a one-dimensional material model that could accurately predict the uniaxial behavior described above is not a particularly difficult task. In fact, the functional representation of the uniaxial test data is the simplest of such models and it can be presented in a number of ways such as, for instance, stress-strain curves plotted for fixed strain-rates as shown in Fig. 1. These curves approach limit curves at the very high and the very low strain rates. The high strain-rate limit curve defines the boundary for instantaneous plasticity. At the lower end of the strain rate spectrum the limit curve defines the creep threshold boundary within which the material behaves purely elastically.

Another way of modeling the strain-rate material data is to plot the stress vs. strain-rate for fixed values of strain as shown in Fig. 2. The data represented in this figure spans the entire range of strain-rate including the steady state creep regime. Although equation (1) has the appearance of enclosing the entire Figure 2, it is theoretically valid only for the extreme right and the extreme left of this figure.

The one-dimensional model displayed in Fig. 2 is constructed using some experimental data curve-fitted and interpolated to cover the entire strain-rate range. Its generalization to three-dimensional stress states and arbitrary time histories requires a theoretical framework that is capable of reproducing the strain-rate dependence of the material in the transitional range discussed above. A convenient method for accomplishing this is to work within the framework of a plasticity theory in which the strain-rate dependence is introduced explicitly. An appropriate constitutive theory can then be constructed whose parameters can be evaluated from available data. In order for this constitutive theory to be of value in practical applications, it should permit an incremental formulation that is suited for currently operational finite element computer programs.

The analytical device employed in the present derivation, treats the yield surface as strain-rate dependent in an attempt to bridge the gap between the creep deformations which characterize quasi-static plastic flow and the so-called instantaneous plasticity. By postulating a strain-rate dependent von Mises potential, valid over the strain-rate range of interest, and working within Prager's kinematic hardening assumption [2], the derivation of an incremental constitutive theory that is valid over the strain-rate range becomes analytically tractable. Its implementation, however, may lead to numerical difficulties in certain instances such as, for example, in the elastic-plastic transition step where the yield surface depends continuously on the instantaneous rate of straining and is therefore changing simultaneously with the changing stresses. Under these conditions, a relatively slow strain-rate during loading from an elastic state to a plastic state shortens the path to the yield surface, which may cause the stress increment to become large in comparison with the yield stress and thus produce solution instability or oscillation. On the other hand, a relatively fast strain-rate during a loading step from a plastic state results in a stress increase, which can either be a state of loading if the yield surface expands at a slower rate than the rate of stress increase or a state of unloading if the reverse is true. It is easy to see, therefore, that various combinations of loading or unloading with slow or fast strain rates can lead to complex numerical algorithms in the general case.

In the next two sections, we will present a basis for treating strain-rate dependence within the framework of the plasticity theory. In Section 4, the incremental stress-strain relations will be derived and numerical examples will be given in Section 5.

The present work is of particular importance to the analysis of light water reactor fuel rod cladding response during loss of coolant accidents.

2. Preliminary Derivations

Prager's kinematic hardening rule [2], makes use of von Mises yield potential in the following form

$$F = \frac{1}{2} S'_{ij} S'_{ij} - \kappa^2 \qquad (3)$$

where

$$S'_{ij} = \sigma'_{ij} - \frac{1}{3} \sigma'_{kk} \delta_{ij}, \qquad (4)$$

$$\sigma'_{ij} = \sigma_{ij} - \alpha_{ij}, \qquad (5)$$

α_{ij} being the translations of the yield surface and κ is a material parameter which may depend on the material environment. By Prager's assumption, the yield surface translates in the direction of the plastic strains, namely,

$$\alpha_{ij} = C \varepsilon^p_{ij} \qquad (6)$$

where C is a constant, related to the slope of a bilinear stress plastic-strain curve, and ε^p_{ij} are the plastic strains.

