ABSTRACT

DAUGHERTY, SPENCER. Schur-like Bases and their Colored Generalizations. (Under the direction of Laura Colmenarejo and Sarah Mason).

We present new results on Schur-like bases of $QSym$ and $NSym$, focusing on bases related by involutions. We introduce the reverse immaculate, row-strict reverse immaculate, and row-strict reverse dual immaculate functions. These bases relate to the immaculate and dual immaculate bases via the involutions ρ and ω on $QSym$ and $NSym$ and share many of their combinatorial properties. Together, the immaculate, row-strict immaculate, reverse immaculate, and row-strict reverse immaculate bases form a closed system under the involutions ψ, ρ, and ω of $NSym$ which generalize the classical involution ω in Sym; the same is true with the respective dual bases in $QSym$. Similarly, we define reverse shin, reverse extended Schur, row-strict reverse shin, and row-strict reverse extended Schur functions that relate similarly to the shin and extended Schur bases. We then prove a variety of properties of the shin, extended Schur, immaculate, and dual immaculate bases as well as their variants. These include basis expansions, skew functions, multiplicative structure, creation operators, and Hopf algebra structure.

We introduce generalizations of the primary Schur-like bases to the dual Hopf algebras $QSym_A$ and $NSym_A$. We define the colored immaculate, colored dual immaculate, colored shin, colored extended Schur, colored Young noncommutative Schur, and colored Young quasisymmetric Schur functions, all of which are bases of $NSym_A$ or $QSym_A$. Each colored Schur-like basis in $QSym_A$ is defined using different types of colored tableaux. The colored Schur-like bases in $NSym_A$ are defined in various ways including duality, creation operators, and a Pieri rule. We prove results relating to expansions to and from other bases, multiplicative structure, skew functions, and Hopf algebra structure. Then, we define two new dual Hopf algebras, one of which is the commutative image of $NSym_A$ and the other a subset of $QSym_A$. Both are colored analogues of Sym and each has a basis that generalizes the Schur functions in terms of colored semistandard Young tableaux.

Schur-like Bases and their Colored Generalizations

by
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BIOGRAPHY

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER

1 INTRODUCTION

The study of symmetric functions forms a cornerstone of algebraic combinatorics. The Schur symmetric functions specifically have robust combinatorics and a wide variety of applications. One of their most important properties is that they are the characters of irreducible representations of the general linear group [78]. The Schur functions appear in representation theory, algebraic geometry, linear algebra, and of course, combinatorics. They lay at the center of many open problems in combinatorics, including Schur positivity problems, the Schur plethysm problem, and problems on the Kronecker coefficients of Schur functions [23, 33, 49, 55, 74, 79].

The quasisymmetric functions introduced by Gessel [35] and the noncommutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [34] are generalizations of the symmetric functions with rich theory and importance in algebraic combinatorics. Their algebras, *QSym* and *NSym*, are dual Hopf algebras that also appear in representation theory, algebraic geometry, and category theory [2, 47, 57, 60]. Notably, *QSym* is the terminal object in the category of combinatorial Hopf algebras [2]. *QSym* contains *Sym*, the algebra of symmetric functions, and *Sym* is the commutative image of *NSym*. A significant amount of work has been done to find quasisymmetric or noncommutative analogues of symmetric function objects, specifically of the Schur basis. One example of this is the development of a variety of *Schur-like bases* such as the immaculate functions, the shin functions, the quasisymmetric Schur functions, and the extended Schur functions [7, 9, 19, 39, 56]. A Schur-like basis of *NSym* is one in which the commutative image of any element indexed by a partition is the Schur function indexed by that partition. Schur-like bases in *QSym* are the bases dual to the aforementioned bases of *NSym* and tend to have combinatorial descriptions in terms of tableaux that are closely related to semistandard Young tableaux.

The first goal of this thesis is to continue developing the theory of these Schur-like bases and to create a comprehensive text studying them. With any Schur-like basis, the properties of the Schur functions we would like to find analogues for include a matrix determinant formula, various types of multiplicative rules, Hopf algebraic structure, and representation theoretic applications, among others. Each of the dual pairs of Schur-like bases reflects these properties in unique ways, making them each suitable for different applications.

The second goal of this thesis is to define colored analogues to Schur-like bases in $QSym_A$ and $NSym_A$, which are generalizations of $QSym$ and $NSym$ introduced by Doliwa [28]. The Hopf algebra $NSym$ has a natural generalization called $NSym_A$ that stems from the relationship between $NSym$ and the algebra of rooted trees [30]. Its dual algebra, $QSym_A$, is defined dually using partially commutative colored variables. The initial goal of these generalizations was to extend the study of the relationship between symmetric functions and integrable systems to a noncommutative setting which is of growing interest in mathematical physics [27, 48]. Additionally, the Hopf algebra of rooted trees has various applications in the field of symbolic computation [38]. Doliwa defines the algebraic structure of $QSym_A$ and $NSym_A$ and analogues to some classic bases. We bring the study of Schur-like bases to this space to continue developing its theory.

In addition to the inherent value in expanding the theory of these spaces, any results on bases in $QSym_A$ and $NSym_A$ specialize immediately to results on their analogous bases in $QSym$ and $NSym$ because they are isomorphic in the case that A is an alphabet of just one letter. In many cases, the combinatorics behind relationships in $QSym$ and $NSym$ can be obscured by cancellation which is significantly reduced when coloring the variables. The techniques we use have great potential for studying various objects related to $QSym$ such as posets, polytopes, and crystal graphs.

1.1 Context

While the algebras introduced by Doliwa in [28] are new, various algebras that generalize Sym, $QSym$ and $NSym$ using colored variables have appeared in the literature. There has also been significant interest generally in exploring new combinatorial Hopf algebras. For examples of each, see [2, 12, 42, 43, 52, 66, 72]. Following these themes, we build on Doliwa's work by expanding the theory of the algebras $NSym_A$ and $QSym_A$, as well as defining similar new algebras of our own. We simultaneously build on a wide variety of work on Schur-like bases by many authors through the introduction of various colored generalizations of Schur-like bases.

The first of the Schur-like bases to be introduced was the quasisymmetric Schur functions in [39] by Haglund, Luoto, Mason, and van Willigenburg. They arose from specializations of nonsymmetric Macdonald polynomials called Demazure atoms. In [56], Luoto, Mykytiuk, and van Willigenburg introduced the Young quasisymmetric Schur functions, which were ultimately more compatible with the Schur basis. These two bases notably generalize the Littlewood-Richardson rule of the Schur functions. They also provide a great example of why it is worthwhile to study Schur-like bases related by involutions. While the combinatorics of these bases are similar, they have very different applications. For example, the quasisymmetric Schur basis and the Young quasisymmetric

Schur basis are related by the involution ρ but the former is much more compatible with Macdonald polynomials while the latter is more useful when working with Schur functions [39, 56]. Both the quasisymmetric Schur and Young quasisymmetric Schur functions have been widely studied, including in [4, 14, 15, 44, 45, 54, 58, 59, 61, 62, 64, 67, 73, 75, 80, 84, 85].

Next to be introduced were the immaculate and dual immaculate functions of Berg, Bergeron, Saliola, Serrano, and Zabrocki in [9]. These functions generalize the Bernstein creation operators and the Jacobi-Trudi rule of the Schur functions. These bases are also widely studied, namely in [1, 4, 5, 10, 11, 13, 18, 21, 22, 32, 36, 51, 53, 63, 69, 68, 70, 71]. Our additional contribution here is to define new variants of the immaculate and dual immaculate functions that are related via involutions, much like the row-strict variants of [69] or the reverse dual immaculate functions of [63].

Last to be introduced were the shin basis of Campbell, Feldman, Light, Shuldiner, and Xu [20] and the extended Schur basis of Assaf and Searles [6]. The shin and extended Schur functions, which are dual bases, are unique among the Schur-like bases for having arguably the most natural relationship with the Schur functions. In *NSym*, the commutative image of a shin function indexed by a partition is the Schur function indexed by that partition, while the commutative image of any other shin function is 0. In $QSym$, the extended Schur function indexed by a partition is equal to the Schur function indexed by that partition [19]. The extended Schur and shin bases are also studied in [17, 58, 76]. We define new variants of the extended Schur and shin bases as we do in the immaculate case. In addition, we define skew and skew-II extended Schur functions as well as their variants related by involutions, and we formalize certain Hopf algebra properties that appear in the process building on [56, 70, 80]. Finally, we find a creation operator construction and related Jacobi-Trudi rule for certain shin functions, inspired by [9].

1.2 Overview

Chapter 3 focuses on the immaculate and dual immaculate functions. In Section 3.2, we introduce a new basis of $QSym$ and two new bases of $NSym$, each of which results from applying the ρ or ω involutions to the immaculate or dual immaculate functions. We call the new basis of $QSym$ the row-strict reverse dual immaculate functions, and we call the two new bases in *NSym* the reverse and row-strict reverse immaculate functions. They are defined combinatorially with tableaux that resemble immaculate tableaux but have different conditions on whether rows increase or decrease and whether that change is weak or strict. The reverse immaculate functions are dual to the reverse dual immaculate functions of Mason and Searles [63], which we study alongside the new bases we define. We also connect these bases to the row-strict immaculate and dual immaculate functions, which result from the application of ψ to the original bases [68].

Together, the involutions ψ, ρ, and ω on $QSym$ and *NSym* generalize the classical involution ω on the symmetric functions. In Sym, the Schur basis is its own image under ω, but in $QSym$ the dual immaculate functions are part of a system of 4 bases that is closed under the three involutions ψ, ρ, and ω. The existence of this system of 4 bases as an analogue to the behavior of the Schur functions under ω is an interesting feature of how these bases generalize the Schur functions. Mason and Searles study various polynomials related by ω in [64], using the quasisymmetric Schur functions and the Young quasisymmetric Schur functions as a starting point.

We use the involutions ρ and ω to state reverse and row-strict reverse analogs to classical results on the immaculate and dual immaculate bases. These include a Pieri rule, a Jacobi-Trudi rule, expansions into other bases, a description of the antipode, a partial Littlewood-Richardson rule, and two different types of skew functions. We set up a framework for skew-II functions using a left action of $NSym$ on $QSym$. We show that the skew-II reverse and row-strict reverse dual immaculate functions are the image of the skew dual immaculate functions under ρ and ω respectively.

In Section 3.3, we generalize the immaculate and dual immaculate bases with the colored immaculate and colored dual immaculate bases of $NSym_A$ and $QSym_A$. The colored dual immaculate functions are defined combinatorially in terms of colored immaculate tableaux. We then give expansions of the colored dual immaculate functions into the colored monomial and colored fundamental bases using the combinatorics of colored immaculate tableaux. Further, we provide an expansion of the colored fundamental functions into the colored dual immaculate basis using a new object we call the colored immaculate descent graph defined using standard tableaux. This result specializes to a new analogous result on the fundamental and dual immaculate bases in $QSym$ that allows for fairly straightforward computation of coefficients.

The colored immaculate functions are defined as the dual basis to the colored dual immaculate functions, but can be defined equivalently with generalizations of the noncommutative Bernstein operators from [9]. These operators allow us to construct the colored immaculate functions as an alternating sum that uses a secondary operator in $NSym_A$ defined to be dual to multiplication in $QSym_A$. We prove a right Pieri rule for the colored immaculate basis and give expansions of the colored complete homogeneous and colored ribbon bases into the colored immaculate basis. Using duality, we obtain an expansion of the colored immaculate functions into the colored ribbon basis using the colored immaculate descent graph. Applying the uncoloring map yields a new combinatorial model for expanding the immaculate functions into the ribbon noncommutative symmetric functions. Additionally, applying the forgetful map yields a new expression for the decomposition of Schur functions into ribbon Schur functions.

Next, we introduce skew colored immaculate tableaux and a partially ordered set on sentences in the style of the immaculate poset. We use this poset to define skew colored dual immaculate functions and find results related to the structure constants of the colored immaculate basis. Finally, we define the colored row-strict immaculate and colored row-strict dual immaculate functions. These two bases are related to the immaculate and dual immaculate bases by an involution on sentences, which we use to translate our results from previous sections to the row-strict case. This material, starting in Section 3.3, also appears in our paper [25].

In Chapter 4, we study the shin and extended Schur bases. The extended Schur bases are defined combinatorially over shin-tableaux which generalize Young tableaux to composition shapes. Their dual, the shin basis, is defined as the unique set of functions that satisfy a multiplicative property called a right Pieri rule. In Section 4.2, we define a creation operator that can be used for a significant number of shin functions. Using this creation operator, we state a Jacobi-Trudi formula for shin functions indexed by strictly increasing compositions. This rule expresses these shin functions as a matrix determinant in terms of the complete homogeneous noncommutative symmetric functions. In Section 4.3, we define the skew extended Schur functions algebraically and in terms of skew shin-tableaux, and note their implications for the multiplicative structure of the shin basis. We also define skew-II extended Schur functions algebraically.

In Section 4.4, we introduce two new bases found by applying the ρ and ω involutions to the extended Schur functions, as well as their dual bases in *NSym*. We call these new bases the reverse extended Schur functions and the row-strict reverse extended Schur functions, while their duals in *NSym* are the reverse and row-strict reverse shin functions, respectively. They are defined combinatorially with tableaux that resemble the shin-tableaux but have different conditions on whether rows increase or decrease and whether that change is weak or strict. We also connect these bases to the row-strict extended Schur and shin functions which result from the application of ψ to the original bases [68].

We use the involutions to state row-strict, reverse, and row-strict reverse analogs to our new and classical results on the extended Schur and shin bases. These include a Pieri rule, a rule for multiplication by a ribbon function, a partial Jacobi-Trudi rule, expansions into other bases, a description of the antipode on the shin and extended Schur bases, and two different types of skew functions. Specifically, we show that the skew-II reverse and row-strict reverse extended Schur functions are the image of the skew extended Schur functions under ρ and ω respectively. The work on shin and extended Schur functions in this chapter up to this point also appears in our paper [26].

Finally, we define colored generalizations of the shin and extended Schur functions in $NSym_A$ and $QSym_A$. The colored extended Schur functions are defined using a colored generalization of shin-tableaux and the colored shin functions are defined as their duals. We are able to generalize various properties from the original bases including, for example, a multiplication rule for the colored ribbon functions.

Chapter 5 focuses on the Young quasisymmetric Schur and Young noncommutative Schur functions as well as bases related by involutions. We define colored generalizations of these bases in $QSym_A$ and $NSym_A$ using colored versions of Young composition tableaux. Then, we prove expansions to and from other colored bases, a right Pieri rule for the colored Young noncommutative Schur functions, and properties of skew colored Young quasisymmetric functions.

In Chapter 6, we defined two new dual Hopf algebras. $PSym_A$ is the commutative image of $NSym_A$ and Sym_A is a subset of $QSym_A$. Both are isomorphic to Sym when A is a unary alphabet. In addition to defining and studying the structure of these two Hopf algebras, we define a pair of dual bases in each that generalize the Schur basis. These are defined in terms of colored semistandard Young tableaux. We close by listing a few open questions.

CHAPTER

2

PRELIMINARIES

We first outline combinatorial basics and terminology, including partitions, compositions, diagrams, and various partial orders. Next, we review the symmetric functions and their bases, with a focus on the Schur basis and its properties following [78]. This is followed by an overview of Hopf algebras based on [28, 37], which leads into material on the quasisymmetric and noncommutative symmetric functions from the previous two sources as well as [9, 60]. We close our background section by presenting Doliwa's colored generalizations of $QSym$ and $NSym$ from [28].

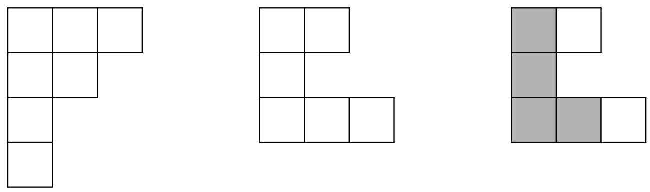
A *composition* of a positive integer n, written $\alpha \vDash n$, is a sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $\sum_i \alpha_i = n$. The *length* of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is the number of parts, $\ell(\alpha) = k$, and the *size* of a composition is the sum of its parts, $|\alpha| = \sum_i \alpha_i = n$. A *weak composition* is a composition that allows zeroes as entries. If β is a weak composition then $\tilde{\beta}$, called the *flattening* [7] of β, is the composition that results from removing all 0's from β. The length of a weak composition is also its number of parts, although it is often implicitly assumed that there are infinitely many zeroes at the end of any weak composition. A *partition* of a positive integer n, written $\lambda \vdash n$, is a composition $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_i \lambda_i = n$. We will typically use λ, μ, and ν to denote partitions and α, β, and γ to denote compositions

Example 2.0.1. The composition $\alpha = (2, 1, 3)$ has size $|\alpha| = 6$ and length $\ell(\alpha) = 3$. The flattening of the weak composition $\beta = (0, 1, 1, 0, 2)$ is $\tilde{\beta} = (1, 1, 2)$. The partition $\lambda = (3, 2, 1, 1)$ has size $|\lambda| = 7$ and length $\ell(\lambda) = 4$.

The *composition diagram* of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a left-justified array of boxes such that row i has α_i boxes. Following the English convention, the top row is considered to be row 1. The composition diagram of a partition is called a *Young diagram*. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and

$\beta = (\beta_1, \ldots, \beta_j)$ be compositions such that $j \leq k$ and $\beta_i \leq \alpha_i$ for $1 \leq i \leq j$. The *skew shape* α/β is a composition diagram of shape α where the first β_i boxes in the i^{th} row are removed for $1 \leq i \leq j$. We represent this removal by shading in the removed boxes.

Example 2.0.2. Let $\lambda = (3, 2, 1, 1)$, $\alpha = (2, 1, 3)$, and $\beta = (1, 1, 2)$. Then the Young diagram of λ, the composition diagram of α, and the skew shape α/β are respectively:



Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_j)$ be two compositions. Under the *refinement order* $\preceq$ on compositions of size n, we say $\alpha \preceq \beta$ if and only if $\{\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \cdots + \beta_j\} \subseteq \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_k\}$. Under the *lexicographic order* $\leq_\ell$ on compositions, $\alpha \leq_\ell \beta$ if and only if $\alpha_i < \beta_i$ where i is the first positive integer such that $\alpha_i \neq \beta_i$. Under the *reverse lexicographic order* $\leq_{r\ell}$ on compositions, $\alpha \leq_{r\ell} \beta$ if and only if $\alpha_i > \beta_i$ where i is the smallest positive integer such that $\alpha_i \neq \beta_i$. Note that, in the last two orders, if such an i does not exist then $\alpha = \beta$. Under the *dominance order* $\subseteq$ on compositions, we say $\alpha \subseteq \beta$ if and only if $k \leq j$, and $\alpha_i \leq \beta_i$ for $1 \leq i \leq k$. The refinement and dominance orders are graded, by length and size respectively.

Example 2.0.3. We have the following chains in the corresponding orders:
Refinement order: $(1, 1, 1, 1) \preceq (1, 2, 1) \preceq (1, 3) \preceq (4)$
Lexicographic order: $(1, 2, 3) \leq_\ell (1, 3, 2) \leq_\ell (2, 1, 3) \leq_\ell (2, 3, 1) \leq_\ell (3, 1, 2) \leq_\ell (3, 2, 1)$
Reverse lexicographic order: $(3, 2, 1) \leq_{r\ell} (3, 1, 2) \leq_{r\ell} (2, 3, 1) \leq_{r\ell} (2, 1, 3) \leq_{r\ell} (1, 3, 2) \leq_{r\ell} (1, 2, 3)$
Dominance order: $(1, 1, 1) \subseteq (2, 1, 1, 1) \subseteq (2, 3, 1, 2)$

There is a natural bijection between subsets of $[n-1] = \{1, 2, \ldots, n-1\}$ and compositions of n. For a set $S = \{s_1, \ldots, s_k\} \subseteq [n-1]$ with $s_1 < \cdots < s_k$, we have $comp(S) = (s_1, s_2 - s_1, \ldots, s_k - s_{k-1}, n - s_k)$ and for a composition $\alpha = (\alpha_1, \ldots, \alpha_j)$, we have $set(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_{j-1}\}$.

Example 2.0.4. For $n = 8$, let $S = \{2, 3, 6, 7\}$ and $\alpha = (1, 2, 1, 4)$. Then $comp(S) = (2, 1, 3, 1, 1)$ and $set(\alpha) = \{1, 3, 4\}$.

We consider three involutions on compositions. The *complement* of a composition α is defined $\alpha^c = comp(set(\alpha)^c)$ where $set(\alpha)^c$ is the (set) complement of $set(\alpha)$ in $[n-1]$. Intuitively, if the composition α is represented as blocks of stars separated by bars, then α^c is the composition given by placing bars in exactly the places where α does not have bars. The *reverse* of $(\alpha_1, \ldots, \alpha_k)$, denoted α^r, is defined as $(\alpha_k, \ldots, \alpha_1)$. The *transpose* of α is defined $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$. We also consider one involution on partitions. The *conjugate* of a partition λ, denoted λ', is obtained by flipping the diagram of λ over the diagonal. The conjugate of a partition is also sometimes called the transpose but, with our notation, λ^t and λ' are different in general.

Example 2.0.5. If $\alpha = (1, 2, 3, 1)$, then $sort(\alpha) = (3, 2, 1, 1)$. Let $\beta = (3, 2)$. Then, we have

$$\beta^c = (1, 1, 2, 1), \quad \beta^r = (2, 3), \quad \beta^t = (1, 2, 1, 1), \quad \text{and} \quad \beta' = (2, 2, 1).$$

We also use three operations on compositions. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_j)$ be two compositions. The *concatenation* of α and β is given by $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \cdots, \beta_j)$ and the *near-concatenation* is given by $\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1, \beta_2, \ldots, \beta_j)$. The *shuffle product* (or shuffle) of α and β is a formal sum given by $\alpha \sqcup\!\sqcup \beta = \sum_{\{i_1,\ldots,i_k\}\subseteq[j+k]} \gamma$ where the sum runs over all subsets such that $i_1 < \cdots < i_k$ and γ is the composition such that $\gamma_{i_1} = \alpha_1$, $\gamma_{i_2} = \alpha_2$, $\ldots \gamma_{i_k} = \alpha_k$, and, if $[j + k] \setminus \{i_1, \ldots, i_k\} = \{b_1, \ldots, b_j\}$ with $b_1 < \cdots, b_j$, then $\gamma_{j_1} = \beta_1$, $\gamma_{j_2} = \beta_2$, etc. In other words, $\alpha \sqcup\!\sqcup \beta$ is the sum of all compositions obtained by interweaving α and β, with multiplicity. Additionally, define $sort(\alpha)$ as the unique partition that can be obtained by rearranging the parts of α into weakly decreasing order [56]. In the literature, $sort(\alpha)$ is sometimes denoted as $\tilde{\alpha}$ which we use to denote the flattening of a composition for consistency with [28].

Example 2.0.6. Let $\alpha = (1, 2, 3, 1)$ and $\beta = (3, 2)$. Then,

$$\alpha \cdot \beta = (1, 2, 3, 1, 3, 2) \qquad \alpha \odot \beta = (1, 2, 3, 4, 2) \qquad sort(\alpha) = (3, 2, 1, 1).$$

$$(1, 2) \sqcup\!\sqcup (3, 2) = (1, 2, 3, 2) + (1, 3, 2, 2) + (3, 1, 2, 2) + (1, 3, 2, 2) + (3, 1, 2, 2) + (3, 2, 1, 2).$$

A *permutation* σ of a set is a bijection from the set to itself. The permutation σ of $[n]$ is written in one-line notation as $\sigma(1)\sigma(2)\cdots\sigma(n)$.

Example 2.0.7. The permutation 312 maps $1 \to 3$, $2 \to 1$, and $3 \to 2$.

For any set I, the *Kroenecker delta* is the function defined for $i, j \in I$ as

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For more basics see [77, 78].

2.1 Symmetric functions

Let $x = (x_1, x_2, \ldots)$ and $c_\alpha \in \mathbb{Q}$. A *symmetric function* $f(x)$ with rational coefficients is a formal power series $f(x) = \sum_\alpha c_\alpha x^\alpha$ where α is a weak composition of a positive integer, $x^\alpha = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$, and $f(x_{\omega(1)}, x_{\omega(2),\ldots}) = f(x_1, x_2, \ldots)$ for all permutations ω of $\mathbb{Z}_{>0}$.

Example 2.1.1. The following function $f(x)$ is a symmetric function.

$$f(x) = x_1^2x_2^3x_3 + x_1^2x_3^3x_2 + x_2^2x_1^3x_3 + x_2^2x_3^3x_1 + x_3^2x_1^3x_2 + x_3^2x_2^3x_1 + \ldots + x_4^2x_5^3x_7 + \ldots.$$

Remark 2.1.2. From here on, we use a common convention and write the symmetric function $f(x)$ as f without the (x). Each of these functions is over the variables $(x_1, x_2, \ldots)$. We do the same for other types of functions in this thesis when clarification is not needed.

The algebra of symmetric functions is denoted *Sym*, and we take $\mathbb{Q}$ as our base field. For a partition $\lambda \vdash n$, the *monomial symmetric function* is defined as

$$m_\lambda = \sum_\alpha x^\alpha,$$

where the sum runs over all unique compositions α that are permutations of the entries of λ. For $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, the *elementary symmetric function* e_λ is defined by

$$e_n = m_{1^n} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \quad \text{and} \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}.$$

The *complete homogeneous symmetric function* is defined by

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \quad \text{with} \quad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}.$$

The *power sum symmetric functions* are defined

$$p_n = m_n = \sum_{i \geq 1} x_i^n \quad \text{with} \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

The sets $\{m_\lambda\}_\lambda$, $\{h_\lambda\}_\lambda$, and $\{p_\lambda\}_\lambda$ are bases of *Sym*. There is a classical endomorphism on the symmetric functions $\omega : Sym \to Sym$ defined by $\omega(e_\lambda) = h_\lambda$. The endomorphism ω is an involution, and it acts on the power sum basis by $\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$.

Sym is *self-dual* (see Section 2.2) with a bilinear form denoted by $\langle \cdot, \cdot \rangle : Sym \times Sym \to \mathbb{Q}$ where $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$. Two bases of *Sym* $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$ are *dual bases* if and only if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$, meaning we have defined the inner product such that $\{m_\lambda\}_\lambda$ and $\{h_\lambda\}_\lambda$ are dual bases. The involution ω is invariant under duality, which is to say $\langle \omega(f), \omega(g) \rangle = \langle f, g \rangle$ for all $f, g \in Sym$.

Remark 2.1.3. We take $\mathbb{Q}$ as our base field throughout this thesis but many of these results hold for any field $\Bbbk$ of characteristic zero.

2.1.1 The Schur Functions

For a partition $\lambda \vdash n$, a *semistandard Young tableau* (SSYT) of shape λ is a filling of the Young diagram of λ with positive integers such that the numbers are weakly increasing from left to right in the rows and strictly increasing from top to bottom in the columns. The *size* of an SSYT is its number of boxes, $|\lambda|$, and its *type* is the weak composition encoding the number of boxes filled with each integer. We write $type(T) = (\beta_1, \ldots, \beta_j)$ if T has β_i boxes containing an i for all $i \in [j]$. Note that "type" is also referred to as "content" in the literature. A *standard Young tableau* (SYT) of size n is a Young tableau in which each number in $[n]$ appears exactly once. A semistandard Young tableaux T of type $\beta = (\beta_1, \ldots, \beta_k)$ is associated with the monomial $x^T = x_1^{\beta_1} \cdots x_k^{\beta_k}$.

Example 2.1.4. The semistandard Young tableaux of shape $(2, 2)$ with entries in $\{1, 2, 3\}$ and their associated monomials are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$$

$$x_1^2x_2^2 \qquad x_1^2x_2x_3 \qquad x_1^2x_3^2 \qquad x_1x_2^2x_3 \qquad x_1x_2x_3^2 \qquad x_2^2x_3^2$$

The standard Young tableaux of shape $(2, 2)$, both of type $(1, 1, 1, 1)$, are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Definition 2.1.5. For a partition λ, the *Schur symmetric function* is defined as

$$s_\lambda = \sum_T x^T,$$

where the sum runs over all semistandard Young tableaux T of shape λ with entries in $\mathbb{Z}_{>0}$. These functions form a basis of *Sym*.

Example 2.1.6. The Schur function $s_{(2,2)}$ is given by

$$s_{(2,2)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x^2x_3 + x_1x_2x_3^3 + x_2^2x_3^2 + \ldots, x_5^2x_6^2 + x_5^2x_6x_7 + \ldots.$$

The Schur functions are an orthonormal basis for *Sym*, meaning $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$. Additionally, the Schur basis maps to itself under ω by $\omega(s_\lambda) = s_{\lambda'}$.

Define the *Kostka number* $K_{\lambda,\alpha}$ as the number of $SSYT$s of shape λ and type α. Then,

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu \qquad \text{and} \qquad h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda.$$

The Kostka numbers have a variety of interesting combinatorics and applications, see [3] for an overview of topics and sources.

Remark 2.1.7. The *Schur polynomials* are defined over finitely many variables $s_\lambda(x_1, \ldots, x_n)$ and correspond to tableaux filled only with integers in $[n]$. This paper deals only with functions in infinitely many variables, but many results restrict to polynomials.

For partitions λ and μ with $\mu \subseteq \lambda$, a *skew semistandard Young tableau* is a skew shape λ/μ filled with positive integers such that the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom. The boxes corresponding to μ are not filled with integers. The notion of type, standard, and associated monomial follow those of the SSYT.

Definition 2.1.8. For partitions λ and μ where $\mu \subseteq \lambda$, the *skew Schur function* is defined as

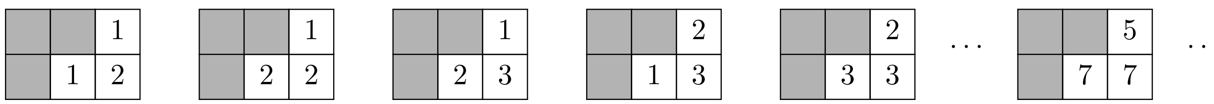
$$s_{\lambda/\mu} = \sum_{T} x^T,$$

where the sum runs over all skew semistandard Young tableaux T of shape λ/μ.

Note that if $\mu = \emptyset$, then $s_{\lambda/\mu} = s_\lambda$.

Example 2.1.9. The skew Schur function $s_{(3,3)/(2,1)}$ is given by

$$s_{(3,3)/(2,1)} = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_2x_3^2 \cdots + x_5x_7^2 + \cdots$$



The Schur and skew Schur functions have multiple equivalent definitions. We will first review the definition of s_α using creation operators in *Sym* and then the definition of $s_{\lambda/\mu}$ using the Jacobi-Trudi rule. For each element $f \in Sym$, there is a *perp operator* $f^\perp : Sym \to Sym$ defined by the relation $\langle fg, h\rangle = \langle g, f^\perp h\rangle$ for all $g, h \in Sym$. This expands in terms of any two dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ of *Sym* as

$$f^\perp(g) = \sum_\lambda \langle g, fa_\lambda\rangle b_\lambda. \tag{2.1}$$

This expression expands the output of a perp operator on a function in terms of any basis with coefficients given by values of the inner product on *Sym*. Note $f^\perp$ is dual to multiplication by f.

Definition 2.1.10. For an integer m, the *Bernstein creation operator* $B_m : Sym_n \to Sym_{m+n}$ is defined by

$$B_m = \sum_{i\geq 0}(-1)^i h_{m+i}e_i^\perp = \sum_{i\geq 0}(-1)^i h_{m+i}m_{1^i}^\perp,$$

where $h_0 = 1$ and $h_{-i} = 0$ for $i > 0$.

As shown below, applying the Bernstein creation operators in sequence produces a Schur symmetric function. This definition also yields Schur functions defined on integer tuples that are not partitions, although these functions are not linearly independent from elements in the Schur basis.

Theorem 2.1.11. *[87] For all tuples $\alpha = [\alpha_1, \ldots, \alpha_m] \in \mathbb{Z}^m$,*

$$s_\alpha = B_{\alpha_1}B_{\alpha_2}\cdots B_{\alpha_m}(1).$$

Example 2.1.12. The Schur function for the composition $(2,1,3)$ is given by

$$s_{(2,1,3)} = B_2B_1B_3(1) = -h_{(2,2,2)} + 2h_{(3,2,1)} - h_{(3,3)} - h_{(4,1,1)} + h_{(4,2)} = -s_{(2,2,2)}$$

The skew Schur functions have an equivalent definition using the *Jacobi-Trudi Rule*. The Jacobi-Trudi rule is a way to express a skew Schur function $s_{\lambda/\mu}$ as the determinant of a matrix with complete homogeneous symmetric functions as entries.

Theorem 2.1.13. *[78] Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell) \subseteq \lambda$. Then,*

$$s_{\lambda/\mu} = det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i,j \leq k},$$

where $h_0 = 1$ and $h_{-i} = 0$ for any $i > 0$.

The skew Schur functions have yet another equivalent definition using a perp operator, by

$$s_\mu^\perp(s_\lambda) = s_{\lambda/\mu}.$$

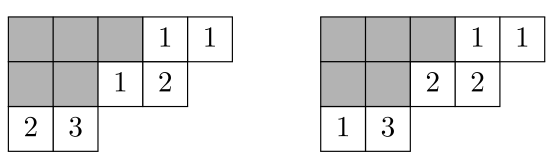
The $s_\mu^\perp$ operator is adjoint to multiplication by s_μ, meaning the skew Schur functions are deeply linked to the multiplicative structure of the Schur basis. It follows from Equation (2.1) that $s_{\lambda/\mu} = \sum_\nu \langle s_\lambda, s_\mu s_\nu \rangle s_\nu$. The coefficient $\langle s_\lambda, s_\mu s_\nu \rangle$ is denoted with $c_{\mu,\nu}^\lambda$ and is called a *Littlewood-Richardson coefficient*. These are the same coefficients that appear in the expansion

$$s_\mu s_\nu = \sum_\lambda \langle s_\lambda, s_\mu s_\nu \rangle s_\lambda.$$

The Littlewood-Richardson coefficients appear in many places in mathematics, especially in representation theory, and have a variety of combinatorial interpretations, many in terms of tableaux. For example, a word is a *reverse lattice word*, or a *Yamanouchi* word, if when read backward from the end to any letter there are at least as many occurrences of i as there are of $i+1$. Now, let $w_{row(T)}$ be the entries of a tableau read from left to right, bottom to top. A semistandard Young tableau T is a *Littlewood-Richardson tableau* if $w_{row}(T)$ is a reverse lattice word.

Proposition 2.1.14. *[31] The Littlewood-Richardson coefficient $c_{\lambda,\mu}^\nu$ counts the number of skew Littlewood-Richardson tableaux of shape ν/λ and type μ.*

Example 2.1.15. The Littlewood-Richardson tableaux of shape $(5,4,2)/(3,2)$ and type $(3,2,1)$ are



and so the term $s_{(5,4,2)}$ has the coefficient 2 in the product $s_{(3,2)}s_{(3,2,1)}$.

When a Schur function is indexed by a row, its multiplication with another Schur function is given by a special case of the Littlewood Richardson rule called the *Pieri rule*. Since $s_k = h_k$ for a positive integer k, we express the rule as the multiplication of h_k by a Schur function.

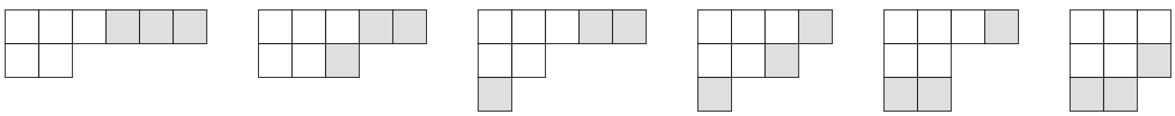
Proposition 2.1.16. *[78] For a positive integer k and a partition λ,*

$$h_k s_\lambda = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \vdash |\lambda| + k$ obtained by adding k boxes to the diagram of λ such that a maximum of one new box is added per column.

Example 2.1.17. The Pieri rule applied to the following expression can be visualized with the diagrams below.

$$h_3 s_{(3,2)} = s_{(6,2)} + s_{(5,3)} + s_{(5,2,1)} + s_{(4,3,1)} + s_{(4,2,2)} + s_{(3,3,2)}.$$



Next, we examine a special class of skew Schur functions indexed by *ribbon shapes*. A skew shape λ/μ is a *ribbon* if it does not contain any 2×2 arrangement of boxes. Ribbons are also called *rim-hooks* and *border strips*, and we say their *height* is their number of rows minus one. With each ribbon skew shape λ/μ, we can associate a composition α by left aligning the rows of λ/μ (removing the shaded-out boxes completely). Then, the *ribbon schur function* is defined as $r_\alpha = s_{\lambda/\mu}$. The ribbon Schur functions expand in terms of the complete homogeneous symmetric functions using the Jacobi-Trudi rule.

Proposition 2.1.18. *[16] For any $\alpha \models n$,*

$$r_\alpha = (-1)^{\ell(\alpha)} \sum_{\beta \geq \alpha} (-1)^{\ell(\beta)} h_{sort(\beta)}.$$

The rule for multiplication of a Schur function by a power sum function, the *Murnaghan-Nakayama rule*, is also expressed in terms of ribbons.

Theorem 2.1.19. *[78] For any partition μ and $r \in \mathbb{N}$, we have*

$$p_r s_\mu = \sum_\lambda (-1)^{height(\lambda/\mu)} s_\lambda,$$

where the sum runs over all partitions $\lambda \supseteq \mu$ for which λ/μ is a border strip of size r.

2.2 Hopf algebras

Hopf algebras are widespread in combinatorics and other fields with notable examples including *Sym*, *QSym*, and *NSym*. We provide a brief overview of the structures needed for our purposes. See [28, 37] for more details.

Definition 2.2.1. Let $\mathbb{k}$ be a field of characteristic zero. An *associative algebra* is a $\mathbb{k}$-module $\mathcal{H}$ with $\mathbb{k}$-linear multiplication $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and a $\mathbb{k}$-linear unit $\eta : \mathbb{k} \to \mathcal{H}$, for which the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} & \xrightarrow{\text{id}\otimes\mu} & \mathcal{H}\otimes\mathcal{H} \\ {\scriptstyle \mu\otimes\text{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\mu} & \mathcal{H} \end{array} \qquad \begin{array}{ccccc} \mathcal{H}\otimes\mathbb{k} & = & \mathcal{H} & = & \mathbb{k}\otimes\mathcal{H} \\ {\scriptstyle \text{id}\otimes\eta}\downarrow & & {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \eta\otimes\text{id}} \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\mu} & \mathcal{H} & \xleftarrow{\mu} & \mathcal{H}\otimes\mathcal{H} \end{array}$$

A *co-associative coalgebra* is a $\mathbb{k}$-module $\mathcal{H}$ with $\mathbb{k}$-linear comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\mathbb{k}$-linear counit $\epsilon : \mathcal{H} \to \mathbb{k}$, satisfying the commutative diagrams:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H}\otimes\mathcal{H} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta\otimes\text{id}} \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\text{id}\otimes\Delta} & \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \end{array} \qquad \begin{array}{ccccc} \mathcal{H}\otimes\mathcal{H} & \xleftarrow{\Delta} & \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H}\otimes\mathcal{H} \\ {\scriptstyle \text{id}\otimes\epsilon}\downarrow & & {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \epsilon\otimes\text{id}} \\ \mathcal{H}\otimes\mathbb{k} & = & \mathcal{H} & = & \mathbb{k}\otimes\mathcal{H} \end{array}$$

Note that in a standard abuse of notation, we often leave out the formal notation for multiplication and just write b_1b_2 to indicate the multiplication of b_1 and b_2, or $\mu(b_1 \otimes b_2)$.

An *algebra (homo)morphism* is a $\mathbb{k}$-linear map $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ between two algebras $\mathcal{A}$ and $\mathcal{B}$ such that for $a_1, a_2 \in \mathcal{A}$ and $k \in \mathbb{k}$,

$$(\varphi \circ \mu_{\mathcal{A}})(a_1 \otimes a_2) = (\mu_{\mathcal{B}} \circ (\varphi \otimes \varphi))(a_1 \otimes a_2) \quad \text{and} \quad (\varphi \circ \eta_{\mathcal{A}})(k) = \eta_{\mathcal{B}}(k).$$

A *coalgebra (homo)morphism* is a linear map $\mathcal{C} \xrightarrow{\varphi} \mathcal{D}$ between two coalgebras $\mathcal{C}$ and $\mathcal{D}$ such that for $c \in \mathcal{C}$,

$$(\Delta_{\mathcal{D}} \circ \varphi)(c) = ((\varphi \otimes \varphi) \circ \Delta_{\mathcal{C}})(c) \quad \text{and} \quad (\epsilon_{\mathcal{D}} \circ \varphi)(c) = \epsilon_{\mathcal{C}}(c). \tag{2.2}$$

The subspace $\mathcal{G}$ is a *subalgebra* of $\mathcal{H}$ if μ restricts to $\mathcal{G}$, and a *subcoalgebra* of $\mathcal{H}$ if Δ restricts to $\mathcal{G}$.

Definition 2.2.2. $(\mathcal{H}, \mu, \Delta, \eta, \epsilon)$ is a *bialgebra* if $(\mathcal{B}, \mu, \eta)$ is an algebra and $(\mathcal{H}, \Delta, \epsilon)$ is a coalgebra and the following hold for $T(x \otimes y) = y \otimes x$ where $k \in \mathbb{k}$ and $h_1, h_2 \in \mathcal{H}$:

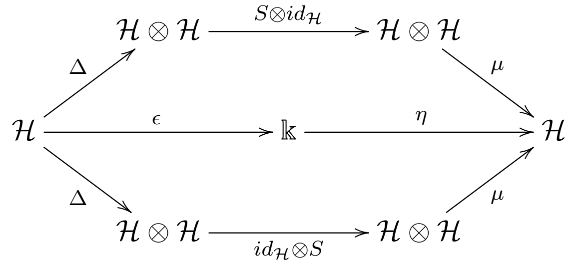
1. $(\Delta \circ \mu)(h_1 \otimes h_2) = ((\mu \otimes \mu) \circ (id \otimes T \otimes id) \circ (\Delta \otimes \Delta))(h_1 \otimes h_2)$
2. $(\mu \circ (\epsilon \otimes \epsilon))(h_1 \otimes h_2) = (\epsilon \circ \mu)(h_1 \otimes h_2)$
3. $(\Delta \circ \eta)(k) = ((\eta \otimes \eta) \circ \Delta)(k)$
4. $id(k) = (\epsilon \circ \eta)(k)$

A bialgebra $\mathcal{H}$ is *graded* if it is graded as a $\mathbb{k}$-module $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}^{(n)}$ with

$$\mathcal{H}^{(n)} \otimes \mathcal{H}^{(m)} \xrightarrow{\mu} \mathcal{H}^{(n+m)}, \qquad \mathcal{H}^{(n)} \xrightarrow{\Delta} \bigoplus_{m+p=n} \mathcal{H}^{(m)} + \mathcal{H}^{(p)}.$$

A graded bialgebra is *connected* if $\mathcal{H}^{(0)} \cong \mathbb{k}$.