In a previous paper [3] the above linear kinematic hardening theory was extended to non-linear kinematic hardening in which

$$\alpha_{ij} = \int C (\alpha_{ij}) \, d\varepsilon^p_{ij} \qquad (7)$$

where the integration in (7) is taken over the most recent plastic path. This derivation led to the following equation for C:

$$C = \frac{2}{3} \frac{d\alpha_e}{d\varepsilon^p_e} = \frac{2}{3} E' \qquad (8)$$

where E´ is the slope of the stress-plastic strain curve, and

$$\alpha_e = \left(\frac{3}{2} \alpha_{ij} \alpha_{ij}\right)^{\frac{1}{2}} \qquad (9)$$

and

$$\varepsilon^p_e = \left(\frac{2}{3} \varepsilon^p_{ij} \varepsilon^p_{ij}\right)^{\frac{1}{2}} \qquad (10)$$

3. Analytical Basis For Strain-Rate Dependence

The formulation in the previous section is valid only for the instantaneous, or strain-rate independent, stress-strain curve. In the following, we introduce the necessary modifications for the strain-rate dependence of the yield surface based on known material data. First, we postulate that the material parameter κ is a function of the total strain-rate, $\dot{\varepsilon}$ and temperature T as follows:

$$\kappa = \kappa (\dot{\varepsilon}, T) \qquad (11)$$

We further postulate that the yield surface translations α_{ij} depend on the plastic strains and the plastic strain-rates as follows:

$$\alpha_{ij} = \alpha_{ij} \left(\varepsilon^p_{11}, \varepsilon^p_{22}, \ldots; \dot{\varepsilon}^p_{11}, \dot{\varepsilon}^p_{22}, \ldots\right) \qquad (12)$$

such that

$$d\alpha_{ij} = C_1 \, d\varepsilon^p_{ij} + C_2 \, d\dot{\varepsilon}^p_{ij} \qquad (13)$$

where C_1 is similar to C, introduced in equation (7) and is strain-rate independent, and C_2 is a new material parameter to be defined later. Defining the quantity:

$$\dot{\varepsilon}^p_e = \left(\frac{2}{3}\, \dot{\varepsilon}^p_{ij}\, \dot{\varepsilon}^p_{ij}\right)^{\frac{1}{2}}, \tag{14}$$

then using the chain rule, we obtain

$$\frac{\partial \dot{\varepsilon}^p_e}{\partial \dot{\varepsilon}^p_{mn}} \frac{\partial \alpha_e}{\partial \dot{\varepsilon}^p_e} = \frac{\partial \alpha_e}{\partial \alpha_{ij}} \frac{\partial \alpha_{ij}}{\partial \dot{\varepsilon}^p_{mn}} \tag{15}$$

From equations (9) and (14),

$$\frac{\partial \alpha_e}{\partial \alpha_{ij}} = \frac{3}{2\,\alpha_e}\ \alpha_{ij}\ , \tag{16}$$

and

$$\frac{\partial \dot{\varepsilon}^p_e}{\partial \dot{\varepsilon}^p_{mn}} = \frac{2}{3\,\dot{\varepsilon}^p_e}\ \dot{\varepsilon}^p_{mn} \tag{17}$$

By virtue of equation (13),

$$\frac{\partial \alpha_{ij}}{\partial \dot{\varepsilon}^p_{mn}} = C_2\ \delta_{im}\ \delta_{jn} \tag{18}$$

Upon substituting equations (16), (17) and (18) into equation (15), and after some manipulation we obtain

$$C_2 = \frac{2}{3} \frac{\partial \alpha_e}{\partial \dot{\varepsilon}^p_e} = \frac{2}{3} E_2' \tag{19}$$

Following similar derivations, it can be shown that

$$C_1 = \frac{2}{3} \frac{\partial \alpha_e}{\partial \varepsilon^p_e} \Bigg|_{\dot{\varepsilon}^p_e = \infty} = \frac{2}{3} E_1' \tag{20}$$

It should be of interest to point out that in the present context, E_1' is the tangent modulus of the instantaneous stress-plastic strain curve, and the new parameter E_2' is the tangent modulus of the stress-plastic strain-rate curve. The new modulus E_2' is obtained by first plotting the uniaxial stress-strain data for various strain rates. Then from these curves one obtains plots of the stress-strain rate at various strains. The tangent of this family of curves is then

$$E_2^* = \frac{d\sigma}{d\dot{\varepsilon}} \Bigg|_{\varepsilon = \text{const.}}$$

and

$$E_2' = \frac{d\sigma}{d\dot{\varepsilon}^p} \Bigg|_{\varepsilon = \text{const.}} = \frac{d\sigma}{d\dot{\varepsilon}} \frac{d\dot{\varepsilon}}{d\dot{\varepsilon}^p} \Bigg|_{\varepsilon = \text{const.}} \tag{22}$$

But

$$\dot{\varepsilon}^p = \dot{\varepsilon} - \frac{\dot{\sigma}}{E} = \dot{\varepsilon}\left(1 - \frac{E_t}{E}\right)_{\dot{\varepsilon} = \text{const.}} \tag{23}$$

where E and E_t are the elastic and tangent moduli of the stress-strain curve respectively. From equations (22) and (23),

$$E_2' = E_2^* \Big/ \left(1 - \frac{E_t}{E}\right) \tag{24}$$

By similar derivation,

$$E_1' = E_1^* \Big/ \left(1 - E_1^* \big/ E\right) \tag{25}$$

In this equation E_1^* is the tangent modulus of the strain-rate independent stress-strain curve. In this analysis, this is the derivative of a stress-strain curve obtained at an arbitrarily determined cutoff strain-rate above which the strain-rate dependence becomes ignorable. However, E_t is the strain-rate dependent tangent modulus.

4. Formulations of the Stress-Strain Relations

The incremental stress-strain relations that depend continuously on the strain-rate can now be formulated. We first recall the strain-rate dependent yield function, namely,

$$F(\dot{\varepsilon},T) = \frac{1}{2} S_{ij}' S_{ij}' - \kappa^2 \tag{26}$$

The associated flow rule is

$$d\varepsilon_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{27}$$

Imposing the condition that the stress point remains on the yield surface during loading, we require that

$$dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial F}{\partial \dot{\varepsilon}_{ij}^p} d\dot{\varepsilon}_{ij}^p + \frac{\partial F}{\partial \kappa} d\kappa = 0 \tag{28}$$

Before we proceed further with the derivation, we consider the stress-strain curves shown in Fig. 1 . In this figure, the upper curve can be treated as a reference curve, valid for all strain-rates above a cutoff rate, $\varepsilon = 1$ say, where the material shows no measurable strain-rate sensitivity. For the reference curve, the change in stress can be written as

$$\Delta\sigma = C_1 \Delta\varepsilon \tag{29}$$

where C_1 represents the secant modulus of the reference (rate-independent curve). If the experiment were to be conducted at variable strain-rate, then the change in stress for a given strain can be assumed to follow two paths: first along the reference curve at constant strain-rate, and then at constant strain to the final position. Mathematically, this can be represented as follows

$$\Delta\sigma = C_1 \Delta\varepsilon + C_2 \Delta\dot{\varepsilon} \tag{30}$$

Generalizing equation (30) to arbitrary strain-rates and ignoring the effects of the elastic strain-rate, we obtain, in the limit,

$$d\sigma = C_1 (\varepsilon^p)\, d\varepsilon^p + C_2(\dot{\varepsilon}^p)\, d\dot{\varepsilon}^p = d\alpha \tag{31}$$

In this equation, $d\alpha$ represents the uniaxial equivalence of the yield surface translation. For three-dimensional stress states, by virtue of equations (12) and (13), we have

$$d\alpha_{ij} = C_1 (\varepsilon_{11}^p, \varepsilon_{22}^p, \ldots)\, d\varepsilon_{ij}^p + C_2 (\dot{\varepsilon}_{11}^p, \dot{\varepsilon}_{22}^p, \ldots)\, d\dot{\varepsilon}_{ij}^p \tag{32}$$

Then from equation (26),

$$\frac{\partial F}{\partial \sigma_{ij}} = S'_{ij} \tag{33}$$

and from equations (32) and (33), making use of equations (4) and (5),

$$\frac{\partial F}{\partial \varepsilon^p_{ij}} = \frac{\partial F}{\partial S'_{mn}} \frac{\partial S'_{mn}}{\partial \varepsilon^p_{ij}} = - S'_{mn} C_1 \delta_{im} \delta_{jn} = - C_1 S'_{ij} \tag{34}$$

and similarly,

$$\frac{\partial F}{\partial \dot{\varepsilon}^p_{ij}} = \frac{\partial F}{\partial S'_{mn}} \frac{\partial S'_{mn}}{\partial \dot{\varepsilon}^p_{ij}} = - S'_{mn} C_2 \delta_{im} \delta_{jn} = - C_2 S'_{ij} \tag{35}$$