Definition 2.2.3. A bialgebra $(\mathcal{H}, \mu, \Delta, \eta, \epsilon)$ is a *Hopf algebra* if there exists a $\Bbbk$-linear anti-endomorphism on $\mathcal{H}$, called the *antipode*, such that the following diagram commutes:



Proposition 2.2.4. *[65] Any graded and connected bialgebra is a Hopf algebra.*

If $\mathcal{H}$ is a graded Hopf algebra, and $x \in \mathcal{H}^{(n)}$ then,

$$S(x) = -\sum_i S(y_i)z_i = -\sum_i y_i S(z_i) \tag{2.3}$$

where $\Delta(x) = \sum_i y_i \otimes z_i$ (Sweedler notation).

A subalgebra $\mathcal{G}$ of $\mathcal{H}$ is a *Hopf subalgebra* if $\mathcal{G}$ is a subcoalgebra of $\mathcal{H}$ and the antipode S of $\mathcal{H}$ restricts to $\mathcal{G}$. A *bialgebra (homo)morphism* is a linear map between two bialgebras that is both an algebra homomorphism and a coalgebra homomorphism.

Corollary 2.2.5. *[37] Let H_1 and H_2 be Hopf algebras with antipodes S_1 and S_2, respectively. Then, any bialgebra morphism $H_1 \xrightarrow{\beta} H_s$ is a Hopf morphism, that is, it commutes with the antipodes $(\beta \circ S_1 = S_2 \circ \beta)$.*

Duality between Hopf algebras is one of the most important tools that we use in this thesis.

Definition 2.2.6. Let $(\mathcal{A}, \mu_\mathcal{A}, \Delta_\mathcal{A}, \eta_\mathcal{A}, \epsilon_\mathcal{A})$ and $(\mathcal{B}, \mu_\mathcal{B}, \Delta_\mathcal{B}, \eta_\mathcal{B}, \epsilon_\mathcal{B})$ be two Hopf Algebras with antipodes $\mathcal{S}_\mathcal{A}$ and $\mathcal{S}_\mathcal{B}$ respectively, where $1_\mathcal{A}$ and $1_\mathcal{B}$ are the multiplicative identity elements. We say that $\mathcal{A}$ and $\mathcal{B}$ are *dually paired* by an inner product $\langle\ ,\ \rangle : \mathcal{B} \otimes \mathcal{A} \to \mathbb{Q}$, if, for all elements $a, a_1, a_2 \in \mathcal{A}$ and $b, b_1, b_2 \in \mathcal{B}$, we have

$$\langle \mu_\mathcal{B}(b_1, b_2), a\rangle = \langle b_1 \otimes_\mathcal{B} b_2, \Delta_\mathcal{A}(a)\rangle, \qquad \langle 1_\mathcal{B}, a\rangle = \epsilon_\mathcal{A}(a),$$

$$\langle b, \mu_\mathcal{A}(a_1, a_2)\rangle = \langle \Delta_\mathcal{B}(b), a_1 \otimes_\mathcal{A} a_2\rangle, \qquad \epsilon_\mathcal{B}(b) = \langle b, 1_\mathcal{A}\rangle, \qquad \langle S_\mathcal{B}(b), a\rangle = \langle b, S_\mathcal{A}(a)\rangle.$$

Two bases $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ of $\mathcal{A}$ and $\mathcal{B}$ respectively are *dual bases* if and only if $\langle a_i, b_i\rangle = \delta_{i,j}$. We say that a map $f : \mathcal{A} \to \mathcal{A}$ is *adjoint* to a map $g : \mathcal{B} \to \mathcal{B}$ if

$$\langle f(b_1), a_1\rangle = \langle b_1, g(a_1)\rangle \tag{2.4}$$

for any $a_1 \in \mathcal{A}, b_1 \in \mathcal{B}$. The following result gives a relation for the change of bases using duality.

Proposition 2.2.7. *[41] Let $\mathcal{A}$ and $\mathcal{B}$ be dually paired algebras and let $\{a_i\}_{i\in I}$ be a basis of $\mathcal{A}$. A basis $\{b_i\}_{i\in I}$ of $\mathcal{B}$ is the unique basis that is dual to $\{a_i\}_{i\in I}$ if and only if the following relationship holds for any pair of dual bases $\{c_i\}_{i\in I}$ in $\mathcal{A}$ and $\{d_i\}_{i\in I}$ in $\mathcal{B}$:*

$$a_i = \sum_{j\in I} k_{i,j} c_j \quad \iff \quad d_j = \sum_{i\in I} k_{i,j} b_i.$$

The coefficients of the multiplication and comultiplication of dual bases have a similar relationship.

Proposition 2.2.8. *[37] The comultiplication of the basis $\{b_i\}_{i\in I}$ in $\mathcal{B}$ is uniquely defined by the multiplication of its dual basis $\{a_i\}_{i\in I}$ in $\mathcal{A}$ in the following way:*

$$a_j a_k = \sum_{i\in I} c^i_{j,k} a_i \quad \iff \quad \Delta(b_i) = \sum_{(j,k)\in I\times I} c^i_{j,k} b_j \otimes b_k.$$

Further, $\Delta : \mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$ is an algebra homomorphism.

Example 2.2.9. For a partition ν, the comultiplication on Schur functions is given by $\Delta(s_\nu) = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda \otimes s_\mu$ where $c^\nu_{\lambda,\mu}$ are the Littlewood Richardson coefficients.

We conclude this subsection by reviewing the Hopf algebra structure of *Sym* from [37] and [86]. Multiplication on the monomial symmetric functions is expressed, for $\lambda \vdash n$ and $\mu \vdash k$, as

$$m_\lambda m_\mu = \sum_{\nu\vdash n+k} r^\nu_{\lambda,\mu} m_\nu \tag{2.5}$$

where $r^\nu_{\lambda,\mu}$ is the number of pairs of sequences (α, β) with $\alpha_i, \beta_i \geq 0$ where $sort(\alpha) = \lambda$ and $sort(\beta) = \mu$ where $\nu = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$. Comultiplication is given by

$$\Delta(m_\lambda) = \sum_{\mu\sqcup\nu=\lambda} m_\mu \otimes m_\nu, \tag{2.6}$$

where $\mu \sqcup \nu$ the multiset union of the parts of μ and ν sorted into a partition. Multiplication on the complete homogeneous symmetric functions is expressed as

$$h_\lambda h_\mu = h_{sort(\lambda\cdot\mu)}, \tag{2.7}$$

for partitions λ and μ. Comultiplication is expressed as

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}. \tag{2.8}$$

2.3 Quasisymmetric functions

For a weak composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. When α has k non-zero entries given by $\alpha_{i_1} = a_1$, $\alpha_{i_2} = a_2$, $\ldots$, $\alpha_{i_k} = a_k$ with $i_1 < \cdots < i_k$, then $x^\alpha = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$. A *quasisymmetric function* $f(x)$ is a formal power series of the form

$$f(x) = \sum_{\alpha} b_\alpha x^\alpha,$$

where the sum runs over weak compositions α and the coefficients of the monomials $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ are equal if $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. We will assume that $b_\alpha \in \mathbb{Q}$.

The quasisymmetric functions, denoted $QSym$, form a Hopf algebra. Given a composition α, the *monomial quasisymmetric function* M_α is defined as

$$M_\alpha = \sum_{i_1 < \dots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k},$$

where the sum runs over strictly increasing sequences of k positive integers $i_1, \dots, i_k \in \mathbb{Z}_{>0}$. The *fundamental quasisymmetric function* F_α is defined as

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta, \quad \text{with} \quad M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} F_\beta.$$

The fundamental functions are also denoted L_α in the literature [78].

Example 2.3.1. The monomial quasisymmetric function indexed by $(2,1)$ is

$$M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + \dots + x_2^2 x_3 + x_2^2 x_4 + \dots + x_3^2 x_4 + x_3^2 x_5 + \dots .$$

The expansion of $F_{(3)}$ into the monomial basis is

$$F_{(3)} = M_{(3)} + M_{(2,1)} + M_{(1,2)} + M_{(1,1,1)}.$$

The monomial basis inherits its multiplication and comultiplication from the quasishuffle and concatenation operations on compositions. The *quasishuffle* $\overset{Q}{\sqcup\!\sqcup}$ of compositions is defined as the sum of shuffles of $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_l)$ where any consecutive pairs α_i and β_j (in that order) may be replaced with $\alpha_i + \beta_j$. Note that the same composition may appear multiple times in the quasishuffle. Multiplication of monomial functions is given by

$$M_\alpha M_\beta = \sum_{\gamma} M_\gamma,$$

where the sum runs over all γ where γ is a summand in $\alpha \overset{Q}{\sqcup\!\sqcup} \beta$ with multiplicity. Comultiplication is given by

$$\Delta(M_\alpha) = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\gamma,$$

where the sum runs over all compositions β, γ such that $\beta \cdot \gamma = \alpha$.

Example 2.3.2. The following expressions show the multiplication and comultiplication on monomial quasisymmetric functions expanded in terms of the monomial basis:

$$M_{(2,1)}M_{(1)} = 2M_{(2,1,1)} + M_{(1,2,1)} + M_{(2,2)} + M_{(3,1)},$$

$$\Delta(M_{(1,2,1)}) = 1 \otimes M_{(1,2,1)} + M_{(1)} \otimes M_{(2,1)} + M_{(1,2)} \otimes M_{(1)} + M_{(1,2,1)} \otimes 1.$$

For more on the quasisymmetric functions, see [60].

2.4 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions*, written *NSym*, is the Hopf algebra dual to *QSym*. *NSym* can be defined as the algebra with generators $\{H_1, H_2, \ldots\}$ and no relations, meaning the generators do not commute. We write *NSym* as

$$NSym = \mathbb{Q}\langle H_1, H_2, \ldots \rangle .$$

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we define $H_\alpha = H_{\alpha_1}H_{\alpha_2}\ldots H_{\alpha_k}$. Then, the set $\{H_\alpha\}_\alpha$ forms a basis of *NSym* called the *complete homogeneous basis*. *NSym* and *QSym* are dually paired by the inner product defined by $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$ for all compositions α, β. This inner product has all the properties listed in Definition 2.2.6 and is the main tool for translating between *QSym* and *NSym*.

Definition 2.4.1. The *forgetful map* $\chi : NSym \to Sym$ is defined as

$$\chi(H_\alpha) = h_{sort(\alpha)},$$

extended linearly to map all elements in *NSym* to their commutative image in *Sym*. The forgetful map is a morphism that is adjoint to the inclusion of *Sym* to *QSym*.

For a composition α, the *ribbon noncommutative symmetric function* is defined as

$$R_\alpha = \sum_{\alpha \preceq \beta} (-1)^{\ell(a)-\ell(\beta)} H_\beta, \quad \text{and so} \quad H_\beta = \sum_{\alpha \preceq \beta} R_\alpha.$$

The ribbon functions form a basis of *NSym* dual to the fundamental basis of *QSym*, meaning $\langle R_\alpha, F_\beta \rangle = \delta_{\alpha,\beta}$. Multiplication in *NSym* on the ribbon and complete homogeneous functions is expressed as

$$H_\alpha H_\beta = H_{\alpha\cdot\beta} \qquad \text{and} \qquad R_\alpha R_\beta = R_{\alpha\cdot\beta} + R_{\alpha\odot\beta}. \tag{2.9}$$

We consider two other bases of *NSym*. For a composition α, the *elementary noncommutative symmetric function* is defined as

$$E_\alpha = \sum_{\beta \preceq \alpha} (-1)^{|\alpha|-\ell(\beta)} H_\beta, \quad \text{and so} \quad H_\beta = \sum_{\alpha \preceq \beta} (-1)^{|\beta|-\ell(\alpha)} E_\alpha.$$

These are also denoted Λ_α in the literature [34]. For $n \in \mathbb{N}$ and a composition $(\alpha = \alpha_1, \ldots, \alpha_k)$, the *noncommutative power sum of the first kind* is defined as

$$\Psi_n = \sum_{i=0}^{n-1} (-1)^i R_{(1^i, n-i)} \quad \text{with} \quad \Psi_\alpha = (\Psi_{\alpha_1}, \ldots, \Psi_{\alpha_k}).$$

This basis is an analogue to the power sum basis of Sym, and in fact $\chi(\Psi_n) = p_n$. For more details on the noncommutative symmetric functions see [34].

Next, consider three pairs of involutions in $QSym$ and $NSym$ defined on the fundamental basis and the ribbon basis, respectively [56]. All six maps are defined as extensions of the involutions given by the complement, reverse, and transpose operations on compositions.

Definition 2.4.2. The involutions ψ, ρ, and ω on $QSym$ and $NSym$ are defined as

$$\psi(F_\alpha) = F_{\alpha^c} \qquad \rho(F_\alpha) = F_{\alpha^r} \qquad \omega(F_\alpha) = F_{\alpha^t},$$

$$\psi(R_\alpha) = R_{\alpha^c} \qquad \rho(R_\alpha) = R_{\alpha^r} \qquad \omega(R_\alpha) = R_{\alpha^t},$$

and extend linearly.

All three maps on $QSym$ and ψ on $NSym$ are automorphisms, while ρ and ω on $NSym$ are anti-automorphisms. Note that we use the same notation for the corresponding involutions on $QSym$ and $NSym$. These automorphisms commute and $\omega = \rho \circ \psi = \psi \circ \rho$. When ω and ψ are restricted to Sym, they are both equivalent to the classical involution $\omega : Sym \to Sym$ while ρ restricts to the identity map. The involutions in $NSym$ act on H_α and E_α in the following way:

$$\psi(H_\alpha) = E_\alpha \qquad \rho(H_\alpha) = H_{\alpha^r} \qquad \rho(E_\alpha) = E_{\alpha^r} \qquad \omega(H_\alpha) = E_{\alpha^r}. \tag{2.10}$$

Jia, Wang, and Yu prove in [44] that ψ, ρ, and ω are in fact the only nontrivial graded algebra automorphisms on $QSym$ that preserve the fundamental basis. Further, ψ is the only nontrivial graded Hopf algebra automorphism on $QSym$ that preserves the fundamental basis.

We conclude our review of $QSym$ and $NSym$ with the concept of Schur-like bases. A basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$ is generally considered to be *Schur-like* if the $\chi(\mathcal{S}_\lambda) = s_\lambda$ for any partition λ. A basis of $QSym$ may be considered Schur-like if it is dual to a Schur-like basis of $NSym$. These bases are usually defined combinatorially in terms of tableaux that resemble or generalize the semistandard Young tableaux. This is not a formal definition and one could certainly claim that other bases qualify as Schur-like. For example, one might include bases of $NSym$ $\{\mathcal{S}_\alpha\}_\alpha$ for which there exists some α for each partition λ such that $\chi(\mathcal{S}_\alpha) = s_\lambda$.

Upon the introduction of $QSym$ and $NSym$, it was the fundamental and ribbon bases respectively that were initially considered to be the appropriate analogues to the Schur functions of $QSym$ and $NSym$. The image of the ribbon noncommutative symmetric functions under the forgetful map χ is the ribbon Schur symmetric functions, given by $\chi(R_\alpha) = r_\alpha$. It will be reasonable to ask if any

Schur-like bases map to themselves under ψ, ρ, and ω, so we note here that the fundamental basis and the ribbon noncommutative basis both have this property and are, in a sense, Schur-like.

2.5 Doliwa's colored $QSym_A$ and $\boldsymbol{NSym}_A$

The algebra of noncommutative symmetric functions and dually the algebra of quasisymmetric functions have natural generalizations isomorphic to algebras of sentences. In [28], Doliwa introduces these generalizations which are built using partially commutative colored variables. Our generalizations of Schur-like bases will belong to these spaces.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be an alphabet of letters, which we call *colors*. *Words* over A are finite sequences of colors written without separating commas. Finite sequences of non-empty words are called *sentences*. The empty word and the empty sentence are both denoted by $\emptyset$. A *weak sentence* may include empty words. The *size* of a word w, denoted $|w|$, is the total number of colors it contains. Note that when we refer to "the number of colors", we are counting repeated colors unless we say "the number of unique colors". The *size* of a sentence $I = (w_1, w_2, \ldots, w_k)$, denoted $|I|$, is also the number of colors it contains. The *length* of a sentence I, denoted $\ell(I)$, is the number of words it contains. The *concatenation* of two words $w = a_1 \cdots a_k$ and $v = b_1 \cdots b_j$ is $w \cdot v = a_1 \cdots a_k b_1 \cdots b_j$, sometimes just denoted wv. The word obtained by concatenating every word in a sentence I is called the *maximal word* of I, denoted $w(I) = w_1 w_2 \ldots w_k$. For our purposes, we also define the *word lengths* of I as $w\ell(I) = (|w_1|, \ldots, |w_k|)$, which gives the underlying composition of the sentence.

Example 2.5.1. Let $a, b, c \in A$ and let $w_1 = ac$, $w_2 = b$, and $w_3 = cab$ be words. Consider the sentence $I = (w_1, w_2, w_3) = (ac, b, cab)$. Then, $|w_1| = 2$, $|w_2| = 1$, $|w_3| = 3$, and $|I| = 6$. The length of I is $\ell(I) = 3$ and the word length of I is $w\ell(I) = (2, 1, 3)$. The maximal word of I is $w(I) = acbcab$.

A sentence I is a refinement of a sentence J, written $I \preceq J$, if J can be obtained by concatenating some adjacent words of I. In other words, $I \preceq J$ if $w(I) = w(J)$ and $w\ell(I) \preceq w\ell(J)$. In this case, I is called a *refinement* of J and J a *coarsening* of I. The *Möbius function* on the poset of sentences ordered by refinement is

$$\mu(J, I) = (-1)^{\ell(J)-\ell(I)} \quad \text{for} \quad J \preceq I. \tag{2.11}$$

Given a total order $\leq$ on A, define the following *lexicographic order* $\preceq_\ell$ on words. For words $w = a_1 \ldots a_k$ and $v = b_1 \ldots b_j$, we say $w \leq_\ell v$ if $a_i < b_i$ for the first positive integer i such that $a_i \neq b_i$. Note that if no such i exists then $w = v$.

Example 2.5.2. Let $A = \{a < b < c\}$ and $I = (abc)$. The refinements of I are (abc), (a, bc), (ab, c), and (a, b, c). Under lexicographic order, $abc \preceq_\ell acb \preceq_\ell bac \preceq_\ell bca \preceq_\ell cab \preceq_\ell cba$.

Let $I = (w_1, \ldots, w_k)$ and $J = (v_1, \ldots, v_h)$ be two sentences. The *concatenation* of I and J is $I \cdot J = (w_1, \ldots, w_k, v_1, \ldots, v_h)$. Their *near-concatenation* is $I \odot J = (w_1, \ldots, w_k \cdot v_1, \ldots, v_h)$ where the words w_k and v_1 are concatenated into a single word. Given $I = (w_1, \ldots, w_k)$ where a_i is the i^{th}

entry in I and a_{i+1} is the $(i+1)^{\text{th}}$ entry in I, we say that I *splits* after the i^{th} entry if $a_i \in w_j$ and $a_{i+1} \in w_{j+1}$ for $j \in [k]$.

Example 2.5.3. Let $I = (a, bc)$ and $J = (ca, b)$. Then, $I \cdot J = (a, bc, ca, b)$ and $I \odot J = (a, bcca, b)$. The sentence $(a, bcca, b)$ splits after the 1^{st} and 5^{th} entries.

Given $I = (w_1, \ldots, w_k)$, the *reversal* of I is $I^r = (w_k, w_{k-1}, \ldots, w_1)$. The *complement* of I, denoted I^c, is the unique sentence such that $w(I) = w(I^c)$ and I^c splits exactly where I does not. Both maps are involutions on sentences.

Example 2.5.4. Let $I = (abc, de)$. Then $I^r = (de, abc)$ and $I^c = (a, b, cd, e)$.

The *flattening* of a weak sentence I, denoted $\tilde{I}$, is the sentence obtained by removing all empty words from I. Further, for a weak sentence $J = (v_1, \ldots, v_k)$ and a sentence $I = (w_1, \ldots, w_k)$, we say that J is *right-contained* in I, denoted $J \subseteq_R I$, if there exists a weak sentence $I/_R J = (u_1, \ldots, u_k)$ such that $w_i = u_i v_i$ for every $i \in [k]$. We say that J is *left-contained* in I, denoted $J \subseteq_L I$, if there exists a weak sentence $I/_L J = (q_1, \ldots, q_k)$ such that $w_i = v_i q_i$ for every $i \in [k]$. Note that right-containment is denoted I/J in [28] but here that notation is used exclusively to denote skew shapes.

Example 2.5.5. Let $I = (ab, cdef)$, $J = (b, ef)$, and $K = (a, cde)$. Then $J \subseteq_R I$ and $I/_R J = (a, cd)$, while $K \subseteq_L I$ and $I/_L K = (b, f)$. Given the weak sentence $I = (\emptyset, a, \emptyset, bc)$, the flattening of I is $\tilde{I} = (a, bc)$.

2.5.1 The Hopf algebra of sentences and colored noncommutative symmetric functions

The algebra of sentences (colored compositions) is a Hopf algebra with the multiplication being the concatenation of sentences, the comultiplication given by

$$\Delta(I) = \sum_{J \subseteq_R I} \widetilde{I/_R J} \otimes \tilde{J},$$

the natural unity map, and the counit and the antipode given by

$$\epsilon(I) = \begin{cases} 1, & \text{if } I = \emptyset, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad S(I) = \sum_{J \preceq I^r} (-1)^{\ell(J)} J.$$

The algebra of sentences taken over an alphabet with only one letter is isomorphic to *NSym*. Thus, the algebra of sentences taken over any alphabet A is a natural extension of *NSym* called *the algebra of colored noncommutative symmetric functions*, denoted $NSym_A$. The linear basis of sentences I is the complete homogeneous basis of $NSym_A$, denoted $\{H_I\}_I$.

$NSym_A$ can also be defined as the algebra freely generated over noncommuting elements H_w for any word in A. The Hopf algebra operations extend to $\{H_I\}_I$ as follows:

$$H_I \cdot H_J = H_{I \cdot J}, \qquad \Delta(H_I) = \sum_{J \subseteq_R I} H_{\widetilde{I/_R J}} \otimes H_{\tilde{J}}, \qquad S(H_I) = \sum_{J \preceq I^r} (-1)^{\ell(J)} H_J.$$

The reversal and complement operations extend as $H_I^r = H_{I^r}$ and $H_I^c = H_{I^c}$.

Definition 2.5.6. The *uncoloring* map $\upsilon : NSym_A \to NSym$ is defined by $\upsilon(H_I) = H_{w\ell(I)}$ and extended linearly. If the alphabet A only contains one color, then υ is an isomorphism.

We say that two bases $\{B_I\}_I$ and $\{C_\alpha\}_\alpha$ in $NSym_A$ and $NSym$ respectively are *analogous* if $\upsilon(B_I) = C_{w\ell(I)}$ for all sentences I when A is an alphabet of one color. For instance, the colored complete homogeneous basis of $NSym_A$ is analogous to the complete homogeneous basis of $NSym$. $NSym_A$ also contains analogues of the elementary and ribbon bases of $NSym$. For a sentence I, the *colored elementary function* is defined by

$$E_I = \sum_{J \preceq I} (-1)^{|I| - \ell(J)} H_J, \qquad \text{and so} \qquad H_I = \sum_{J \preceq I} (-1)^{|I| - \ell(J)} E_J.$$

The *colored ribbon function* is defined by

$$R_I = \sum_{J \succeq I} (-1)^{\ell(J) - \ell(I)} H_J, \qquad \text{and so} \qquad H_I = \sum_{J \succeq I} R_J. \tag{2.12}$$

For more details, see [28].

2.5.2 The colored quasisymmetric functions and duality

The colored quasisymmetric functions, which constitute the algebra dual to $NSym_A$, are constructed using partially commutative colored variables. For a color $a \in A$, define the set of infinite colored variables $x_a = \{x_{a,1}, x_{a,2}, \ldots\}$ and let $x_A = \bigcup_{a \in A} x_a$. These variables are assumed to be partially commutative in the sense that variables only commute if the second indices are different. That is, for $a, b \in A$,

$$x_{a,i} x_{b,j} = x_{b,j} x_{a,i} \text{ for } i \neq j \qquad \text{and} \qquad x_{a,i} x_{b,i} \neq x_{b,i} x_{a,i} \text{ if } a \neq b.$$

As a result, every monomial in variables $x_{a,i}$ can be uniquely re-ordered so that the sequence of the second indices of the variables is weakly increasing, at which point any first indices sharing the same color can be combined into a single word. Every monomial has a sentence $(w_1, \ldots, w_m)$ defined by its re-ordered and combined form $x_{w_1,j_1} \cdots x_{w_m,j_m}$ where $j_1 < \ldots < j_m$. Similar notions of coloring with different assumptions of partial commutativity can be found in [12, 72].

Example 2.5.7. The monomial $x_{a,2} x_{b,3} x_{b,1} x_{c,2}$ can be reordered as $x_{b,1} x_{a,2} x_{c,2} x_{b,3}$ and combined as $x_{b,1} x_{ac,2} x_{b,3}$. Then, the sentence associated to this monomial is (b, ac, b).

$QSym_A$ is a subset of $\mathbb{Q}[x_A]$ defined as the set of formal power series such that the coefficients of the monomials indexed by the same sentence are equal.

Example 2.5.8. The following function $f(x_A)$ is in $QSym_A$:

$$f(x_A) = 3x_{a,1}x_{bc,2} + 3x_{a,1}x_{bc,3} + \ldots + 3x_{a,2}x_{bc,3} + 3x_{a,2}x_{bc,4} + \ldots .$$

Bases in $QSym$ extend naturally to bases in $QSym_A$. For a sentence $I = (w_1, w_2, \ldots, w_m)$, the *colored monomial quasisymmetric function* M_I is defined as

$$M_I = \sum_{1 \leq j_1 < j_2 < \ldots < j_m} x_{w_1,j_1} x_{w_2,j_2} \ldots x_{w_m,j_m},$$

where the sum runs over strictly increasing sequences of m positive integers $j_1, \ldots, j_m \in \mathbb{Z}_{>0}$.

Example 2.5.9. The colored monomial quasisymmetric function for the sentence (a, bc) is

$$M_{(a,bc)} = x_{a,1}x_{bc,2} + x_{a,1}x_{bc,3} + \ldots + x_{a,2}x_{bc,3} + x_{a,2}x_{bc,4} + \ldots + x_{a,3}x_{bc,4} + \ldots .$$

Proposition 2.5.10. *[28] $QSym_A$ and $NSym_A$ are dual Hopf algebras with the inner product $\langle H_I, M_J \rangle = \delta_{I,J}$.*

$QSym_A$ and $NSym_A$ inherit the multiplication and comultiplication from the Hopf algebra of sentences. The quasishuffle $I \overset{Q}{\sqcup\!\sqcup} J$ is defined as the sum of all shuffles of sentences I and J and shuffles of sentences I and J with any number of pairs $w_i v_j$ of consecutive words $w_i \in I$ and $v_j \in J$ concatenated.

Example 2.5.11. The usual shuffle operation on (ab, c) and (d, e) is

$$(ab, c) \sqcup\!\sqcup (d, e) = (ab, c, d, e) + (ab, d, c, e) + (d, ab, c, e) + (ab, d, e, c) + (d, ab, e, c) + (d, e, ab, c).$$

The quasishuffle of (ab, c) and (d, e) is

$$\begin{aligned}(ab, c) \overset{Q}{\sqcup\!\sqcup} (d, e) &= (ab, c, d, e) + (ab, cd, e) + (ab, d, c, e) + (abd, c, e) + (ab, d, ce) + (abd, ce) + \\ &+ (d, ab, c, e) + (d, ab, ce) + (ab, d, e, c) + (abd, e, c) + (d, ab, e, c) + (d, abe, c) + (d, e, ab, c).\end{aligned}$$

Multiplication in $QSym_A$ is dual to the comultiplication Δ in $NSym_A$, and given by

$$M_I M_J = \sum_K M_K,$$

where the sum runs over all summands K in $I \overset{Q}{\sqcup\!\sqcup} J$. Comultiplication in $QSym_A$ is defined by multiplication in $NSym_A$ using deconcatenation

$$\Delta(M_I) = \sum_{I=J\cdot K} M_J \otimes M_K, \tag{2.13}$$

where the sum runs over all sentences J and K such that $I = J \cdot K$. Finally, the antipode S^* in $QSym_A$ is given by

$$S^*(M_I) = (-1)^{\ell(I)} \sum_{J^r \succeq I} M_J.$$

Definition 2.5.12. The *uncoloring* map $\upsilon : QSym_A \to QSym$ is defined by $\upsilon(x_{w_1,1} \cdots x_{w_k,k}) = x_1^{|w_1|} \cdots x_k^{|w_k|}$ and extends linearly. If the alphabet A contains only one color, υ is an isomorphism.

Note that we use υ to denote the uncoloring maps on both $QSym_A$ and $NSym_A$, and often refer to these together as if they are one map. We say two bases $\{B_I\}_I$ and $\{C_\alpha\}_\alpha$ of $QSym_A$ and $QSym$ are *analogous* if $\upsilon(B_I) = C_{w\ell(I)}$ for all sentences I when A is an alphabet of one color. By definition, the colored monomial functions are analogues for the monomial quasisymmetric functions. The fundamental quasisymmetric functions have a colored analogue, called the *colored fundamental quasisymmetric functions*, that are defined as

$$F_I = \sum_{J \preceq I} M_J, \quad \text{and so} \quad M_I = \sum_{J \preceq I} (-1)^{\ell(J)-\ell(I)} F_J, \tag{2.14}$$

where the sums run over all sentences J that are refinements of I. The colored fundamental basis is dual to the colored ribbon basis with $\langle R_I, F_J \rangle = \delta_{I,J}$.

CHAPTER

3

THE IMMACULATE AND DUAL IMMACULATE FUNCTIONS

We define a new pair of dual bases that generalize the immaculate and dual immaculate bases to the colored algebras $QSym_A$ and $NSym_A$. The colored dual immaculate functions are defined combinatorially via tableaux, and we present results on their Hopf algebra structure, expansions to and from other bases, and their extension to the framework of skew functions. This includes a combinatorial method for expanding the colored fundamental quasisymmetric functions into the colored dual immaculate basis that specializes to a new result in the uncolored case. For the colored immaculate functions, defined using creation operators, we study expansions to and from other bases and provide a right Pieri rule. We use the same methods to define colored generalizations of the row-strict immaculate and row-strict dual immaculate functions with similar results.

We also introduce two new bases of *NSym* and a new basis of *QSym* that relate to the immaculate and dual immaculate bases by the involutions ρ and ω. For each of these bases and the reverse dual immaculate functions of Mason and Searles, we state results analogous to those of the immaculate and dual immaculate bases. Then, we define what we call 'skew-II' functions for the reverse dual immaculate and row-strict reverse dual immaculate bases. These are defined via a right action of *NSym* on *QSym* as opposed to the left action of *NSym* on *QSym* that defines 'skew' functions. We show that the skew-II reverse dual immaculate functions and the skew-II row-strict reverse dual immaculate functions are the image of the skew dual immaculate functions under ρ and ω respectively.

3.1 Background

Berg, Bergeron, Saliola, and Serrano introduced the immaculate basis and its dual in [9]. They have been widely studied since then, with applications including the noncommutative Hall-Littlewood basis of *NSym*[9], the noncommutative Bell polynomials [71], and 0-Hecke algebras [10, 22]. The immaculate functions are distinguished among Schur-like bases for having a Jacobi-Trudi rule and for mapping under the forgetful map to the Schur functions defined for compositions in all cases.

3.1.1 Dual immaculate quasisymmetric functions

The dual immaculate functions are defined combinatorially as the sum of monomials associated to certain tableaux that generalize semistandard Young tableaux.

Definition 3.1.1. Let α and β be a composition and weak composition respectively. An *immaculate tableau* of shape α and type β is a labelling of the boxes of the diagram of α by positive integers in such a way that:

1. the number of boxes labelled by i is β_i,

2. the sequence of entries in each row, from left to right, is weakly increasing, and

3. the sequence of entries in the first column, from top to bottom, is strictly increasing.

An immaculate tableau T of type $\beta = (\beta_1, \dots, \beta_h)$ is associated with the monomial $x^T = x_1^{\beta_1} x_2^{\beta} \cdots x_h^{\beta_h}$, which may also be written as x^β.

Example 3.1.2. The immaculate tableaux of shape $\alpha = (2, 3)$ and type $\beta = (1, 2, 2)$ are:

$$\begin{array}{|c|c|c} \hline 1 & 2 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 1 & 3 & \\ \hline 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Both tableaux are associated with the monomial $x_1 x_2^2 x_3^2$.

Definition 3.1.3. For a composition α, the *dual immaculate function* is defined by

$$\mathfrak{S}^*_\alpha = \sum_T x^T,$$

where the sum runs over all immaculate tableaux T of shape α.

Example 3.1.4. The dual immaculate function $\mathfrak{S}^*_{(2,2)}$ corresponds to immaculate tableaux of shape $(2, 2)$:

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\ \hline 3&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\ \hline 2&2\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2&2\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\ \hline 3&3\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\ \hline 3&3\\ \hline\end{array}\quad \cdots$$

Therefore,

$$\mathfrak{S}^*_{(2,2)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2 + x_1x_3^3 + x_2^2x_3^2 + \dots.$$

Dual immaculate functions have positive expansions into the monomial and fundamental bases.

Proposition 3.1.5. *[9] The dual immaculate functions are monomial positive. In fact,*

$$\mathfrak{S}^*_\alpha = \sum_{\beta \le_\ell \alpha} \mathfrak{K}_{\alpha,\beta} M_\beta,$$

where $\mathfrak{K}_{\alpha,\beta}$ is the number of immaculate tableaux of shape α and type β.

Example 3.1.6. Let $\alpha = (2,2)$. The set of compositions β such that $|\alpha| = |\beta|$ and $\beta \le_\ell \alpha$ is $\{(2,2),(2,1,1),(1,3),(1,2,1),(1,1,2),(1,1,1,1)\}$. The two immaculate tableaux of shape $(2,2)$ and type $(1,1,2)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 3 \\ \hline \end{array}$$

Thus, $\mathfrak{K}_{(2,2),(1,1,2)} = 2$. Repeating repeating this calculation for each β in the set above yields

$$\mathfrak{S}^*_{(2,2)} = M_{(2,2)} + M_{(2,1,1)} + M_{(1,3)} + 2M_{(1,2,1)} + 2M_{(1,1,2)} + 3M_{(1,1,1,1)}.$$

The expansion of the dual immaculate functions into the fundamental basis relies on the following subset of immaculate tableaux.

Definition 3.1.7. A *standard immaculate tableau* of shape $\alpha \models n$ is an immaculate tableau in which each integer 1 through n appears exactly once.

Example 3.1.8. The standard immaculate tableaux of shape $\alpha = (2,3)$ are:

$$\begin{array}{|c|c|c}\cline{1-2} 1 & 2 & \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\cline{1-2} 1 & 3 & \\ \hline 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\cline{1-2} 1 & 4 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\cline{1-2} 1 & 5 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Each immaculate tableau is associated with a standard immaculate tableau by standardization.

Definition 3.1.9. Given an immaculate tableau T of shape α, form a standard immaculate tableau $std(T) = U$ of shape α by relabeling the boxes of T with the integers 1 through n in the following way. Begin with all boxes filled with 1's then continue on to the boxes filled with 2's, then 3's, and so on, ignoring boxes we have already relabelled. Starting from the lowest row containing each value, move through boxes filled with the same value from left to right and bottom to top, relabelling each with the next integer from $[n]$, starting the very first box with 1. The resultant tableau U is a standard immaculate tableau which we call the *standardization* of T.

Example 3.1.10. The two immaculate tableaux below both have shape $(2,3)$ and type $(1,2,2)$ but different standardizations:

$$T_1 = \begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c} \hline 1 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$std(T_1) = \begin{array}{|c|c|c} \hline 1 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad std(T_2) = \begin{array}{|c|c|c} \hline 1 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Definition 3.1.11. [9, 78] A standard immaculate tableau U has a *descent* in position i if $(i+1)$ is in a row strictly lower than i in U. We denote the set of all descents in U as $Des_{\mathfrak{S}}(U)$, called the *descent set* of U. If $Des_{\mathfrak{S}}(U) = \{d_1, \ldots, d_{k-1}\}$ then the *descent composition* of U is defined as $co_{\mathfrak{S}}(U) = comp(Des_{\mathfrak{S}}(U)) = (d_1, d_2 - d_1, d_3 - d_2, \ldots, n - d_{k-1})$.

Proposition 3.1.12. *[9] The dual immaculate functions $\mathfrak{S}^*_\alpha$ are fundamental positive. In fact,*

$$\mathfrak{S}^*_\alpha = \sum_{\beta \leq_\ell \alpha} \mathfrak{L}_{\alpha,\beta} F_\beta,$$

where $\mathfrak{L}_{\alpha,\beta}$ is the number of standard immaculate tableaux with shape α and descent composition β.

Example 3.1.13. Let $\alpha = (2,2)$. The standard immaculate tableaux of shape $(2,2)$, listed with their descent sets and descent compositions, are

$$S_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad S_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad S_3 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}$$

$Des_{\mathfrak{S}}(S_1)=\{2\}$, $co_{\mathfrak{S}}(S_1)=(2,2)$ $\qquad$ $Des_{\mathfrak{S}}(S_2)=\{1,3\}$, $co_{\mathfrak{S}}(S_2)=(1,2,1)$ $\qquad$ $Des_{\mathfrak{S}}(S_3)=\{1\}$, $co_{\mathfrak{S}}(S_3)=(1,3)$

Therefore, $\mathfrak{L}_{(2,2)(2,2)} = 1$, $\mathfrak{L}_{(2,2),(1,2,1)} = 1$, and $\mathfrak{L}_{(2,2),(1,3)} = 1$, meaning

$$\mathfrak{S}^*_{(2,2)} = F_{(1,2,1)} + F_{(1,3)} + F_{(2,2)}.$$

The Schur functions expand into an alternating sum of dual immaculate functions. As a result, the expansion of any symmetric function into dual immaculate functions translates directly to the Schur basis expansion of that function.

Theorem 3.1.14. *[9] For a partition λ with $\ell(\lambda) = k$,*

$$s_\lambda = \sum_{\sigma \in S_k} (-1)^\sigma \mathfrak{S}^*_{\lambda_{\sigma_1}+1-\sigma_1, \lambda_{\sigma_2}+2-\sigma_2, \cdots, \lambda_{\sigma_k}+k-\sigma_k},$$

where the sum runs over permutations σ such that $\lambda_{\sigma_i} + i - \sigma_i > 0$ for all $i \in [k]$.

Corollary 3.1.15. *[9] Let f be a symmetric function. Then,*

$$f = \sum_{\alpha \models n} d_\alpha \mathfrak{S}^*_\alpha \Longrightarrow f = \sum_{\lambda \vdash n} d_\lambda s_\lambda.$$

Allen, Hallam, and Mason also define exactly when dual immaculate functions will be symmetric.

Proposition 3.1.16. *[4] Let α be a composition of n of length $k+1$. Then $\mathfrak{S}^*_\alpha$ is symmetric if and only if $\alpha = (n-k, 1^k)$.*

3.1.2 Immaculate noncommutative symmetric functions

The dual immaculate functions were originally developed as the duals to the immaculate functions in *NSym* [9]. The immaculate functions are defined constructively by creation operators that generalize the Bernstein operators used to define the Schur functions.

For $F, G \in QSym$, the *perp operator* $F^\perp$ acts on elements $H \in NSym$ based on the relation $\langle H, FG\rangle = \langle F^\perp H, G\rangle$. This expands as

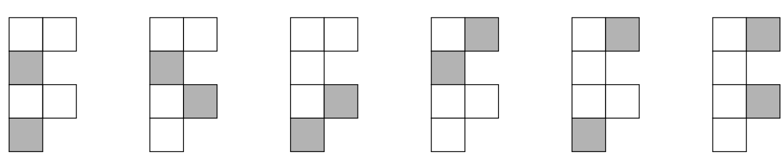
$$F^\perp(H) = \sum_\alpha \langle H, FA_\alpha\rangle B_\alpha \tag{3.1}$$

for dual bases $\{A_\alpha\}_\alpha$ of *QSym* and $\{B_\alpha\}_\alpha$ of *NSym*. Most important for our purposes is the specialization of this operator to the fundamental basis [9], where $F^\perp_{1^i}$ acting on H_α is

$$F^\perp_{1^i}(H_\alpha) = \sum_{\substack{\beta \in \mathbb{N}^m \\ |\beta| = |\alpha| - i \\ \alpha_j - 1 \le \beta_j \le \alpha_j}} H_{\tilde{\beta}}.$$

We interpret the action of this operator on the indices of H_α. This operator acts on the composition α by taking a diagram of shape α and returning the sum of all diagrams (as indices of H_βs) whose shape is obtained by removing i boxes from the right-hand side with no more than 1 box being removed from each row.

Example 3.1.17. For instance, $F^\perp_{(1,1)}(H_{(2,1,2,1)}) = H_{(2,2)} + 2H_{(2,1,1)} + H_{(1,2,1)} + H_{(1,1,2)} + H_{(1,1,1,1)}$ can be visualized with the following tableaux, where the gray blocks are removed and all rows are moved up to fill any entirely empty rows.



$H_{(2,2)}$ $H_{(2,1,1)}$ $H_{(2,1,1)}$ $H_{(1,2,1)}$ $H_{(1,1,2)}$ $H_{(1,1,1,1)}$

Definition 3.1.18. For $m \in \mathbb{Z}$, the *noncommutative Bernstein operator* $\mathbb{B}_m$ is defined as

$$\mathbb{B}_m = \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^{\perp}.$$

These operators generalize the Bernstein operators used to construct the Schur functions [78] and thus allow for the construction of a noncommutative generalization of the Schur functions.

Definition 3.1.19. For $\alpha = [\alpha_1, \ldots, \alpha_m] \in \mathbb{Z}^m$, the *immaculate noncommutative symmetric function* $\mathfrak{S}_\alpha$ is defined as

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1} \cdots \mathbb{B}_{\alpha_m}(1).$$

The *immaculate basis* of *QSym* is the set of immaculate functions $\{\mathfrak{S}_\alpha\}_\alpha$ where $\alpha \models n$ for $n \in \mathbb{Z}_{>0}$.

Example 3.1.20. If $\alpha = (\alpha_1, \alpha_2)$, then $\mathfrak{S}_{(\alpha_1,\alpha_2)} = \mathbb{B}_{\alpha_1}(H_{\alpha_2}) = H_{\alpha_1} H_{\alpha_2} - H_{\alpha_1+1} H_{\alpha_2-1}$.