Equation (28) can be utilized as a constraint condition to force the stress state to remain on the yield surface. The same constraint condition can also be imposed on the yield surface to conform to material data. More specifically if,

$$\dot{\varepsilon}^p_{ij} = \dot{J} S_{ij} \tag{36}$$

where

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}, \tag{37}$$

and $\dot{J}$ is the time-rate of change of the creep compliance obtained in a creep test, then the $d\dot{\varepsilon}^p_{ij}$ in equation (28) can be written as

$$d\dot{\varepsilon}^p_{ij} = \dot{J}\, dS_{ij} + d\dot{J}\, S_{ij} \tag{38}$$

At this point it would be convenient to discuss the creep compliance J. In a high temperature creep experiment, the creep strain may be expressed as

$$\varepsilon^c = A\sigma^n t^\alpha e^{-Q/RT} \tag{39}$$

Where A, n and α are constants, Q is the activation energy for high temperature creep (may be a function of temperature), R is the gas constant and T is the temperature. Most material data indicate that for high temperature $\alpha = 1$, implying steady state creep. This is sufficient for our present purposes. From equation (39),

$$\dot{\varepsilon}^c_e = A\sigma^n_e e^{-Q/RT} \tag{40}$$

from which

$$\dot{J} = \frac{3}{2\sigma_e} \dot{\varepsilon}^c_e \tag{41}$$

consequently

$$d\dot{J} = \frac{3}{2\sigma_e} \left[\frac{\partial \dot{\varepsilon}^c_e}{\partial \sigma_e} d\sigma_e + \frac{\partial \dot{\varepsilon}^c_e}{\partial T} dT \right] \tag{42}$$

which, after simplification, yields

$$d\dot{J} = \dot{J} \left(\frac{n}{\sigma_e} d\sigma_e + \frac{Q}{RT^2} dT \right) \tag{43}$$

Substituting for $d\sigma_e$, from the relation

$$d\sigma_e = d\left(\frac{3}{2} S_{ij} S_{ij}\right)^{\frac{1}{2}} = S_{ij} dS_{ij}/\sigma_e = S_{ij} d\sigma_{ij}/\sigma_e \tag{44}$$

Combining equations (38), (43), and (44) we obtain

$$d\dot{\varepsilon}^p_{ij} = \dot{J}\, dS_{ij} + \frac{\dot{J}n}{\sigma_e^2} S_{mn}\, d\sigma_{mn}\, S_{ij} + \frac{\dot{J}Q}{RT^2} dT\, S_{ij} \tag{45}$$

From equation (27) and (33), we have

$$d\varepsilon_{ij}^{p} = \lambda S'_{ij} \tag{46}$$

Next, we assume that the strains can be decomposed into its three components, elastic, plastic (strain-rate independent) and creep. This assumption may not be strictly valid for large strains that could occur in high temperature deformations, but because in such a case one deals with small elastic strains compared to the inelastic strains, the assumption is acceptable. On this basis then,

$$d\varepsilon_{ij} = C_{ijkl}\, d\sigma_{kl} + \lambda\, S'_{kl} + \dot{J}\, dt\, S_{kl} + d\theta_{ij} \tag{47}$$

where C_{ijkl} is the elastic material tensor and $d\theta_{ij}$ are the thermal strains. From equation (46) we have

$$d\sigma_{ij} = H_{ijkl} \left[d\varepsilon_{kl} - \lambda S'_{kl} - \dot{J}\, dt\, S_{kl} - d\theta_{kl}\right] \tag{48}$$

Substituting equations (33), (34), (35), (45), (46) and (49) into equation (28) yields

$$S'_{ij}\, d\sigma_{ij} - C_1 S'_{ij} \lambda S'_{ij}$$
$$- C_2\, S'_{ij}\, (\dot{J} dS_{ij} + \frac{\dot{J}n}{\sigma_e^2}\, S_{mn} d\sigma_{mn} S_{ij} + \frac{\dot{J}Q}{RT^2}\, dT\, S_{ij}) - 2\kappa\, d\kappa = 0 \tag{49}$$

Grouping terms, we have

$$(S'_{ij} - C_2 \dot{J} S'_{ij} - \frac{nJ}{\sigma_e^2}\; C_2 S'_{mn} S_{mn} S_{ij})\; d\sigma_{ij}$$
$$- C_1 \lambda S'_{ij} S'_{ij} - \frac{\dot{J}Q}{RT^2}\, dT\, C_2\, S'_{ij}\, S_{ij} - 2\kappa\, d\kappa = 0 \tag{50}$$