Properties of these Bernstein operators lead to a right Pieri rule for the immaculate functions, but first, we need additional notation from [9].

Definition 3.1.21. For a positive integer s and compositions α and β, we write $\alpha \subset_s^{\mathfrak{S}} \beta$ if $\beta \models |\alpha| + s$ and $\alpha \subseteq \beta$ and $\ell(\beta) \leq \ell(\alpha) + 1$.

The relationship $\subset_1^{\mathfrak{S}}$ constitutes a partial order on compositions.

Example 3.1.22. The compositions β for which $(1,2) \subset_2^{\mathfrak{S}} \beta$ are

$$(1,2) \subset_2^{\mathfrak{S}} (2,3) \text{ and } (1,2) \subset_2^{\mathfrak{S}} (2,2,1) \text{ and } (1,2) \subset_2 (1,3,1).$$

Theorem 3.1.23. *[9] For a composition α and an integer s,*

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s^{\mathfrak{S}} \beta} \mathfrak{S}_\beta,$$

where the sum runs over all compositions β such that $\alpha \subset_s^{\mathfrak{S}} \beta$.

Example 3.1.24. Applying the Pieri rule for $\alpha = (2,1)$ and $s = 2$ yields

$$\mathfrak{S}_{(2,1)} H_{(2)} = \mathfrak{S}_{(2,1,2)} + \mathfrak{S}_{(2,2,1)} + \mathfrak{S}_{(3,1,1)} + \mathfrak{S}_{(2,3)} + \mathfrak{S}_{(3,2)} + \mathfrak{S}_{(4,1)}.$$

Iteration of this Pieri rule yields the following positive expansion of the complete homogeneous basis, and subsequently the ribbon basis, in terms of the immaculate basis:

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{K}_{\alpha,\beta} \mathfrak{S}_\alpha \qquad \text{and} \qquad R_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{L}_{\alpha,\beta} \mathfrak{S}_\alpha. \tag{3.2}$$

Notice that these expansions relate to those in Propositions 3.1.5 and 3.1.12 via Proposition 2.2.7. Thus, the dual immaculate functions are dual to the immaculate functions.

Proposition 3.1.25. *[9] For any compositions α and β, we have $\langle \mathfrak{S}_\alpha, \mathfrak{S}^*_\beta \rangle = \delta_{\alpha,\beta}$.*

The expansion of the immaculate functions into the complete homogeneous basis follows a Jacobi-Trudi rule much like the Schur functions.

Theorem 3.1.26. *[9] For $\alpha = [\alpha_1, \ldots, \alpha_m] \in \mathbb{Z}^m$,*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \ldots, \alpha_m+\sigma_m-m},$$

with $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$. This is equivalent to taking the noncommutative analogue of the determinant of the matrix below obtained by expanding the determinant of the matrix along the first row and multiplying those elements on the left:

$$\begin{bmatrix} H_{\alpha_1} & H_{\alpha_1+1} & \cdots & H_{\alpha_1+\ell-1} \\ H_{\alpha_2-1} & H_{\alpha_2} & \cdots & H_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_\ell-\ell+1} & H_{\alpha_\ell-\ell+2} & \cdots & H_{\alpha_\ell} \end{bmatrix}.$$

The commutative image of the immaculate functions follows from this Jacobi-Trudi rule.

Corollary 3.1.27. *[9] For any composition α, we have $\chi(\mathfrak{S}_\alpha) = s_\alpha$.*

Certain classes of immaculate functions also have simpler expansions in terms of the complete homogeneous basis [9]. For instance, for a positive integer n,

$$\mathfrak{S}_{1^n} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} H_\alpha.$$

There is another right Pieri rule for multiplication by these immaculate functions. For a composition α and a positive integer s,

$$\mathfrak{S}_\alpha \mathfrak{S}_{1^s} = \sum_{\substack{\beta \models |\alpha|+s \\ \alpha_i \leq \beta_i \leq \alpha_i+1}} \mathfrak{S}_\beta. \tag{3.3}$$

In certain cases, as shown by Campbell as well as Allen and Mason, an immaculate function has a Jacob-Trudi rule in terms of the ribbon functions as well. A more general ribbon Jacobi-Trudi rule for the immaculate functions is not yet known.

Theorem 3.1.28. *[18] Let α be a composition of the form (a^b, c^d) where $b \leq c$ and $b \leq a$. Then,*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_{\ell(\alpha)}} (-1)^\sigma R_{(\alpha_1-1+\sigma_1, \alpha_2-2+\sigma_2, \ldots, \alpha_{\ell(\alpha)}-\ell(\alpha)+\sigma_{\ell(\alpha)}}.$$

Theorem 3.1.29. *[5] Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of positive integers for which there exists j with $1 \leq j \leq k$ such that $\alpha_i \geq i$ for $1 \leq i \leq j$, $\alpha_{j+1} \geq j$, and $\alpha_\ell = \alpha_{j+1}$ for $j+2 \leq \ell \leq k$.*

Then,

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} (-1)^\sigma R_{(\alpha_1 - 1 + \sigma_1, \ldots, \alpha_k - k + \sigma_k)},$$

where R_α vanishes if α contains any nonpositive parts.

There is also a Murnaghan-Nakayama-type rule for immaculate functions.

Theorem 3.1.30. *[11] For a composition α and a positive integer k,*

$$\mathfrak{S}_\alpha \Psi_k = -\sum_{\beta \models k} (-1)^{\ell(\beta)} \mathfrak{S}_{(\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \ldots, \beta_{\ell(\beta)})} + \sum_{j=1}^{\ell(\alpha)} \mathfrak{S}_{(\alpha_1, \ldots, \alpha_j + k, \ldots, \alpha_{\ell(\alpha)})}.$$

This yields the following expansion of a power sum function into the immaculate basis.

Lemma 3.1.31. *[11] For $k \geq 1$,*

$$\Psi_k = \sum_{\beta \models k} (-1)^{\ell(\beta)+1} \mathfrak{S}_\beta.$$

The expansion of immaculate functions into other bases is generally less well understood. In [53], Loehr and Niese study the immaculate inverse Kostka matrix, which contains the coefficients in the expansion of an immaculate function into the H-basis, in terms of determinants, raising operators, special rim hook tableaux, and tournaments. In [5], Allen and Mason give a complete combinatorial description of the inverse Kostka matrix in terms of tunnel hooks, which generalize the special rim hooks of Eğecioğlu and Remmel [29].

Of major interest with any basis, but especially Schur-like bases, are the coefficients that appear in the expansion of the product of two basis elements. For compositions α and β,

$$\mathfrak{S}_\alpha \mathfrak{S}_\beta = \sum_\gamma = C^\gamma_{\alpha,\beta} \mathfrak{S}_\gamma$$

where $C^\gamma_{\alpha,\beta}$ are called the *structure coefficients* of the immaculate basis or the *immaculate Littlewood-Richardson coefficients.* A combinatorial interpretation of these coefficients for the general case is unknown. In certain cases, these coefficients have a combinatorial interpretation that generalizes the usual Littlewood-Richardson coefficients in a fairly straightforward way. The reading word of a tableau T, denoted by $read(T)$, is the word of entries read from left to right starting in the top row and moving down. A tableau T is called *Yamanouchi* if in $read(T)$ there are as many occurrences of any integer j and there are of integer $j+1$ among the first d letters of $read(T)$ for all d.

Theorem 3.1.32. *[11] For a composition α and partition λ,*

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\gamma = |\alpha| + |\lambda|} C^\gamma_{\alpha,\lambda} \mathfrak{S}_\gamma,$$

where $C^\gamma_{\alpha,\lambda}$ is the number of skew immaculate Yamanouchi tableaux of shape γ/α and type λ.

The immaculate structure coefficients have the following translation invariance.

Theorem 3.1.33. *[51] For compositions $\alpha, \beta, \gamma, \nu$ with $\ell(\nu) \leq \ell(\alpha)$, we have $C^{\gamma}_{\alpha,\beta} = C^{\gamma+\nu}_{\alpha+\nu,\beta}$.*

The classical Littlewood-Richardson coefficients can be expressed as an alternating sum of immaculate structure coefficients using the convolution product. For a sequence of integers ζ of length m and a permutation $\sigma \in S_m$, let $\zeta \star \sigma = [\zeta_{\sigma_1} - \sigma_1 + 1, \zeta_{\sigma_2} - \sigma_2 + 2, \ldots, \zeta_{\sigma_m} - \sigma_m + m]$.

Corollary 3.1.34. *[11] For partitions λ, μ, ν such that $|\lambda| + |\mu| = |\nu|$,*

$$c^{\nu}_{\mu,\lambda} = \sum_{\sigma \in S_{\ell(\nu)}} (-1)^{\sigma} C^{\nu\star\sigma}_{\mu,\lambda},$$

with the convention that $C^{\nu\star\sigma}_{\mu,\lambda} = 0$ if $\nu \star \sigma$ is not a composition of length $m = \ell(\nu)$ that contains μ.

Like the Littlewood-Richardson coefficients, certain immaculate structure coefficients appear geometrically as the number of integral points inside a polytope as shown in [11]. Further multiplicative results, specifically Pieri rules and skew Pieri rules, can be found in [9, 11, 13, 51, 70].

Recall that the antipode is an anti-endomorphism which is an important part of Hopf algebra structure. No general formula for the antipode of the immaculate functions is known, but Benedetti and Sagan prove the following formula for immaculate functions indexed by hook shapes.

Theorem 3.1.35. *[8] For all $n \geq 1$ and $k \geq 0$,*

$$S(\mathfrak{S}_{(n,1^k)}) = (-1)^{n+k} \mathfrak{S}_{(k+1,1^{n-1})}.$$

Campbell also gives the expansion of the antipode of various immaculate functions expanded into the ribbon basis. One such result is included below.

Theorem 3.1.36. *[21] The cancellation-free formula*

$$S(\mathfrak{S}_\alpha) = \sum_{\sigma \in S_{\ell(\alpha)}} (-1)^{\sigma+ab+cd} R_{((\alpha_1-1+\sigma_1, \alpha_2-2+\sigma_2, \ldots, \alpha_{\ell(\alpha)}-\ell(\alpha)+\sigma_{\ell(\alpha)})^t)^r},$$

holds true for compositions $\alpha = (a^b, c^d)$ such that $b \leq c$ and $b \leq a$.

3.1.3 Skew dual immaculate functions

The *immaculate poset* $\mathfrak{P}^{\mathfrak{S}}$, also defined in [9], is a labelled poset on compositions where α covers β if $\beta \subset^{\mathfrak{S}}_1 \alpha$. In other words, α covers β if α can be obtained by adding 1 to any part of β or to the end of β as a new part. In terms of diagrams, this is equivalent to adding a box to the right of any row or adding a box at the bottom of the tableau.

In the Hasse diagram of $\mathfrak{P}^{\mathfrak{S}}$, label the arrow from β to α with m, where m is the number of the row where the new box is added. Maximal chains from $\emptyset$ to α are equivalent to standard immaculate tableaux of shape α, and maximal chains from β to α define skew standard immaculate tableaux of shape α/β. A path $\{\beta = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \ldots \xrightarrow{m_k} \beta^{(k)} = \alpha\}$ corresponds to the skew shape α/β where the boxes are filled with positive integers in the order they were added following the path.

Example 3.1.37. Consider two paths $\mathcal{P}_1 = \{\emptyset \xrightarrow{1} (1) \xrightarrow{2} (1,1) \xrightarrow{2} (1,2) \xrightarrow{1} (2,2)\}$ and $\mathcal{P}_2 = \{\emptyset \xrightarrow{1} (1) \xrightarrow{1} (2) \xrightarrow{2} (2,1) \xrightarrow{2} (2,2)\}$. These paths correspond to the standard immaculate tableaux T_1 and T_2 below, respectively. The path $\mathcal{P}_3 = \{(1) \xrightarrow{2} (1,1) \xrightarrow{1} (2,1) \xrightarrow{2} (2,2)\}$ corresponds to the skew standard immaculate tableau T_3.

$$T_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|c|}\hline \blacksquare & 2 \\ \hline 1 & 3 \\ \hline \end{array}$$

Definition 3.1.38. [69] Let α and β be compositions where $\beta \subseteq \alpha$. A *skew immaculate tableau* of shape α/β is a skew shape α/β filled with positive integers such that the entries in the first column of α are strictly increasing from top to bottom and the entries in rows are weakly increasing from left to right. We say T is a *skew standard immaculate tableau* if it contains the entries $1, \ldots, |\alpha| - |\beta|$ with each appearing exactly once.

Definition 3.1.39. [9] Given compositions α and β with $\beta \subseteq \alpha$, the *skew dual immaculate function* is defined as

$$\mathfrak{S}^*_{\alpha/\beta} = \sum_{\gamma} \langle \mathfrak{S}_\beta H_\gamma, \mathfrak{S}^*_\alpha \rangle M_\gamma,$$

where the sum runs over all γ such that $|\alpha| - |\beta| = |\gamma|$.

The coefficient $\langle \mathfrak{S}_\beta H_\gamma, \mathfrak{S}^*_\alpha \rangle$ is exactly equal to the number of skew standard immaculate tableaux of shape α/β with type γ. Thus, the skew dual immaculate functions can also be defined by a sum over skew immaculate tableaux.

Theorem 3.1.40. *[69] Let α and β be compositions with $\beta \subseteq \alpha$. Then*

$$\mathfrak{S}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over all skew immaculate tableaux of shape α/β.

Expansions of the skew dual immaculate functions into the fundamental and dual immaculate bases yield coefficients with connections to the multiplicative structure of the immaculate functions.

Proposition 3.1.41. *[9] Given compositions α and β with $\beta \subseteq \alpha$,*

$$\mathfrak{S}^*_{\alpha/\beta} = \sum_{\gamma} \langle \mathfrak{S}_\beta R_\gamma, \mathfrak{S}^*_\alpha \rangle F_\gamma = \sum_{\gamma} \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}^*_\alpha \rangle \mathfrak{S}^*_\gamma,$$

where both sums run over all compositions γ such that $|\alpha| - |\beta| = |\gamma|$. Moreover, the coefficients $C^\alpha_{\beta,\gamma} = \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}^*_\alpha \rangle$ are the immaculate structure constants that appear in the expansion

$$\mathfrak{S}_\beta \mathfrak{S}_\gamma = \sum_{\alpha} C^\alpha_{\beta,\gamma} \mathfrak{S}_\alpha.$$

The coproduct of the dual immaculate functions can be described using skew dual immaculate functions.

Definition 3.1.42. [69] Given $\alpha \models n$, the comultiplication on $\mathfrak{S}^*_\alpha$ is defined by

$$\Delta(\mathfrak{S}^*_\alpha) = \sum_\beta \mathfrak{S}^*_\beta \otimes \mathfrak{S}^*_{\alpha/\beta},$$

where the sum runs over all compositions β such that $\beta \subseteq \alpha$.

The right multiplication of an immaculate function by a ribbon function can also be described combinatorially in terms of skew immaculate tableaux.

Theorem 3.1.43. *[11] For compositions α and β,*

$$\mathfrak{S}_\alpha R_\beta = \sum_T \mathfrak{S}_\gamma$$

where the sum runs over standard immaculate tableaux of shape γ/α and descent composition β.

In [70], Niese, Sundaram, van Willigenburg, and Wang also give Pieri rules for the multiplication of skew dual immaculate functions. For more on the immaculate and dual immaculate bases and immaculate tableaux, see [1, 10, 22, 32, 36, 63, 71].

3.1.4 Row-strict immaculate and dual immaculate functions

Niese, Sundaram, Van Willigenburg, Vega, and Wang define a pair of dual bases in *QSym* and *NSym*, the row-strict dual immaculate and row-strict immaculate bases, in [69] by applying the involution ψ to the immaculate and dual immaculate bases. The row-strict dual immaculate basis has extensive representation theoretic applications, specifically to 0-Hecke algebras. The combinatorics of this basis involve a variation of immaculate tableaux with different conditions on the rows and columns. Note that the original paper uses French notation for diagrams (the bottom row is row 1) so the definitions here have been adapted to English notation. We begin by recalling several definitions and results from [69].

Definition 3.1.44. Given a composition α, a *row-strict immaculate tableau* (RSIT) is a filling of the diagram of α such that the entries in the leftmost column weakly increase from top to bottom and the entries in each row strictly increase from left to right. The *type* of a row-strict immaculate tableau is defined the same way as the type of an immaculate tableau. A row-strict immaculate tableau with n boxes is *standard* if each integer 1 through n appears exactly once.

Monomials are associated with row-strict immaculate tableaux according to their type in the same fashion as immaculate tableaux.

Example 3.1.45. The following are row-strict immaculate tableaux of shape $(2,3)$:

$$\begin{array}{|c|c|l}\hline 1 & 2 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \begin{array}{|c|c|l}\hline 1 & 2 & \\ \hline 1 & 3 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \qquad \begin{array}{|c|c|l}\hline 1 & 2 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \qquad \begin{array}{|c|c|l}\hline 1 & 2 & \\ \hline 3 & 4 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \qquad \begin{array}{|c|c|l}\hline 1 & 3 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \begin{array}{|c|c|l}\hline 1 & 4 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

Definition 3.1.46. For a composition α, the *row-strict dual immaculate function* is defined as

$$\mathfrak{RS}^*_\alpha = \sum_T x^T,$$

where the sum runs over all row-strict immaculate tableaux T of shape α.

To standardize a row-strict immaculate tableau T, replace the 1's in T with $1, 2, \ldots$ moving left to right and top to bottom, then continue with the 2's, etc. Note also that the set of standard row-strict immaculate tableaux is the same as the set of standard immaculate tableaux.

Definition 3.1.47. A positive integer i is a *descent* of a standard row-strict immaculate tableau U if U contains the entry $i+1$ in a weakly higher row than entry i. The *descent set* of a standard row-strict immaculate tableau U is

$$Des_{\mathfrak{RS}}(U) = \{i : i+1 \text{ is weakly above } i \text{ in } U\}.$$

The *descent composition* of a standard row-strict immaculate tableaux U is defined as

$$co_{\mathfrak{RS}}(U) = comp(Des_{\mathfrak{RS}}(U)).$$

Proposition 3.1.48. *[69] For a composition α, $\mathfrak{RS}^*_\alpha$ is expressed as*

$$\mathfrak{RS}^*_\alpha = \sum_S F_{co_{\mathfrak{RS}}(S)},$$

where the sum is taken over all standard row-strict immaculate tableaux of shape α.

It becomes clear with the expansion of the row-strict dual immaculate functions into the fundamental basis that they are the image of the usual dual immaculate under the involution ψ.

Theorem 3.1.49. *[69] Let α be a composition. Then, $\psi(\mathfrak{S}^*_\alpha) = \mathfrak{RS}^*_\alpha$. Moreover, $\{\mathfrak{RS}^*_\alpha\}_\alpha$ is a basis for QSym.*

For a composition α and weak composition β, let $\mathfrak{K}^{\mathfrak{RS}}_{\alpha,\beta}$ be the number of row-strict immaculate tableaux of shape α and type β, and $\mathfrak{L}^{\mathfrak{RS}}_{\alpha,\beta}$ be the number of standard row-strict immaculate tableaux of shape α with row-strict descent composition β.

Theorem 3.1.50. *[69] For a composition α, the row-strict dual immaculate function expands as*

$$\mathfrak{RS}^*_\alpha = \sum_\beta \mathfrak{K}^{\mathfrak{RS}}_{\alpha,\beta} M_\beta = \sum_\beta \mathfrak{L}^{\mathfrak{RS}}_{\alpha,\beta} F_\beta,$$

where the sums run over compositions β such that $|\beta| = |\alpha|$.

The row-strict dual immaculate functions have a dual basis that can be constructed similarly to the immaculate basis.

Definition 3.1.51. For $m \in \mathbb{Z}$, the *noncommutative row-strict Bernstein operator* is defined by

$$\mathbb{B}_m^{\mathfrak{RS}} = \sum_{i \geq 0} (-1)^i E_{m+i} F_{(i)}^{\perp}.$$

Definition 3.1.52. For a composition $\alpha = (\alpha_1 \dots, \alpha_k)$, the *row-strict immaculate function* $\mathfrak{RS}_\alpha$ is defined as

$$\mathfrak{RS}_\alpha = \mathbb{B}_{\alpha_1}^{\mathfrak{RS}} \dots \mathbb{B}_{\alpha_k}^{\mathfrak{RS}}(1).$$

These functions are dual to the row-strict dual immaculate functions, $\langle \mathfrak{RS}_\alpha, \mathfrak{RS}_\beta^* \rangle = \delta_{\alpha,\beta}$, and they are the image of the immaculate basis under ψ with $\psi(\mathfrak{S}_\alpha) = \mathfrak{RS}_\alpha$.

Applying ψ to various results from [9] yields similar results for the row-strict immaculate and row-strict dual immaculate bases, which are summarized in the following theorem.

Theorem 3.1.53. *[69] For compositions $\alpha, \beta \models n$, $s \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}$, and $f \in NSym$,*

1. $$\mathbb{B}_m(f)H_s = \mathbb{B}_{m+1}(f)H_{s-1} + \mathbb{B}_m(fH_s) \quad \overset{\psi}{\Longleftrightarrow} \quad \mathbb{B}_{m+1}^{\mathfrak{RS}}(f)E_{s-1} + \mathbb{B}_m^{\mathfrak{RS}}(fE_s).$$

2. *Multiplicity-free right Pieri rule:*

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta \quad \overset{\psi}{\Longleftrightarrow} \quad \mathfrak{RS}_\alpha E_s = \sum_{\alpha \subset_s \beta} \mathfrak{RS}_\beta.$$

3. *Multiplicity-free right Pieri rule:*

$$\mathfrak{S}_\alpha \mathfrak{S}_{(1^s)} = \mathfrak{S}_\alpha E_s = \sum_{\beta} \mathfrak{S}_\beta \quad \overset{\psi}{\Longleftrightarrow} \quad \mathfrak{RS}_\alpha \mathfrak{RS}_{(1^s)} = \mathfrak{RS}_\alpha H_s = \sum_{\beta} \mathfrak{RS}_\beta,$$

   where the sum runs over compositions $\beta \models |\alpha| + s$ such that $\alpha_i \leq \beta_i \leq \alpha_i + 1$ and $\alpha_i = 0$ for $i > \ell(\alpha)$.

4. $$\mathfrak{S}_{(1^n)} = \textstyle\sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} H_\alpha = E_n \quad \overset{\psi}{\Longleftrightarrow} \quad \mathfrak{RS}_{(1^n)} = \textstyle\sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} E_\alpha = H_n.$$

5. *Complete homogeneous and elementary expansions:*

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{K}_{\alpha,\beta} \mathfrak{S}_\alpha \quad \overset{\psi}{\Longleftrightarrow} \quad E_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{K}_{\alpha,\beta} \mathfrak{RS}_\alpha.$$

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{K}_{\alpha,\beta}^{\mathfrak{RS}} \mathfrak{RS}_\alpha \quad \overset{\psi}{\Longleftrightarrow} \quad E_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{K}_{\alpha,\beta}^{\mathfrak{RS}} \mathfrak{S}_\alpha.$$

6. Ribbon basis expansions:

$$R_\beta = \sum_{\alpha \geq_\ell \beta} \mathfrak{L}_{\alpha,\beta} \mathfrak{S}_\alpha \quad \overset{\psi}{\Longleftrightarrow} \quad R_{\beta^c} = \sum_{\alpha \geq_\ell \beta} \mathfrak{L}_{\alpha,\beta} \mathfrak{R}\mathfrak{S}_\alpha.$$

The immaculate poset also represents a poset of the standard row-strict immaculate tableaux as a result of the equivalence between standard immaculate tableaux and standard row-strict immaculate tableaux, thus results for the row-strict skew case follow those of the dual immaculate functions closely.

Definition 3.1.54. Let α and β be compositions with $\beta \subseteq \alpha$. A *skew row-strict immaculate tableau* is a skew shape α/β filled with positive integers such that the entries in the first column are weakly increasing from top to bottom and the entries in each row strictly increase from left to right.

Definition 3.1.55. For compositions α, β such that $\beta \subseteq \alpha$, the *skew row-strict dual immaculate functions* are defined as

$$\mathfrak{R}\mathfrak{S}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{R}\mathfrak{S}_\beta H_\gamma, \mathfrak{R}\mathfrak{S}^*_\alpha \rangle M_\gamma,$$

where the sum runs over all compositions γ such that $|\alpha| - |\beta| = |\gamma|$.

As with the skew dual immaculate functions, these functions connect to the multiplication of the row-strict immaculate functions and the comultiplication of the row-strict dual immaculate functions.

Theorem 3.1.56. *[69] Let α and β be compositions with $\beta \subseteq \alpha$. Then,*

$$\mathfrak{R}\mathfrak{S}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over all skew row-strict immaculate tableaux T of shape α/β. Moreover,

$$\mathfrak{R}\mathfrak{S}^*_{\alpha/\beta} = \psi(\mathfrak{S}^*_{\alpha/\beta}) = \sum_\gamma \langle \mathfrak{R}\mathfrak{S}_\beta R_\gamma, \mathfrak{R}\mathfrak{S}^*_\alpha \rangle F_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{S}_\beta \mathfrak{R}\mathfrak{S}_\gamma, \mathfrak{R}\mathfrak{S}^*_\alpha \rangle \mathfrak{R}\mathfrak{S}^*_\gamma,$$

where the sums run over all compositions γ such that $|\alpha| - |\beta| = |\gamma|$.

Comultiplication on the row-strict dual immaculate functions is can be expressed in terms of skew row-strict dual immaculate functions.

Definition 3.1.57. For a composition α,

$$\Delta(\mathfrak{R}\mathfrak{S}^*_\alpha) = \sum_\beta \mathfrak{R}\mathfrak{S}^*_\beta \otimes \mathfrak{R}\mathfrak{S}^*_{\alpha/\beta},$$

where the sum runs over all compositions β such that $\beta \subseteq \alpha$.

3.2 Involutions on the immaculate and dual immaculate functions

In this section, we introduce the reverse immaculate functions, row-strict reverse immaculate functions, and row-strict reverse dual immaculate functions, and study them together with the reverse dual immaculate functions of Mason and Searles [63]. These functions relate to the immaculate and dual immaculate functions via the involutions ρ and ω.

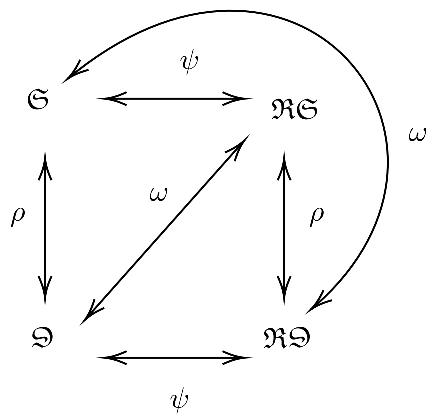


Figure 3.1: Mappings between immaculate variants in *NSym*.

These functions are all defined on tableaux that resemble immaculate tableaux. Table 3.1 below summarizes the definitions and conditions on these variations of immaculate tableaux. For these bases, we present results such as left Pieri rules, Jacobi-Trudi rules, partial Littlewood-Richardson rules, Hopf algebra structure, and expansions to and from other bases.

	Descent	**Reading Word**	**Rows**	**First Column**
Immaculate	strictly below	L to R, B to T	weakly inc.	strictly inc.
Row-strict	weakly above	L to R, T to B	strictly inc.	weakly inc.
Reverse	strictly below	R to L, B to T	weakly dec.	strictly inc.
Row-strict Reverse	weakly above	R to L, T to B	strictly dec.	weakly inc.

Table 3.1: Variations on immaculate tableaux.

Because *NSym* is noncommutative, there are two reasonable ways to define skew functions in *QSym*. The skew dual immaculate functions can be defined as $\mathfrak{S}_\beta^\perp(\mathfrak{S}_\alpha^*) = \mathfrak{S}_{\alpha/\beta}^*$ using the (left) perp operator. For $H \in NSym$, the operator $H^\perp$ acts on elements $F \in QSym$ based on the relation $\langle HG, F\rangle = \langle G, H^\perp F\rangle$. This expands as

$$H^\perp(F) = \sum_\alpha \langle HA_\alpha, F\rangle B_\alpha, \tag{3.4}$$

for dual bases $\{A_\alpha\}_\alpha$ of *NSym* and $\{B_\alpha\}_\alpha$ of *QSym*. For our second type of skew function, we need a variation on the operator of Equation (3.4).

Definition 3.2.1. For $H \in NSym$, the *right-perp operator* $H^{\pm}$ acts on elements $F \in QSym$ based on the relation $\langle GH, F\rangle = \langle G, H^{\pm}F\rangle$. This expands as

$$H^{\pm}(F) = \sum_{\alpha} \langle A_\alpha H, F\rangle B_\alpha,$$

for dual bases $\{A_\alpha\}_\alpha$ of $NSym$ and $\{B_\alpha\}_\alpha$ of $QSym$.

The usual perp operator $H^{\perp}$ of $QSym$ is dual to *left* multiplication by H in $NSym$, whereas the right-perp operator $H^{\pm}$ of $QSym$ is dual to *right* multiplication by H in $NSym$. Because Sym is commutative and self-dual, the perp and right-perp operators for Sym are equivalent.

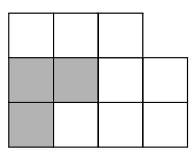
Remark 3.2.2. In [70] and [82], the left and right actions of $NSym$ on $QSym$ are used in place of the perp operators. In that framework, $\mathfrak{S}^{\pm}_{\beta}(\mathfrak{S}^{*}_{\alpha})$ is the left action of $NSym$ on $QSym$.

We define skew-II functions for the reverse and row-strict reverse dual immaculate functions using the right-perp operator. The skew dual immaculate functions relate to the skew-II reverse and row-strict reverse immaculate functions via the involutions ρ, and ω.

The shapes used to index our skew-II functions were introduced in [56], although we use a different name for continuity within this paper. Note that skew-II functions have also appeared before, but not in the perp operator framework.

Definition 3.2.3. For compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_j)$ such that $\beta^r \subseteq \alpha^r$, the *skew-II shape* $\alpha /\!\!/ \beta$ is the composition diagram of α where the left-most β_i boxes are removed from row α_{k+1-i} for $1 \le i \le j$. This removal is often represented by shading the boxes in.

For example, the following diagram is the skew-II shape $(3, 4, 4) /\!\!/ (2, 1)$.



We define one final concept for use in our proofs. Define the *flattening* of a tableau T, denoted $\tilde{T}$, as the tableau obtained by replacing all the smallest entries in T with 1's, all the next smallest entries in T with 2's, etc. We will also extend this definition of flattening to the other types of tableaux later in this thesis.

Example 3.2.4. The immaculate tableaux below flatten as follows.

$$T_1 = \begin{array}{|c|c|c} \hline 2 & 6 & \\ \hline 6 & 7 & \multicolumn{1}{c|}{7} \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c} \hline 1 & 5 & \\ \hline 3 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

$$\tilde{T}_1 = \begin{array}{|c|c|c} \hline 1 & 2 & \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \tilde{T}_2 = \begin{array}{|c|c|c} \hline 1 & 3 & \\ \hline 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

3.2.1 Reverse immaculate and dual immaculate functions

The reverse dual immaculate functions were introduced by Mason and Searles in [63]. They used these analogues to find a new relationship between the dual immaculate functions and Demazure atoms as well as extend prior results on positive expansions of the dual immaculate functions to the full polynomial ring. The reverse dual immaculate functions are also the stable limit of reverse dual immaculate slide polynomials.

Definition 3.2.5. Let α be a composition and β a weak composition. A *(semistandard) reverse immaculate tableau* (RIT) of shape α and type β is a diagram α filled with positive integers that weakly decrease along the rows from left to right and strictly increase along the first column from top to bottom, where each positive integer i appears β_i times.

A *standard* reverse immaculate tableau (SRIT) of shape $\alpha \models n$ is one containing the entries 1 through n each exactly once.

Example 3.2.6. A few reverse immaculate tableaux of shape $(2, 3)$ are:

$$\begin{array}{|c|c|c}\hline 1 & 1 & \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 3 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 3 & 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 3 & 2 & \\ \hline 4 & 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 3 & 2 & \\ \hline 4 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Definition 3.2.7. For a composition α, the *reverse dual immaculate function* is defined as

$$\mathfrak{S}^*_\alpha = \sum_T x^T,$$

where the sum runs over all reverse immaculate tableaux T of shape α.

The *descent set* of a standard reverse immaculate tableau S is defined as $Des_{\mathfrak{S}}(S) = \{i : i+1 \text{ is strictly below } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{S}}(S)$ is called a *descent* of S. The *descent composition* of S is defined as $co_{\mathfrak{S}}(S) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{S}}(S) = \{i_1, \ldots, i_d\}$. The *reverse immaculate reading word* of a reverse immaculate tableau T, denoted $rw_{\mathfrak{S}}(T)$, is the word obtained by reading the rows of T from right to left, starting with the bottom row and moving up. To *standardize* a reverse immaculate tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{S}}(T)$, then the 2's starting with the next consecutive number, etc.

Example 3.2.8. The tableaux in Example 3.2.6 above have the following standardizations and descent compositions:

$$\begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 4 & 1 & \\ \hline 5 & 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 4 & 3 & \\ \hline 5 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 4 & 3 & \\ \hline 5 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

$$(2,3) \qquad (2,3) \qquad (1,3,1) \qquad (4,1) \qquad (4,1)$$

Proposition 3.2.9. *[63] For a composition α,*

$$\reflectbox{$\mathfrak{S}$}^*_\alpha = \sum_S F_{co_{\reflectbox{$\mathfrak{S}$}}(S)},$$

where the sum runs over standard reverse immaculate tableaux S of shape α.

Example 3.2.10. The expansion of the reverse dual immaculate function $\reflectbox{$\mathfrak{S}$}^*_{(3,2)}$ into the fundamental basis and the standard reverse immaculate tableaux of shape $(3,2)$ are:

$$\reflectbox{$\mathfrak{S}$}^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)} + F_{(4,1)} + F_{(1,3,1)} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \\ \cline{1-2}\end{array}$$

Mason and Searles defined a map on standard tableaux, which they call the *flip-and-reverse* map, that relate the dual immaculate and reverse dual immaculate functions via tableaux.

Definition 3.2.11. Define the map $flip$ on standard tableaux so that $flip(S)$ is the tableau U obtained by flipping S horizontally (in other words, reversing the order of the rows of S) and then replacing each entry i with $n+1-i$.

$$flip\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\cline{1-2} 4 & 1 & \\ \hline 5 & 3 & 2 \\ \hline\end{array}$$

Observe that $flip$ is an involution from standard immaculate tableaux of shape α to standard reverse immaculate tableaux of shape α^r. It is shown using this bijection in [63] that

$$\mathfrak{S}^*_\alpha(x_1, \ldots, x_\ell) = \reflectbox{$\mathfrak{S}$}^*_{\alpha^r}(x_\ell, \ldots, x_1),$$

for any positive integer ℓ. Similarly, we use $flip$ to show that the reverse dual immaculate functions are the image of the immaculate functions under the involution ρ.

Theorem 3.2.12. *For a composition α,*

$$\rho(\mathfrak{S}^*_\alpha) = \reflectbox{$\mathfrak{S}$}^*_{\alpha^r} \quad \textit{and} \quad \omega(\mathfrak{R}\mathfrak{S}^*_\alpha) = \reflectbox{$\mathfrak{S}$}^*_{\alpha^r}.$$

Moreover, $\{\reflectbox{$\mathfrak{S}$}^*_\alpha\}_\alpha$ is a basis of QSym.

Proof. Let U be a standard immaculate tableau of shape α and S be a standard reverse immaculate tableau of shape α^r such that $flip(U) = S$. By construction, there will be a descent at i in U if and only if there is a descent at $|\alpha| - i$ in S, thus $(co_{\mathfrak{S}}(U))^r = co_{\reflectbox{$\mathfrak{S}$}}(S)$. Therefore,

$$\begin{aligned}\rho(\mathfrak{S}^*_\alpha) &= \sum_U \rho(F_{co_{\mathfrak{S}}(U)}) = \sum_U F_{co_{\mathfrak{S}}(U)^r} \\ &= \sum_{flip(U)} F_{co_{\reflectbox{$\mathfrak{S}$}}(flip(U))} = \sum_S F_{co_{\reflectbox{$\mathfrak{S}$}}(S)} = \reflectbox{$\mathfrak{S}$}^*_{\alpha^r},\end{aligned}$$

where the sums run over SIT U of shape α and SRIT S of shape α^r. The fact that the reverse dual immaculate functions are a basis follows from ρ being an automorphism. □

We proceed to introduce various new results on the reverse dual immaculate functions. Let $\mathfrak{K}^{\mathfrak{D}}_{\alpha,\beta}$ be the number of reverse immaculate tableaux of shape α and type β, and let $\mathfrak{L}^{\mathfrak{D}}_{\alpha,\beta}$ be the number of reverse immaculate tableaux with shape α and descent composition β. Using Proposition 3.2.9, it is straightforward to show that the reverse dual immaculate functions have the following positive expansions into the monomial and fundamental bases.

Proposition 3.2.13. *For a composition α,*

$$\mathfrak{D}^*_\alpha = \sum_\beta \mathfrak{K}^{\mathfrak{D}}_{\alpha,\beta} M_\beta \quad \textit{and} \quad \mathfrak{D}^*_\alpha = \sum_\beta \mathfrak{L}^{\mathfrak{D}}_{\alpha,\beta} F_\beta.$$

Remark 3.2.14. The reverse dual immaculate basis is not equivalent to the dual immaculate basis or the row-strict dual immaculate basis. The reverse dual immaculate function $\mathfrak{D}^*_{(1,2,1)}$ can be expanded as linear combinations of the dual immaculate and row-strict dual immaculate bases, which indicates that reverse immaculate functions exist that are not in the immaculate basis.

$$\begin{aligned}\rho(\mathfrak{S}^*_{(1,2,1)}) &= \mathfrak{D}^*_{(1,2,1)} = \mathfrak{S}^*_{(2,1,1)} - \mathfrak{S}^*_{(1,1,2)}\\ \omega(\mathfrak{R}\mathfrak{S}^*_{(1,2,1)}) &= \mathfrak{D}^*_{(1,2,1)} = \mathfrak{R}\mathfrak{S}^*_{(1,1,2)} - \mathfrak{R}\mathfrak{S}^*_{(1,2,1)} + \mathfrak{R}\mathfrak{S}^*_{(2,2)}.\end{aligned}$$

Next, we introduce the basis of *NSym* that is dual to the reverse dual immaculate basis.

Definition 3.2.15. Define the *reverse immaculate basis* of *NSym* as the unique basis $\{\mathfrak{D}_\alpha\}_\alpha$ that is dual to the reverse dual immaculate basis. That is, $\langle \mathfrak{D}_\alpha, \mathfrak{D}^*_\beta\rangle = \delta_{\alpha,\beta}$ for all compositions α, β.

Like in the dual case, the reverse immaculate functions are related to the immaculate functions via the involution ρ.

Proposition 3.2.16. *For a composition α, we have*

$$\mathfrak{D}_\alpha = \rho(\mathfrak{S}_{\alpha^r}) \quad \textit{and} \quad \mathfrak{D}_\alpha = \omega(\mathfrak{R}\mathfrak{S}_{\alpha^r}).$$

Proof. The involution ρ is invariant under duality because

$$\langle R_\alpha, F_\beta\rangle = \langle R_{\alpha^r}, F_{\beta^r}\rangle = \langle \rho(R_\alpha), \rho(F_\beta)\rangle.$$

Thus, our claim is implied by Theorem 3.2.12 and Definition 3.2.15. □

The expansions of the complete homogeneous functions and the monomial functions of *NSym* into the reverse immaculate basis follow via duality from Proposition 3.2.13. For a composition β,

$$H_\beta = \sum_\alpha \mathfrak{K}^{\mathfrak{D}}_{\alpha,\beta} \mathfrak{D}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathfrak{L}^{\mathfrak{D}}_{\alpha,\beta} \mathfrak{D}_\alpha. \tag{3.5}$$

We use ρ to construct an equivalent creation operator definition of the reverse immaculate basis.

Definition 3.2.17. For a positive integer m and $G \in NSym$, define the *noncommutative reverse Bernstein operator* on G by

$$\mathbb{B}^{\mathfrak{D}}_m(G) = \sum_{i \geq 0} (-1)^i F^{\perp}_{1^i}(G) H_{m+i}.$$

Proposition 3.2.18. *For a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$,*

$$\mathfrak{D}_\alpha = \mathbb{B}^{\mathfrak{D}}_{\alpha_k} \mathbb{B}^{\mathfrak{D}}_{\alpha_{k-1}} \cdots \mathbb{B}^{\mathfrak{D}}_{\alpha_1}(1).$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha^\natural = (\alpha_1, \ldots, \alpha_{k-1})$. It is sufficient to show that

$$\mathbb{B}^{\mathfrak{D}}_{\alpha_k}(\mathfrak{D}_{\alpha^\natural}) = \mathfrak{D}_\alpha.$$

First, we show that for $F \in QSym$ and $H \in NSym$, we have $\rho(F^\perp(H)) = [\rho(F)]^\perp(\rho(H))$. Let $\{A_\alpha\}_\alpha$ and $\{B_\alpha\}_\alpha$ be dual bases of $NSym$ and $QSym$ respectively. Then, using the invariance of ρ under duality and the fact that it is an anti-automorphism,

$$\begin{aligned}\rho(F^\perp(H)) &= \sum_\alpha \langle H, FB_\alpha \rangle \rho(A_\alpha) = \sum_\alpha \langle \rho(H), \rho(FB_\alpha) \rangle \rho(A_\alpha) \\ &= \sum_\alpha \langle \rho(H), \rho(F)\rho(B_\alpha) \rangle \rho(A_\alpha) = [\rho(F)]^\perp(\rho(H)) \quad \text{By Equation 3.1.}\end{aligned}$$

By Definition 3.1.19,

$$\mathfrak{S}_{\alpha^r} = \mathbb{B}_{\alpha_k}(\mathfrak{S}_{(\alpha_{k-1},\ldots,\alpha_1)}) = \sum_{i \geq 0} (-1)^i H_{\alpha_k + i} F^{\perp}_{1^i}(\mathfrak{S}_{(\alpha_{k-1},\ldots,\alpha_1)}).$$

Applying ρ yields

$$\begin{aligned}\rho(\mathfrak{S}_{\alpha^r}) = \mathfrak{D}_\alpha &= \sum_{i \geq 0} (-1)^i \rho[H_{\alpha_k+i} F^{\perp}_{1^i}(\mathfrak{S}_{(\alpha_{k-1},\ldots,\alpha_1)})] = \sum_{i \geq 0} (-1)^i \rho(F^{\perp}_{1^i}(\mathfrak{S}_{(\alpha_{k-1},\ldots,\alpha_1)}))\rho(H_{\alpha_k+i}) \\ &= \sum_{i \geq 0} (-1)^i F^{\perp}_{1^i}(\mathfrak{D}_{(\alpha_1,\ldots,\alpha_{k-1})}) H_{\alpha_k+i} = \mathbb{B}^{\mathfrak{D}}_{\alpha_k}(\mathfrak{D}_{(\alpha_1,\ldots,\alpha_{k-1})}).\end{aligned}$$

□

By applying ρ, we can translate many of the results on the immaculate functions to the reverse immaculate functions.

Theorem 3.2.19. *For compositions α, β, a partition λ, and a positive integer m,*

1. $\chi(\mathfrak{D}_{\lambda^r}) = s_\lambda$, *and*

$$s_\lambda = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^\sigma \mathfrak{D}^*_{(\lambda_{\sigma_k}+k-\sigma_k, \cdots, \lambda_{\sigma_2}+2-\sigma_2, \lambda_{\sigma_1}+1-\sigma_1)},$$

where the sum is over all permutations σ such that $\lambda_{\sigma_i} + i - \sigma_i > 0$ for all $i \in [k]$.

2. *(A left Pieri rule)*
$$H_s\mathfrak{D}_\alpha = \sum_{\alpha^r \subset_s^{\mathfrak{S}} \beta^r} \mathfrak{D}_\beta.$$

3. *(Another left Pieri rule)*
$$\mathfrak{D}_{1^s}\mathfrak{D}_\alpha = \sum_{\beta \models |\alpha|+s,\ \alpha_i \le \beta_i \le \alpha_i+1} \mathfrak{D}_\beta$$

4. *(Jacobi-Trudi Rule)*
$$\mathfrak{D}_\alpha = \sum_{\sigma \in S_{\ell(\alpha)}} (-1)^\sigma H_{(\alpha_1+\sigma_\ell-\ell,\ldots,\alpha_{\ell-1}+\sigma_2-2,\alpha_\ell+\sigma_1-1)}.$$

5. *(Partial Littlewood-Richardson Rule)*
$$\mathfrak{D}_{\lambda^r}\mathfrak{D}_\alpha = \sum_{\gamma=|\alpha|+|\lambda|} C^{\gamma^r}_{\alpha^r,\lambda}\mathfrak{D}_\gamma,$$
where $C^{\gamma^r}_{\alpha^r,\lambda}$ *counts skew immaculate Yamanouchi tableaux of shape* γ^r/α^r *and type* λ*.*

6. $$H_\beta = \sum_\alpha \mathfrak{K}_{\alpha^r,\beta^r}\mathfrak{D}_\alpha \quad \textit{and} \quad R_\beta = \sum_\alpha \mathfrak{L}_{\alpha^r,\beta^r}\mathfrak{D}_\alpha.$$

7. $$H_\beta = \sum_\alpha \mathfrak{K}^{\mathfrak{D}}_{\alpha^r,\beta^r}\mathfrak{S}_\alpha \quad \textit{and} \quad R_\beta = \sum_\alpha \mathfrak{L}^{\mathfrak{D}}_{\alpha^r,\beta^r}\mathfrak{S}_\alpha.$$

Proof.

1. Apply ρ to Theorem 3.1.14 and Corollary 3.1.27.
2. Apply ρ to Theorem 3.1.23.
3. Apply ρ to Equation (3.3).
4. Apply ρ to Theorem 3.1.26.
5. Apply ρ to Theorem 3.1.32
6. Apply ρ to Equation (3.2).
7. Apply ρ to Equation (3.5). □

Next, we define skew-II reverse immaculate functions algebraically and in terms of tableaux.

Definition 3.2.20. For compositions $\beta^r \subseteq \alpha^r$, a *skew-II reverse immaculate function* is defined by
$$\mathfrak{D}^*_{\alpha/\!\!/\beta} = \mathfrak{D}^{\perp}_\beta(\mathfrak{D}^*_\alpha).$$

By Equation 3.2.1, the function $\mathfrak{D}^*_{\alpha /\!\!/ \beta}$ expands into various bases as follows,

$$\mathfrak{D}^*_{\alpha /\!\!/ \beta} = \sum_{\gamma} \langle H_\gamma \mathfrak{D}_\beta, \mathfrak{D}^*_\alpha \rangle M_\gamma = \sum_{\gamma} \langle R_\gamma \mathfrak{D}_\beta, \mathfrak{D}^*_\alpha \rangle F_\gamma = \sum_{\gamma} \langle \mathfrak{D}_\gamma \mathfrak{D}_\beta, \mathfrak{D}^*_\alpha \rangle \mathfrak{D}^*_\gamma. \tag{3.6}$$

Proposition 3.2.21. *For a composition α and β where $\beta \subseteq \alpha$,*

$$\rho(\mathfrak{S}^*_{\alpha/\beta}) = \mathfrak{D}^*_{\alpha^r /\!\!/ \beta^r}.$$

Proof. First, observe that

$$\langle \rho(\mathfrak{S}_\beta \mathfrak{S}_\gamma), \rho(\mathfrak{S}^*_\alpha) \rangle = \langle \rho(\mathfrak{S}_\gamma)\rho(\mathfrak{S}_\beta), \rho(\mathfrak{S}^*_\alpha) \rangle = \langle \mathfrak{D}_{\gamma^r} \mathfrak{D}_{\beta^r}, \mathfrak{D}^*_{\alpha^r} \rangle,$$

because ρ is invariant under duality and an anti-isomorphism. Then by Equation (3.6),

$$\rho(\mathfrak{S}^*_{\alpha/\beta}) = \sum_{\gamma} \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}^*_\alpha \rangle \rho(\mathfrak{S}^*_\gamma) = \sum_{\gamma} \langle \mathfrak{D}_{\gamma^r} \mathfrak{D}_{\beta^r}, \mathfrak{D}^*_{\alpha^r} \rangle \mathfrak{D}^*_{\gamma^r} = \mathfrak{D}^*_{\alpha^r /\!\!/ \beta^r}. \quad \square$$

The first line of the proof above yields the following.

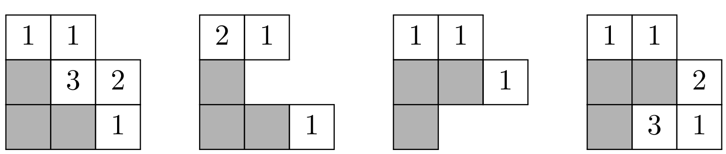
Corollary 3.2.22. *For compositions $\alpha, \beta,$ and γ,*

$$\mathfrak{D}_\gamma \mathfrak{D}_\beta = \sum_{\alpha} C^{\alpha^r}_{\beta^r, \gamma^r} \mathfrak{D}_\alpha.$$

Skew-II reverse immaculate functions are defined combinatorially by tableaux of skew-II shapes.

Definition 3.2.23. For compositions β and α such that $\beta^r \subseteq \alpha^r$, a *skew-II reverse immaculate tableau* of shape $\alpha /\!\!/ \beta$ is a skew-II diagram $\alpha /\!\!/ \beta$ filled with positive integers such that each row is weakly decreasing from left to right and the first column is strictly increasing from top to bottom. A skew-II reverse immaculate tableau is *standard* if it contains the numbers 1 through $|\alpha| - |\beta|$ each exactly once.

Example 3.2.24. The following are examples of skew-II reverse immaculate tableaux:



Proposition 3.2.25. *For compositions α and β with $\beta^r \subseteq \alpha^r$,*

$$\mathfrak{D}^*_{\alpha /\!\!/ \beta} = \sum_{T} x^T,$$

where the sum runs over skew-II reverse immaculate tableaux T of shape $\alpha /\!\!/ \beta$.

Proof. We can extend the bijection $flip$ to map from the set of skew RIT of shape α/β to the set of skew-II SRIT of shape $\alpha^r /\!\!/ \beta^r$. Given a skew immaculate tableau T, the skew-II reverse immaculate tableau $flip(T)$ is obtained by flipping T horizontally and replacing each entry i with $|\alpha| - |\beta| - i$. The descent composition of T will be the reverse of the descent composition of $flip(T)$ because there is a descent at $|\alpha| - |\beta| - i$ in $flip(T)$ whenever there is a descent at i in T. Thus,

$$\reflectbox{$\mathfrak{S}$}^*_{\alpha/\!\!/\beta} = \rho(\mathfrak{S}^*_{\alpha^r/\beta^r}) = \sum_S F_{co_{\mathfrak{S}}(S)^r} = \sum_S F_{co_{\reflectbox{$\mathfrak{S}$}}(flip(S))} = \sum_U F_{co_{\reflectbox{$\mathfrak{S}$}}(U)},$$

where the sums run over skew standard immaculate tableaux S of shape α^r/β^r and skew-II standard reverse immaculate tableaux U of shape $\alpha /\!\!/ \beta$. We can expand each $F_{co_{\reflectbox{$\mathfrak{S}$}}(U)}$ as follows. For any $\gamma \preceq co_{\reflectbox{$\mathfrak{S}$}}(U)$, there is exactly one skew-II reverse immaculate tableau of shape $\alpha /\!\!/ \beta$ of type γ that standardizes to U. Therefore,

$$F_{co_{\reflectbox{$\mathfrak{S}$}}(U)} = \sum_{\gamma \preceq co_{\reflectbox{$\mathfrak{S}$}}(U)} M_\gamma = \sum_T M_{type(T)} = \sum_Q \sum_{\tilde{Y}=Q} x^Y = \sum_T x^T,$$

where the sums run over flat skew-II reverse immaculate tableaux Q that standardize to U, skew-II reverse immaculate tableaux Y that flatten to Q, and skew-II reverse immaculate tableaux T of shape α/β with $std(T) = U$. The sum of fundamental functions expands into the desired result. □

3.2.2 Row-strict reverse immaculate and dual immaculate functions

Definition 3.2.26. Let α be a composition and β a weak composition. A *row-strict reverse immaculate tableau* (RSRIT) of shape α and type β is a diagram α filled with positive integers that strictly decrease along the rows from left to right and weakly increase along the first column from top to bottom, where each positive integer i appears β_i times. A *standard* row-strict reverse immaculate tableau (SRSRIT) of shape $\alpha \models n$ contains the entries 1 through n each exactly once.

Example 3.2.27. A few row-strict reverse immaculate tableaux of shape $(2, 3)$ are:

2	1	
3	2	1

3	1	
3	2	1

3	2	
3	2	1

3	2	
4	3	2

4	3	
5	2	1

Definition 3.2.28. For $\alpha \models n$, the *row-strict reverse dual immaculate function* is defined as

$$\mathfrak{R}\reflectbox{$\mathfrak{S}$}^*_\alpha = \sum_T x^T,$$

where the sum runs over all row-strict reverse immaculate tableaux T of shape α.

The *descent set* of a standard row-strict reverse immaculate tableau S is defined as $Des_{\mathfrak{R}\reflectbox{$\mathfrak{S}$}}(S) = \{i : i+1 \text{ is weakly above } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{R}\reflectbox{$\mathfrak{S}$}}(S)$ is called a *descent* of S. The *descent composition* of S is defined as $co_{\mathfrak{R}\reflectbox{$\mathfrak{S}$}}(S) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{R}\reflectbox{$\mathfrak{S}$}}(S) = \{i_1, \ldots, i_d\}$.

The *row-strict reverse immaculate reading word* of a row-strict reverse immaculate tableau T, denoted $rw_{\mathfrak{RS}}(T)$, is the word obtained by reading the rows of T from right to left, starting with the top row and moving down. To *standardize* a row-strict reverse immaculate tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{RS}}(T)$, then the 2's starting with the next consecutive number, etc.

Example 3.2.29. The standardizations of the tableaux in Example 3.2.27 and their descent compositions are:

$$\begin{array}{ccccc}
\begin{array}{|c|c|c}\hline 3 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array} &
\begin{array}{|c|c|c}\hline 4 & 1 & \\ \hline 5 & 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} &
\begin{array}{|c|c|c}\hline 4 & 2 & \\ \hline 5 & 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} &
\begin{array}{|c|c|c}\hline 3 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array} &
\begin{array}{|c|c|c}\hline 4 & 3 & \\ \hline 5 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \\
(2,2,1) & (2,1,2) & (1,2,2) & (2,2,1) & (1,1,1,2)
\end{array}$$

Proposition 3.2.30. *For a composition α,*

$$\mathfrak{RS}^*_\alpha = \sum_S F_{co_{\mathfrak{RS}}(S)},$$

where the sum runs over standard row-strict reverse immaculate tableaux S of shape α.

Proof. For any given standard row-strict reverse immaculate tableau S with descent composition γ, there is exactly one row-strict reverse immaculate tableaux T that standardizes to S and has a type β for any $\beta \preceq \alpha$. For such an S,T, and β, we can write $M_\beta = \sum_{\tilde{Y}=T,\ std(Y)=S} x^Y$ where the sum runs over reverse immaculate tableaux Y such that Y flattens to T and standardizes to S. Then,

$$\mathfrak{RS}^*_\alpha = \sum_S \sum_{\beta \preceq co_{\mathfrak{RS}}(S)} M_\beta = \sum_S F_{co_{\mathfrak{RS}}(S)},$$

where the sum runs over standard row-strict reverse immaculate tableaux S of shape α. □

Example 3.2.31. The F-expansion of the row-strict reverse dual immaculate function $\mathfrak{RS}^*_{(3,2)}$ and the standard row-strict reverse immaculate tableaux of shape $(3,2)$ are:

$$\mathfrak{RS}^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)} + F_{(1,1,1,2)} + F_{(2,1,2)}$$

$$\begin{array}{cccc}
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}
\end{array}$$

Let $\mathfrak{K}^{\mathfrak{RS}}_{\alpha,\beta}$ be the number of row-strict reverse immaculate tableaux of shape α and type β, and let $\mathfrak{L}^{\mathfrak{RS}}_{\alpha,\beta}$ be the number of row-strict reverse immaculate tableaux with shape α and descent composition β. Using Proposition 3.2.30, it is straightforward to show that the row-strict reverse dual immaculate functions have the following positive expansions into the monomial and fundamental bases.

Proposition 3.2.32. *For a composition α,*

$$\mathfrak{RS}^*_\alpha = \sum_\beta \mathfrak{K}^{\mathfrak{RS}}_{\alpha,\beta} M_\beta \quad \textit{and} \quad \mathfrak{RS}^*_\alpha = \sum_\beta \mathfrak{L}^{\mathfrak{RS}}_{\alpha,\beta} F_\beta.$$

Now we show that the row-strict reverse immaculate basis is the image of the immaculate, row-strict immaculate, and reverse immaculate bases under ω, ρ, and ψ respectively.

Theorem 3.2.33. *For a composition α,*

$$\omega(\mathfrak{S}^*_\alpha) = \mathfrak{RD}^*_{\alpha^r} \quad and \quad \rho(\mathfrak{RS}^*_\alpha) = \mathfrak{RD}^*_{\alpha^r} \quad and \quad \psi(\mathfrak{D}^*_\alpha) = \mathfrak{RD}^*_\alpha.$$

Moreover, $\{\mathfrak{RD}^*_\alpha\}_\alpha$ is a basis of $QSym$.

Proof. Given a standard reverse immaculate tableaux S (which is also a standard row-strict reverse tableaux) we have $F_{co_{\mathfrak{D}}(S)^c} = F_{co_{\mathfrak{RD}}(S)}$ due to the complementary definitions of descents for reverse and row-strict reverse tableaux. Then applying ψ to Proposition 3.2.9 yields

$$\psi(\mathfrak{D}^*_\alpha) = \sum_S \psi(F_{co_{\mathfrak{D}}(S)}) = \sum_S F_{co_{\mathfrak{D}}(S)^c} = \sum_S F_{co_{\mathfrak{RD}}(S)} = \mathfrak{RD}^*_\alpha,$$

where the sums run over standard reverse immaculate tableaux S of shape α or equivalently standard row-strict reverse immaculate tableaux S of shape α. The rest follows from the fact that $\psi \circ \rho = \omega$. □

Remark 3.2.34. The row-strict reverse dual immaculate basis is not equivalent to the dual immaculate basis, the row-strict dual immaculate basis, or the reverse dual immaculate basis. The row-strict reverse dual immaculate function $\mathfrak{RD}^*_{(1,2,1)}$ can be written as a linear combination of each of the other three bases in question. This indicates that there are row-strict reverse immaculate functions that are not elements of the immaculate basis.

$$\begin{aligned}
\omega(\mathfrak{S}^*_{(1,2,1)}) &= \mathfrak{RD}^*_{(1,2,1)} = \mathfrak{S}^*_{(1,1,2)} - \mathfrak{S}^*_{(1,2,1)} + \mathfrak{S}^*_{(2,2)} \\
\rho(\mathfrak{RS}^*_{(1,2,1)}) &= \mathfrak{RD}^*_{(1,2,1)} = \mathfrak{RS}^*_{(2,1,1)} - \mathfrak{RS}^*_{(1,1,2)} \\
\psi(\mathfrak{D}^*_{(1,2,1)}) &= \mathfrak{RD}^*_{(1,2,1)} = \mathfrak{D}^*_{(1,3)} - \mathfrak{D}^*_{(3,1)}.
\end{aligned}$$

Next, we define the basis of *NSym* that is dual to the row-strict reverse dual immaculate basis.

Definition 3.2.35. Define the *row-strict reverse immaculate basis* of *NSym* to be the unique basis $\{\mathfrak{RD}_\alpha\}_\alpha$ that is dual to the row-strict reverse dual immaculate basis. That is, $\langle \mathfrak{RD}_\alpha, \mathfrak{RD}^*_\beta \rangle$ for all compositions α, β.

Like in the dual case, the row-strict reverse immaculate functions are related to the immaculate functions via the involution ω.

Proposition 3.2.36. *For a composition α, we have*

$$\mathfrak{RD}_\alpha = \omega(\mathfrak{S}_{\alpha^r}) \quad and \quad \mathfrak{RD}_\alpha = \rho(\mathfrak{RS}_{\alpha^r}) \quad and \quad \mathfrak{RD}_\alpha = \psi(\mathfrak{D}_\alpha).$$

Proof. The involution ω is invariant under duality because

$$\langle R_\alpha, F_\beta \rangle = \langle R_{\alpha^t}, F_{\beta^t} \rangle = \langle \omega(R_\alpha), \omega(F_\beta) \rangle.$$

Thus, our claim follows from Theorem 3.2.33 and Definition 3.2.35. $\square$

The expansions of the complete homogeneous functions and the monomial functions of *NSym* into the row-strict reverse immaculate basis follow via duality from Proposition 3.2.32. For $\beta \models n$,

$$H_\beta = \sum_\alpha \mathfrak{K}^{\mathfrak{R}\mathfrak{D}}_{\alpha,\beta} \mathfrak{R}\mathfrak{D}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathfrak{L}^{\mathfrak{R}\mathfrak{D}}_{\alpha,\beta} \mathfrak{R}\mathfrak{D}_\alpha. \tag{3.7}$$

We can also use ω to construct an equivalent creation operator definition of the row-strict reverse immaculate functions.

Definition 3.2.37. For a positive integer m and $G \in NSym$, define the *noncommutative row-strict reverse Bernstein operator* on G by

$$\mathbb{B}^{\mathfrak{R}\mathfrak{D}}_m(G) = \sum_{i \geq 0} (-1)^i F^{\perp}_{(i)}(G) E_{m+i}.$$

Proposition 3.2.38. *For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$,*

$$\mathfrak{R}\mathfrak{D}_\alpha = \mathbb{B}^{\mathfrak{R}\mathfrak{D}}_{\alpha_k} \cdots \mathbb{B}^{\mathfrak{R}\mathfrak{D}}_{\alpha_2} \mathbb{B}^{\mathfrak{R}\mathfrak{D}}_{\alpha_1}(1).$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\alpha^\natural = (\alpha_1, \dots, \alpha_{k-1})$. It suffices to show that

$$\mathbb{B}^{\mathfrak{R}\mathfrak{D}}_{\alpha_k}(\mathfrak{R}\mathfrak{D}_{\alpha^\natural}) = \mathfrak{R}\mathfrak{D}_\alpha.$$

The effect of ψ on $F^\perp(H)$, which mirrors the effect of ρ, is given by $\psi(F^\perp(H)) = [\psi(F)]^\perp(\psi(H))$ from [69]. By Proposition 3.2.18, $\mathfrak{D}_\alpha = \mathbb{B}^{\mathfrak{D}}_{\alpha_k}(\mathfrak{D}_{\alpha^\natural})$. Applying ψ yields

$$\begin{aligned}\psi(\mathfrak{D}_\alpha) = \mathfrak{R}\mathfrak{D}_\alpha &= \sum_{i \geq 0} (-1)^i \psi[F^\perp_{1^i}(\mathfrak{D}_{\alpha^\natural}) H_{\alpha_k+i}] = \sum_{i \geq 0} (-1)^i \psi(F^\perp_{1^i}(\mathfrak{D}_{\alpha^\natural}))\psi(H_{\alpha_k+i}) \\ &= \sum_{i \geq 0} (-1)^i F^\perp_{(i)}(\mathfrak{R}\mathfrak{D}_{\alpha^\natural}) E_{\alpha_k+i} = \mathbb{B}^{\mathfrak{R}\mathfrak{D}}_{\alpha_k}(\mathfrak{R}\mathfrak{D}_{\alpha^\natural}).\end{aligned}$$ $\square$

By applying ω, we can translate many of the results on the immaculate functions to the row-strict reverse immaculate functions.

Theorem 3.2.39. *For compositions α, β, a partition λ, and a positive integer m,*

1. $\chi(\mathfrak{R}\mathfrak{D}_{\lambda^r}) = s_{\lambda'}$*, and*

$$s_{\lambda'} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^\sigma \mathfrak{R}\mathfrak{D}^*_{(\lambda_{\sigma_k}+k-\sigma_k, \cdots, \lambda_{\sigma_2}+2-\sigma_2, \lambda_{\sigma_1}+1-\sigma_1)},$$

where the sum is over all permutations σ such that $\lambda_{\sigma_i} + i - \sigma_i > 0$ for all $i \in [k]$.

2. *(A left Pieri rule)*
$$E_s \mathfrak{RD}_\alpha = \sum_{\alpha^r \subset_s^{\mathfrak{S}} \beta^r} \mathfrak{RD}_\beta.$$

3. *(Another left Pieri rule)*
$$\mathfrak{RD}_{1^s} \mathfrak{RD}_\alpha = \sum_{\beta \models |\alpha|+s,\ \alpha_i \leq \beta_i \leq \alpha_i + 1} \mathfrak{RD}_\beta$$

4. *(Jacobi-Trudi Rule)*
$$\mathfrak{RD}_\alpha = \sum_{\sigma \in S_{\ell(\alpha)}} (-1)^\sigma E_{(\alpha_1+\sigma_\ell-\ell,\dots,\alpha_{\ell-1}+\sigma_2-2,\alpha_\ell+\sigma_1-1)}.$$

5. *(Partial Littlewood-Richardson Rule)*
$$\mathfrak{RD}_{\lambda^r} \mathfrak{RD}_\alpha = \sum_{\gamma = |\alpha|+|\lambda|} C^{\gamma^r}_{\alpha^r,\lambda} \mathfrak{RD}_\gamma,$$

   where $C^{\gamma^r}_{\alpha^r,\lambda}$ counts skew immaculate Yamanouchi tableaux of shape γ^r/α^r and type λ.

6. $$E_\beta = \sum_\alpha \mathfrak{K}_{\alpha^r,\beta^r} \mathfrak{RD}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathfrak{L}_{\alpha^r,\beta^t} \mathfrak{RD}_\alpha.$$

7. $$E_\beta = \sum_\alpha \mathfrak{K}^{\mathfrak{RD}}_{\alpha^r,\beta^r} \mathfrak{S}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathfrak{L}^{\mathfrak{RD}}_{\alpha^r,\beta^t} \mathfrak{S}_\alpha.$$

Proof. 1. Apply ω to Theorem 3.1.14 and Corollary 3.1.27.

2. Apply ω to Theorem 3.1.23.

3. Apply ω to Equation (3.3).

4. Apply ω to Theorem 3.1.26.

5. Apply ω to Theorem 3.1.32

6. Apply ω to Equation (3.2).

7. Apply ω to Equation (3.7). □

Important parts of the Hopf algebra structures of $QSym$ and $NSym$ are the antipodes. We write the antipode of $NSym$ as $S : NSym \to NSym$ and the antipode of $QSym$ as $S^* : QSym \to QSym$.

Formulas for the antipodes of *QSym* and *NSym* on the fundamental and ribbon bases were given by Malvenuto and Reutenauer [57] and Benedetti and Sagan [8], respectively. For a composition α,

$$S^*(F_\alpha) = (-1)^{|\alpha|} F_{\alpha^t} \quad \text{and} \quad S(R_\alpha) = (-1)^{|\alpha|} R_{\alpha^t}. \tag{3.8}$$

Formulas for the antipodes of the immaculate basis have been studied in various papers [8, 21], but there is not yet a general formula. Theorem 3.2.33 and Proposition 3.2.36 allow us to express the antipode on these bases in terms of the row-strict reverse dual immaculate basis and the row-strict reverse immaculate basis. This result reduces the problem of studying the expansion of the row-strict reverse dual immaculate functions into the dual immaculate functions and vice versa, which may be interesting to approach using tableaux combinatorics.

Corollary 3.2.40. *For a composition α,*

$$S(\mathfrak{S}_\alpha) = (-1)^{|\alpha|} \mathfrak{RD}_{\alpha^r} \quad \textit{and} \quad S^*(\mathfrak{S}^*_\alpha) = (-1)^{|\alpha|} \mathfrak{RD}^*_{\alpha^r}.$$

Next, we define skew-II row-strict reverse immaculate functions algebraically and in terms of tableaux.

Definition 3.2.41. For compositions $\beta^r \subseteq \alpha^r$, the *skew-II row-strict reverse immaculate functions* are defined by

$$\mathfrak{RD}^*_{\alpha /\!\!/ \beta} = \mathfrak{RD}^{\perp}_\beta(\mathfrak{RD}^*_\alpha).$$

By Equation 3.2.1, $\mathfrak{RD}^*_{\alpha/\!\!/\beta}$ expands into various bases as follows. For compositions $\beta^r \subseteq \alpha^r$,

$$\mathfrak{RD}^*_{\alpha/\!\!/\beta} = \sum_\gamma \langle H_\gamma \mathfrak{RD}_\beta, \mathfrak{RD}^*_\alpha \rangle M_\gamma = \sum_\gamma \langle R_\gamma \mathfrak{RD}_\beta, \mathfrak{RD}^*_\alpha \rangle F_\gamma = \sum_\gamma \langle \mathfrak{RD}_\gamma \mathfrak{RD}_\beta, \mathfrak{RD}^*_\alpha \rangle \mathfrak{RD}^*_\gamma. \tag{3.9}$$

Proposition 3.2.42. *For a composition α and β with $\beta \subseteq \alpha$,*

$$\omega(\mathfrak{S}^*_{\alpha/\beta}) = \mathfrak{RD}^*_{\alpha^r /\!\!/ \beta^r} \quad \textit{and} \quad \psi(\mathfrak{D}^*_{\alpha^r/\!\!/\beta^r}) = \mathfrak{RD}^*_{\alpha^r/\!\!/\beta^r}.$$

Proof. First, observe that

$$\langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}^*_\alpha \rangle = \langle \omega(\mathfrak{S}_\beta \mathfrak{S}_\gamma), \omega(\mathfrak{S}^*_\alpha) \rangle = \langle \omega(\mathfrak{S}_\gamma)\omega(\mathfrak{S}_\beta), \omega(\mathfrak{S}^*_\alpha) \rangle = \langle \mathfrak{RD}_{\gamma^r} \mathfrak{RD}_{\beta^r}, \mathfrak{RD}^*_{\alpha^r} \rangle, \tag{3.10}$$

because ω is invariant under duality and an anti-isomorphism. Then by Equation (3.9),

$$\omega(\mathfrak{S}^*_{\alpha/\beta}) = \sum_\gamma \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}^*_\alpha \rangle \omega(\mathfrak{S}^*_\gamma) = \sum_\gamma \langle \mathfrak{RD}_{\gamma^r} \mathfrak{RD}_{\beta^r}, \mathfrak{RD}^*_{\alpha^r} \rangle \mathfrak{RD}^*_{\gamma^r} = \mathfrak{RD}^*_{\alpha^r/\!\!/\beta^r}.$$

Similarly, by the invariance of ψ we have $\langle \mathfrak{D}_{\gamma^r} \mathfrak{D}_{\beta^r}, \mathfrak{D}^*_{\alpha^r} \rangle = \langle \mathfrak{RD}_{\gamma^r} \mathfrak{RD}_{\beta^r}, \mathfrak{RD}^*_{\alpha^r} \rangle$ so

$$\psi(\mathfrak{D}^*_{\alpha^r/\!\!/\beta^r}) = \sum_\gamma \langle \mathfrak{D}_{\gamma^r} \mathfrak{D}_{\beta^r}, \mathfrak{D}^*_{\alpha^r} \rangle \psi(\mathfrak{S}^*_\gamma) = \sum_\gamma \langle \mathfrak{RD}_{\gamma^r} \mathfrak{RD}_{\beta^r}, \mathfrak{RD}^*_{\alpha^r} \rangle \mathfrak{RD}^*_{\gamma^r} = \mathfrak{RD}^*_{\alpha^r/\!\!/\beta^r}. \quad \square$$

The relationship between the coefficients in Equation (3.10) above yields the following.

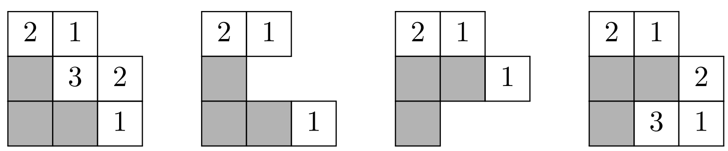
Corollary 3.2.43. *For compositions α, β, and γ,*

$$\mathfrak{RS}_{\gamma}\mathfrak{RS}_{\beta} = \sum_{\alpha} C^{\alpha^r}_{\beta^r,\gamma^r}\mathfrak{RS}_{\alpha}.$$

The skew-II row-strict reverse immaculate functions are expressed combinatorially by tableaux of skew-II shapes.

Definition 3.2.44. For compositions β and α such that $\beta^r \subseteq \alpha^r$, a *skew-II row-strict reverse immaculate tableau* of shape $\alpha /\!\!/ \beta$ is a skew-II diagram $\alpha /\!\!/ \beta$ filled with positive integers such that each row is strictly decreasing from left to right and the first column is weakly increasing from top to bottom. A skew-II row-strict reverse immaculate tableau is *standard* if it contains the numbers 1 through $|\alpha| - |\beta|$ each exactly once.

Example 3.2.45. The following are examples of skew-II row-strict reverse immaculate tableaux:



Proposition 3.2.46. *For compositions α and β where $\beta^r \subseteq \alpha^r$,*

$$\mathfrak{RS}^*_{\alpha /\!\!/ \beta} = \sum_{T} x^T,$$

where the sum runs over skew-II row-strict reverse immaculate tableaux T of shape $\alpha /\!\!/ \beta$.

Proof. In the proof of Proposition 3.2.25, we show that $\mathfrak{S}^*_{\alpha /\!\!/ \beta} = \sum_U F_{co_{\mathfrak{S}}(U)}$ where the sum runs over skew-II standard reverse immaculate tableaux U of shape $\alpha /\!\!/ \beta$. By Proposition 3.2.42,

$$\mathfrak{RS}^*_{\alpha /\!\!/ \beta} = \psi(\mathfrak{S}^*_{\alpha /\!\!/ \beta}) = \sum_{S} \psi(F_{co_{\mathfrak{S}}(S)}) = \sum_{S} F_{co_{\mathfrak{S}}(S)^c} = \sum_{S} F_{co_{\mathfrak{RS}}(S)}$$

where the sum runs over standard skew-II reverse immaculate tableau of shape $\alpha /\!\!/ \beta$, or equivalently standard skew-II row-strict reverse immaculate tableau of shape $\alpha /\!\!/ \beta$. Expanding this sum further as in the proof of proposition 3.2.25 ends the proof of our claim. □

3.3 Colored generalizations of the immaculate functions and dual immaculate functions

To generalize the dual immaculate functions to $QSym_A$, we first define a colored generalization of tableaux. These allow for a combinatorial definition of the colored dual immaculate functions, which then expand positively into the colored monomial and colored fundamental bases. Additionally, we define the colored immaculate descent graph and use it to give an expansion of the colored fundamental functions into the colored dual immaculate functions. Note that the remaining content of this chapter also appears in our paper [25].

In [63], Mason and Searles study a lift of the dual immaculate functions to the full polynomial ring. Our generalization of the dual immaculate functions is more aligned with the Hopf algebra-related aspects of the original functions whereas Mason and Searles' lift relates closely to slide polynomials, key polynomials, and Demazure atoms. The dual immaculate functions are the stable limit of their lifts while they are isomorphic to a special case of our lift.

3.3.1 The colored dual immaculate basis of $QSym_A$

Definition 3.3.1. For a sentence $J = (w_1, \ldots, w_k)$, the *colored composition diagram* of shape J is a composition diagram of $w\ell(J)$ where the j^{th} box in row i is colored with the j^{th} color in w_i.

Example 3.3.2. The colored composition diagram of shape $J = (aba, cb)$ for $a, b, c \in A$ is

a	b	a
c	b	

Definition 3.3.3. For a sentence I, a *colored immaculate tableau* (CIT) of shape I is a colored composition diagram of I filled with positive integers such that the integer entries in each row are weakly increasing from left to right and the entries in the first column are strictly increasing from top to bottom.

Definition 3.3.4. The *type* of a CIT T is a sentence $B = (u_1, \ldots, u_j)$ that indicates how many boxes of each color are filled with each integer and in what order those boxes appear. That is, each word u_i in B is defined by starting in the lowest box containing an i and reading the colors of all boxes containing i's going from left to right, bottom to top. If no box is filled with the number i, then $u_i = \emptyset$.

For a CIT T of type $B = (u_1, \ldots, u_j)$, the monomial x_T is defined $x_T = x_{u_1,1}x_{u_2,2}\cdots x_{u_j,j}$, which may also be denoted x_B.

Example 3.3.5. The CIT of shape $J = (aba, cb)$ and type $B = (a, c, \emptyset, b, ba)$ are

$a,1$	$b,5$	$a,5$
$c,2$	$b,4$	

$a,1$	$b,4$	$a,5$
$c,2$	$b,5$	

Both are associated with the monomial $x_{a,1}x_{c,2}x_{b,4}x_{ba,5}$ and have the flat type $\tilde{B} = (a, c, b, ba)$.

Definition 3.3.6. For a sentence J, the *colored dual immaculate function* is defined as

$$\mathfrak{S}^*_J = \sum_T x_T,$$

where the sum is taken over all colored immaculate tableaux T of shape J.

Example 3.3.7. For $J = (aba, cb)$, the colored dual immaculate function is

$$\mathfrak{S}^*_{(aba,cb)} = x_{aba,1}x_{cb,2} + x_{ab,1}x_{cba,2} + x_{aba,1}x_{c,2}x_{b,3} + \ldots + 2x_{a,1}x_{c,2}x_{b,3}x_{ba,4} + \ldots.$$

The colored dual immaculate functions map to the dual immaculate functions in $QSym$ under the uncoloring map υ, thus we say the two bases are analogous.

Proposition 3.3.8. *Let A be an alphabet of one color, $A = \{a\}$, and I be a sentence. Then,*

$$\upsilon(\mathfrak{S}^*_I) = \mathfrak{S}^*_{w\ell(I)}.$$

Moreover, $\{\mathfrak{S}^*_I\}_I$ in $QSym_A$ is analogous to $\{\mathfrak{S}^*_\alpha\}_\alpha$ in $QSym$.

Proof. If T be a colored immaculate tableau of shape I then let T' be the immaculate tableau of shape $w\ell(I)$ with the same integer entries as T. Observe that υ acts on a monomial x_T by mapping it to the monomial $x^{T'}$. Thus, $\upsilon(\mathfrak{S}^*_I) = \mathfrak{S}^*_{w\ell(I)}$ for all alphabets A and more specifically alphabets A containing only one color. $\square$

We now introduce further results on colored immaculate tableaux to provide a foundation for the expansions of the colored dual immaculate functions into other bases of $QSym_A$.

Definition 3.3.9. The *flattening* of a colored immaculate tableau T is the colored immaculate tableau $\tilde{T}$ obtained by replacing the smallest entries in T with 1's, the next smallest entries with 2's, and so on. T is called *flat* if $T = \tilde{T}$. The *flat type* of T is given by the type of $\tilde{T}$ or equivalently the flattening of *type*(T), denoted again by $\widetilde{type}(T)$.

Example 3.3.10. The colored immaculate tableaux T below has type $(\emptyset, \emptyset, a, \emptyset, bc, b)$, flat type (a, bc, b), and flattens to $\tilde{T}$.

$T =$

$a,3$	$b,5$
$c,5$	$b,6$

$\tilde{T} =$

$a,1$	$b,2$
$c,2$	$b,3$

Definition 3.3.11. A *standard colored immaculate tableau* (SCIT) of size n is a colored immaculate tableau in which the integers 1 through n each appear exactly once. The *standardization* of a CIT T, denoted *std*(T), is a standard colored immaculate tableau obtained by renumbering the boxes of T in the order they appear in its type.

Example 3.3.12. A few colored immaculate tableaux of shape $J = (ab, cb)$ together with their standardizations, which are the only standard colored immaculate tableaux of shape J, are:

$T_1 =$

$a,1$	$b,2$
$c,3$	$b,3$

$T_2 =$

$a,1$	$b,2$
$c,2$	$b,3$

$T_3 =$

$a,1$	$b,3$
$c,2$	$b,2$

$$std(T_1) = \begin{array}{|c|c|} \hline a,1 & b,2 \\ \hline c,3 & b,4 \\ \hline \end{array} \qquad std(T_2) = \begin{array}{|c|c|} \hline a,1 & b,3 \\ \hline c,2 & b,4 \\ \hline \end{array} \qquad std(T_3) = \begin{array}{|c|c|} \hline a,1 & b,4 \\ \hline c,2 & b,3 \\ \hline \end{array}$$

Standard colored immaculate tableaux share certain statistics and properties with non-colored standard immaculate tableaux. The number of SCIT of shape J is the same as the number of standard immaculate tableaux of shape $w\ell(J)$, meaning both are counted by the same hook length formula from [9]. Additionally, the notions of *descent* and *descent composition* for SCIT are the same as those in Definition 3.1.11, simply disregarding color. However, we define an additional concept of the colored descent composition.

Definition 3.3.13. Let T be a standard colored immaculate tableau of type B with descent set $Des_{\mathfrak{S}}(T) = \{i_1, \dots, i_k\}$ for some $k \in \mathbb{Z}_{>0}$. The *colored descent composition* of T, denoted $co_A^{\mathfrak{S}}(T)$, is the unique sentence obtained by splitting $w(B)$ after the i_j^{th} entry for each $j \in [k]$.

The colored descent composition can also be defined as the sentence obtained by reading through the colors of the tableau in the order that the boxes are numbered and splitting into a new word each time the next box is in a strictly lower row. Note that for a SCIT T of type B, the colored descent composition is the unique sentence for which $w\ell(co_A^{\mathfrak{S}}(T)) = co_{\mathfrak{S}}(T)$ and $w(co_A^{\mathfrak{S}}(T)) = w(B)$.

Example 3.3.14. The standard colored immaculate tableaux of shape (ab, cb), along with their descent sets and colored descent compositions, are:

$$T_1 = \begin{array}{|c|c|} \hline a,1 & b,2 \\ \hline c,3 & b,4 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|} \hline a,1 & b,3 \\ \hline c,2 & b,4 \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|c|} \hline a,1 & b,4 \\ \hline c,2 & b,3 \\ \hline \end{array}$$

$Des_{\mathfrak{S}}(T_1)=\{2\}$, $co_A^{\mathfrak{S}}(T_1)=(ab,cb)$ $\qquad$ $Des_{\mathfrak{S}}(T_1)=\{1,3\}$, $co_A^{\mathfrak{S}}(T_2)=(a,cb,b)$ $\qquad$ $Des_{\mathfrak{S}}(T_3)=\{1\}$, $co_A^{\mathfrak{S}}(T_3)=(a,cbb)$

Proposition 3.3.15. *Let T_1 and T_2 be colored immaculate tableaux of shape J and type B. Then, $T_1 = T_2$ if and only if $std(T_1) = std(T_2)$.*

Proof. It is trivial that $T_1 = T_2$ implies $std(T_1) = std(T_2)$. Now, let $std(T_1) = std(T_2) = U$, meaning by definition that the boxes of T_1 appear in B in the same order as the boxes of T_2. Each box (i, j) in row i and column j in both tableaux is filled with the same integer k and with the k^{th} color in $w(B)$, thus $T_1 = T_2$. □

Proposition 3.3.16. *Let U be a standard colored immaculate tableau of shape J. For a weak sentence B, there exists a colored immaculate tableau T of shape J and type B that standardizes to U if and only if $\tilde{B} \preceq co_A^{\mathfrak{S}}(U)$.*

Proof. ($\Rightarrow$) Let T be a colored immaculate tableau of shape J and type B such that $std(T) = U$. Both B and $co_A^{\mathfrak{S}}(U)$ are defined by the order that boxes appear in the type of T, thus they have the same maximum words $w(B) = w(co_A^{\mathfrak{S}}(U))$. Note that this also means the i^{th} letter in $\tilde{B}$ and the i^{th}

letter in $co_A^{\mathfrak{S}}(U)$ correspond to the same box in J. Recall that $co_A^{\mathfrak{S}}(U)$ splits only after descents, and suppose that $co_A^{\mathfrak{S}}(U)$ splits after the i^{th} letter. Then the $(i+1)^{\text{th}}$ letter is on a strictly lower row. Given that these entries correspond exactly to the i^{th} and $(i+1)^{\text{th}}$ letter in $\tilde{B}$, this tells us that $\tilde{B}$ must also split since the following entry is on a lower row. Thus $\tilde{B}$ also splits after every descent which implies that $\tilde{B} \preceq co_A^{\mathfrak{S}}(U)$.