Substituting for $d\sigma_{ij}$ from (48), yields

$$(S'_{ij} - C_2 \dot{J} S'_{ij} - \frac{n\dot{J}}{\sigma_e^2}\, C_2 S'_{mn} S_{mn} S_{ij})\;\; H_{ijkl}\, (d\varepsilon_{kl} - \lambda S'_{kl} - \dot{J} dt\, S_{kl} - d\theta_{kl})$$
$$- \lambda C_1 S'_{ij} S'_{ij} - \frac{\dot{J}Q\, dT}{RT^2}\;\, C_2\, S'_{ij} S'_{ij} - 2\kappa\, d\kappa = 0 \tag{51}$$

Define

$$\beta_{ij} = S'_{ij} - C_2 \dot{J} S'_{ij} - \frac{\eta J}{\sigma_e^2}\, C_2 S'_{mn} S_{mn} S_{ij} \tag{52}$$

and

$$d\varepsilon^{*}_{kl} = d\varepsilon_{kl} - \dot{J}\, dt\, S_{kl} - d\theta_{kl} \tag{53}$$

equation (51) becomes

$$\beta_{ij} H_{ijkl}\, (d\varepsilon^{*}_{kl} - \lambda S'_{kl}) - \lambda C_1 S'_{ij} S'_{ij}$$
$$\frac{-\dot{J}Q\, dT}{RT^2}\; C_2\;\; S'_{ij} S_{ij} - 2\kappa\, d\kappa = 0 \tag{54}$$

Solving for λ from (54) gives

$$\lambda = \Lambda_{kl}\, d\varepsilon^{*}_{kl} + G \tag{55}$$

where

$$\Lambda_{kl} = \beta_{ij} H_{ijkl}/R \tag{56}$$

$$G = -\frac{\dot{J}Q}{RT^2} dT\, C_2 S'_{ij} S'_{ij} - 2\kappa\, d\kappa \tag{57}$$

$$R = \beta_{ij} H_{ijkl} S'_{kl} + C_1 S'_{ij} S'_{ij} \tag{58}$$

Finally, the incremental stress-strain relations can be obtained from equations (48) and (55) as follows:

$$d\sigma_{ij} = H_{ijkl} \left[d\varepsilon^*_{kl} - \eta S'_{kl} (\Lambda_{mn} d\varepsilon^*_{mn} + G) \right] \tag{59}$$

where $\eta = 0$ or 1, for elastic or plastic state respectively. Equation (59) can be further broken down as follows:

$$\begin{aligned} d\sigma_{ij} = {} & H_{ijkl} \left[d\varepsilon_{kl} - \eta S'_{kl} \Lambda_{mn} d\varepsilon_{mn} \right] \\ & - H_{ijkl} \left[\dot{J}\, dT\, S_{kl} + d\theta_{kl} + \eta S_{kl} G \right. \\ & \left. - \eta S_{kl} \Lambda_{mn} (\dot{J}\, dT\, S_{mn} + d\theta_{mn}) \right] \end{aligned} \tag{60}$$

The yield surface translation is given by[4] :

$$d\alpha_{ij} = \sigma'_{ij}\, du \tag{61}$$

where σ'_{ij} is given by equation (5). The scaler du is determined by the condition that the stress point remains on the yield surface during loading. This condition is stated as follows:

$$(d\sigma_{ij} - d\alpha_{ij}) \frac{\partial F}{\partial \sigma_{ij}} + \frac{\partial F}{\partial \kappa} \frac{d\kappa}{d\dot{\varepsilon}} d\dot{\varepsilon} + \frac{\partial F}{\partial \kappa} \frac{d\kappa}{dT} dT = 0 \tag{62}$$

The first term in equation (62) gives the projection of the change in stress on the normal to the yield surface and the second and third terms give the changes in the yield surface due to the changes in the strain-rate and the temperature respectively. Setting the sum of these three terms to zero satisfies the condition that the changes in the calculated stresses be compatible with the yield surface motion.

Equations (61) and (62) can be solved simultaneously to give the expression for the yield surface translation as follows:

$$d\alpha_{ij} = \sigma'_{ij} \left[S'_{mn} d\sigma_{mn} - 2\kappa \left(\frac{\partial \kappa}{\partial \dot{\varepsilon}} d\dot{\varepsilon} + \frac{\partial \kappa}{\partial T} dT \right) \right] \Big/ \sigma'_{kl} S'_{kl} \tag{63}$$

It might be more useful at this point to recast the above equations in matrix form for easier implementation. We first combine equations (52) and (56),

$$\underset{\sim}{\Lambda} = \left(\underset{\sim}{S}'^{T} - C_2 \dot{J} \underset{\sim}{S}'^{T} - \frac{\eta \dot{J} C_2}{\sigma_e^2} \underset{\sim}{S}'^{T} \underset{\sim}{S}\, \underset{\sim}{S}'^{T} \right) \underset{\sim}{H} \Big/ R \tag{64}$$

Then equation (60) becomes

$$\begin{aligned} d\underset{\sim}{\sigma} = {} & \underset{\sim}{H} \left[\underset{\sim}{I} - \eta \underset{\sim}{S}' \Lambda^T \right] d\underset{\sim}{\varepsilon} \\ & - \underset{\sim}{H} \left[\dot{J}\, dt\, \underset{\sim}{S} + d\underset{\sim}{\theta} + \eta G \underset{\sim}{S}' \right. \\ & \left. - \eta \dot{J}\, dt\, \underset{\sim}{S}' \underset{\sim}{\Lambda}^T \underset{\sim}{S} - \eta \underset{\sim}{S}' \underset{\sim}{\Lambda}^T d\underset{\sim}{\theta} \right] \end{aligned} \tag{65}$$

Equation (65) together with equations (57), (58), (63) and (64), constitute the incremental stress strain relations for materials exhibiting the following characteristics:

(a) anisotropic elasticity

(b) kinematic hardening

(c) temperature and strain-rate dependent yield function

(d) strain-rate sensitivity (strain-rate dependent stress-strain curve)

(e) creep

The material data in these equations appear in the parameters:

κ, C_1, C_2 and $\dot{J}$.

5. Numerical Examples

The constitutive relations presented in the previous sections were tested using two examples: The first is a simple example of uniaxial tension under step changes in strain-rate from 10^{-4} in/in/hr to 10^{0} in/in/hr. The analysis was carried out through a two-dimensional finite element computer program applying equal strain increments of 0.05%. The strain increments at the jumps were applied instantaneously. Fig. 3 shows the solution compared with the material data. As can be seen in the figure, the solution follows the jumps between the constant strain-rate curves elastically as expected. The true test of the theory, however, is a comparison with variable strain-rate tests in which the strain-rate is varied arbitrarily in the test. However, such data does not exist in an ideal manner, and the second example makes use of some of the limited data reported in reference [5]. The results are shown in Fig. 4 , which compares the present analysis with experiment (see Fig. 4 , 6 specimen T-25, Ref. 5). This example gives a qualitative verification of the theory in the sense that the trends in the material behavior seem to be fairly well represented although the actual values are not. Quantitative matching of the stress-strain curve is not expected here because of the differences in the material data used in the analysis and the experiment.

References

[1] RABATNOV, Y. N., "On the Equation of State for Creep", Progress in Applied Mechanics, Prager Anniversary volume, McMillan, New York, (1963).

[2] PRAGER, N., "A New Method of Analyzing Stress and Strain in Work Hardening Plastic Solids", J. Appl. Mech. 23, 493-496, (1956).

[3] RASHID, Y. R., "On Nonlinear Kinematic Hardening Plasticity Theory", Nucl. Eng. Design 29, 135-140 (1974).

[4] ZIEGLER, H., "A Modification of Prager's Hardening Rule", Quart. Appl. Math., 17, 55-65, (1959).

[5] AZZARTO, F. J., BALDWIN, E. E., WIESINGER, F.W., LEWIS, D. M., "Unirradiated, In-Pile and Post-Irradiation Low Strain Rate Tensile Properties of Zircaloy-4", J. Nuc. Mat. 30, 208-218, (1969).