($\Leftarrow$) Let $\tilde{B} = (v_1, \ldots, v_j) \preceq co_A^{\mathfrak{S}}(U)$ and let v_i be the n_i^{th} word in B. We create a colored immaculate tableau T of shape J and type B that standardizes to U by filling the boxes of T in the order they are numbered in U. The first $|v_1|$ boxes are labeled with n_1's, the next $|v_2|$ boxes are labeled with n_2's, and continue this process until the last $|v_j|$ boxes are labeled with n_j's. Since $\tilde{B} \preceq co_A^{\mathfrak{S}}(U)$, each time there is a descent in U the number being filled in must increase. This maintains the order of the boxes in the type from U, meaning T standardizes to U. This filling also maintains the strictly increasing condition on the first column and the weakly increasing condition on each row by construction. Therefore, T is a colored immaculate tableau of shape J and type B with $std(T) = U$. □

The colored dual immaculate functions have positive expansions into the colored monomial and colored fundamental bases. Their coefficients are determined combinatorially using colored immaculate tableaux. To see this, we establish the relationship between the colored monomial quasisymmetric functions and colored immaculate tableaux. Then, we define coefficients counting colored immaculate tableaux and prove our expansion. Finally, the transition matrix of these coefficients leads to a proof that the colored dual immaculate functions are indeed a basis.

Proposition 3.3.17. *For a sentence B, consider a standard colored immaculate tableau U where $B \preceq co_A^{\mathfrak{S}}(U)$. Then,*

$$M_B = \sum_T x_T,$$

where the sum runs over all colored immaculate tableaux T such that $std(T) = U$ and $\widetilde{type(T)} = B$. Equivalently, given any colored immaculate tableau T such that $T = \tilde{T}$, the colored monomial function expands as $M_{type(T)} = \sum_{\tilde{Y}=T} x_T$.

Proof. Consider a standard colored immaculate tableau U and a sentence $B = (v_1, \ldots, v_h)$ such that $B \preceq co_A^{\mathfrak{S}}(U)$. By definition,

$$M_B = \sum_{1 \leq j_1 < \ldots < j_h} x_{v_1, j_1} \ldots x_{v_h, j_h}.$$

Each monomial $x_{v_1,j_1} \ldots x_{v_h,j_h}$ is equal to x_T where T is the unique (by Proposition 3.3.15) colored immaculate tableau such that $std(T) = U$ and its type $C = (u_1, \ldots, u_g)$ is the sentence where word u_{j_i} is equal to v_i for $1 \leq i \leq h$ and all other words are empty. This includes a tableau T for every sentence C such that $\tilde{C} = B$. Thus, the above sum is equivalent to summing x_T over all CIT T with type C such that $std(T) = U$ and $\tilde{C} = B$. □

Example 3.3.18. The CIT of shape $J = (ab, cb)$ and type $B = (a, cb, b)$ are

$$T_1 = \begin{array}{|c|c|}\hline a,1 & b,3 \\ \hline c,2 & b,2 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|}\hline a,1 & b,2 \\ \hline c,2 & b,3 \\ \hline \end{array}$$

The tableaux T_1 and T_2 have the same shape and type, but different standardizations (see Example 3.3.14). Now, consider all tableaux with types that flatten to B and standardizations equal to $std(T_1)$:

$$\begin{array}{|c|c|}\hline a,1 & b,3 \\ \hline c,2 & b,2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,4 \\ \hline c,2 & b,2 \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|c|}\hline a,1 & b,4 \\ \hline c,3 & b,3 \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|c|}\hline a,2 & b,5 \\ \hline c,4 & b,4 \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|c|}\hline a,5 & b,9 \\ \hline c,7 & b,7 \\ \hline \end{array}$$

A single monomial function $M_{(a,cb,b)}$ can be associated with this set of tableaux. The tableaux of flat type B that standardize to T_2 are also represented by a function $M_{(a,cb,b)}$. Thus, when finding the overall monomial expansion for $\mathfrak{S}^*_{(ab,cb)}$, the tableaux of flat type (a, cb, b) contribute to the sum as the term $2M_{(a,cb,b)}$.

Definition 3.3.19. For a sentence J and weak sentence B, define the *colored immaculate Kostka coefficient* $\mathfrak{K}_{J,B}$ as the number of colored immaculate tableaux of shape J and type B.

Proposition 3.3.20. *Let $J = (w_1, \ldots, w_k)$ and $C = (u_1, \ldots, u_g)$ be sentences. Then, $\mathfrak{K}_{J,C} = \mathfrak{K}_{J,\tilde{C}}$.*

Proof. Suppose $\tilde{C} = (v_1, \ldots, v_h)$ where $u_{i_1} = v_1, \ldots, u_{i_h} = v_h$ for some $i_1 < \ldots < i_h$, with all other $u_j = \emptyset$. We define a map from the colored immaculate tableaux of shape J and type $\tilde{C}$ to the colored immaculate tableaux of shape J and type C. Given a colored immaculate tableau T of shape J and type $\tilde{C}$, replace each 1 with i_1, 2 with i_2, …, and h with i_h. This produces a tableau T' of shape J and type C. The inverse map takes a tableau T' of shape J and type C and changes each i_1 to 1, i_2 to 2, …, and i_h to h, which yields the initial tableau T of shape J and type $\tilde{C}$. This is a bijection, meaning $\mathfrak{K}_{J,C} = \mathfrak{K}_{J,\tilde{C}}$. □

Example 3.3.21. Let $J = (ab, cb)$ and $B = (\emptyset, a, \emptyset, cb, b)$. Then, $\mathfrak{K}_{J,B} = 2$ because the colored immaculate tableaux of shape J and type B are:

$$\begin{array}{|c|c|}\hline 2,a & 5,b \\ \hline 4,c & 4,b \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2,a & 4,b \\ \hline 4,c & 5,b \\ \hline \end{array}$$

Notice that $\mathfrak{K}_{J,\tilde{B}} = 2$ as well, since the colored immaculate tableaux of shape J and type $\tilde{B}$ are:

$$\begin{array}{|c|c|}\hline 1,a & 3,b \\ \hline 2,c & 2,b \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1,a & 2,b \\ \hline 2,c & 3,b \\ \hline \end{array}$$

Theorem 3.3.22. *For a sentence J, the colored dual immaculate function $\mathfrak{S}^*_J$ expands positively into the colored monomial basis as*

$$\mathfrak{S}^*_J = \sum_B \mathfrak{K}_{J,B} M_B,$$

where the sum is taken over all sentences B such that $|B| = |J|$.

Proof. Let $B_1, \ldots, B_j$ be all possible flat types of colored immaculate tableaux of shape J. Then arrange the sum $\mathfrak{S}^*_J = \sum_T x_T$ into parts based on the flat types of the tableaux T as

$$\mathfrak{S}^*_J = \sum_{\widetilde{type(T)}=B_1} x_T + \ldots + \sum_{\widetilde{type(T)}=B_j} x_T.$$

Consider the sum of x_T over T such that $\widetilde{type(T)} = B_i$. By Proposition 3.3.20, for any C such that $\tilde{C} = B$ we have $\mathfrak{K}_{J,B_i} = \mathfrak{K}_{J,C}$. By definition, for any flat sentence B,

$$M_B = \sum_{\tilde{C}=B} x_C.$$

Thus, we can write

$$\sum_{\widetilde{type(T)}=B_i} x_T = \sum_{\tilde{C}=B_i} \mathfrak{K}_{J,C} x_C = \mathfrak{K}_{J,B_i} \left(\sum_{\tilde{C}=B_i} x_C \right) = \mathfrak{K}_{J,B_i} M_{B_i}.$$

Therefore the overall sum becomes

$$\mathfrak{S}^*_J = \mathfrak{K}_{J,B_1} M_{B_1} + \ldots + \mathfrak{K}_{J,B_j} M_{B_j} = \sum_B \mathfrak{K}_{J,B} M_B,$$

where the sum runs over all flat types B of the colored immaculate tableaux of shape J. For all other B such that $|B| = |J|$, we have $\mathfrak{K}_{J,B} = 0$ and we can extend this sum to be over all sentences B such that $|B| = |J|$. □

Theorem 3.3.23. *The set of colored dual immaculate functions forms a basis for $QSym_A$.*

Proof. Let A be an alphabet with a total ordering, and consider the transition matrix $\mathcal{K}$ from $\{\mathfrak{S}^*_I\}_I$ to $\{M_I\}_I$. By Theorem 3.3.22, the entry of $\mathcal{K}$ in row J and column C is $\mathfrak{K}_{J,C}$. We want to prove that $\mathcal{K}$ is upper unitriangular and thus invertible when the rows and columns are ordered first by the reverse lexicographic order of compositions applied to word lengths, then by lexicographic order on words. For example, row $(a_1a_2a_3, a_4a_5)$ would come before row (a_1a_2, a_3, a_4a_5) because $(3,2) \preceq_{r\ell} (2,1,2)$, and, given $a_1 < a_2 < \ldots < a_5$, row $(a_1a_2a_2, a_1a_3)$ would come before row $(a_1a_2a_3, a_4a_5)$ because $(3,2) = (3,2)$ and $(a_1a_2a_2) \preceq_\ell (a_1a_2a_3)$,

Let $J = (w_1, \ldots, w_k)$ and $C = (v_1, \ldots, v_h)$ be sentences with $|J| = |C|$. We claim that if $wl(J) \succeq_{rl} wl(C)$ and $\mathfrak{K}_{J,C} \neq 0$ then $J = C$. Assume there exists a tableau T of shape J and type

C with $wl(J) \succeq_{rl} wl(C)$ and $wl(J) \neq wl(C)$. Then $|w_1| \leq |v_1|$. Observe that the first row of the tableau has $|w_1|$ boxes and so if $|w_1| < |v_1|$, there would have to be a 1 placed in a box somewhere below row 1. This is impossible by the conditions on colored immaculate tableaux so $|w_1| = |v_1|$ and every box in row 1 is filled with 1's. Next, $|w_2| \leq |v_2|$ and so the second row must start with a 2 for any 2's to exist in T. This implies that the first entry in each subsequent row is greater than 2 meaning that no other row can contain 2's. If every 2 is in the second row then and the number of 2's is at least w_2, then $|w_2| = |v_2|$. Continuing this reasoning, $|w_i| = |v_i|$ for $1 \leq i \leq k$. Thus, $wl(J) = wl(C)$. Further, by this method we have filled the first row with 1's, the second row with 2's, the i^{th} row with i's, etc. to construct a colored immaculate tableau such that $w_i = v_i$ for all i. Therefore, $J = C$. By construction, this is the only tableau of shape J and type J so $\mathfrak{K}_{J,J} = 1$. To summarize, we have shown that $\mathfrak{K}_{J,C} = 0$ when $wl(J) \succeq_l wl(C)$ unless $J = C$, in which case the entry of the matrix lies on the diagonal and $\mathfrak{K}_{J,J} = 1$. Thus, we have proved $\mathcal{K}$ is upper unitriangular. □

To expand the colored dual immaculate functions into the colored fundamental basis we first define coefficients counting SCIT. Relating these to our earlier coefficients counting colored immaculate tableaux, we reformulate our expansion in Theorem 3.3.22 to an expansion in terms of the colored fundamental basis.

Definition 3.3.24. For sentences J and C, define $\mathfrak{L}_{J,C}$ as the number of standard colored immaculate tableaux of shape J that have colored descent composition C.

Example 3.3.25. Let $J = (ab, cb, b)$ and $C = (a, cb, bb)$. The standard colored immaculate tableaux of shape J with colored descent composition C are

$U_1 =$

$a,1$	$b,3$
$c,2$	$b,5$
$b,4$	

$U_2 =$

$a,1$	$b,5$
$c,2$	$b,3$
$b,4$	

Thus, $\mathfrak{L}_{(ab,cb,b),(a,cb,bb)} = 2$.

Proposition 3.3.26. *For sentences J and B,*

$$\mathfrak{K}_{J,B} = \sum_{C \succeq B} \mathfrak{L}_{J,C}.$$

Proof. Recall that $\mathfrak{K}_{J,B}$ is the number of colored immaculate tableaux of shape J and type B and $\mathfrak{L}_{J,C}$ is the number of standard colored immaculate tableaux of shape J and descent composition C. For this proof, let $\mathcal{T}$ be the set of all colored immaculate tableaux of shape J and type B, and let $\mathcal{U}$ be the set of standard colored immaculate tableaux U of shape J and descent composition C with $C \succeq B$. We need to show that the map $std : \mathcal{T} \to \mathcal{U}$, where std is the standardization map from Definition 3.3.11, is a bijection on these sets. By Proposition 3.3.15, colored immaculate tableaux with the same shape and type must have different standardizations or they would be the same tableau, thus our map is injective. By Proposition 3.3.16, the map is surjective. Thus, *std* is a bijection and we have shown that $\mathfrak{K}_{J,B} = \sum_{C \succeq B} \mathfrak{L}_{J,C}$. □

Theorem 3.3.27. *For a sentence J, the colored dual immaculate function $\mathfrak{S}^*_J$ expands positively into the fundamental basis as*

$$\mathfrak{S}^*_J = \sum_C \mathfrak{L}_{J,C} F_C,$$

where the sum runs over sentences C such that $|C| = |J|$.

Proof. Let J be a sentence. Observe that applying the Möbius inversion to Proposition 3.3.26 yields

$$\mathfrak{L}_{J,C} = \sum_{C \preceq B} (-1)^{\ell(C)-\ell(B)} \mathfrak{K}_{J,B}.$$

Then, by Theorem 3.3.22 and Equation (2.14),

$$\begin{aligned} \mathfrak{S}^*_J = \sum_B \mathfrak{K}_{J,B} M_B &= \sum_B \mathfrak{K}_{J,B} \left(\sum_{C \preceq B} (-1)^{\ell(C)-\ell(B)} F_C \right) \\ &= \sum_C \left(\sum_{C \preceq B} (-1)^{\ell(C)-\ell(B)} \mathfrak{K}_{J,B} \right) F_C = \sum_C \mathfrak{L}_{J,C} F_C. \end{aligned}$$ □

This expansion can be written as a sum over all standard colored immaculate tableaux of a certain shape instead of using coefficients to count tableaux based on their colored descent compositions.

Corollary 3.3.28. *For a sentence J,*

$$\mathfrak{S}^*_J = \sum_U F_{co^{\mathfrak{S}}_A(U)},$$

where the sum runs over all standard colored immaculate tableaux U of shape J.

Next, we define the colored immaculate descent graph to directly determine the expansion of the colored fundamental functions into the colored dual immaculate basis. Additionally, our result specializes to a new combinatorial expansion of the fundamental quasisymmetric functions into the dual immaculate functions.

Definition 3.3.29. Define the *colored immaculate descent graph*, denoted $\mathfrak{D}^n_A$, as an edge-weighted directed simple graph such that the vertex set is the set of sentences in A of size n, and there is a directed edge from I to J if there exists a standard colored immaculate tableau of shape I with colored descent composition J. The edge from I to J is weighted with the coefficient $\mathfrak{L}_{I,J}$ from Definition 3.3.24. For a path $\mathcal{P}$ in $\mathfrak{D}^n_A$, let *prod*$(\mathcal{P})$ denote the product of the edge-weights in $\mathcal{P}$ and let *prod*$(\emptyset) = 1$. We define the length of a path $\mathcal{P}$ as its number of vertices, $\ell(\mathcal{P})$.

Example 3.3.30. In Figure 1 we illustrate the subgraph of $\mathfrak{D}^5_{\{a,b,c\}}$ with top vertex (ab, cbb). In this subgraph, all edges are weighted 1 because $\mathfrak{L}_{I,J} = 1$ for each I and J (and thus *prod*$(\mathcal{P}) = 1$ for all paths) but, for example, the edge from (ab, cb, b) to (a, cb, bb) would be 2 since $\mathfrak{L}_{(ab,cb,b)(a,cb,bb)} = 2$ as in Example 3.3.25.

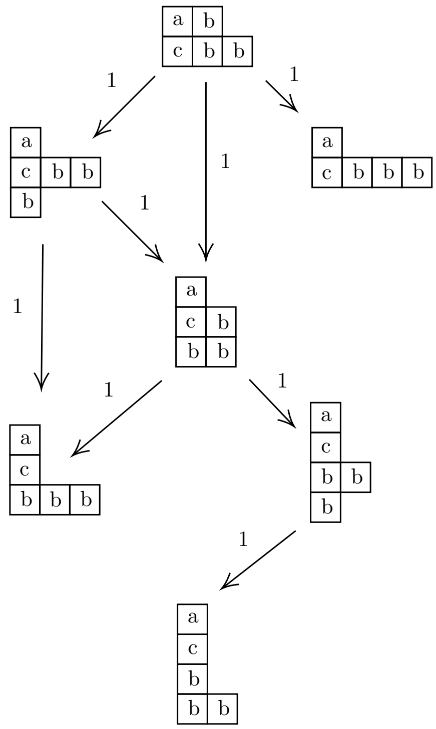


Figure 3.2: A subgraph of $\mathfrak{D}^5_{\{a,b,c\}}$.

The element $(ab, cbb) \in \mathfrak{D}^5_{\{a,b,c\}}$ has edges going down to elements (a, cbb, b), (a, cb, bb), and $(a, cbbb)$ because these sentences represent possible descent compositions (with the exception of (ab, cbb) itself) of colored standard immaculate tableaux of shape (ab, cbb) as shown below.

$a,1$	$b,4$	
$c,2$	$b,3$	$b,5$

(a, cbb, b)

$a,1$	$b,3$	
$c,2$	$b,4$	$b,5$

(a, cb, bb)

$a,1$	$b,5$	
$c,2$	$b,3$	$b,4$

$(a, cbbb)$

$a,1$	$b,2$	
$c,3$	$b,4$	$b,5$

(ab, cbb)

We say a sentence K is *reachable* from a sentence I if there is a directed path from I to K. This includes the empty path, meaning that I is reachable from itself.

Theorem 3.3.31. *For a sentence I of size n, the colored fundamental functions expand into the colored dual immaculate basis as*

$$F_I = \sum_K \mathfrak{L}^{-1}_{I,K} \mathfrak{S}^*_K \qquad \textit{with coefficients} \qquad \mathfrak{L}^{-1}_{I,K} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums run over all sentences K reachable from I in $\mathfrak{D}^n_A$ and directed paths $\mathcal{P}$ from I to K in $\mathfrak{D}^n_A$, respectively.

Proof. We proceed by induction on the length of the longest path starting at I in $\mathfrak{D}^n_A$, denoted here with k. If $k = 1$, there are no elements reachable from I so $F_I = \mathfrak{S}^*_I$ which agrees with Theorem 3.3.27. Now for some positive integer k, assume the statement is true for any path of length $\leq k$. Consider a sentence I where the length of the longest path starting at I is $k+1$. By Theorem 3.3.27,

$$F_I = \mathfrak{S}^*_I - \sum_J \mathfrak{L}_{I,J} F_J,$$

where the sum runs over all sentences $J \neq I$ such that $|J| = |I|$. We only need to consider, however, sentences J that are descent compositions of a SCIT of shape I because otherwise $\mathfrak{L}_{I,J} = 0$. Since there is an edge from I to each of these J's, the length of the longest path starting at any J is at most k. Thus, by induction,

$$F_I = \mathfrak{S}^*_I - \sum_J \mathfrak{L}_{I,J} \sum_K \mathfrak{L}^{-1}_{J,K} \mathfrak{S}^*_K,$$

for all sentences K reachable from J and $\mathfrak{L}^{-1}_{J,K} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} \mathfrak{L}_{K_1,K_2} \cdots \mathfrak{L}_{K_{j-1},K_j}$ for paths $\mathcal{P} = \{K = K_j \leftarrow K_{j-1} \leftarrow \ldots \leftarrow K_1 = J\}$ from K to J. Note that

$$-\sum_J \mathfrak{L}_{I,J} \sum_K \mathfrak{L}^{-1}_{J,K} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} \mathfrak{L}_{I,J} \mathfrak{L}_{K_1,K_2} \mathfrak{L}_{K_2,K_3} \cdots \mathfrak{L}_{K_{j-1},K_j} = \mathfrak{L}^{-1}_{I,K},$$

where the sum runs over all paths $\mathcal{P} = \{K = K_j \leftarrow \ldots \leftarrow K_1 = J \leftarrow I\}$ from K to I. Then,

$$F_I = \sum_K \mathfrak{L}^{-1}_{I,K} \mathfrak{S}^*_K,$$

summing over all sentences K reachable from I. □

Example 3.3.32. The subgraph in Figure 1 yields the following expansion of $F_{(ab,cbb)}$:

$$F_{(ab,cbb)} = \mathfrak{S}^*_{(ab,cbb)} - \mathfrak{S}^*_{(a,cbb,b)} + \mathfrak{S}^*_{(a,c,bbb)} - \mathfrak{S}^*_{(a,cbbb)}.$$

Similarly, the (non-colored) immaculate descent graph $\mathfrak{D}^n$ can be defined as the graph with a vertex set of compositions of size n where there is an edge from α to β if there exists a standard immaculate tableau of shape α with descent composition β. The edge from α to β will be weighted with coefficient $\mathfrak{L}_{\alpha,\beta}$. This leads to an analogous result that follows from the proof above.

Corollary 3.3.33. *For a composition $\alpha \models n$, the fundamental quasisymmetric functions expand into the dual immaculate functions as*

$$F_\alpha = \sum_\beta \mathfrak{L}^{-1}_{\alpha,\beta} \mathfrak{S}^*_\beta \qquad \textit{with coefficients} \qquad \mathfrak{L}^{-1}_{\alpha,\beta} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums runs over all β reachable from α in $\mathfrak{D}^n$ and over paths $\mathcal{P}$ going from α to β in $\mathfrak{D}^n$.

3.3.2 The colored immaculate functions in $\boldsymbol{NSym}_A$

A colored generalization of the immaculate basis can be defined by first introducing a colored version of noncommutative Bernstein creation operators. Various properties of these operators and extensions of our earlier results via duality lead to results on the colored immaculate functions. These notably include a right Pieri rule and an expansion of the colored immaculate functions into the colored ribbon functions. As a corollary, we provide a new combinatorial model for the expansion of an immaculate function into the ribbon basis. It remains an open problem to find a cancellation-free expansion of the immaculate functions into the ribbon functions, but our formula does provide a straightforward and explicit way to compute the entries in the transition matrix between the immaculate and ribbon bases. Applying the forgetful map to our expression also yields a new expansion of Schur functions into the ribbon Schur functions.

The process for constructing our generalization of the noncommutative Bernstein operators mirrors that done in [9] with some adjustments to account for the use of sentences in place of compositions.

Definition 3.3.34. For $M \in QSym_A$, define the action of the linear *perp operator* $M^{\perp}$ on $H \in NSym_A$ as $\langle M^{\perp}H, G\rangle = \langle H, MG\rangle$ for all $G \in QSym_A$. We define the action of the linear *right-perp operator* $M^{\pm}$ on $H \in NSym_A$ as $\langle M^{\pm}H, G\rangle = \langle H, GM\rangle$ for all $G \in QSym_A$. Thus, for dual bases $\{A_I\}_I$ of $QSym_A$ and $\{B_I\}_I$ of $NSym_A$, we have

$$M^{\perp}(H) = \sum_I \langle H, MA_I\rangle B_I \quad \text{and} \quad M^{\pm}(H) = \sum_I \langle H, A_I M\rangle B_I.$$

These operators are dual to the left and right multiplication by M in $QSym_A$. Note that the analogues to these operators in $QSym$ are equivalent due to commutativity.

Proposition 3.3.35. *For sentences $I = (w_1, \dots, w_k)$ and $J = (v_1, \dots, v_h)$,*

$$M_I^{\pm}(H_J) = \sum_K H_{\widetilde{J/_R K}},$$

where the sum runs over all sentences K such that $\tilde{K} = I$ and $K \subseteq_R J$. Moreover, each $\widetilde{J/_R K}$ appearing in this sum is equivalent to the shape of a colored composition diagram originally of shape J with boxes corresponding to each word in I uniquely removed from its righthand side such that each word w_j is removed from a single row strictly lower than the row from which w_{j+1} is removed.

Proof. Let $I = (w_1, \dots, w_k)$ and $J = (v_1, \dots, v_h)$. We have that

$$M_I^{\pm}(H_J) = \sum_L \langle H_J, M_L M_I\rangle H_L = \sum_L \langle H_J, \sum_R M_R\rangle H_L = \sum_L \sum_R \langle H_J, M_R\rangle H_L,$$

where the sums run over all sentences L of size $|J|-|I|$ and each summand R in $L \overset{Q}{\sqcup\!\sqcup} I$, respectively. Note that each sentence R may occur multiple times in $L \overset{Q}{\sqcup\!\sqcup} I$ and we account for the multiplicity in the summations. The sum $\sum_R \langle H_J, M_R\rangle$ is equal to the number of times that J appears as a summand in $L \overset{Q}{\sqcup\!\sqcup} I$. Recall that in $L \overset{Q}{\sqcup\!\sqcup} I$, each summand is a sentence made up of words from L, words from I, and concatenated pairs of words from L and I (in that order) where all words from L and all words from I are present and in the same relative order, respectively. For each time J is a summand in $L \overset{Q}{\sqcup\!\sqcup} I$ there exists a unique weak sentence $K^\natural$ such that $\tilde{K^\natural} = I$ and $\widetilde{J/_R K^\natural} = L$. Further, the set of all $K^\natural$ obtained for J in $L \overset{Q}{\sqcup\!\sqcup} I$ considered across every possible L is simply the set of weak sentences K such that $\tilde{K} = I$ and $K \subseteq_R J$, and so we can rewrite

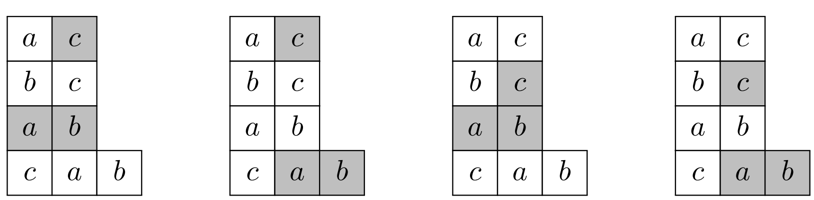
$$M_I^{\perp}(H_J) = \sum_L \sum_{K^\natural} H_L = \sum_K H_{\widetilde{J/_R K}},$$

where the sums run over all sentences L of size $|J|-|I|$, all weak sentences $K^\natural$ such that $\tilde{K^\natural} = I$ and $\widetilde{J/_R K^\natural} = L$, and all weak sentences K such that $\tilde{K} = I$ and $K \subseteq_R J$, respectively.

Visualizing sentences as colored composition diagrams, we see that each weak sentence K can be viewed as a unique set of boxes being removed from the right-hand side of the colored composition diagram of J where the first word in K (including empty words) is removed from the first row of J and so on. Thus, the set of indices $\widetilde{J/_R K}$ of H in the sum can also be viewed as the set of colored composition diagrams resulting from all possible ways of removing boxes corresponding to I from a colored composition diagram of shape J then moving rows up to fill empty rows, where each w_j in I is removed from a single row strictly lower than the single row from which w_{j+1} in I is removed. $\square$

Example 3.3.36. In this example we show the action of $M_{c,ab}^{\perp}$ on colored diagrams:

$$M_{(c,ab)}^{\perp}(H_{(ac,bc,ab,cab)}) = H_{(a,bc,cab)} + H_{(a,bc,ab,c)} + H_{(ac,b,cab)} + H_{(ac,b,ab,c)}.$$



Next, we prove various properties of the $M^{\perp}$ operator that will be key in constructing creation operators for the colored immaculate basis.

Lemma 3.3.37. *Let J, K be sentences, $A_I \in QSym_A$ and $G, H \in NSym_A$. Then,*

$$\langle G \otimes H, \Delta(A_I)(M_J \otimes M_K)\rangle = \langle M_J^{\perp}(G) \otimes M_K^{\perp}(H), \Delta(A_I)\rangle.$$

Proof. Let $a, b \in NSym_A$ and $c, d \in QSym_A$. The inner product on $NSym_A \times QSym_A$ extends to

$NSym_A \otimes NSym_A \times QSym_A \otimes QSym_A$ as

$$\langle \cdot, \cdot \rangle : NSym_A \otimes NSym_A \times QSym_A \otimes QSym_A \to \mathbb{Q} \quad \text{where} \quad \langle a \otimes b, c \otimes d \rangle \to \langle a, c \rangle \langle b, d \rangle$$

In Sweedler notation, $\Delta(A_I) = \sum_i A^{(i)} \otimes A_{(i)}$. Thus, we write

$$\begin{aligned}
\langle G \otimes H, \Delta(A_I)(M_J \otimes M_K) \rangle &= \left\langle G \otimes H, \sum_i A^{(i)} M_J \otimes A_{(i)} M_K \right\rangle = \sum_i \langle G \otimes H, A^{(i)} M_J \otimes A_{(i)} M_K \rangle \\
&= \sum_i \langle G, A^{(i)} M_J \rangle \langle H, A_{(i)} M_K \rangle \\
&= \sum_i \langle M_J^{\pm}(G), A^{(i)} \rangle \langle M_K^{\pm}(H), A_{(i)} \rangle \quad \text{by Definition 3.3.34} \\
&= \sum_i \langle M_J^{\pm}(G) \otimes M_K^{\pm}(H), A^{(i)} \otimes A_{(i)} \rangle \\
&= \left\langle M_J^{\pm}(G) \otimes M_K^{\pm}(H), \sum_i A^{(i)} \otimes A_{(i)} \right\rangle \\
&= \langle M_J^{\pm}(G) \otimes M_K^{\pm}(H), \Delta(A_I) \rangle.
\end{aligned}$$

□

Proposition 3.3.38. *For a sentence $Q = (q_1, \ldots, q_i)$ and $G, H \in NSym_A$,*

$$M_Q^{\pm}(GH) = \sum_{0 \leq j \leq i} M^{\pm}_{(q_1, \ldots, q_j)}(G) M^{\pm}_{(q_{j+1}, \ldots, q_i)}(H).$$

In particular, for a word w,

$$M_Q^{\pm}(GH_w) = M_Q^{\pm}(G) H_w + M^{\pm}_{(q_1, \ldots, q_{i-1})}(G) M^{\pm}_{q_i}(H_w).$$

Proof. Let $\{A_I\}_I$ and $\{B_I\}_I$ be dual bases of $QSym_A$ and $NSym_A$ respectively, and let $Q =$

$(q_1, \dots, q_i)$. Then,

$$\begin{aligned}
M_Q^{\pm}(GH) &= \sum_I \langle GH, A_I M_Q\rangle B_I \quad \text{by Definition 3.3.34}\\
&= \sum_I \langle G\otimes H, \Delta(A_I M_Q)\rangle B_I = \sum_I \langle G\otimes H, \Delta(A_I)\Delta(M_Q)\rangle B_I \quad \text{by Definition 2.2.6}\\
&= \sum_I \sum_{Q=J\cdot K} \langle G\otimes H, \Delta(A_I)(M_J\otimes M_K)\rangle B_I \quad \text{by Equation (2.13)}\\
&= \sum_I \sum_{Q=J\cdot K} \langle M_J^{\pm}(G)\otimes M_K^{\pm}(H), \Delta(A_I)\rangle B_I \quad \text{by Lemma 3.3.37}\\
&= \sum_I \sum_{Q=J\cdot K} \langle M_J^{\pm}(G) M_K^{\pm}(H), A_I\rangle B_I \quad \text{by Definition 2.2.6}\\
&= \sum_I \langle \sum_{Q=J\cdot K} M_J^{\pm}(G) M_K^{\pm}(H), A_I\rangle B_I = \sum_{Q=J\cdot K} M_J^{\pm}(G) M_K^{\pm}(H) \quad \text{by Definition 3.3.34}\\
&= \sum_{j=0}^{i} M^{\pm}_{(q_1,\dots,q_j)}(G) M^{\pm}_{(q_{j+1},\dots,q_i)}(H).
\end{aligned}$$

In the case of $H = H_w$, the term $M^{\pm}_{(q_{j+1},\dots,q_i)}(H_w)$ is 0 whenever $i-(j+1) > 0$ because boxes corresponding to $(q_j, \dots, q_i)$ must each be removed from separate rows but w has only one row. Thus, the equation simplifies as

$$M_Q^{\pm}(GH_w) = M_Q^{\pm}(G)H_w + M^{\pm}_{(q_1,\dots,q_{i-1})}(G) M^{\pm}_{q_i}(H_w). \qquad \square$$

Definition 3.3.39. For a word v, the *colored noncommutative Bernstein operator* $\mathbb{B}_v$ is defined as

$$\mathbb{B}_v = \sum_u \sum_{w(Q^r)=u} (-1)^i H_{v\cdot u} \left(\sum_{Q\preceq S} M_S^{\pm} \right),$$

where the sums run over all words u, all sentences $Q = (q_1, \dots, q_i)$ such that $q_i \cdot \ldots \cdot q_1 = u$, and all sentences S that are coarsenings of Q.

Notice that, by the definition of $M^{\pm}$, the only values of u that could yield a nonzero summand in $\mathbb{B}_v(H_I)$ for a sentence I are those for which there is some permutation of the letters in u that yields a subword of $w(I)$. Thus, this sum always has a finite number of terms.

Definition 3.3.40. For a sentence $J = (v_1, \dots, v_h)$, define the *colored immaculate function* $\mathfrak{S}_J$ as

$$\mathfrak{S}_J = \mathbb{B}_{v_1}\mathbb{B}_{v_2}\ldots\mathbb{B}_{v_h}(1).$$

Example 3.3.41. The colored immaculate functions $\mathfrak{S}_{(def)}$ and $\mathfrak{S}_{(abc,def)}$ are obtained using creation operators as follows:

$$\mathfrak{S}_{(def)} = \mathbb{B}_{def}(1) = \sum_u \sum_{w(Q^r)=u} (-1)^i H_{(def\cdot u)} \left(\sum_{Q\preceq S} M_S^{\pm}(1) \right) = (-1)^0 H_{(def)} M_\emptyset^{\pm}(1) = H_{(def)}.$$

$$\mathfrak{S}_{(abc,def)} = \mathbb{B}_{abc}(\mathfrak{S}_{(def)}) = \mathbb{B}_{abc}(H_{(def)}) = \sum_u \sum_{w(Q^r)=u} (-1)^i H_{(abc\cdot u)} \left(\sum_{Q\preceq S} M_S^{\pm}(H_{(def)}) \right)$$

$$= (-1)^0 H_{(abc)} M_\emptyset^{\pm}(H_{(def)}) + (-1)^1 H_{(abcf)} M_{(f)}^{\pm}(H_{(def)}) + (-1)^1 H_{(abcef)} M_{(ef)}^{\pm}(H_{(def)})$$
$$+ (-1)^2 H_{(abcfe)} M_{(ef)}^{\pm}(H_{(def)}) + (-1)^1 H_{(abcdef)} M_{(def)}^{\pm}(H_{(def)}) + (-1)^2 H_{(abcefd)} M_{(def)}^{\pm}(H_{(def)})$$
$$+ (-1)^2 H_{(abcfde)} M_{(def)}^{\pm}(H_{(def)}) + (-1)^3 H_{(abcfed)} M_{(def)}^{\pm}(H_{(def)})$$

$$= H_{(abc,def)} - H_{(abcf,de)} - H_{(abcef,d)} + H_{(abcfe,d)} - H_{(abcdef)} + H_{(abcefd)} + H_{(abcfde)} - H_{(abcfed)}.$$

To get the term $H_{(abcfe,d)}$, for example, we look at $u = fe$. The possible values of Q for this u are $Q = (fe)$ and $Q = (e, f)$, meaning the possible S values are $S = (fe)$, $S = (e, f)$, and $S = (ef)$. Observe that $M_{(fe)}^{\pm}(H_{(def)})$ and $M_{(e,f)}^{\pm}(H_{(def)})$ are both zero because S is not right-contained in def. Thus, the only remaining term for these values is $S = (ef)$ for which $M_{(ef)}^{\pm}(H_{(def)}) = H_{(d)}$. Thus the term of the sum given by $u = fe$, $Q = (e, f)$, and $S = ef$ is $(-1)^2 H_{(abcfe,d)}$, which is also the only term for $u = fe$. Many values of u will yield entirely zero terms.

Before proving that this basis is indeed analogous to the immaculate functions in *NSym*, we must prove that it is dual to the colored dual immaculate basis. The following property of the colored noncommutative Bernstein operators leads to a right Pieri rule, which illuminates the structure of the colored immaculate functions to this end.

Proposition 3.3.42. *Let $w = a_1 \dots a_k$ and $f, H \in NSym_A$, then*

$$\mathbb{B}_v(f)H_w = \sum_{0\le j\le k} \mathbb{B}_{(v\cdot(a_{j+1}\cdots a_k))}(fH_{(a_1\cdots a_j)}).$$

Proof. Given a sentence $Q = (q_1, \dots, q_i)$, we write $Q^\natural = (q_1, \dots, q_{i-1})$. Let $f \in NSym_A$ and let v and $w = a_1 \dots a_k$ be words. Then,

$$\mathbb{B}_v(fH_w) = \sum_u \sum_{w(Q^r)=u} (-1)^i H_{v\cdot u} \left(\sum_{Q\preceq S} M_S^{\pm}(fH_w) \right) \text{ by Definition 3.3.39}$$
$$= \sum_u \sum_{w(Q^r)=u} (-1)^i H_{v\cdot u} \left(\sum_{Q\preceq S} \left[M_S^{\pm}(f)H_w + M_{S^\natural}^{\pm}(f) M_{s_t}^{\pm}(H_w) \right] \right) \text{ by Proposition 3.3.38}$$
$$= \mathbb{B}_v(f)H_w + \sum_u \sum_{w(Q^r)=u} (-1)^i H_{v\cdot u} \left(\sum_{Q\preceq S} M_{S^\natural}^{\pm}(f) M_{s_t}^{\pm}(H_w) \right) \text{ by factoring.}$$

We want to consider the cases in which $M^{\pm}_{s_t}(H_w)$ is non-zero. This only happens whenever $s_t \subseteq_R w$ because in our combinatorial interpretation, we visualize $M^{\pm}_{s_t}$ as removing s_t from the righthand side of $w = a_1 \cdots a_k$ to get $H_{(a_1 \ldots a_h)}$ for some $h \leq k$. Note that because $Q \preceq S$ and q_i and s_t are the final words in Q and S respectively, $q_i \subseteq_R s_t$. It follows that $q_i \subseteq_R w$ and thus $q_i = a_{j+1} \cdots a_k$ for a non-negative integer $j < k$. Recalling that $u = q_i \cdots q_1$, let $u^\natural = q_{i-1} \cdots q_1$ so that we can write $u = a_{j+1} \cdots a_k \cdot u^\natural$. Rewriting the last equation in terms of $u^\natural$ and $Q^\natural$ yields

$$\mathbb{B}_v(fH_w) = \mathbb{B}_v(f)H_w + \sum_{0 \leq j < k} \sum_{u^\natural} \sum_{Q^\natural} (-1)^i H_{(v \cdot (a_{j+1} \cdots a_k) \cdot u^\natural)} \sum_{(Q^\natural \cdot (a_{j+1} \cdots a_k)) \preceq S} M^{\pm}_{S^\natural}(f) M^{\pm}_{s_t}(H_{(a_1 \cdots a_j)}).$$

Next, the sum can be split into two parts by separating out the cases where $q_i = s_t$ and those where $q_i \neq s_t$. If $q_i = s_t$ for $q_i = a_{j+1} \cdots a_k$ then $M^{\pm}_{s_t}(H_w) = M^{\pm}_{(a_{j+1} \cdots a_k)}(H_w) = H_{(a_1 \cdots a_j)}$. Otherwise, there must exist a non-negative integer $\iota < i - 1$ such that $s_t = q_{\iota+1} \cdots q_{i-1} q_i$. We can rearrange the part of the sum by substituting s_t with $q_{\iota+1} \cdots q_i$ and summing over the possible ι. Then,

$$\begin{aligned}\mathbb{B}_v(fH_w) = \mathbb{B}_v(f)H_w - \sum_{0 \leq j < k} \sum_{u^\natural} \sum_{Q^\natural} \Bigg(& (-1)^{i-1} H_{(v(a_{j+1} \cdots a_k)u^\natural)} \left[\sum_{Q^\natural \preceq S^\natural} M^{\pm}_{S^\natural}(f) H_{(a_1 \cdots a_j)} \right] \\ & + \left[\sum_{0 \leq \iota < i-1} \sum_{(q_1, \ldots, q_\iota) \preceq S^\natural} M^{\pm}_{S^\natural}(f) M^{\pm}_{(q_{\iota+1} \cdot \; \cdots \; \cdot q_{i-1} \cdot (a_{j+1} \cdots a_k))}(H_w) \right] \Bigg).\end{aligned}$$

Using the visualization of $M^{\pm}$, observe that $M^{\pm}_{(q_{\iota+1} \cdots q_{i-1}(a_{j+1} \cdots a_k))}(H_w) = M^{\pm}_{(q_{\iota+1} \cdots q_{i-1})}(H_{(a_1 \cdots a_j)})$ so

$$\begin{aligned}\mathbb{B}_v(fH_w) = \mathbb{B}_v(f)H_w - \sum_{0 \leq j < k} \sum_{u^\natural} \sum_{Q^\natural} \Bigg(& (-1)^{i-1} H_{(v(a_{j+1} \ldots a_k)u^\natural)} \left[\sum_{Q^\natural \preceq S^\natural} M^{\pm}_{S^\natural}(f) H_{(a_1 \ldots a_j)} \right] \\ & + \left[\sum_{0 \leq \iota < i-1} \sum_{(q_1, \ldots, q_\iota) \preceq S^\natural} M^{\pm}_{S^\natural}(f) M^{\pm}_{(q_{\iota+1} \cdots q_{i-1})}(H_{(a_1 \cdots a_j)}) \right] \Bigg).\end{aligned}$$

Next, rename every $S^\natural$ to $R = (r_1, \ldots, r_\tau)$ in the first section of the sum. In the second section, rename $S^\natural$ to $R^\natural = (r_1, \ldots, r_{\tau-1})$ and let $q_{\iota+1} \cdots q_{i-1} = r_\tau$.

$$\begin{aligned}\mathbb{B}_v(fH_w) = \mathbb{B}_v(f)H_w - \sum_{0 \leq j < k} \sum_{u^\natural} \sum_{Q^\natural} \Bigg(& (-1)^{i-1} H_{(v(a_{j+1} \ldots a_k)u^\natural)} \left[\sum_{Q^\natural \preceq R} M^{\pm}_{R}(f) H_{(a_1 \ldots a_j)} \right] \\ & + \left[\sum_{0 \leq \iota < i-1} \sum_{(q_1, \ldots, q_\iota) \preceq R^\natural} M^{\pm}_{R^\natural}(f) M^{\pm}_{r_\tau}(H_{(a_1 \cdots a_j)}) \right] \Bigg).\end{aligned}$$

In the second part of the sum, notice that considering $R^\natural \cdot r_\tau$ where $R^\natural = (q_1, \ldots, q_\iota)$ and $r_\tau = q_{\iota+1} \cdots q_{i-1}$ for $1 \leq \iota \leq i-1$ is equivalent to considering $R^\natural \cdot r_\tau = R \succeq (q_1, \ldots, q_{i-1}) = Q^\natural$. Then,

$$\mathbb{B}_v(fH_w) = \mathbb{B}_v(f)H_w - \sum_{0\leq j<k} \sum_{u^\natural} \sum_{Q^\natural} \left((-1)^{i-1} H_{(v(a_{j+1}\ldots a_k)u^\natural)} \left[\sum_{Q^\natural \preceq R} M_R^\pm(f) H_{(a_1\ldots a_j)} \right] \right.$$
$$\left. + \left[\sum_{Q^\natural \preceq R} M_{R^\natural}^\pm(f) M_{r_\tau}^\pm(H_{(a_1\cdots a_j)}) \right] \right).$$

Now in both parts of the sum, we are looking at sentences R such that $Q^\natural \preceq R$, and combining them we get

$$\begin{aligned}
\mathbb{B}_v(fH_w) &= \mathbb{B}_v(f)H_w - \sum_{0\leq j<k} \sum_{u^\natural} \sum_{Q^\natural} (-1)^{i-1} H_{(v(a_{j+1}\ldots a_k)u^\natural)} \left(\sum_{Q^\natural \preceq R} \left[M_R^\pm(f) H_{(a_1\cdots a_j)} \right.\right. \\
&\quad \left.\left. + M_{R^\natural}^\pm(f) M_{r_\tau}^\pm(H_{(a_1\cdots a_j)}) \right] \right) \\
&= \mathbb{B}_v(f)H_w - \sum_{0\leq j<k} \sum_{u^\natural} \sum_{Q^\natural} (-1)^{i-1} H_{(v(a_{j+1}\ldots a_k)u^\natural)} \sum_{Q^\natural \preceq R} M_R^\pm(fH_{(a_1\ldots a_j)}) \text{ by Prop 3.3.38} \\
&= \mathbb{B}_v(f)H_w - \sum_{0\leq j<k} \mathbb{B}_{(v\cdot(a_{j+1}\cdots a_k))}(fH_{(a_1\cdots a_j)}) \text{ by Definition 3.3.39.}
\end{aligned}$$

□

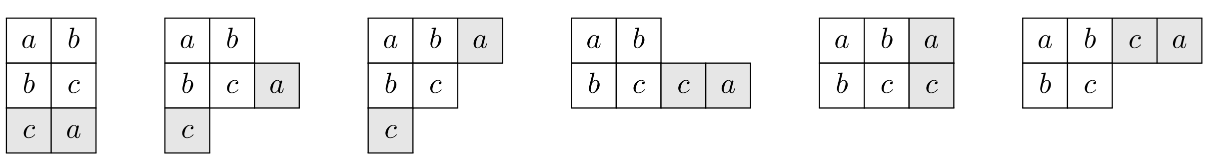
Theorem 3.3.43 (Right Pieri Rule)**.** *For the sentence $J = (v_1, \ldots, v_h)$ and the word $w = a_1 \ldots a_i$,*

$$\mathfrak{S}_J H_w = \sum_{J \subset_w K} \mathfrak{S}_K,$$

where $J \subset_w K = (u_1, \ldots, u_g)$ if $u_j = v_j \cdot q_j$ for $1 \leq j \leq g$ such that $q_g \cdot q_{g-1} \cdot \ldots \cdot q_2 \cdot q_1 = w$ and $g \leq h+1$ where $v_{h+1} = \emptyset$.

For a sentence I and word w, the product $\mathfrak{S}_I H_w$ given by the Pieri rule can be visualized in terms of the indices of colored immaculate functions in the resulting sum. The indices correspond to all diagrams obtained by adding colored boxes below or to the right of the diagram of I, such that when reading the colors of boxes left to right from bottom to top they correspond exactly to w.

Example 3.3.44. The product below can be visualized using the following tableaux:



$$\mathfrak{S}_{(ab,bc)} H_{(ca)} = \mathfrak{S}_{(ab,bc,ca)} + \mathfrak{S}_{(ab,bca,c)} + \mathfrak{S}_{(aba,bc,c)} + \mathfrak{S}_{(ab,bcca)} + \mathfrak{S}_{(aba,bcc)} + \mathfrak{S}_{(abca,bc)}.$$

Proof of Theorem 3.3.43. We proceed by induction on $|w| + \ell(J)$. There are two base cases where $|w| + \ell(J) = 1$.

1. If $|w| = 1$ and $\ell(J) = 0$, then $\mathfrak{S}_\emptyset H_w = \sum_{1 \subset_w K} \mathfrak{S}_K = \mathfrak{S}_w = H_w$.
2. If $\ell(J) = 1$ and $|w| = 0$ then $\mathfrak{S}_J H_\emptyset = \sum_{J \subset_\emptyset K} \mathfrak{S}_K = \mathfrak{S}_J$.

Next, assume the statement is true when $|w|+\ell(J) \leq k$ and let $\bar{J} = (v_2, \ldots, v_h)$. Let $|w|+\ell(J) = k+1$.

$$\mathfrak{S}_J H_w = \mathbb{B}_{v_1}(\mathfrak{S}_{\bar{J}}) H_w = \sum_{0 \leq j < i} \mathbb{B}_{(v_1 \cdot a_{j+1} \ldots a_i)} \left(\mathfrak{S}_{\bar{J}} H_{a_1 \ldots a_j} \right) \text{ by Def 3.3.40 and Prop 3.3.42,}$$
$$= \sum_{0 \leq j < i} \mathbb{B}_{)v_1 \cdot a_{j+1}) \ldots a_i} \left(\sum_{\bar{J} \subset_{a_1 \ldots a_j} G} \mathfrak{S}_G \right) \qquad \text{by induction,}$$
$$= \sum_{0 \leq j < i} \left(\sum_{\bar{J} \subset_{a_1 \ldots a_j} G} \mathfrak{S}_{(v_1 a_{j+1} \ldots a_i) \cdot G} \right) = \sum_{J \subset_w K} \mathfrak{S}_K. \qquad \square$$

The expansion of the colored complete homogeneous functions into the colored immaculate functions follows from repeated application of the right Pieri rule.

Theorem 3.3.45. *For a sentence C, the colored complete homogeneous function expands positively into the colored immaculate basis as*

$$H_C = \sum_J \mathfrak{K}_{J,C} \mathfrak{S}_J,$$

where the sum runs over sentences J such that $|J| = |C|$.

Proof. Let $C = (t_1, \ldots, t_k)$ and $C^\natural = (t_1, \ldots, t_{k-1})$. First we claim that $\mathfrak{K}_{J,C} = \sum_{G \subset_{t_k} J} \mathfrak{K}_{G,C^\natural}$ where the sum runs over sentences G such that $G \subset_{t_k} J$. For any colored immaculate tableau of shape J and type C, we can remove the boxes of T filled with the number k, all of which will be on the right-hand side of T, to obtain a colored immaculate tableau of shape G with type $C^\natural$. Thus the sum of $\mathfrak{K}_{G,C^\natural}$ for all the $G \subset_{t_k} J$ gives $\mathfrak{K}_{J,C}$. With this fact, we proceed by induction on the length of C.

$$H_C = H_{C^\natural} H_{t_k} = \left(\sum_G \mathfrak{K}_{G,C^\natural} \mathfrak{S}_G \right) H_{t_k} \quad \text{by induction}$$
$$= \sum_G \mathfrak{K}_{G,C^\natural} \mathfrak{S}_G H_{t_k} = \sum_G \mathfrak{K}_{G,C^\natural} \sum_{G \subset_{t_k} J} \mathfrak{S}_J \quad \text{by Theorem 3.3.43}$$
$$= \sum_J \left(\sum_{G \subset_{t_k} J} \mathfrak{K}_{G,C^\natural} \right) \mathfrak{S}_J \quad \text{by rearranging the sums}$$
$$= \sum_J \mathfrak{K}_{J,C} \mathfrak{S}_J,$$

where the final two sums run over all sentences J such that there exists a colored immaculate tableau of shape J and type C. If there is no such CIT of shape J and type C then $\mathfrak{K}_{J,C} = 0$, and it is equivalent to taking this sum over all sentences J such that $|J| = |C|$. $\square$

Note that this unique expansion satisfies Proposition 2.2.7 and in fact verifies the duality of the colored immaculate and colored dual immaculate bases.

Corollary 3.3.46. *The colored immaculate basis is dual to the colored dual immaculate basis.*

With this duality verified, we can prove that the colored immaculate functions are analogous to the original noncommutative Bernstein operators because they are isomorphic under υ in the case of a unary alphabet A.

Proposition 3.3.47. *Let $G \in NSym_A$ and $F \in QSym_A$. If $A = \{a\}$, then $\langle G, F\rangle = \langle \upsilon(G), \upsilon(F)\rangle$.*

Proof. Let $A = \{a\}$, and let $G = \sum_J c_J H_J$ and $F = \sum_I b_I M_I$ where the sums run over all sentences I, J, respectively. Then,

$$\langle G, F\rangle = \left\langle \sum_J c_J H_J, \sum_I b_I M_I \right\rangle = \sum_{I,J} c_J b_I \langle H_J, M_I\rangle = \sum_I c_I b_I.$$

Next, for $\upsilon(G) \in NSym$ and $\upsilon(F) \in QSym$, we have that

$$\begin{aligned}\langle \upsilon(G), \upsilon(F)\rangle &= \left\langle \sum_J c_J \upsilon(H_J), \sum_I b_I \upsilon(M_I) \right\rangle \\ &= \left\langle \sum_J c_J H_{w\ell(J)}, \sum_I b_I M_{w\ell(I)} \right\rangle = \sum_{I,J} c_J b_I \langle H_{w\ell(J)}, M_{w\ell(I)}\rangle.\end{aligned}$$

The inner product $\langle H_{w\ell(J)}, M_{w\ell(I)}\rangle$ is zero unless $w\ell(I) = w\ell(J)$ which happens exactly when $I = J$ because the alphabet A is made up of only one color. In other words, there is exactly one sentence I such that $w\ell(I) = \alpha$ for each composition α in this case. Thus,

$$\langle \upsilon(G), \upsilon(F)\rangle = \sum_I c_I b_I = \langle G, F\rangle. \qquad \square$$

Proposition 3.3.48. *Let $A = \{a\}$, and let I be a sentence. Then, $\upsilon(\mathfrak{S}_I) = \mathfrak{S}_{w\ell(I)}$. Moreover, $\{\mathfrak{S}_I\}_I$ in $NSym_A$ is analogous to $\{\mathfrak{S}_\alpha\}_\alpha$ in NSym.*

Proof. Let $A = \{a\}$ and let I and J be sentences. By Proposition 3.3.47,

$$\langle \mathfrak{S}_I, \mathfrak{S}^*_J\rangle = \langle \upsilon(\mathfrak{S}_I), \upsilon(\mathfrak{S}^*_J)\rangle = \langle \upsilon(\mathfrak{S}_I), \mathfrak{S}^*_{w\ell(J)}\rangle.$$

Because A is unary, $I = J$ if and only if $w\ell(I) = w\ell(J)$ and thus $\delta_{I,J} = \delta_{w\ell(I),w\ell(J)}$. As a result,

$$\langle \mathfrak{S}_I, \mathfrak{S}^*_J\rangle = \langle \mathfrak{S}_{w\ell(I)}, \mathfrak{S}^*_{w\ell(J)}\rangle = \langle \upsilon(\mathfrak{S}_I), \mathfrak{S}^*_{w\ell(J)}\rangle,$$

for all sentences I and J. Therefore, $\upsilon(\mathfrak{S}_I) = \mathfrak{S}_{w\ell(I)}$. $\square$

The expansion of the colored ribbon functions into the colored immaculate functions now follows from the application of Proposition 2.2.7 to Theorem 3.3.27.

Corollary 3.3.49. *For a sentence C, the colored ribbon noncommutative symmetric functions expand positively into the colored immaculate functions as*

$$R_C = \sum_J \mathfrak{L}_{J,C} \mathfrak{S}_J,$$

where the sum runs over all sentences J such that $|J| = |C|$.

This corollary allows us to define the expansion of the colored immaculate function indexed by a sentence of the form $(a_1, \ldots, a_k)$ in terms of the $\{H_I\}_I$ basis.

Proposition 3.3.50. *For a sentence $(a_1, \ldots a_k)$,*

$$\mathfrak{S}_{(a_1,\ldots,a_k)} = \sum_{(a_1,\ldots,a_k) \preceq J} (-1)^{k-\ell(J)} H_J.$$

Proof. Let $C = (a_1, \ldots, a_k)$, and notice that $\mathfrak{L}_{J,(a_1,\ldots,a_k)} = 0$ unless $J = (a_1, \ldots, a_k)$ in which case $\mathfrak{L}_{(a_1,\ldots,a_k),(a_1,\ldots,a_k)} = 1$. Then by Corollary 3.3.49, we have $\mathfrak{S}_{(a_1,\ldots,a_k)} = R_{(a_1,\ldots,a_k)}$. Then, expanding $R_{(a_1,\ldots,a_k)}$ into the $\{H_I\}$ basis yields the desired formula. $\square$

Applying Proposition 2.2.7 to Theorem 3.3.31 also yields an expansion of the colored immaculate functions into the colored ribbon basis using the colored immaculate descent graph of Definition 3.3.29.

Corollary 3.3.51. *For a sentence J, the colored immaculate functions expand into colored ribbon functions as*

$$\mathfrak{S}_J = \sum_I \mathfrak{L}^{-1}_{I,J} R_I \qquad \textit{with coefficients} \qquad \mathfrak{L}^{-1}_{I,J} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums run over all sentences I in $\mathfrak{D}^n_A$ and over directed paths $\mathcal{P}$ from I to J in $\mathfrak{D}^n_A$.

Example 3.3.52. The colored immaculate function $\mathfrak{S}_{(a,cb,b)}$ expands in to the colored ribbon functions as

$$\mathfrak{S}_{(a,cb,b)} = R_{(a,cb,b)} - R_{(ab,cb)} + R_{(abb,c)} - R_{(ab,c,b)}.$$

The term $R_{(abb,c)}$ has a coefficient of 1 because the only path from (abb, c) to (a, cb, b) is

$$(abb, c) \xrightarrow{1} (ab, cb) \xrightarrow{1} (a, cb, b).$$

Proposition 2.2.7 can be applied to Corollary 3.3.33 to get a result in *NSym* analogous to Corollary 3.3.51. It is actually an open question to find a cancellation-free combinatorial way of expanding immaculate functions into the ribbon basis. Campbell defines formulas for a few special cases in [19]. In [5], Allen and Mason give a complete combinatorial description of the expansion of immaculate functions into the complete homogeneous basis in terms of tunnel hooks, which generalize the special rim hooks of Eğecioğlu and Remmel [29]. This becomes a somewhat complicated expansion of any immaculate function into the ribbon basis, but for certain immaculate functions, the expression simplifies to a Jacobi-Trudi-Like formula. While our formula is not cancellation-free, it does provide a concise way to compute the coefficients in the expansion for every case. Additionally, it is relatively easy to compute a single coefficient without calculating the entire expression or the entire transition matrix.

Corollary 3.3.53. *For a composition $\beta \models n$, the immaculate functions expand into the ribbon functions as*

$$\mathfrak{S}_\beta = \sum_{\alpha \models n} \mathfrak{L}^{-1}_{\alpha,\beta} R_\alpha \qquad \text{with coefficients} \qquad \mathfrak{L}^{-1}_{\alpha,\beta} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums run over all $\alpha \models n$ and directed paths $\mathcal{P}$ from α to β in $\mathfrak{D}^n$, respectively.

Applying the forgetful map χ to Corollary 3.3.53 produces a new expansion of the Schur functions into ribbon Schur functions. The question of expressing Schur functions in terms of ribbon Schur functions was notably studied by Lascoux and Pragacz in [50] as well as Hamel and Goulden in [40]. One advantage of this expression compared to the matrix determinant expressions of Lascoux and Pragacz or Hamel and Goulden is that we can compute single coefficients without computing the entire expansion.

Corollary 3.3.54. *For a partition $\lambda \vdash n$, a Schur function can be decomposed into ribbon Schur functions as*

$$s_\lambda = \sum_{\alpha \models n} L^{-1}_{\alpha,\lambda} r_\alpha \quad \text{with} \quad L^{-1}_{\alpha,\lambda} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums run over compositions α and over directed paths $\mathcal{P}$ from α to λ in $\mathfrak{D}^n$.

While the colored immaculate functions mirror many of the properties of the immaculate functions, the Jacobi-Trudi formula does not generalize naturally. This is in part due to the challenges of a deletion operation on words which would be needed to generalize integer subtraction. Future work may investigate such a formula.

3.3.3 The colored immaculate poset and skew colored immaculate tableaux

Colored composition diagrams admit a natural partial ordering similar to that of Young's lattice and the immaculate poset. The elements of this poset can be thought of as sentences or colored composition diagrams, which gives a more visual representation. This poset has a combinatorial relationship with standard colored immaculate tableaux and leads to a natural definition of skew colored immaculate tableaux which in turn leads to the skew colored dual immaculate functions.

Additionally, the right Pieri rule on colored immaculate functions connects this poset and these skew functions to the structure constants of the colored immaculate functions as it does in the non-colored case.

Definition 3.3.55. The *colored immaculate poset* $\mathfrak{P}_A^{\mathfrak{S}}$ is the set of all sentences on A with the partial order defined by the cover relation that I covers J if $J \subset_a I$ for some $a \in A$.

This cover relation means that I covers J if I differs from J by the addition of a box colored with a placed on the right side of, or below, J. In this case, arrows from J to I in the Hasse diagram of $\mathfrak{P}_A^{\mathfrak{S}}$ are labeled with (m, a) where m is the number of the row to which a is added in J.

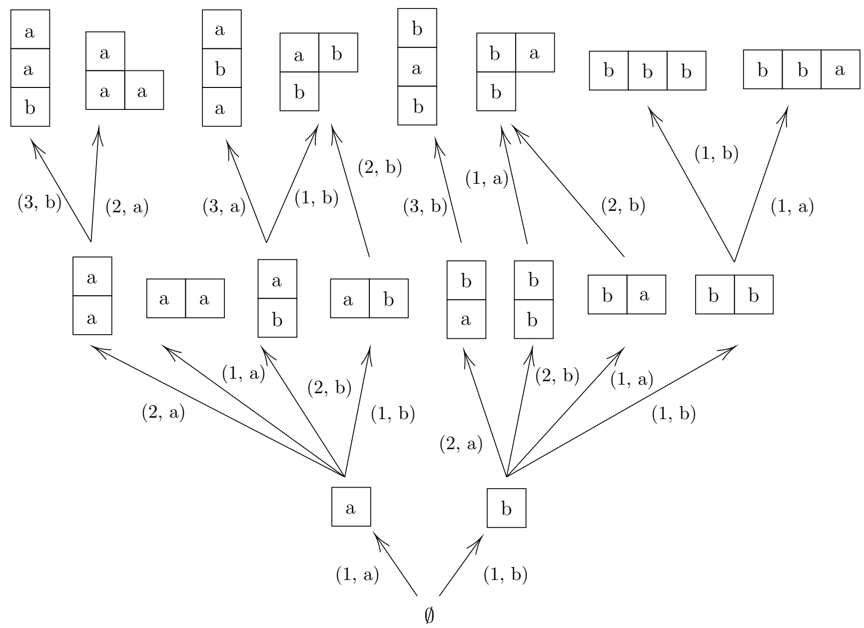


Figure 3.3: A portion of the colored immaculate poset $\mathfrak{P}_A^{\mathfrak{S}}$ on the alphabet $A = \{a, b\}$.

The maximal chains on $\mathfrak{P}_A^{\mathfrak{S}}$ from $\emptyset$ to I are equivalent to the standard colored immaculate tableaux of shape I. The maximal chain $C = \{\emptyset = J_0 \xrightarrow{(m_1,a_1)} J_1 \xrightarrow{(m_2,a_2)} \cdots \xrightarrow{(m_k,a_k)} J_k = I\}$ is associated with the standard colored immaculate tableau of shape I whose boxes are filled with the integers 1 through n in the order they appear in the path. That is, the box added from $J_j \xrightarrow{(m_{j+1},a_{j+1})} J_{j+1}$, which is added to row m_{j+1} and colored with a_{j+1}, is filled with $j + 1$.

Example 3.3.56. The maximal chain

$$C = \{\emptyset \xrightarrow{(1,a)} [a] \xrightarrow{(2,d)} [a, d] \xrightarrow{(2,e)} [a, de] \xrightarrow{(1,b)} [ab, de] \xrightarrow{(2,f)} [ab, def] \xrightarrow{(1,c)} [abc, def]\}$$

is associated with the following tableaux:

$a,1$	$b,4$	$c,6$
$d,2$	$e,3$	$f,5$

Maximal chains starting from a non-empty sentence J going to I lead to a natural definition of skew standard colored immaculate tableaux.

Definition 3.3.57. For sentences I and $J = (v_1, \dots, v_h)$ with $J \subseteq_L I$, the *colored skew shape* I/J is the colored composition diagram of I where, for $1 \le i \le h$, the first $|v_i|$ boxes of the i^{th} row are inactive. The inactive boxes are shaded gray to indicate that they have in a sense been "removed", however the colors filling them are still relevant.

Definition 3.3.58. For sentences I and J with $J \subseteq_L I$, a *skew colored immaculate tableau* of shape I/J is a colored skew shape I/J filled with integers such that the sequence of integer entries in the first column is strictly increasing from top to bottom and the sequence of integer entries in each row is weakly increasing from left to right. Here the inactive boxes of I/J are not filled, and we consider the first column of a colored skew shape I/J to be the one corresponding to the first column of I.

The maximal chain $C = \{J = J_0 \xrightarrow{(m_1,a_1)} J_1 \xrightarrow{(m_2,a_2)} \cdots \xrightarrow{(m_k,a_k)} J_k = I\}$ is associated with the skew standard colored immaculate tableau of shape I/J whose boxes are filled with the integers $1, \dots, k$ in the order they appear in the path.

Example 3.3.59. The maximal chain $C = \{[a, de] \xrightarrow{(1,b)} [ab, de] \xrightarrow{(2,f)} [ab, def] \xrightarrow{(1,c)} [abc, def]\}$ is associated with the following skew colored immaculate tableau:

a	$b,1$	$c,3$
d	e	$f,2$

Definition 3.3.60. For sentences I, J such that $J \subseteq_L I$, define the *skew colored dual immaculate function* as

$$\mathfrak{S}^*_{I/J} = \sum_K \langle \mathfrak{S}_J H_K, \mathfrak{S}^*_I \rangle M_K,$$

where the sum runs over all sentences K such that $|I| - |J| = |K|$.

Proposition 3.3.61. *The coefficient $\langle \mathfrak{S}_J H_K, \mathfrak{S}^*_I \rangle$ is equal to the number of skew colored immaculate tableaux of shape I/J with type K.*

Proof. Let $K = (u_1, \dots, u_g)$ be a sentence. Notice that $\mathfrak{S}_J H_K = (((\mathfrak{S}_J H_{u_1}) H_{u_2}) \cdots H_{u_g})$ and by Theorem 3.3.43, we have

$$\mathfrak{S}_J H_K = \sum_{J \subset_{u_1} J_1 \subset_{u_2} \cdots J_{g-1} \subset_{u_g} L} \mathfrak{S}_L,$$

for some sentences $J_1, \dots J_{g-1}$. Thus,

$$\langle \mathfrak{S}_J H_K, \mathfrak{S}_I^* \rangle = \left\langle \sum_{J \subset_{u_1} J_1 \subset_{u_2} \dots J_{g-1} \subset_{u_g} L} \mathfrak{S}_L, \mathfrak{S}_I^* \right\rangle = \sum_{J \subset_{u_1} J_1 \subset_{u_2} \dots J_{g-1} \subset_{u_g} L} \langle \mathfrak{S}_L, \mathfrak{S}_I^* \rangle$$

for some sentences $J_1, \dots, J_{g-1}$. Therefore, for $J_1, \dots, J_{g-1}$, this inner product is equivalent to the number of times that the sentence I appears when summing over all sentences L such that $J \subset_{u_1} J_1 \subset_{u_2} \dots J_{g-1} \subset_{u_g} L$. Each occurrence of I can be associated with a unique sequence of sentences $(J, J_1, \dots, J_{g-1})$ that appear in the sum, and each sequence can be associated with a unique skew colored immaculate tableau of shape I/J and type K. Starting with the colored skew shape I/J, first fill the boxes corresponding to those in J_1/J with 1's. Then fill the boxes corresponding to J_2/J_1 with 2's and continue repeating this process until the remaining boxes in I/J_{g-1} are filled with $(g-1)$'s. Note that because $J \subset_{u_1} J_1 \subset_{u_2} \dots J_{g-1} \subset_{u_g} I$, the colors of the boxes filled with each number j, read from left to right and bottom to top, correspond exactly to the word u_j. Through this construction, each sequence $J, J_1, \dots, J_{g-1}$ corresponds to a unique skew colored immaculate tableau of shape I/J and type K. Additionally, each skew CIT T of shape I/J and type K can be associated with a unique sequence $J, J_1, \dots, J_{g-1}$ such that $J \subset_{u_1} J_1 \subset_{u_2} \dots J_{g-1} \subset_{u_g} I$ by taking T and removing all boxes filled with integers greater than j, for each $1 \leq j < g$, to get a colored tableau of shape J_j. Therefore, $\langle \mathfrak{S}_J H_K, \mathfrak{S}_I^* \rangle$ counts the number of skew CIT with shape I/J and type K. □

The use of the perp operator allows for the expansions of the skew colored dual immaculate functions into the colored fundamental basis and the colored dual immaculate basis with inner product coefficients.

Proposition 3.3.62. *For an interval $[J, I]$ in $\mathfrak{P}_A^{\mathfrak{S}}$,*

$$\mathfrak{S}_{I/J}^* = \sum_K \langle \mathfrak{S}_J R_K, \mathfrak{S}_I^* \rangle F_K = \sum_K \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}_I^* \rangle \mathfrak{S}_K^*,$$

where the sums run over all sentences K such that $|I| - |J| = |K|$. The coefficients $\langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}_I^* \rangle$ are equal to the structure coefficients $C_{J,K}^I$ for colored immaculate multiplication,

$$\mathfrak{S}_J \mathfrak{S}_K = \sum_I C_{J,K}^I \mathfrak{S}_I = \sum_I \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}_I^* \rangle \mathfrak{S}_I,$$

where the sums run over all sentences I.

Proof. Observe that by Definition 3.3.34,

$$\mathfrak{S}_{I/J}^* = \sum_K \langle \mathfrak{S}_J H_K, \mathfrak{S}_I^* \rangle M_K = \mathfrak{S}_J^{\perp}(\mathfrak{S}_I^*) = \sum_K \langle \mathfrak{S}_J R_K, \mathfrak{S}_I^* \rangle F_K = \sum_K \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}_I^* \rangle \mathfrak{S}_K^*. \quad \square$$

The skew colored dual immaculate functions can also be defined explicitly in terms of skew colored immaculate tableaux following Definition 3.3.60.

Proposition 3.3.63. *Let $I = (w_1, \ldots, w_k)$ and $J = (v_1, \ldots, v_h)$ be sentences such that $J \subseteq_L I$. Then*

$$\mathfrak{S}^*_{I/J} = \sum_T x_T,$$

where the sum is taken over all skew colored immaculate tableaux of shape I/J.

Proof. By Definition 3.3.60, $\mathfrak{S}^*_{I/J} = \sum_K \langle \mathfrak{S}_J H_K, \mathfrak{S}^*_I \rangle M_K$, where the sum runs over all sentences $K \in \mathfrak{P}^{\mathfrak{S}}_A$. By Proposition 3.3.61, $\langle \mathfrak{S}_J H_K, \mathfrak{S}^*_I \rangle$ is equal to the number of skew colored immaculate tableaux of shape I/J and type K. Thus, following Proposition 3.3.20, $\langle \mathfrak{S}_J H_K, \mathfrak{S}^*_I \rangle M_K = \sum_{T'} x_{T'}$ where the sum runs over all skew CIT T' of shape I/J and flat type K. Therefore,

$$\mathfrak{S}^*_{I/J} = \sum_K \sum_{T'} x_{T'} = \sum_T x_T$$

where the sums run over sentences K such that $|I| - |J| = |K|$, skew CIT T' of shape I/J and flat type K, and all skew CIT T of shape I/J and type T. □

Additionally, comultiplication on the colored dual immaculate basis can be defined in terms of skew functions following Propositions 2.2.8 and 3.3.62.

Proposition 3.3.64. *For a sentence I,*

$$\Delta(\mathfrak{S}^*_I) = \sum_J \mathfrak{S}^*_J \otimes \mathfrak{S}^*_{I/J},$$

where the sum runs over all sentences J such that $J \subseteq_L I$.

Proof. Let J and K be sentences, and observe that $\mathfrak{S}_J \mathfrak{S}_K = \sum_I \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}^*_I \rangle \mathfrak{S}_I$. By Proposition 2.2.8, this implies

$$\begin{aligned}\Delta(\mathfrak{S}^*_I) &= \sum_{J,K} \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}^*_I \rangle \mathfrak{S}^*_J \otimes \mathfrak{S}^*_K = \sum_J \left(\mathfrak{S}^*_J \otimes \sum_K \langle \mathfrak{S}_J \mathfrak{S}_K, \mathfrak{S}^*_I \rangle \mathfrak{S}^*_K \right) \\ &= \sum_J \mathfrak{S}^*_J \otimes \mathfrak{S}^*_{I/J} \quad \text{by Proposition 3.3.62.}\end{aligned}$$

□

As in the non-colored case, finding general combinatorial formulas for multiplication and the antipode of the colored dual immaculate functions remains an open problem. As shown in the example below, the product of two colored dual immaculate functions does not have exclusively positive structure constants, and their combinatorial description is not yet evident.

$$\mathfrak{S}^*_{(ab)} \mathfrak{S}^*_{(c)} = \mathfrak{S}^*_{(abc)} + \mathfrak{S}^*_{(c,ab)} + \mathfrak{S}^*_{(ac,b)} - \mathfrak{S}^*_{(a,bc)}.$$

3.3.4 Colored row-strict dual immaculate functions in $QSym_A$

To generalize the row-strict definitions and results to the colored case, we first define a lift of the involution ψ to $QSym_A$ and $NSym_A$. Note that we technically define two separate dual involutions

ψ, one on $QSym_A$ and one on $NSym_A$, but we treat them as a single map on both spaces.

Definition 3.3.65. For a sentence J, define the linear maps $\psi : QSym_A \to QSym_A$ and $\psi : NSym_A \to NSym_A$ by

$$\psi(F_J) = F_{J^c} \quad \text{and} \quad \psi(R_J) = R_{J^c}.$$

Proposition 3.3.66. *The maps ψ are involutions, and the duality between $QSym_A$ and $NSym_A$ is invariant under ψ, meaning that*

$$\langle G, F\rangle = \langle \psi(G), \psi(H)\rangle.$$

Furthermore, the map $\psi : NSym_A \to NSym_A$ is an isomorphism.

Proof. To see that ψ is invariant under duality, it suffices to observe that $\langle R_I, F_J\rangle = \langle R_{I^c}, F_{J^c}\rangle = \langle \psi(R_I), \psi(F_J)\rangle$. The map ψ is an involution because $\psi(\psi(F_I)) = F_{(I^c)^c} = F_I$ and $\psi(\psi(R_I)) = R_{(I^c)^c} = R_I$ and the map extends linearly. Next, we show that ψ is an isomorphism on $NSym_A$. For sentences I and J, we have $R_I R_J = R_{I\cdot J} + R_{I\odot J}$ [28] and thus $\psi(R_I R_J) = \psi(R_{I\cdot J}) + \psi(R_{I\odot J})$. Observe that $(I \cdot J)^c = I^c \odot J^c$ and $(I \odot J)^c = I^c \cdot J^c$. Therefore, $\psi(R_I R_J) = R_{I^c \odot J^c} + R_{I^c\cdot J^c} = R_{I^c} R_{J^c} = \psi(R_I)\psi(R_J)$. □

Note that $\psi : QSym_A \to QSym_A$ is not an isomorphism because it fails to preserve multiplication. Now, we prove that ψ maps the complete homogenous basis to the elementary basis in *NSym* and vice versa, which will allow us to apply ψ to both these bases.

Proposition 3.3.67. *For a sentence J, $\psi(E_J) = H_J$.*

Proof. First, for a sentence J, we expand E_J in terms of the colored ribbon basis as

$$E_J = \sum_{K\preceq J} (-1)^{|J|-\ell(K)} H_K = \sum_{K\preceq J} (-1)^{|J|-\ell(K)} \left[\sum_{I\succeq K} R_I\right] = \sum_{I} \left[\sum_{K\preceq J,I} (-1)^{|J|-\ell(K)}\right] R_I.$$

Next, we split the sum into two pieces according to I: one where $I \succ J^c$ and the other where $I \preceq J^c$,

$$E_J = \sum_{I\succ J^c} \left[\sum_{K\preceq J,I} (-1)^{|J|-\ell(K)}\right] R_I + \sum_{I\preceq J^c} \left[\sum_{K\preceq J,I} (-1)^{|J|-\ell(K)}\right] R_I.$$

In the first case, observe that $I \succ J^c$ implies that $J \succ I$. Thus, $K \preceq J, I$ becomes $K \preceq J$. Also notice that because J is constant we can write $(-1)^{|J|} = (-1)^{|J|-\ell(J)}(-1)^{\ell(J)}$ and factor the first term out of the sum. In the second case, $I \preceq J^c$ means that $K \preceq I, J$ becomes $K \preceq J, J^c$. The only way for K to be a refinement of a sentence and its complement is if K is a sentence made up of only single letters. That is, $|K| = \ell(K)$. Thus the inner sum has only one summand, which is $(-1)^{|J|-\ell(K)} = (-1)^{|J|-|K|} = 1$. As a result, the equation simplifies as

$$E_J = \sum_{I\succ J^c} (-1)^{|J|-\ell(J)} \left[\sum_{J^c\preceq K\preceq I} (-1)^{\ell(J)-\ell(K)}\right] R_I + \sum_{I\preceq J^c} R_I.$$

By properties of the Möbius function [28], the sum in brackets above is 0 for all K and we have

$$E_J = \sum_{I \preceq J^c} R_I.$$

Therefore, applying ψ to E_J and noticing that $I \preceq J^c$ if and only if $J \preceq I^c$, yields

$$\psi(E_J) = \sum_{I \preceq J^c} \psi(R_I) = \sum_{I \preceq J^c} R_{I^c} = \sum_{J \preceq I^c} R_{I^c} = H_J. \qquad \square$$

We continue by defining colored row-strict immaculate tableaux. Their combinatorics in relation to those of the colored immaculate tableaux will allow us to define the colored row-strict dual immaculate basis and verify its relationship to the colored dual immaculate basis via ψ.

Definition 3.3.68. A *colored row-strict immaculate tableau* (CRSIT) of shape I is a colored composition diagram of shape I in which the sequence of integer entries is strictly increasing from left to right in each row, and weakly increasing top to bottom in the leftmost column. The *type* of a colored row-strict immaculate tableau T is the sentence $C = (u_1, \ldots, u_g)$ such that for each $i \in [g]$ the word u_i lists the colors of all boxes in T filled with the integer i in the order they appear when entries in T are read from left to right and top to bottom. A *standard colored row-strict immaculate tableau* is a colored row-strict immaculate tableau of size n with the integer entries $1, \ldots, n$ each appearing exactly once. To *standardize* a colored row-strict tableau, replace its integer entries with the numbers $1, 2, \ldots$ based on the order they appear in the type, first replacing all entries equal to 1, then 2, etc. just as in the standardization of non-colored row-strict immaculate tableaux.

We also use the same notion of *row-strict descents* and the *row-strict descent set* $Des_{\mathfrak{RS}}$ from row-strict immaculate tableaux, but define an additional concept of colored row-strict descent composition.

Definition 3.3.69. The *colored row-strict descent composition* of a standard colored row-strict immaculate tableau U, denoted $co_A^{\mathfrak{RS}}(U)$, is the sentence obtained by reading the colors in each box in order of their number and splitting into a new word after each row-strict descent.

Example 3.3.70. Below are a few CRSIT of shape (ab, bca) along with their types and standardization, as well as the row-strict descent sets and colored row-strict descent compositions of these standardizations.

$$T_1 = \begin{array}{|c|c|c} \hline a,1 & b,2 & \multicolumn{1}{c}{} \\ \hline b,1 & c,3 & \multicolumn{1}{c|}{a,4} \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c} \hline a,1 & b,3 & \multicolumn{1}{c}{} \\ \hline b,1 & c,2 & \multicolumn{1}{c|}{a,4} \\ \hline \end{array} \qquad T_3 = \begin{array}{|c|c|c} \hline a,1 & b,4 & \multicolumn{1}{c}{} \\ \hline b,2 & c,3 & \multicolumn{1}{c|}{a,3} \\ \hline \end{array}$$

$$(ab, b, c, a) \qquad\qquad (ab, c, b, a) \qquad\qquad (a, b, bc, a)$$

$$U_2 = \begin{array}{|c|c|c} \hline a,1 & b,3 & \multicolumn{1}{c}{} \\ \hline b,2 & c,4 & \multicolumn{1}{c|}{a,5} \\ \hline \end{array} \qquad U_3 = \begin{array}{|c|c|c} \hline a,1 & b,4 & \multicolumn{1}{c}{} \\ \hline b,2 & c,3 & \multicolumn{1}{c|}{a,5} \\ \hline \end{array} \qquad U_4 = \begin{array}{|c|c|c} \hline a,1 & b,5 & \multicolumn{1}{c}{} \\ \hline b,2 & c,3 & \multicolumn{1}{c|}{a,4} \\ \hline \end{array}$$

$Des_{\mathfrak{RS}}(U_2)$={2,4} $co_A^{\mathfrak{RS}}(U_2)$=($ab$,$bc$,$a$)

$Des_{\mathfrak{RS}}(U_3)$={2,3} $co_A^{\mathfrak{RS}}(U_3)$=($ab$,$c$,$ba$)

$Des_{\mathfrak{RS}}(U_4)$={2,3,4} $co_A^{\mathfrak{RS}}(U_4)$=($ab$,$c$,$a$,$b$)

Definition 3.3.71. For a sentence J, the *colored row-strict dual immaculate function* is defined as

$$\mathfrak{RS}^*_J = \sum_T x_T,$$

where the sum is taken over all colored row-strict immaculate tableaux T of shape J.

Example 3.3.72. For $J = (ab, bca)$, the colored row-strict dual immaculate function is

$$\mathfrak{RS}^*_{ab,bca} = x_{ab,1}x_{b,2}x_{c,3}x_{a,4} + x_{ab,1}x_{c,2}x_{b,3}x_{a,4} + x_{a,1}x_{b,2}x_{bc,3}x_{a,4} + \ldots + 2x_{a,1}x_{b,2}x_{b,3}x_{c,4}x_{a,5} + \ldots.$$

Proposition 3.3.73. *For a sentence J,*

$$\mathfrak{RS}^*_J = \sum_S F_{co_A^{\mathfrak{RS}}(S)},$$

where the sum runs over all standard colored row-strict immaculate tableaux S of shape J.

Proof. Let T be a colored row-strict immaculate tableau of shape J that standardizes to the standard colored row-strict immaculate tableau S. The flattening of the type of T must be a refinement of the colored row-strict descent composition of S, which can be shown by applying the same reasoning used in the proof of Proposition 3.3.16. In fact, each sentence B that flattens to a refinement of $co_A^{\mathfrak{RS}}(S)$ corresponds to a unique colored row-strict immaculate tableau of type B that standardizes to S. Therefore,

$$F_{co_A^{\mathfrak{RS}}(S)} = \sum_{T_S} x_{T_S},$$

where the sum runs over all colored row-strict immaculate tableaux T_S of shape J that standardize to S. It follows that

$$\mathfrak{RS}^*_J = \sum_T x_T = \sum_S \sum_{T_S} x_{T_S} = \sum_S F_{co_A^{\mathfrak{RS}}(S)},$$

where the sums run over all CRSIT T of shape J, all standard CRSIT S of shape J, and all CRSIT T_S of shape J that standardize to S. □

Theorem 3.3.74. *Let J be a sentence. Then,*

$$\psi(\mathfrak{S}^*_J) = \mathfrak{RS}^*_J.$$

Proof. For a sentence J,

$$\psi(\mathfrak{S}^*_J) = \psi(\sum_U F_{co^{\mathfrak{S}}_A(U)}) = \sum_U F_{(co^{\mathfrak{S}}_A(U))^c}.$$

The complement of the colored descent composition of a standard colored immaculate tableau U splits exactly where U does not have a descent. These are exactly the locations of the row-strict descents in U, thus $(co^{\mathfrak{S}}_A(U))^c = co^{\mathfrak{RS}}_A(U)$, and

$$\psi(\mathfrak{S}^*_J) = \sum_U F_{(co^{\mathfrak{RS}}_A(U))} = \mathfrak{RS}^*_J. \qquad \square$$

Because $\{\mathfrak{S}^*_J\}_J$ is a basis and ψ is an involution, Theorem 3.3.74 also implies the following.

Corollary 3.3.75. *$\{\mathfrak{RS}^*_J\}_J$ is a basis for $QSym_A$.*

Using ψ, we extend each of our results on the colored dual immaculate functions to the colored row-strict dual immaculate functions.

Definition 3.3.76. For sentences J, C, and weak sentence B, define $\mathfrak{K}^{\mathfrak{RS}}_{J,B}$ as the number of colored row-strict immaculate tableaux of shape J and type B, and $\mathfrak{L}^{\mathfrak{RS}}_{J,C}$ as the number of standard colored row-strict immaculate tableaux of shape J with row-strict descent composition C.

Proposition 3.3.77. *For a sentence J,*

$$\mathfrak{RS}^*_J = \sum_B \mathfrak{K}^{\mathfrak{RS}}_{J,B} M_B \qquad \text{and} \qquad \mathfrak{RS}^*_J = \sum_C \mathfrak{L}^{\mathfrak{RS}}_{J,C} F_C,$$

where the sums run over sentences B and C such that $|B| = |J|$ and $|C| = |J|$.

The above proposition follows from Definition 3.3.71 in the manner of Theorem 3.3.22. The results of Section 4.3 also extend nicely to the row-strict case under the involution ψ.

Definition 3.3.78. The *colored row-strict immaculate descent graph*, denoted $\mathfrak{D}^{\mathfrak{RS}_n}_A$, is the edge-weighted directed graph with the set of sentences on A of size n as its vertex set and an edge from each sentence I to J if there exists a standard CRSIT of shape I with colored row-strict descent composition J. The edge from I to J is weighted with the coefficient $\mathfrak{L}^{\mathfrak{RS}}_{I,J}$.

Due to the differing definitions of descents and descent compositions in row-strict tableaux, the neighbors of I in $\mathfrak{D}^{\mathfrak{RS}_n}_A$ are exactly the (sentence) complements of I's neighbors in $\mathfrak{D}^n_A$ along with the complement of I itself. Here, we say two vertices are *neighbors* if they are adjacent by an edge in either direction.