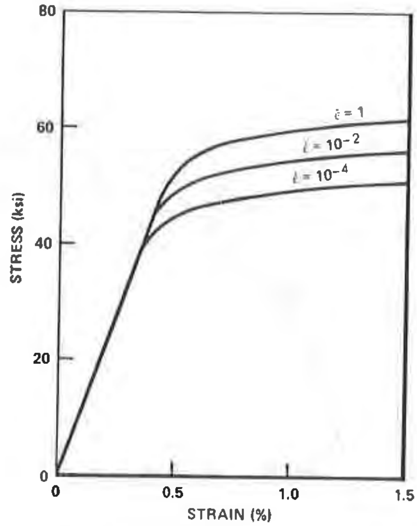

Fig. 1 Stress vs. Strain Curves For Different Strain-Rates

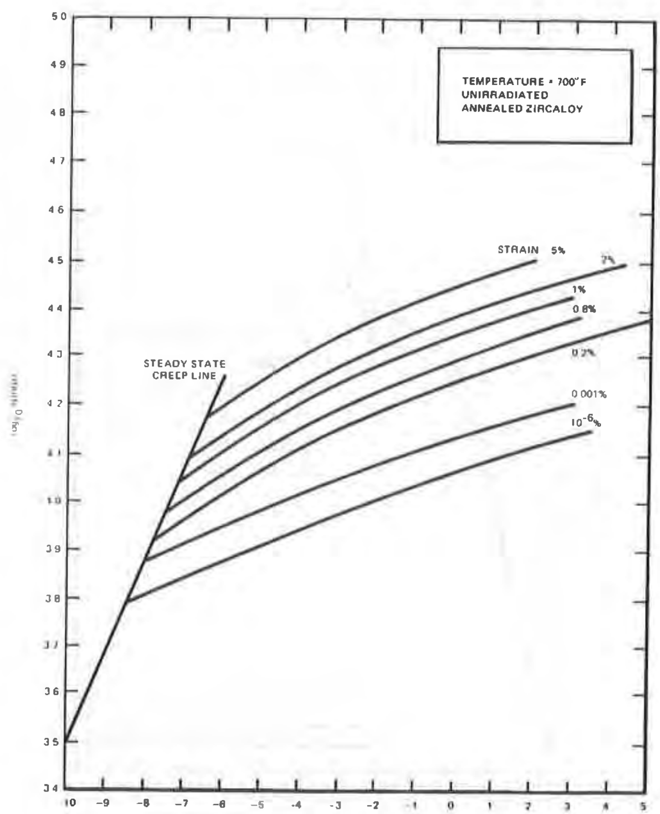

Fig. 2 Stress vs. Strain-Rate Curves For Different Strains

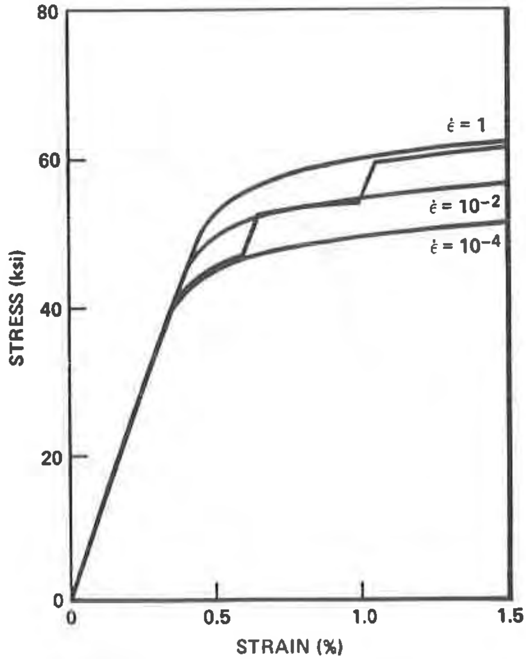

Fig. 3 Analysis of Step Changes in Strain-Rates Compared with Constant Strain-Rate Data

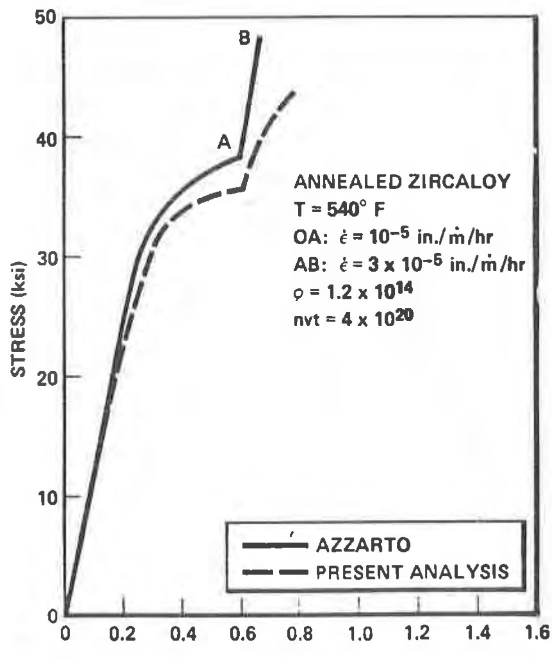

Fig. 4 Comparison of Analysis With Variable Slow Strain-Rate Experimental Data