Example 3.3.79. The standard colored row-strict immaculate tableaux of shape (ab, cbb) have colored row-strict descent compositions (a, bc, b, b), (ac, bb, b), (ac, b, bb), and (ac, b, bb), so (ab, cbb) has outgoing edges to these sentences in $\mathfrak{D}^{\mathfrak{RS}_5}_{\{a,b,c\}}$. Notice that if we take the complement of each of these sentences we get (ab, cbb), (a, cb, bb), $(a, cbbb)$, and (a, cbb, b) which are exactly (ab, cbb) itself and the sentences to which it has outgoing edges to in $\mathfrak{D}^5_{\{a,b,c\}}$, as seen in Figure 1.

Proposition 3.3.80. *For a sentence I, the colored fundamental functions expand into the colored row strict immaculate basis as*

$$F_I = \sum_J \mathfrak{L}_{I,J}^{\mathfrak{RS}(-1)} \mathfrak{RS}_J^* \qquad \text{with coefficients} \qquad \mathfrak{L}_{I,J}^{\mathfrak{RS}(-1)} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P}),$$

where the sums run over all sentences J and all directed paths $\mathcal{P}$ from I to J in $\mathfrak{D}_A^{\mathfrak{RS}_n}$.

The proof follows that of Theorem 3.3.31 using Proposition 3.3.77 in place of Theorem 3.3.27. Similarly, this proposition specializes to the non-colored case in the same manner as Corollary 3.3.33.

3.3.5 Colored row-strict immaculate functions

We define the colored row-strict immaculate functions as the image of the colored immaculate functions under ψ, and thus also as the basis dual to the colored row-strict dual immaculate functions.

Definition 3.3.81. For a sentence J, the *colored row-strict immaculate function* is defined as

$$\mathfrak{RS}_J = \psi(\mathfrak{S}_J).$$

Equivalently, due to the invariance of ψ under duality, we have $\langle \mathfrak{RS}_I, \mathfrak{RS}_J^* \rangle = \delta_{I,J}$.

Applying ψ to the colored immaculate functions yields row-strict versions of our earlier results and colored generalizations of the results in Theorem 3.1.53. Note that certain results from Theorem 3.1.53 are not generalized here because we lack the corresponding result on the colored immaculate functions or due to the fact that ψ is not an isomorphism on $QSym_A$. The non-colored analogues of ψ are automorphisms on both $QSym$ and $NSym$.

Theorem 3.3.82. *For words w and v, sentences J and C, and $f \in NSym_A$*

1. *(Right Pieri rule)*
$$\mathfrak{S}_J H_w = \sum_{J \subset_w K} \mathfrak{S}_K \overset{\psi}{\Leftrightarrow} \mathfrak{RS}_J E_w = \sum_{J \subset_w K} \mathfrak{RS}_K.$$

2. *(Colored complete homogeneous and colored elementary expansions)*
$$H_C = \sum_J \mathfrak{K}_{J,C} \mathfrak{S}_J \overset{\psi}{\Leftrightarrow} E_C = \sum_J \mathfrak{K}_{J,C} \mathfrak{RS}_J, \qquad H_C = \sum_J \mathfrak{K}_{J,C}^{\mathfrak{RS}} \mathfrak{RS}_J \overset{\psi}{\Longleftrightarrow} E_C = \sum_J \mathfrak{K}_{J,C}^{\mathfrak{RS}} \mathfrak{S}_J.$$

3. *(Colored ribbon expansions)*
$$R_C = \sum_J \mathfrak{L}_{J,C} \mathfrak{S}_J \overset{\psi}{\Longleftrightarrow} R_{C^c} = \sum_J \mathfrak{L}_{J,C} \mathfrak{RS}_J.$$

The application of Proposition 2.2.7 to Proposition 3.3.80 also yields the following result. The analogous result is also true in *NSym*, as in Corollary 3.3.53.

Corollary 3.3.83. *For a sentence J, the colored row-strict immaculate functions expand into the colored ribbon basis as*

$$\mathfrak{RS}_J = \sum_I \mathfrak{L}^{\mathfrak{RS}(-1)}_{I,J} R_I \qquad \text{with coefficients} \qquad \mathfrak{L}^{\mathfrak{RS}(-1)}_{I,J} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P})-1} prod(\mathcal{P})$$

where the sums run over sentences I and all paths $\mathcal{P}$ from I to J in $\mathfrak{D}^{\mathfrak{RS}_n}_A$.

3.3.6 Skew colored row-strict dual immaculate functions

Definition 3.3.84. For sentences I and J where $J \subseteq_L I$, a *skew colored row-strict immaculate tableau* of shape I/J is a colored skew shape I/J filled with positive integers such that the sequences of entries in the first column is weakly increasing top to bottom and the sequence of integers in each row is strictly increasing left to right.

Definition 3.3.85. For sentences I and J where $J \subseteq_L I$, define the *skew colored row-strict dual immaculate function* as

$$\mathfrak{RS}^*_{I/J} = \sum_K \langle \mathfrak{RS}_J H_K, \mathfrak{RS}^*_I \rangle M_K,$$

where the sum runs over all sentences $K \in \mathfrak{P}^{\mathfrak{S}}_A$ such that $|I| - |J| = |K|$.

Applying ψ to the equations in Proposition 3.3.62 yields the following results.

Theorem 3.3.86. *For sentences I and J with $J \subseteq_L I$,*

$$\mathfrak{RS}^*_{I/J} = \sum_K \langle \mathfrak{RS}_J R_K, \mathfrak{RS}^*_I \rangle F_K = \sum_K \langle \mathfrak{RS}_J \mathfrak{RS}_K, \mathfrak{RS}^*_I \rangle \mathfrak{RS}^*_K,$$

where the sums run over all sentences $K \in \mathfrak{P}^{\mathfrak{S}}_A$ such that $|I| - |J| = |K|$.

Proposition 3.3.87. *For sentences I and J such that $J \subseteq_L I$,*

$$\psi(\mathfrak{S}^*_{I/J}) = \mathfrak{RS}^*_{I/J}.$$

Proof. Let I and J be sentences such that $J \subseteq_L I$. Then,

$$\begin{aligned}
\psi(\mathfrak{RS}^*_{I/J}) &= \sum_K \langle \mathfrak{RS}_J R_K, \mathfrak{RS}^*_I \rangle \psi(F_K) = \sum_K \langle \mathfrak{RS}_J R_K, \mathfrak{RS}^*_I \rangle F_{K^c} \\
&= \sum_K \langle \psi(\mathfrak{S}_J R_{K^c}), \psi(\mathfrak{S}^*_I) \rangle F_{K^c} \quad \text{by Theorem 3.3.74} \\
&= \sum_K \langle \mathfrak{S}_J R_{K^c}, \mathfrak{S}^*_I \rangle F_{K^c} = \mathfrak{S}^*_{I/J}. \quad \text{By Proposition 3.3.66}
\end{aligned}$$

□

Comultiplication on the colored row-strict immaculate basis can be defined in terms of skew colored row-strict immaculate functions. The proof follows Proposition 3.3.64 using Theorem 3.3.86.

Proposition 3.3.88. *Let I be a sentence. Then,*

$$\Delta(\mathfrak{RS}^*_I) = \sum_J \mathfrak{RS}^*_J \otimes \mathfrak{RS}^*_{I/J},$$

where the sum runs over all sentences J such that $J \subseteq_L I$.

Multiplication and antipode of the colored row-strict dual immaculate functions are closely related to the multiplication and antipode of colored dual immaculate functions, and thus also remain open.

CHAPTER

4 THE SHIN AND EXTENDED SCHUR FUNCTIONS

We introduce two new bases of $QSym$, the reverse extended Schur functions and the row-strict reverse extended Schur functions, as well as their duals in $NSym$, the reverse shin functions and row-strict reverse shin functions. These bases are the images of the extended Schur functions and shin bases under the involutions ρ and ω, which generalize the classical involution ω on the symmetric functions. In addition, we prove a Jacobi-Trudi rule for certain shin functions using creation operators. We also define two different types of skew extended Schur functions based on left and right actions of $NSym$ and $QSym$. We then use the involutions ρ and ω to translate these and other known results to our reverse and row-strict reverse bases.

4.1 Background

The shin and extended Schur functions, which are dual bases, are unique among the Schur-like bases for having arguably the most natural relationship with the Schur functions. In $NSym$, the commutative image of a shin function indexed by a partition is the Schur function indexed by that partition, while the commutative image of any other shin function is 0. In $QSym$, the extended Schur function indexed by a partition is equal to the Schur function indexed by that partition [19].

4.1.1 The extended Schur functions

The extended Schur functions of $QSym$ were introduced by Assaf and Searles in [6] as the stable limits of polynomials related to Kohnert diagrams, and defined independently by Campbell, Feldman, Light, Shuldiner, and Xu in [19] as the duals to the shin functions. We use the name "extended Schur functions" but otherwise retain the notation and terminology of the dual shin functions. We denote these functions with ש, the Hebrew character *Shin*.

Definition 4.1.1. Let α and β be a composition and a weak composition of n, respectively. A *shin-tableau* of shape α and type β is a labeling of the boxes of the diagram of α by positive integers such that the number of boxes labeled by i is β_i, the sequence of entries in each row is weakly increasing from left to right, and the sequence of entries in each column is strictly increasing from top to bottom.

Shin-tableaux are a direct generalization of semistandard Young tableaux to composition shapes.

Example 4.1.2. The shin-tableaux of shape $(3, 4)$ and type $(1, 2, 1, 1, 2)$ are

1	2	2	
3	4	5	5

1	2	3	
2	4	5	5

1	2	4	
2	3	5	5

A shin-tableau of shape $\alpha \models n$ is *standard* if each number 1 through n appears exactly once. The *descent set* is defined as $Des_{\text{ש}}(U) = \{i : i+1 \text{ is strictly below } i \text{ in } U\}$ for a standard shin-tableau U. Each entry i in $Des_{\text{ש}}(U)$ is called a *descent* of U. The *descent composition* of U is defined as $co_{\text{ש}}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\text{ש}}(U) = \{i_1, \ldots, i_d\}$. Equivalently, the descent composition is found by counting the number of entries in U (in numbered order) between each descent and the next descent, including the latter. The *shin reading word* of a shin-tableau T, denoted $rw_{\text{ש}}(T)$ is the word obtained by reading the rows of T from left to right starting with the bottom row and moving up. To *standardize* a shin-tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\text{ש}}(T)$, then the 2's starting with the next consecutive number, etc.

Definition 4.1.3. For a composition α, the *extended Schur function* is defined as

$$\text{ש}^*_\alpha = \sum_T x^T,$$

where the sum runs over shin-tableaux of shape α.

The extended Schur functions have positive expansions into the monomial and fundamental bases in terms of shin-tableaux [6, 19]. For a composition α,

$$\text{ש}^*_\alpha = \sum_\beta \mathcal{K}_{\alpha,\beta} M_\beta \qquad \text{and} \qquad \text{ש}^*_\alpha = \sum_\beta \mathcal{L}_{\alpha,\beta} F_\beta, \tag{4.1}$$

where $\mathcal{K}_{\alpha,\beta}$ denotes the number of shin-tableaux of shape α and type β, and $\mathcal{L}_{\alpha,\beta}$ denotes the number of standard shin-tableaux of shape α with descent composition β. Additionally, Assaf and Searles showed in [6] that $\mathfrak{w}^*_\alpha = F_\alpha$ if and only if α is a reverse hook shape $(1^k, m)$ for $k, m \geq 0$.

Example 4.1.4. The F-expansion of $\mathfrak{w}^*_{(2,3)}$ and its relevant standard shin-tableaux are:

$$\mathfrak{w}^*_{(2,3)} = F_{(2,3)} + F_{(1,2,2)}$$

1	2	
3	4	5

1	3	
2	4	5

From these definitions, it follows that $\mathfrak{w}^*_\lambda = s_\lambda$ for a partition λ. One consequence of this fact is that the product of any two elements of the extended Schur basis indexed by partitions expands positively into the extended Schur basis with the Littlewood-Richardson coefficients. That is, for partitions λ and μ,

$$\mathfrak{w}^*_\lambda \mathfrak{w}^*_\mu = \sum_{\nu \vdash |\lambda|+|\mu|} c^\nu_{\lambda,\mu} \mathfrak{w}^*_\nu,$$

where $c^\nu_{\lambda,\mu}$ are the Littlewod-Richardson coefficients.

4.1.2 The shin functions

The shin basis of *NSym* was introduced in [19] by Campbell, Feldman, Light, Shuldiner, and Xu. Let α and β be compositions. Then β is said to differ from α by a *shin-horizontal strip* of size r, denoted $\alpha \subset^{\mathfrak{w}}_r \beta$, provided for all i, we have $\beta_i \geq \alpha_i$, $|\beta| = |\alpha| + r$, and for all indices $i \in \mathbb{N}$ if $\beta_i > \alpha_i$ then for all $j > i$, we have $\beta_j \leq \alpha_i$. The shin functions are defined recursively with a right Pieri rule using shin-horizontal strips.

Definition 4.1.5. The shin basis $\{\mathfrak{w}_\alpha\}_\alpha$ is defined as the unique set of functions such that

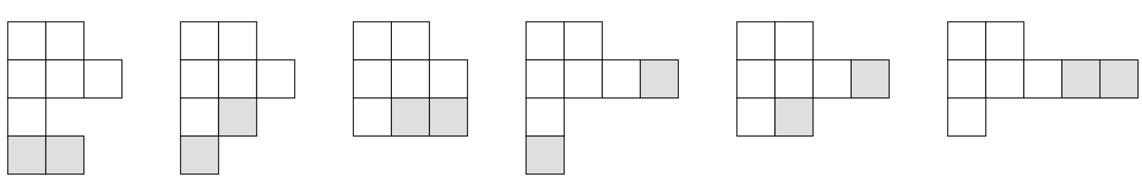
$$\mathfrak{w}_\alpha H_r = \sum_{\alpha \subset^{\mathfrak{w}}_r \beta} \mathfrak{w}_\beta, \tag{4.2}$$

where the sum runs over all compositions β which differ from α by a shin-horizontal strip of size r.

Intuitively, the β in the right-hand side are given by taking the diagram of α and adding r blocks on the right such that if you add boxes to some row i then no row below i is longer than the original row i. This is referred to as the *overhang rule*.

Example 4.1.6. The following example can be pictured with the diagrams below.

$$\mathfrak{w}_{(2,3,1)} H_{(2)} = \mathfrak{w}_{(2,3,1,2)} + \mathfrak{w}_{(2,3,2,1)} + \mathfrak{w}_{(2,3,3)} + \mathfrak{w}_{(2,4,1,1)} + \mathfrak{w}_{(2,4,2)} + \mathfrak{w}_{(2,5,1)}$$



Remark 4.1.7. One computes $\text{ш}_{(3,2)}$ recursively in terms of the complete homogeneous basis starting with $\text{ш}_{\emptyset}H_{(n)} = \text{ш}_{(n)}$ so $H_{(n)} = \text{ш}_{(n)}$ for all n. Then, $\text{ш}_{(4)}H_{(1)} = \text{ш}_{(5)} + \text{ш}_{(4,1)}$ so $\text{ш}_{(4,1)} = H_{(4)}H_{(1)} - H_{(5)}$. Next, $\text{ш}_{(3)}H_{(2)} = \text{ш}_{(5)} + \text{ш}_{(4,1)} + \text{ш}_{(3,2)}$ so $\text{ш}_{(3,2)} = H_{(3)}H_{(2)} - H_{(5)} - H_{(4)}H_{(1)} + H_{(5)} = H_{(3,2)} - H_{(4,1)}$.

Repeated application of this right Pieri rule yields the expansion of a complete homogeneous noncommutative symmetric function in terms of the shin functions. This expansion verifies that the extended Schur functions and the shin functions are dual bases by Proposition 2.2.7, and allows for the expansion of the ribbon functions into the shin basis dually to the expansion of the extended Schur functions into the fundamental basis [19]. For any composition β,

$$H_\beta = \sum_{\beta \leq_\ell \alpha} \mathcal{K}_{\alpha,\beta} \text{ш}_\alpha \qquad \text{and} \qquad R_\beta = \sum_{\beta \leq_\ell \alpha} \mathcal{L}_{\alpha,\beta} \text{ш}_\alpha. \tag{4.3}$$

Recall that the extended Schur functions have the special property that $\text{ш}^*_\lambda = s_\lambda$ for partitions λ. Since the forgetful map χ is dual to the inclusion map from Sym to $QSym$, we have

$$\chi(\text{ш}_\alpha) = \sum_\lambda (\text{coefficient of } \text{ш}^*_\alpha \text{ in } s_\lambda) s_\lambda = \begin{cases} s_\lambda & \text{if } \alpha = \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

It follows that $\chi(\text{ш}_\lambda) = s_\lambda$ when λ is a partition and $\chi(\text{ш}_\alpha) = 0$ otherwise. Another interesting feature of the shin functions is their relationship with the other two primary Schur-like bases: the immaculate functions and the Young noncommutative Schur functions. For any partition λ, the immaculate function $\mathfrak{S}_\lambda$ always equals the Young noncommutative Schur function $\hat{\mathbf{s}}_\lambda$, however the shin function ш_λ often differs. The first examples of this appear for some partitions λ with $|\lambda| = 7$, for example $\text{ш}_{(2,2,2,1)}$.

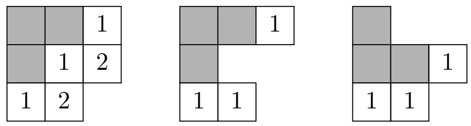
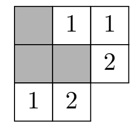
The shin basis has a few other nice combinatorial properties that rely on shin-tableaux defined on skew shapes. These include a rule for the multiplication of a shin function by a ribbon function.

Definition 4.1.8. For compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$ such that $\beta \subseteq \alpha$, a filling of the skew shape α/β with positive integers is a *skew shin-tableau* if:

1. each row is weakly increasing from left to right,
2. each column is strictly increasing from top to bottom, and
3. if $\alpha_i > \beta_i$ for any $1 \leq i \leq k$, then $\beta_j < \beta_i$ for all $j > i$.

A skew shin-tableau is *standard* if it contains the numbers $1, \ldots, |\alpha| - |\beta|$ each exactly once. Note that the boxes associated with β are not filled.

Example 4.1.9. The three leftmost diagrams are skew shin-tableaux while the rightmost diagram is not because there are boxes in the first row sitting directly above skewed-out boxes in the second row, which violates condition 3 of Definition 4.1.8.

Skew shin:  Not skew shin: 

The notions of type, descents, and descent composition for skew shin-tableaux are the same as those for shin-tableaux. There are combinatorial rules for the multiplication of the shin basis by ribbon functions in terms of these skew shin-tableaux.

Theorem 4.1.10. *[19] For all compositions α and β,*

$$\boldsymbol{\mathfrak{w}}_\alpha R_\beta = \sum_{\gamma \models |\alpha|+|\beta|} \sum_{\mathcal{T}} \boldsymbol{\mathfrak{w}}_\gamma,$$

where the inner sum is over all standard skew shin-tableaux $\mathcal{T}$ of skew shape γ/α such that $co_{\boldsymbol{\mathfrak{w}}}(\mathcal{T}) = \beta$.

The shin functions also have a Murnagahn-Nakayama rule based on objects called *shin-slinkies* that are defined in [19].

Theorem 4.1.11. *[19] Let α be a composition and $n \in \mathbb{N}$, then*

$$\boldsymbol{\mathfrak{w}}_\alpha \Psi_n = \sum_{\beta} (-1)^{height(\beta/\alpha)-1} \boldsymbol{\mathfrak{w}}_\beta,$$

where the sum is over all compositions $\beta \models |\alpha| + n$ such that β/α is a shin-slinky.

For more details on the shin and extended Schur functions, see [6, 19, 76].

4.1.3 Row-strict extended Schur and shin functions

The row-strict extended Schur and row-strict shin bases were introduced by Niese, Sundaram, van Willigenburg, Vega, and Wang in [68], motivated by the representation theory of 0-Hecke modules. Let α be a composition and let β be a weak composition. A *row-strict shin-tableau* (RSST) of shape α and type β is a filling of the composition diagram of α with positive integers such that each row strictly increases from left to right, each column weakly increases from top to bottom, and each integer i appears β_i times. A *standard* row-strict shin-tableau (SRSST) with n boxes is one containing the entries 1 through n each exactly once.

Definition 4.1.12. For a composition α, define the *row-strict extended Schur function* as

$$\mathfrak{R}\boldsymbol{\mathfrak{w}}^*_\alpha = \sum_T x^T,$$

where the sum runs over all row-strict shin-tableaux T of shape α. The *row-strict shin functions* $\mathfrak{R}\boldsymbol{\mathfrak{w}}_\alpha$ are defined as the duals in *NSym* to the row-strict extended Schur functions in *QSym*.

The *descent* set is defined to be $Des_{\mathfrak{Rw}}(U) = \{i : i+1 \text{ is weakly above } i \text{ in } U\}$ for a standard row-strict shin-tableau U. Each entry i in $Des_{\mathfrak{Rw}}(U)$ is called a *descent* of U. The *descent composition* of U is defined to be $co_{\mathfrak{Rw}}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{Rw}}(U) = \{i_1, \ldots, i_d\}$. Note that the set of standard row-strict shin-tableaux is exactly the same as the set of standard shin-tableaux.

Using the framework of standard row-strict shin-tableaux, it is shown in [68] that for a composition α, the row-strict extended Schur function expands into the fundamental basis as

$$\mathfrak{Rw}^*_\alpha = \sum_U F_{co_{\mathfrak{Rw}}(U)}, \tag{4.5}$$

where the sum runs over all standard row-strict shin-tableaux.

Example 4.1.13. The F-expansion of the row-strict extended Schur function $\mathfrak{Rw}^*_{(2,3)}$ and the standard row-strict shin-tableaux of shape $(2,3)$ are:

$$\mathfrak{Rw}^*_{(2,3)} = F_{(1,2,1,1)} + F_{(2,2,1)}$$

1	2	
3	4	5

1	3	
2	4	5

The extended Schur and row-strict extended Schur functions are related by ψ, most easily seen via their F-expansions. This relationship follows from the fact that the set of standard tableaux is the same but the definitions of descent sets are in a sense complementary and the map ψ extends the complement map on compositions.

Proposition 4.1.14. *[68] For a composition α,*

$$\psi(\mathfrak{w}^*_\alpha) = \mathfrak{Rw}^*_\alpha \quad \textit{and} \quad \psi(\mathfrak{w}_\alpha) = \mathfrak{Rw}_\alpha.$$

Moreover, $\{\mathfrak{Rw}^*_\alpha\}_\alpha$ is a basis of QSym and $\{\mathfrak{Rw}_\alpha\}_\alpha$ is a basis of NSym.

Recall that ψ on $QSym$ restricts to ω on Sym, so it follows that $\mathfrak{Rw}^*_\lambda = s_{\lambda'}$ for a partition λ. This is also easily seen from the (cancellation-free) expansions of the row-strict extended Schur functions into the monomial and fundamental bases, which follow from Equation (4.5). For a composition α,

$$\mathfrak{Rw}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{Rw}}_{\alpha,\beta} M_\beta \quad \text{and} \quad \mathfrak{Rw}^*_\alpha = \sum_\beta \mathcal{L}^{\mathfrak{Rw}}_{\alpha,\beta} F_\beta,$$

where $\mathcal{K}^{\mathfrak{Rw}}_{\alpha,\beta}$ is the number of row-strict shin tableaux of shape α and type β, and $\mathcal{L}^{\mathfrak{Rw}}_{\alpha,\beta}$ is the number of standard row-strict shin tableaux of shape α with descent composition β. Dually, for $\beta \models n$,

$$H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{Rw}}_{\alpha,\beta} \mathfrak{Rw}_\alpha \qquad \text{and} \qquad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{Rw}}_{\alpha,\beta} \mathfrak{Rw}_\alpha. \tag{4.6}$$

4.2 A Jacobi-Trudi rule for certain shin functions

The Schur functions and the immaculate functions can both be defined in terms of creation operators [9]. It is using these operators that one can prove various properties of the immaculate basis including the Jacobi-Trudi rule [9], a left Pieri rule [13], a combinatorial interpretation of the inverse Kostka matrix [5], and a partial Littlewood Richardson rule [11]. Here we give similar creation operators for certain shin functions which allow us to define a Jacobi-Trudi rule. This rule is especially useful because there is currently no other combinatorial way to expand shin functions into the complete homogeneous basis. We denote our operators with $\beth$, the Hebrew character *Bet*.

Definition 4.2.1. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $k \geq 1$ and a positive integer m, define the linear operator $\beth_m$ on the complete homogeneous basis by

$$\beth_m(H_\alpha) = H_{(m,\alpha_1,\alpha_2,\ldots)} - H_{(\alpha_1,m,\alpha_2,\ldots)}, \qquad \beth_m(1) = H_m.$$

Example 4.2.2. For example, $\beth_2(H_{(3,1)}) = H_{(2,3,1)} - H_{(3,2,1)}$.

Using this definition, we describe how these operators act on certain shin functions.

Theorem 4.2.3. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $k \geq 1$ and $0 < m < \alpha_1$, then $\beth_m(\text{ש}_\alpha) = \text{ש}_{(m,\alpha)}$.*

Proof. Define $f_\alpha = \beth_{\alpha_1}(\text{ש}_{\bar{\alpha}})$ for any nonempty $\alpha = (\alpha_1, \ldots, \alpha_k)$ where $\alpha_1 < \alpha_2$ and $\bar{\alpha} = (\alpha_2, \ldots, \alpha_k)$. We want to show that $f_\alpha = \text{ש}_\alpha$ for all such α. In the case that $\ell(\alpha) = 1$, we simply have $f_\alpha = \beth_{\alpha_1}(1) = H_{\alpha_1} = \text{ש}_{\alpha_1}$.

Now, consider only α where $\ell(\alpha) > 1$. First, we show that the f_α's satisfy the right Pieri rule for shin functions. Observe that, for a positive integer m and a nonempty composition $\gamma = (\gamma_1, \gamma_2, \ldots)$,

$$\beth_m(H_\gamma)H_s = H_{m,\gamma_1,\gamma_2,\ldots,s} - H_{\gamma_1,m,\gamma_2,\ldots,s} = \beth_m(H_{\gamma\cdot s}).$$

Since $\beth_m$ is a linear operator, $\beth_m(\text{ש}_\gamma)$ is equivalent to the sum of $\beth_m$ applied to each term in the H-expansion of ש_γ. Thus, $\beth_m(\text{ש}_\gamma)H_s = \beth_m(\text{ש}_\gamma H_s)$ for any composition γ and positive integer s. For $\alpha = (\alpha_1, \ldots, \alpha_k)$ where $\alpha_1 < \alpha_2$ and $\ell(\alpha) > 1$, it follows that $f_\alpha H_m = \beth_{\alpha_1}(\text{ש}_{\bar{\alpha}})H_m = \beth_{\alpha_1}(\text{ש}_{\bar{\alpha}} H_m)$. We expand this expression further using the shin right Pieri rule (Definition 4.1.5). Notice that if $\alpha_1 < \alpha_2$ and $\bar{\alpha} \subset^{\text{ש}}_m \beta$ for a composition $\beta = (\beta_1, \ldots, \beta_{k-1})$, then $\alpha_2 \leq \beta_1$ which implies $\alpha_1 < \beta_1$. Thus,

$$f_\alpha H_m = \beth_{\alpha_1}\left(\sum_{\bar{\alpha} \subset^{\text{ש}}_m \beta} \text{ש}_\beta\right) = \sum_{\bar{\alpha} \subset^{\text{ש}}_m \beta} \beth_{\alpha_1}(\text{ש}_\beta) = \sum_{\bar{\alpha} \subset^{\text{ש}}_m \beta} f_{\alpha_1 \cdot \beta}.$$

Observe that every γ where $\alpha \subset^{\text{ש}}_m \gamma$ has $\alpha_1 \leq \gamma_1$, and we can only have $\alpha_1 < \gamma_1$ if $\alpha_j \leq \gamma_j < \alpha_1$ for all $j > 1$. For α, we know $\alpha_2 > \alpha_1$, implying that γ_2 could never be less than α_1. Thus, $\gamma_1 > \alpha_1$ is not an option because it would violate the overhang rule, meaning has $\gamma_1 = \alpha_1$. It follows that the set of γ such that $\alpha \subset_m^{\shin} \gamma$ is the same as the set of $\alpha_1 \cdot \beta$ such that $\bar{\alpha} \subset_m^{\shin} \beta$. Therefore,

$$f_\alpha H_m = \sum_{\alpha \subset_m^{\shin} \gamma} f_\gamma. \tag{4.7}$$

Next, we show by recursive calculation that $f_\alpha = \shin_\alpha$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ where $\ell(\alpha) > 2$ and $\alpha_1 < \alpha_2$. Let $\alpha^\natural = (\alpha_1, \ldots, \alpha_{k-1})$ and $\bar{\alpha}^\natural = (\alpha_2, \ldots, \alpha_{k-1})$, and note that because we assumed $\alpha_1 < \alpha_2$, we have $\beth_{\alpha_1}(\shin_{\bar{\alpha}^\natural}) = f_{\alpha^\natural}$. Observe that $f_{\alpha^\natural} H_{\alpha_k} = \sum_{\alpha^\natural \subset_{\alpha_k}^{\shin} \beta} f_\beta$ by Equation (4.7), and rearranging this expression yields

$$f_\alpha = f_{\alpha^\natural} H_{\alpha_k} - \sum_{\substack{\alpha^\natural \subset_{\alpha_k}^{\shin} \beta \\ \beta \neq \alpha}} f_\beta. \tag{4.8}$$

Thus, we have a recursive formula for f_α in terms of $f_{\alpha^\natural}$ and f_β where either $\ell(\beta) < \ell(\alpha)$ or $\ell(\beta) = \ell(\alpha)$ and $\beta_k < \alpha_k$. That is to say, our recursive formula is defined in terms of f indexed either by a composition shorter than α or the same length as α but with a shorter final row.

It simply remains to show that $f_\alpha = \shin_\alpha$ when $\ell(\alpha) = 2$ as our base case and this recursive definition implies $f_\alpha = \shin_\alpha$ because it matches the recursive definition for $\shin_\alpha$ given by manipulating the right Pieri rule of Definition 4.1.5 in the same way (and has the same base case). Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 < \alpha_2$. Then we have

$$f_\alpha = \beth_{\alpha_1}(\shin_{\alpha_2}) = \beth_{\alpha_1}(H_{\alpha_2}) = H_{\alpha_1} H_{\alpha_2} - H_{\alpha_2} H_{\alpha_1}.$$

Next, observe by Definition 4.1.5 we have

$$H_{\alpha_1} H_{\alpha_2} - H_{\alpha_2} H_{\alpha_1} = \shin_{\alpha_1} H_{\alpha_2} - \shin_{\alpha_2} H_{\alpha_1} = \sum_{\alpha_1 \subset_{\alpha_2}^{\shin} \beta} \shin_\beta - \sum_{\alpha_2 \subset_{\alpha_1}^{\shin} \gamma} \shin_\gamma.$$

The set of β such that $\alpha_1 \subset_{\alpha_2}^{\shin} \beta$ is given by either $\beta = (\beta_1, \beta_2)$ where $\beta_1 \geq \alpha_2$ and $\beta_2 = \alpha_1 + \alpha_2 - \beta_1$ or $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$, due to the overhang rule. The set of γ such that $\alpha_2 \subset_{\alpha_1}^{\shin} \gamma$ is given by $\gamma = (\gamma_1, \gamma_2)$ where $\gamma_1 \geq \alpha_1$ and $\gamma_2 = \alpha_1 + \alpha_2 - \gamma_1$. Then $\sum_{\alpha_1 \subset_{\alpha_2}^{\shin} \beta} \shin_\beta - \sum_{\alpha_2 \subset_{\alpha_1}^{\shin} \gamma} \shin_\gamma = \shin_{(\alpha_1, \alpha_2)}$. Thus, we have shown that $f_\alpha = \shin_\alpha$ when $\ell(\alpha) = 2$. □

Example 4.2.4. Applying $\beth_2$ to the H-expansion of $\shin_{(3,1)}$ yields the H-expansion of $\shin_{(2,3,1)}$.

$$\shin_{(3,1)} = H_{(3,1)} - H_{(4)} \qquad \text{and} \qquad \shin_{(2,3,1)} = \beth_2(\shin_{(3,1)}) = H_{(2,3,1)} - H_{(3,2,1)} - H_{(2,4)} + H_{(4,2)}.$$

The creation operators also allow us to construct shin functions indexed by strictly increasing compositions from the ground up.

Corollary 4.2.5. *Let $\beta = (\beta_1, \ldots, \beta_k)$ be a strictly increasing sequence where $\beta_i < \beta_{i+1}$ for all i. Then,*

$$\shin_\beta = \beth_{\beta_1} \cdots \beth_{\beta_k}(1).$$

Example 4.2.6. Creation operators are used to build up a shin function as follows.

$$\text{ש}_{(1,3,4)} = \beth_1\beth_3\beth_4(1) = \beth_1\beth_3(H_{(4)}) = \beth_1(H_{(3,4)} - H_{(4,3)}) = H_{(1,3,4)} - H_{(1,4,3)} - H_{(3,1,4)} + H_{(4,1,3)}.$$

We also define a Jacobi-Trudi rule to express these same shin functions as matrix determinants. This formula is computationally much simpler and combinatorially more straightforward to work with. Let $S_k^{\geq}(-1)$ be the set of permutations $\sigma \in S_k$ such that $\sigma(i) \geq i-1$ for all $i \in [k]$.

Theorem 4.2.7. *Let $\beta = (\beta_1, \ldots, \beta_k)$ be a composition such that $\beta_i < \beta_{i+1}$ for all i. Then,*

$$\text{ש}_\beta = \sum_{\sigma \in S_k^{\geq}(-1)} (-1)^\sigma H_{\beta_{\sigma(1)}} \cdots H_{\beta_{\sigma(k)}}.$$

Equivalently, ש_β can be expressed as the matrix determinant of

$$\text{ש}_\beta = det \begin{vmatrix} H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & 0 & \cdots & 0 & H_{\beta_{k-1}} & H_{\beta_k} \end{vmatrix}$$

using the noncommutative analogue to the determinant obtained by expanding along the first row.

Proof. We proceed by induction on the length of β. If $\ell(\beta) = 1$ then $\text{ש}_\beta = H_\beta = det|H_{\beta_1}|$, and our claim holds. Assume that our claim holds for all β with $\ell(\beta) = k-1$. Now consider $\beta = (\beta_1, \ldots, \beta_k)$. Since $\beta_1 < \beta_2$ we have $\text{ש}_\beta = \beth_{\beta_1}(\text{ש}_{\bar{\beta}})$ where $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_{k-1}) = (\beta_2, \ldots, \beta_k)$. By our inductive assumption,

$$\text{ש}_{\bar{\beta}} = \sum_{\pi \in S_{k-1}^{\geq}(-1)} (-1)^\pi H_{\bar{\beta}_{\pi(1)}} \cdots H_{\bar{\beta}_{\pi(k-1)}},$$

where the sum runs over π such that $\pi(i) \geq i-1$ for all $i \in [k-1]$. Then, applying $\beth_{\beta_1}$ yields

$$\text{ש}_\beta = \sum_{\pi \in S_{k-1}^{\geq}(-1)} (-1)^\pi H_{\beta_1} H_{\bar{\beta}_{\pi(1)}} \cdots H_{\bar{\beta}_{\pi(k-1)}} - \sum_{\pi \in S_{k-1}^{\geq}(-1)} (-1)^\pi H_{\bar{\beta}_{\pi(1)}} H_{\beta_1} H_{\bar{\beta}_{\pi(2)}} \cdots H_{\bar{\beta}_{\pi(k-1)}}$$

Now we rewrite our sums in terms of some subset of permutations σ of k and β with terms that look like $H_{\beta_{\sigma(1)}} H_{\beta_{\sigma(2)}} \cdots H_{\beta_{\sigma(k)}}$. We need to find the appropriate subset of S_k so that we correctly rewrite our sum. For the first portion, we need $H_{\beta_{\sigma(1)}} = H_{\beta_1}$ so we set $\sigma(1) = 1$. Note that for all $i \in [k-1]$ we have $\bar{\beta}_i = \beta_{i+1}$. We also need $H_{\beta_{\sigma(i)}} = H_{\bar{\beta}_{\pi(i-1)}}$ which would mean $\sigma(i) = \pi(i-1)+1$. Then $\sigma(i) \geq i-1$ for any $1 < i \leq k$ because $\pi \in S_{k-1}^{\geq}(-1)$. Since we already set $\sigma(1) = 1 \geq 0$, we have shown that the permutations σ satisfy $\sigma(i) \geq i-1$ for all i. For the right-hand term, we need $H_{\beta_{\sigma(2)}} = H_{\beta_1}$ so we limit our sum to permutations where $\sigma(2) = 1$. Note that we need $\beta_{\sigma(1)} = \bar{\beta}_{\pi(1)} = \beta_{\pi(1)+1}$ so $\sigma(1) \geq 2$, and $\beta_{\sigma(i)} = \bar{\beta}_{\pi(i-1)} = \beta_{\pi(i-1)+1}$ so $\sigma(i) \geq i-1$ for all $i \geq 3$. These are exactly the permutations σ such that $\sigma(i) \geq i-1$ and $\sigma(2) = 1$. Therefore,

$$\begin{aligned}
\mathfrak{w}_\beta &= \sum_{\substack{\sigma \in S_k^{\geq}(-1), \\ \sigma(1)=1}} (-1)^\sigma H_{\beta_{\sigma(1)}} H_{\beta_{\sigma(2)}} \cdots H_{\beta_{\sigma(k)}} + \sum_{\substack{\sigma \in S_k^{\geq}(-1) \\ \sigma(2)=1}} (-1)^\sigma H_{\beta_{\sigma(1)}} H_{\beta_{\sigma(2)}} \cdots H_{\beta_{\sigma(k)}} \\
&= \sum_{\sigma \in S_k^{\geq}(-1)} (-1)^\sigma H_{\beta_{\sigma(1)}} H_{\beta_{\sigma(2)}} \cdots H_{\beta_{\sigma(k)}}.
\end{aligned}$$

This is equivalent to the matrix determinant described using the typical expansion of determinants in terms of permutations, excluding those terms that would be multiplied by 0. □

Example 4.2.8. The function from Example 4.2.6 expands as a matrix determinant as

$$\mathfrak{w}_{(1,3,4)} = det \begin{vmatrix} H_1 & H_3 & H_4 \\ H_1 & H_3 & H_4 \\ 0 & H_3 & H_4 \end{vmatrix} = H_1H_3H_4 - H_1H_4H_3 - H_3H_1H_4 + H_4H_1H_3.$$

It remains open to find a combinatorial or algebraic way of understanding the expansion of the shin basis into the complete homogeneous basis for the general case. We show by counterexample that there is not a matrix rule of this form for every shin function, including those indexed by partitions. The smallest counterexample for partitions is $\mathfrak{w}_{(2,2,2,1)}$. The argument for it is too long, and so instead we present a smaller example using similar logic.

Example 4.2.9. We have $\mathfrak{w}_{(2,2,4)} = H_{(2,2,4)} - H_{(2,4,2)} - H_{(3,1,4)} + H_{(4,3,1)} + H_{(5,1,2)} - H_{(5,2,1)}$. For the determinant of a 3×3 matrix of the form $(H_{b_{(i,j)}})_{1 \leq i,j \leq 3}$ with $b_{(i,j)} \in \mathbb{Z}$ to yield this expression, we would need H_2, H_3, H_4, and H_5 to be in the first row. This is impossible, so such a matrix does not exist.

4.3 Skew extended Schur functions

We define skew extended Schur functions algebraically and then connect to tableaux combinatorics.

Definition 4.3.1. For compositions α and β with $\beta \subseteq \alpha$, the *skew extended Schur functions* are defined as

$$\mathfrak{w}^*_{\alpha/\beta} = \mathfrak{w}^\perp_\beta(\mathfrak{w}^*_\alpha).$$

By Equation (3.4), $\boldsymbol{\mathfrak{w}}^*_{\alpha/\beta}$ expands into various bases as follows. For $\beta \subseteq \alpha$,

$$\boldsymbol{\mathfrak{w}}^*_{\alpha/\beta} = \sum_{\gamma} \langle \boldsymbol{\mathfrak{w}}_\beta H_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle M_\gamma = \sum_{\gamma} \langle \boldsymbol{\mathfrak{w}}_\beta R_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle F_\gamma = \sum_{\gamma} \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\gamma. \tag{4.9}$$

The coefficients $\mathcal{C}^\alpha_{\beta,\gamma} = \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle$ appear when multiplying shin functions,

$$\boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma = \sum_{\alpha} \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}_\alpha.$$

Many of the shin structure coefficients are either 0 or equal to the Littlewood Richardson coefficients as a result of the shin basis' relationship with the Schur functions.

Proposition 4.3.2. *Let α, β, and γ be compositions that are not partitions and let λ, μ, and ν be partitions. Then,*

$$\mathcal{C}^\lambda_{\beta,\gamma} = \mathcal{C}^\lambda_{\beta,\nu} = \mathcal{C}^\lambda_{\mu,\gamma} = 0 \quad \textit{and} \quad \mathcal{C}^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu},$$

where $c^\lambda_{\mu,\nu}$ are the usual Littlewood-Richardson coefficients.

Proof. For any compositions α and β,

$$\boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma = \sum_{\alpha} \mathcal{C}^\alpha_{\beta,\gamma} \boldsymbol{\mathfrak{w}}_\alpha.$$

Observe that if either β or γ is a composition that is not a partition then

$$\chi(\boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma) = \chi(\boldsymbol{\mathfrak{w}}_\beta)\chi(\boldsymbol{\mathfrak{w}}_\gamma) = 0 = \sum_{\alpha \models |\beta|+|\gamma|} \mathcal{C}^\alpha_{\beta,\gamma} \chi(\boldsymbol{\mathfrak{w}}_\alpha) = \sum_{\lambda \vdash |\beta|+|\gamma|} \mathcal{C}^\lambda_{\beta,\gamma} s_\lambda.$$

Thus, $\mathcal{C}^\lambda_{\beta,\gamma} = 0$ for all partitions λ when either β or γ is not a partition. If β and γ are both partitions, which we instead call μ and ν, we have

$$\chi(\boldsymbol{\mathfrak{w}}_\mu \boldsymbol{\mathfrak{w}}_\nu) = \chi(\boldsymbol{\mathfrak{w}}_\mu)\chi(\boldsymbol{\mathfrak{w}}_\nu) = s_\mu s_\nu = \sum_{\alpha \models |\mu|+|\nu|} \mathcal{C}^\alpha_{\mu,\nu} \chi(\boldsymbol{\mathfrak{w}}_\alpha) = \sum_{\lambda \vdash |\mu|+|\nu|} \mathcal{C}^\lambda_{\mu,\nu} s_\lambda,$$

so $\mathcal{C}^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu}$, the classic Littlewood-Richardson coefficients, for partitions λ. □

Example 4.3.3. Below are different shin products that result by varying the left and right indicies are partitions or compositions (computed in Sagemath [83]).

$$\begin{aligned}
\boldsymbol{\mathfrak{w}}_{(1,3)}\boldsymbol{\mathfrak{w}}_{(2,2)} &= \boldsymbol{\mathfrak{w}}_{(1,3,2,2)} + \boldsymbol{\mathfrak{w}}_{(1,4,1,2)} + \boldsymbol{\mathfrak{w}}_{(1,4,2,1)} + \boldsymbol{\mathfrak{w}}_{(1,5,2)} \\
\boldsymbol{\mathfrak{w}}_{(3)}\boldsymbol{\mathfrak{w}}_{(2,3,2)} &= \boldsymbol{\mathfrak{w}}_{(3,2,3,2)} + \boldsymbol{\mathfrak{w}}_{(4,1,3,2)} - \boldsymbol{\mathfrak{w}}_{(4,2,4)} - \boldsymbol{\mathfrak{w}}_{(5,1,4)} \\
\boldsymbol{\mathfrak{w}}_{(1,3)}\boldsymbol{\mathfrak{w}}_{(1,2,3)} &= \boldsymbol{\mathfrak{w}}_{(1,3,1,2,3)} - \boldsymbol{\mathfrak{w}}_{(1,4,1,3,1)} - \boldsymbol{\mathfrak{w}}_{(1,4,1,4)} - \boldsymbol{\mathfrak{w}}_{(1,5,1,3)} \\
\boldsymbol{\mathfrak{w}}_{(2,1,1)}\boldsymbol{\mathfrak{w}}_{(1,1)} &= \boldsymbol{\mathfrak{w}}_{(2,1,1,1,1)} + \boldsymbol{\mathfrak{w}}_{(2,1,2,1)} + \boldsymbol{\mathfrak{w}}_{(2,2,1,1)} + \boldsymbol{\mathfrak{w}}_{(2,2,2)} + \boldsymbol{\mathfrak{w}}_{(3,1,1,1)} + \boldsymbol{\mathfrak{w}}_{(3,1,2)} + \boldsymbol{\mathfrak{w}}_{(3,2,1)}.
\end{aligned}$$

Note that in larger cases, $\mathfrak{w}_\lambda\mathfrak{w}_\mu$ (for λ, μ partitions) can contain negative terms. For example, $\mathfrak{w}_{(3,1)}\mathfrak{w}_{(2,2,1)}$ has the term $-\mathfrak{w}_{(4,5)}$ in its shin basis expansion.

The skew shin functions are defined combinatorially over skew shin-tableaux as follows.

Proposition 4.3.4. *For compositions α and β such that $\beta \subseteq \alpha$, the coefficient $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha \rangle$ is equal to the number of skew shin-tableaux of shape α/β and type γ. Moreover,*

$$\mathfrak{w}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over skew shin-tableaux T of shape α/β.

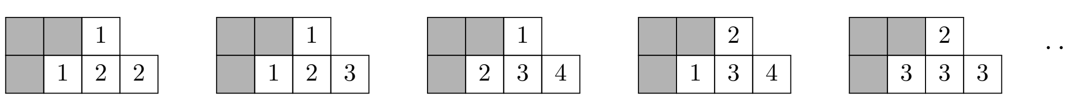
Proof. Observe that

$$\mathfrak{w}_\beta H_\gamma = \sum_{\beta=\beta^{(0)} \subset^{\mathfrak{w}}_{\gamma_1} \beta^{(1)} \subset^{\mathfrak{w}}_{\gamma_2} \beta^{(2)} \cdots \subset^{\mathfrak{w}}_{\gamma_j} \beta^{(j)}=\alpha} \mathfrak{w}_\alpha$$

via repeated applications of the right Pieri rule. Then, $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha \rangle$ counts the number of unique sequences $(\beta_0, \beta_1, \ldots, \beta_j)$ such that $\beta = \beta^{(0)} \subset^{\mathfrak{w}}_{\gamma_1} \beta^{(1)} \subset^{\mathfrak{w}}_{\gamma_2} \beta^{(2)} \cdots \subset^{\mathfrak{w}}_{\gamma_j} \beta^{(j)} = \alpha$. Each of these sequences is associated with a unique skew shin-tableau of shape α/β with type γ by filling with $i's$ the blocks in $\beta^{(i)}/\beta^{(i-1)}$. The containment condition of the Pieri rule ensures that rows are weakly increasing and the overhang axiom ensures that columns are strictly increasing. The overhang axiom also ensures that if $\alpha_i > \beta_i$ for any $1 \le i \le k$, then $\beta_j < \beta_i$ for all $j > i$. Thus, our tableau T is indeed a skew shin-tableaux. It is simple to see that any skew shin-tableau can be expressed by such a sequence, showing that $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha \rangle$ is equal to the number of skew shin-tableaux of shape α/β and type γ. It now follows from Equation (4.9) that $\mathfrak{w}^*_{\alpha/\beta} = \sum_T x^T$ where the sum runs over all skew shin-tableaux of shape α/β. $\square$

Note that in many cases these functions are equal to 0 because skew shin tableaux are not defined on all skew shapes α/β.

Example 4.3.5. The skew extended Schur function indexed by $(3, 4)/(2, 1)$ is given by



$$\mathfrak{w}^*_{(3,4)/(2,1)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_3x_4 + x_2x_3^3 + \ldots$$

Skew shin-tableaux of shape λ/μ for partitions λ, μ are skew semistandard Young tableaux and thus by Proposition 4.3.4, the skew extended Schur functions are equal to the skew Schur functions. That is, for partitions λ and μ where $\mu \subseteq \lambda$,

$$\mathfrak{w}^*_{\lambda/\mu} = s_{\lambda/\mu}.$$

Remark 4.3.6. It also follows from Proposition 4.3.4 that $\mathcal{C}^{\alpha}_{\beta,\gamma} = 0$ if there exists some i and j with $i < j$ where $\alpha_i > \beta_i$ but $\beta_j > \beta_i$. This tells us that many terms that do not appear in $\mathfrak{w}_\beta \mathfrak{w}_\gamma$.

Standard skew shin-tableaux are also closely related to the poset structure on composition diagrams whose cover relations are determined by the addition of a shin-horizontal-strip of size 1.

Definition 4.3.7. Define the *shin poset*, denoted $\mathcal{P}^{\mathfrak{w}}$, as the poset on compositions with the cover relation $\subset^{\mathfrak{w}}_1$. In the Hasse diagram, label an edge between two elements α and β with the integer m if α differs from β by the addition of a box to row m.

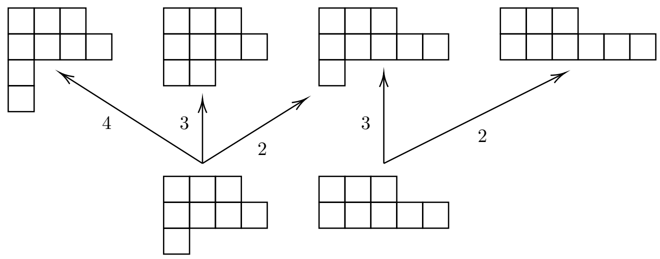
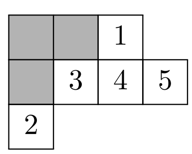


Figure 4.1: A section of the shin poset $\mathcal{P}^{\mathfrak{w}}$.

That is to say, β covers α in the shin poset if β differs from α by the addition of a shin-horizontal strip of size 1 (a single box). Note that the subposet of $\mathcal{P}^{\mathfrak{w}}$ induced on partitions corresponds to Young's Lattice. Further, we generally visualize the elements of this poset as the diagrams associated with each composition.

Maximal chains from β to α in $\mathcal{P}^{\mathfrak{w}}$ are associated with a skew standard shin-tableau of shape α/β. The chain $C = \{\beta = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \cdots \xrightarrow{m_k} \beta^{(k)} = \alpha\}$ is associated with the skew standard shin-tableau T of shape α/β where the boxes are filled with the integers 1 through k in the order they are added on the chain. Thus, the box added from $\beta^{(j)} \xrightarrow{m_{j+1}} \beta^{(j+1)}$ is filled with $j+1$. If $\beta = \emptyset$ then the chain C is associated with a standard shin-tableau that is not skew.

Example 4.3.8. The maximal chain $C = \{(2,1) \xrightarrow{1} (3,1) \xrightarrow{3} (3,1,1) \xrightarrow{2} (3,2,1) \xrightarrow{2} (3,3,1) \xrightarrow{2} (3,4,1)\}$ is associated with the skew standard shin-tableau:



These skew functions also have the usual relationship with comultiplication.

Proposition 4.3.9. *For a composition α,*

$$\Delta \mathfrak{w}^*_\alpha = \sum_{\beta \subseteq \alpha} \mathfrak{w}^*_\beta \otimes \mathfrak{w}^*_{\alpha/\beta},$$

where the sum runs over compositions β.

Proof. The product of the shin functions uniquely defines the coproduct of the extended Schur functions due to Hopf algebra properties [37]. Specifically,

$$\begin{aligned}\Delta(\boldsymbol{\mathfrak{w}}^*_\alpha) &= \sum_{\beta,\gamma} \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\beta \otimes \boldsymbol{\mathfrak{w}}^*_\gamma = \sum_\beta \left(\boldsymbol{\mathfrak{w}}^*_\beta \otimes \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\gamma \right) \\ &= \sum_\beta \boldsymbol{\mathfrak{w}}^*_\beta \otimes \boldsymbol{\mathfrak{w}}^*_{\alpha/\beta} \quad \text{by Equation (4.9).}\end{aligned}$$

□

We can also defined a skew-II variant using the right-perp operator.

Definition 4.3.10. For compositions α and β, the *skew-II extended Schur function* is defined as

$$\boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta} = \boldsymbol{\mathfrak{w}}^{\perp}_\beta(\boldsymbol{\mathfrak{w}}^*_\alpha).$$

By Definition 3.2.1, $\boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta}$ expands into various bases as follows. For compositions α and β,

$$\boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta} = \sum_\gamma \langle H_\gamma \boldsymbol{\mathfrak{w}}_\beta, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle M_\gamma = \sum_\gamma \langle R_\gamma \boldsymbol{\mathfrak{w}}_\beta, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle F_\gamma = \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\gamma \boldsymbol{\mathfrak{w}}_\beta, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\gamma. \tag{4.10}$$

In terms of the shin structure coefficients, we have $\boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta} = \sum_\gamma \mathcal{C}^\alpha_{\gamma,\beta} \boldsymbol{\mathfrak{w}}^*_\gamma$.

Remark 4.3.11. According to calculations done in Sagemath [83], the skew-II extended Schur function $\boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta}$ does not expand positively into the monomial basis. For example, $\boldsymbol{\mathfrak{w}}^*_{(2,1,3) /\!\!/ (1,2,1)}$ has the term $-M_{(1,1)}$ in its expansion. Thus, these functions cannot be expressed as positive sums of a skew-II shin-tableaux.

The comultiplication of the extended Schur basis can also be expressed in terms of the skew-II extended Schur functions. Note that a similar formula could be given for any basis using analogous skew-II functions defined via the perp operator.

Proposition 4.3.12. *For a composition α,*

$$\Delta(\boldsymbol{\mathfrak{w}}^*_\alpha) = \sum_\beta \boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta} \otimes \boldsymbol{\mathfrak{w}}^*_\beta,$$

where the sum runs over compositions β where $\beta^r \subseteq \alpha^r$.

Proof. For a composition α,

$$\Delta(\boldsymbol{\mathfrak{w}}^*_\alpha) = \sum_{\beta,\gamma} \langle \boldsymbol{\mathfrak{w}}_\gamma \boldsymbol{\mathfrak{w}}_\beta, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\gamma \otimes \boldsymbol{\mathfrak{w}}^*_\beta = \sum_\beta \left(\sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\gamma \boldsymbol{\mathfrak{w}}_\beta, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \boldsymbol{\mathfrak{w}}^*_\gamma \right) \otimes \boldsymbol{\mathfrak{w}}^*_\beta = \sum_\beta \boldsymbol{\mathfrak{w}}^*_{\alpha /\!\!/ \beta} \otimes \boldsymbol{\mathfrak{w}}^*_\beta.$$

□

where the sum runs over compositions β.

4.4 Involutions on the extended Schur and shin bases

We introduce two new pairs of dual bases ($\boldsymbol{\mathfrak{m}}$, $\boldsymbol{\mathfrak{m}}^*$ and $\mathfrak{R}\boldsymbol{\mathfrak{m}}$, $\mathfrak{R}\boldsymbol{\mathfrak{m}}^*$) in $QSym$ and $NSym$ by applying ρ and ω to the extended Schur and shin functions. Applying ψ to the extended Schur and shin functions recovers the row-strict shin and row-strict extended Schur functions ($\mathfrak{R}\boldsymbol{\mathfrak{w}}$, $\mathfrak{R}\boldsymbol{\mathfrak{w}}^*$) of Niese, Sundaram, van Willigenburg, Vega, and Wang from [68]. Specifically, for a composition α, we have

$$\psi(\boldsymbol{\mathfrak{w}}^*_\alpha) = \mathfrak{R}\boldsymbol{\mathfrak{w}}^*_\alpha \qquad \rho(\boldsymbol{\mathfrak{w}}^*_\alpha) = \boldsymbol{\mathfrak{m}}^*_{\alpha^r} \qquad \omega(\boldsymbol{\mathfrak{w}}^*_\alpha) = \mathfrak{R}\boldsymbol{\mathfrak{m}}^*_{\alpha^r},$$

$$\psi(\boldsymbol{\mathfrak{w}}_\alpha) = \mathfrak{R}\boldsymbol{\mathfrak{w}}_\alpha \qquad \rho(\boldsymbol{\mathfrak{w}}_\alpha) = \boldsymbol{\mathfrak{m}}_{\alpha^r} \qquad \omega(\boldsymbol{\mathfrak{w}}_\alpha) = \mathfrak{R}\boldsymbol{\mathfrak{m}}_{\alpha^r}.$$

We give combinatorial interpretations of these 2 new pairs of bases in terms of variations on shin-tableaux. While specific definitions are to follow, we describe intuitively how ψ, ρ, and ω act on the tableaux defining each basis, which is the same as how they act on immaculate tableaux. Recall that shin-tableaux have weakly increasing columns and strictly increasing rows. The ψ map switches whether the strictly changing condition is on rows or columns (the other is allowed to change weakly). The ρ map switches the row condition from increasing to decreasing or vice versa. The ω map does both. Through this combinatorial interpretation, each of the four pairs of dual bases is related to any other by one of the three involutions ψ, ρ, or ω as shown in the figure below.

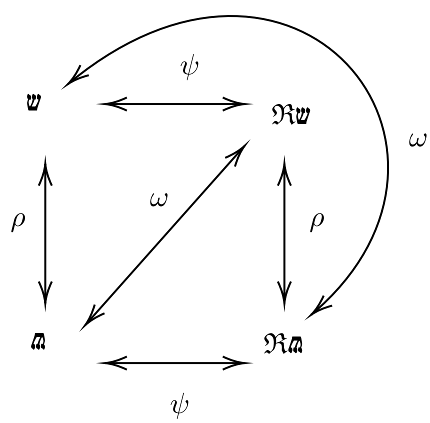


Figure 4.2: Mappings between shin variants in *NSym*.

Again our Schur-like bases are part of a system of 4 related bases that is closed with respect to ψ, ρ and ω. Table 4.1 summarizes the tableaux defined over the course of this section. It lists each type of tableaux, the position of $i+1$ relative to i that makes i a descent, the order the boxes appear in the reading word (Left, Right, Top, Bottom), the condition on entries of each row, and the condition on entries in each column.

Before we move on to our new bases, we apply ψ to various results on the shin and extended Schur bases to find analogous results on the row-strict shin and row-strict extended Schur bases.

	Descent	Reading Word	Rows	Columns
Shin	strictly below	L to R, B to T	weakly increasing	strictly inc.
Row-strict	weakly above	L to R, T to B	strictly increasing	weakly inc.
Reverse	strictly below	R to L, B to T	weakly decreasing	strictly inc.
Row-strict reverse	weakly above	R to L, T to B	strictly decreasing	weakly inc.

Table 4.1: Variations on shin tableaux.

Theorem 4.4.1. *Let α and β be compositions and let m be a positive integer.*

1. *(Right Pieri Rule)*
$$\mathfrak{Rw}_\alpha E_m = \sum_{\alpha \subset^{\mathfrak{w}}_m \beta} \mathfrak{Rw}_\beta.$$

2. *(Right Ribbon Multiplication)*
$$\mathfrak{Rw}_\alpha R_\beta = \sum_{\gamma \models |\alpha|+|\beta|} \sum_U \mathfrak{Rw}_\gamma,$$
where the sum runs over all skew standard row-strict shin-tableaux U of shape γ/α with $co_{\mathfrak{Rw}}(U) = \beta$.

3. $E_\beta = \sum_\alpha \mathcal{K}_{\alpha,\beta} \mathfrak{Rw}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha,\beta^c} \mathfrak{Rw}_\alpha$.

4. $\chi(\mathfrak{Rw}_\lambda) = s_{\lambda'}$ *for a partition λ and* $\chi(\mathfrak{Rw}_\alpha) = 0$ *when α is not a partition.*

5. *Let γ be a composition such that $\gamma_i < \gamma_{i+1}$ for all $1 \leq i < \ell(\gamma)$. Then,*
$$\mathfrak{Rw}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma E_{\gamma_{\sigma(1)}} E_{\gamma_{\sigma(2)}} \cdots E_{\gamma_{\sigma(\ell(\gamma))}},$$
where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \geq i-1$ for all $i \in [\ell(\gamma)]$.

6. $E_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{Rw}}_{\alpha,\beta} \mathfrak{w}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{Rw}}_{\alpha,\beta^c} \mathfrak{w}_\alpha$.

Proof. For a composition α, note that $\psi(H_\alpha) = E_\alpha$.

1. Apply ψ to Definition 4.1.5.

2. Applying ψ to the LHS of Theorem 4.1.10 yields $\mathfrak{Rw}_\alpha R_{\beta^c}$ while the RHS yields $\sum_{\gamma \models |\alpha|+|\beta|} \sum_U \mathfrak{Rw}_\gamma$ where the sum runs over all skew standard shin-tableaux U of shape γ/α and descent composition β. By switching β with β^c everywhere, we rewrite this equality as $\mathfrak{Rw}_\alpha R_\beta = \sum_{\gamma \models |\alpha|+|\beta|} \sum_U \mathfrak{Rw}_\gamma$ where the sum runs over all skew standard shin-tableaux U of shape γ/α with descent composition β^c. Recall that these are equivalent to the skew standard row-strict shin-tableaux of shape γ/α with descent composition β. Our claim follows.

3. Apply ψ to Equation (4.3).

4. Based on the restriction of ω to *Sym*, we have $\psi(\mathfrak{w}^*_\lambda) = \omega(s_\lambda) = s_{\lambda'}$. Because χ is dual to the inclusion map from *Sym* to *QSym* [19], we have $\chi(\mathfrak{R}\mathfrak{w}_\alpha) = \sum_\lambda(\text{coefficient of } \mathfrak{R}\mathfrak{w}^*_\alpha \text{ in } s_\lambda)s_\lambda$. Our claim follows.

5. Apply ψ to Theorem 4.2.7.

6. Apply ψ to Equation (4.6). □

We also develop row-strict versions of our results from Section 4.3.

Definition 4.4.2. For compositions α and β with $\beta \subseteq \alpha$, the *skew row strict extended Schur functions* are defined by

$$\mathfrak{R}\mathfrak{w}^*_{\alpha/\beta} = \mathfrak{R}\mathfrak{w}^\perp_\beta(\mathfrak{R}\mathfrak{w}^*_\alpha).$$

By Equation (3.4), $\mathfrak{R}\mathfrak{w}^*_{\alpha/\beta}$ expands into various bases as follows. For compositions $\beta \subseteq \alpha$,

$$\mathfrak{R}\mathfrak{w}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{R}\mathfrak{w}_\beta H_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle M_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{w}_\beta R_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle F_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle \mathfrak{R}\mathfrak{w}^*_\gamma. \tag{4.11}$$

Like before, the coefficients $\langle \mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle$ are the coefficients that appear when multiplying row-strict shin functions,

$$\mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma = \sum_\alpha \langle \mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle \mathfrak{R}\mathfrak{w}_\alpha.$$

Because ψ is an automorphism in both *QSym* and *NSym*, it maps the skew row-strict extended Schur to the extended Schur functions.

Proposition 4.4.3. *For compositions α and β with $\beta \subseteq \alpha$,*

$$\psi(\mathfrak{w}^*_{\alpha/\beta}) = \mathfrak{R}\mathfrak{w}^*_{\alpha/\beta}.$$

Proof. First, observe that

$$\langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle = \langle \psi(\mathfrak{w}_\beta \mathfrak{w}_\gamma), \psi(\mathfrak{w}^*_\alpha)\rangle = \langle \mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle,$$

because ψ is invariant under duality and an automorphism in *NSym*. Then,

$$\psi(\mathfrak{w}^*_{\alpha/\beta}) = \sum_\gamma \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle \psi(\mathfrak{w}^*_\gamma) = \sum_\gamma \langle \mathfrak{R}\mathfrak{w}_\beta \mathfrak{R}\mathfrak{w}_\gamma, \mathfrak{R}\mathfrak{w}^*_\alpha\rangle \mathfrak{R}\mathfrak{w}^*_\gamma.$$

By Equation (4.11), this implies $\psi(\mathfrak{w}^*_{\alpha/\beta}) = \mathfrak{R}\mathfrak{w}^*_{\alpha/\beta}$. □

The first line of the proof above yields the following.

Corollary 4.4.4. *For compositions α, β and γ,*

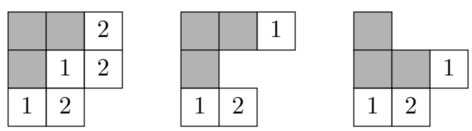
$$\mathfrak{Rw}_\beta \mathfrak{Rw}_\gamma = \sum_\alpha \mathcal{C}^\alpha_{\beta,\gamma} \mathfrak{Rw}_\alpha.$$

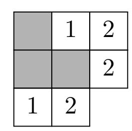
By Proposition 4.3.2, many of the coefficients of the skew row-strict extended Schur functions expanded into the row-strict extended Schur basis are zero while others equal certain Littlewood-Richardson coefficients. Thus, the same is true for the structure coefficients of the row-strict shin functions.

Like the skew extended Schur functions, the skew row-strict extended Schur functions are defined combinatorially via a class of skew tableaux.

Definition 4.4.5. For compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, b_\ell)$ such that $\beta \subseteq \alpha$, a *row-strict* skew shin-tableau of skew shape α/β is a diagram α/β filled with integers such that each row is strictly increasing left to right, each column is weakly increasing top to bottom, and if $\alpha_i > \beta_i$ for any $1 \leq i \leq k$, then $\beta_j < \beta_i$ for all $j > i$. A skew row-strict shin-tableau is *standard* if it contains the numbers 1 through $|\alpha| - |\beta|$ each exactly once.

Example 4.4.6. The three leftmost diagrams are skew row-strict shin-tableaux while the rightmost diagram is not.

Skew row-strict shin: 

Not: 

Proposition 4.4.7. *For compositions α and β where $\beta \subseteq \alpha$,*

$$\mathfrak{Rw}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over all skew row-strict shin-tableaux T of shape α/β.

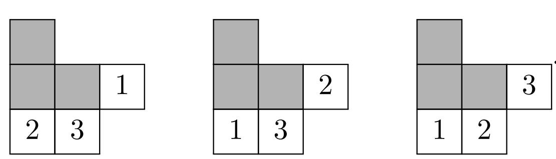
Proof. By Theorem 4.4.1 (2), the coefficient $\langle \mathfrak{Rw}_\beta R_\gamma, \mathfrak{Rw}^*_\alpha \rangle$ is the number of skew row-strict shin-tableaux of shape α/β with descent composition γ, which we denote $\mathcal{L}^{\mathfrak{Rw}}_{\alpha/\beta,\gamma}$. Then by Equation (4.11), we have

$$\mathfrak{Rw}^*_{\alpha/\beta} = \sum_\gamma \mathcal{L}^{\mathfrak{Rw}}_{\alpha/\beta,\gamma} F_\gamma.$$

From here it is simple to expand F_γ into a over tableaux and obtain our claim. □

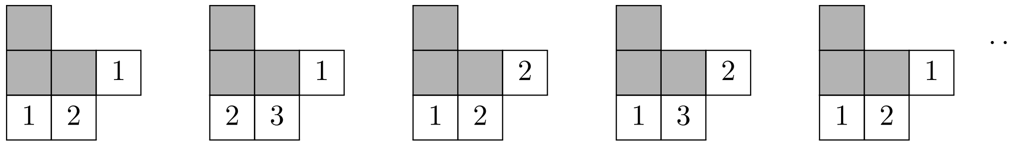
Example 4.4.8. The skew row-strict extended Schur function $\mathfrak{Rw}^*_{(1,3,2)/(1,2)}$ expands in terms of the fundamental basis as

$$\mathfrak{Rw}^*_{(1,3,2)/(1,2)} = F_{2,1} + F_{1,2} + F_{1,1,1}.$$



This further expands in terms of skew row-strict shin tableaux as

$$\mathfrak{Rw}^*_{(1,3,2)/(1,2)} = x_1^2x_2 + x_1x_2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 \cdots$$

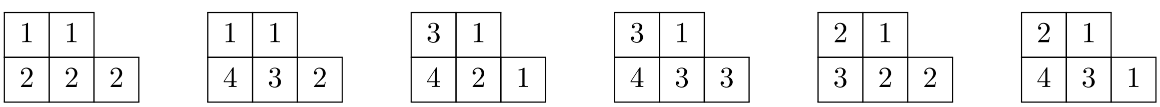


Now, we introduce the reverse extended Schur basis and row-strict reverse extended Schur basis in *QSym*, as well as their dual bases in *NSym*. Using the three involutions, we translate many results from the shin and extended Schur functions to our new bases, including the Jacobi-Trudi rule and two different types of skew functions. We also note connections between the row-strict reverse extended Schur basis and the antipode of *QSym* on the extended Schur functions, as well as the dual connection in *NSym*.

4.4.1 Reverse extended Schur and shin functions

Let α be a composition and β a weak composition. A *reverse shin-tableau* (RST) of shape α and type β is a diagram α filled with positive integers that weakly decrease along the rows from left to right and strictly increase along the columns from top to bottom, where each positive integer i appears β_i times. A *standard* reverse shin-tableau (SRST) of shape $\alpha \models n$ is one containing the entries 1 through n each exactly once.

Example 4.4.9. A few reverse shin-tableaux of shape $(2, 3)$ are



Definition 4.4.10. For a composition α, the *reverse extended Schur function* is defined as

$$\reflectbox{$\mathfrak{s}$}^*_\alpha = \sum_T x^T,$$

where the sum runs over all reverse shin-tableaux T of shape α.

The *descent set* is defined as $Des_{\reflectbox{$\mathfrak{s}$}}(S) = \{i : i+1 \text{ is strictly below } i \text{ in } S\}$ for a standard reverse shin-tableau S. Each entry i in $Des_{\reflectbox{$\mathfrak{s}$}}(S)$ is called a *descent* of S. The *descent composition* of S is defined $co_{\reflectbox{$\mathfrak{s}$}}(S) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d)$ for $Des_{\reflectbox{$\mathfrak{s}$}}(S) = \{i_1, \dots, i_d\}$. The *reverse shin-reading word* of a reverse shin-tableau T, denoted $rw_{\reflectbox{$\mathfrak{s}$}}(T)$ is obtained by reading the rows of T from right to left starting with the bottom row and moving up. To *standardize* a reverse shin-tableau T, replace the 1's in T with $1, 2, \dots$ in the order they appear in $rw_{\reflectbox{$\mathfrak{s}$}}(T)$, then the 2's starting with the next consecutive number, etc.

Proposition 4.4.11. *For a composition α,*

$$\shin^*_\alpha = \sum_S F_{co_{\shin}(S)},$$

where the sum runs over standard reverse shin-tableaux U of shape α.

Proof. We can write $\shin^*_\alpha = \sum_S \sum_{std(T)=S} x^T$ where the sums run over standard reverse shin-tableaux S of shape α and reverse shin-tableaux T that standardize to S. Now we want to show that, given a SRST S, we can write $F_{co_{\shin}(S)} = \sum_{std(T)=S} x^T$ where the sum runs over RST T that standardize to S. First, observe that given T such that $std(T) = S$, if $i \in Des_{\shin}(S)$ then the box in T corresponding to $i+1$ is strictly below the box corresponding to i. Therefore, by the order of standardization (right to left, bottom to top), the box corresponding to $\widetilde{i+1}$ must be filled with a strictly higher number than the box corresponding to i. It follows that $type(T)$ is a refinement of the descent composition of S. In fact, any refinement B of the descent composition of S is a possible type of a tableau T such that $std(T) = S$ because it is associated with a valid filling. Each box of S corresponds to a letter in B. If a box of S corresponds to a letter in word v_i of B, then fill that same box in T with an i. This is equivalent to reading through the boxes T in the order they appear in S, and filling them based on the location of the corresponding letter in B. Note that this method creates the unique tableau T with a specific type that standardizes to S because we have used the only possible order of filling to maintain our desired type and standardization. Thus,

$$F_{co_{\shin}(S)} = \sum_{\beta \preceq co_{\shin}(S)} M_\beta = \sum_{\beta \preceq co_{\shin}(S)} \sum_{\widetilde{type(T)}=\beta} x^T = \sum_{std(T)=S} x^T.$$

Therefore, we have $\shin^*_\alpha = \sum_S F_{co_{\shin}(S)}$ where the sum runs over SRSTs of shape α. □

Example 4.4.12. The expansion of the reverse extended Schur function $\shin^*_{(3,2)}$ into the fundamental basis and the standard reverse shin-tableaux of shape $(3,2)$ are:

$$\shin^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)}$$

3	2	1
5	4	

4	2	1
5	3	

Let $\mathcal{K}^{\shin}_{\alpha,\beta}$ be the number of RST of shape α and type β, and let $\mathcal{L}^{\shin}_{\alpha,\beta}$ be the number of SRST with shape α and descent composition β. Using Proposition 4.4.11, it is straightforward to show that the reverse extended Schur functions have the following positive (cancellation-free) expansions into the monomial and fundamental bases. For a composition α,

$$\shin^*_\alpha = \sum_\beta \mathcal{K}^{\shin}_{\alpha,\beta} M_\beta \quad \text{and} \quad \shin^*_\alpha = \sum_\beta \mathcal{L}^{\shin}_{\alpha,\beta} F_\beta. \tag{4.12}$$

Standard shin-tableaux and standard reverse shin-tableaux are again related by the $flip$ map, which is a bijection $flip$: {standard shin-tableaux} $\to$ {standard reverse shin-tableaux}. By construction, the descent composition of a standard shin-tableau U is the reverse of the descent composition of the standard reverse shin-tableau given by $flip(U)$. Using this fact, we show that the reverse extended Schur functions are the image of the extended Schur functions under ρ.

Theorem 4.4.13. *For a composition α,*

$$\rho(\mathfrak{w}^*_\alpha) = \reflectbox{$\mathfrak{w}$}^*_{\alpha^r} \quad \textit{and} \quad \omega(\mathfrak{R}\mathfrak{w}^*_\alpha) = \reflectbox{$\mathfrak{w}$}^*_{\alpha^r}.$$

Moreover, $\{\reflectbox{$\mathfrak{w}$}^*_\alpha\}_\alpha$ is a basis of $QSym$.

Proof. Let $\alpha \models n$. First, observe that given a standard shin-tableau U of shape α and a standard reverse shin-tableau S of shape α^r with $flip(U) = S$, we have $(co_{\mathfrak{w}}(U))^r = co_{\reflectbox{$\mathfrak{w}$}}(S)$. Therefore,

$$\begin{aligned}\rho(\mathfrak{w}^*_\alpha) &= \rho(\sum_U F_{co_{\mathfrak{w}}(U)}) = \sum_U \rho(F_{co_{\mathfrak{w}}(U)})\\ &= \sum_U F_{co_{\mathfrak{w}}(U)^r} = \sum_{flip(U)} F_{co_{\reflectbox{$\mathfrak{w}$}}(flip(U))} = \sum_S F_{co_{\reflectbox{$\mathfrak{w}$}}(S)},\end{aligned}$$

where the sums run over SST U of shape α and standard reverse shin-tableaux S of shape α^r. Now, reverse extended Schur functions are a basis because ρ is an automorphism in $QSym$. □

Remark 4.4.14. The reverse extended Schur basis is not equivalent to the extended Schur basis or the row-strict extended Schur basis. From Example 4.4.12 above, we see that there is no β such that $\mathfrak{w}^*_\beta = \reflectbox{$\mathfrak{w}$}^*_{(3,2)}$. The only standard shin-tableaux with descent composition $(3,2)$ are tableaux of shape $(3,2)$ or shape $(4,1)$ meaning $\mathfrak{w}^*_{(3,2)}$ and $\mathfrak{w}^*_{(4,1)}$ are the only extended Schur functions in which $F_{(3,2)}$ appears but neither of them equal $\reflectbox{$\mathfrak{w}$}^*_{(3,2)}$. Now consider $\reflectbox{$\mathfrak{w}$}^*_{(1,2,1)} = F_{(1,2,1)} + F_{(2,1,1)}$. The only β for which there exists a standard row-strict shin-tableaux of shape β with descent compositions $(1,2,1)$ and $(2,1,1)$ is $\beta = (3,1)$. However, $\mathfrak{R}\mathfrak{w}^*_{(3,1)} = F_{(3,1)} + F_{(1,2,1)} + F_{(1,1,2)}$. Thus, there is no β such that $\reflectbox{$\mathfrak{w}$}^*_{(1,2,2)} = \mathfrak{R}\mathfrak{w}^*_\beta$.

Next, we introduce the basis of $NSym$ that is dual to the reverse extended Schur functions and its relationship with the shin functions.

Definition 4.4.15. Define the *reverse shin basis* $\{\reflectbox{$\mathfrak{w}$}_\alpha\}_\alpha$ as the unique basis of $NSym$ that is dual to the reverse extended Schur basis. Equivalently, $\langle \reflectbox{$\mathfrak{w}$}_\alpha, \reflectbox{$\mathfrak{w}$}^*_\beta \rangle = \delta_{\alpha,\beta}$ for all compositions α and β.

The expansions of the complete homogeneous functions and the monomial functions of $NSym$ into the reverse shin basis follow via duality from Equation (4.12). For a composition β,

$$H_\beta = \sum_\alpha \mathcal{K}^{\reflectbox{$\mathfrak{w}$}}_{\alpha,\beta} \reflectbox{$\mathfrak{w}$}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\reflectbox{$\mathfrak{w}$}}_{\alpha,\beta} \reflectbox{$\mathfrak{w}$}_\alpha. \tag{4.13}$$

As in the dual case, the reverse shin functions relate to the shin functions via the involution ρ.

Proposition 4.4.16. *For a composition α, we have*

$$\reflectbox{$\mathfrak{w}$}_\alpha = \rho(\mathfrak{w}_{\alpha^r}) \quad \reflectbox{$\mathfrak{w}$}_\alpha = \omega(\mathfrak{R}\mathfrak{w}_{\alpha^r}).$$

Proof. The map ρ is invariant under duality. By definition, $\langle R_\alpha, F_\beta\rangle = \delta_{\alpha,\beta} = \langle R_{\alpha^r}, F_{\beta^r}\rangle = \langle \rho(R_\alpha), \rho(F_\beta)\rangle$, thus for $G \in QSym$ and $H \in NSym$, $\langle H, G\rangle = \langle \rho(H), \rho(G)\rangle$ due to the bilinearity of the inner product. Then the first part of our claim follows from Theorem 4.4.13. The relationship with the row-strict shin functions follows from $\omega = \rho \circ \psi$ and Proposition 4.1.14. □

By applying ρ, we translate many of the results on the shin functions to the reverse shin functions.

Theorem 4.4.17. *Let α, β be compositions, λ a partition, and m a positive.*

1. *(Left Pieri Rule)*
$$H_m \reflectbox{$\shin$}_\alpha = \sum_{\alpha^r \subset_m^{\shin} \beta^r} \reflectbox{$\shin$}_\beta.$$

2. $H_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \reflectbox{$\shin$}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^r} \reflectbox{$\shin$}_\alpha$.

3. $\reflectbox{$\shin$}^*_{\lambda^r} = s_\lambda$. *Also,* $\chi(\reflectbox{$\shin$}_{\lambda^r}) = s_\lambda$ *and* $\chi(\reflectbox{$\shin$}_\alpha) = 0$ *when α is not weakly increasing.*

4. *Let γ be a composition such that $\gamma_i > \gamma_{i+1}$ for all $1 \leq i \leq \ell(\gamma)$. Then,*
$$\reflectbox{$\shin$}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma H_{\gamma_{\sigma(1)}} \cdots H_{\gamma_{\sigma(\ell(\gamma))}},$$
where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \geq i-1$ for all $i \in [\ell(\gamma)]$.

5. $H_\beta = \sum_\alpha \mathcal{K}^{\reflectbox{$\shin$}}_{\alpha^r,\beta^r} \shin_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}^{\reflectbox{$\shin$}}_{\alpha^r,\beta^r} \shin_\alpha$.

Proof. For a composition α, note that $\rho(H_\alpha) = H_{\alpha^r}$.

1. Apply ρ to Definition 4.1.5.

2. Apply ρ to Equation (4.3).

3. Since ρ restricts to the identity map on the Schur functions, we have $\reflectbox{$\shin$}^*_{\lambda^r} = \rho(\shin^*_\lambda) = \rho(s_\lambda) = s_\lambda$. Because χ is dual to the inclusion map from Sym to $QSym$ [19], we have $\chi(\reflectbox{$\shin$}_\alpha) = \sum_\lambda($ coefficient of $\reflectbox{$\shin$}^*_\alpha$ in $s_\lambda)s_\lambda$. Our claim follows.

4. Apply ρ to Theorem 4.2.7.

5. Apply ρ to Equation (4.13). □

Next, we define skew reverse extended Schur functions algebraically, and then we define skew-II reverse extended Schur functions algebraically and in terms of tableaux.

Definition 4.4.18. For compositions $\beta \subseteq \alpha$, *skew reverse extended Schur functions* are defined by
$$\reflectbox{$\shin$}^*_{\alpha/\beta} = \reflectbox{$\shin$}^\perp_\beta(\reflectbox{$\shin$}^*_\alpha).$$

As before, $\shin^*_{\alpha/\beta}$ expands into various bases according to Equation (3.4). For compositions $\beta \subseteq \alpha$,

$$\shin^*_{\alpha/\beta} = \sum_\gamma \langle \shin_\beta H_\gamma, \shin^*_\alpha \rangle M_\gamma = \sum_\gamma \langle \shin_\beta R_\gamma, \shin^*_\alpha \rangle F_\gamma = \sum_\gamma \langle \shin_\beta \shin_\gamma, \shin^*_\alpha \rangle \shin^*_\gamma. \tag{4.14}$$

The coefficients $\langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle$ also appear when multiplying row-strict shin functions,

$$\shin_\beta \shin_\gamma = \sum_\alpha \langle \shin_\beta \shin_\gamma, \shin^*_\alpha \rangle \shin_\alpha.$$

Because ρ is an anti-automorphism in *NSym*, it does not map the skew extended Schur functions to the skew reverse extended Schur functions. Instead, the images of these functions are the skew-II reverse extended Schur functions.

Definition 4.4.19. For compositions α and β where $\beta^r \subseteq \alpha^r$, the *skew-II reverse extended Schur functions* are defined by

$$\shin^*_{\alpha /\!\!/ \beta} = \shin^{\perp}_\beta(\shin^*_\alpha).$$

Using Definition 3.2.1, $\shin^*_{\alpha/\!\!/\beta}$ expands into various bases as follows. For compositions α, β with $\beta^r \subseteq \alpha^r$,

$$\shin^*_{\alpha/\!\!/\beta} = \sum_\gamma \langle H_\gamma \shin_\beta, \shin^*_\alpha \rangle M_\gamma = \sum_\gamma \langle R_\gamma \shin_\beta, \shin^*_\alpha \rangle F_\gamma = \sum_\gamma \langle \shin_\gamma \shin_\beta, \shin^*_\alpha \rangle \shin^*_\gamma. \tag{4.15}$$

Theorem 4.4.20. *For compositions α and β where $\beta \subseteq \alpha$,*

$$\rho(\boldsymbol{\mathfrak{w}}^*_{\alpha/\beta}) = \shin^*_{\alpha^r /\!\!/ \beta^r}.$$

Proof. First, observe that

$$\langle \rho(\boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma), \rho(\boldsymbol{\mathfrak{w}}^*_\alpha) \rangle = \langle \rho(\boldsymbol{\mathfrak{w}}_\gamma)\rho(\boldsymbol{\mathfrak{w}}_\beta), \rho(\boldsymbol{\mathfrak{w}}^*_\alpha) \rangle = \langle \shin_{\gamma^r} \shin_{\beta^r}, \shin_{\alpha^r} \rangle,$$

because ρ is invariant under duality and an anti-automorphism in *NSym*. Then,

$$\rho(\boldsymbol{\mathfrak{w}}^*_{\alpha/\beta}) = \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \rho(\boldsymbol{\mathfrak{w}}^*_\gamma) = \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha \rangle \shin^*_{\gamma^r} = \sum_\gamma \langle \shin_{\gamma^r} \shin_{\beta^r}, \shin_{\alpha^r} \rangle \shin^*_{\gamma^r} = \shin^*_{\alpha^r /\!\!/ \beta^r},$$

by Equation (4.15). □

The first line of the proof above yields the following.

Corollary 4.4.21. *For compositions β and γ,*

$$\shin_\gamma \shin_\beta = \sum_\alpha \mathcal{C}^{\alpha^r}_{\beta^r, \gamma^r} \shin_\alpha.$$

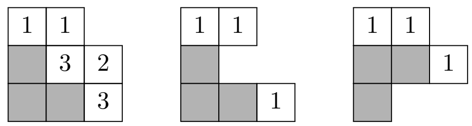
The skew-II reverse extended Schur functions are defined combinatorially via a special class of tableaux with skew-II shapes.

Definition 4.4.22. For compositions $\beta = (\beta_1, \ldots, b_\ell)$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $\beta^r \subseteq \alpha^r$, a *skew-II reverse shin-tableau* of shape $\alpha /\!/ \beta$ is a skew-II diagram $\alpha /\!/ \beta$ filled with integers such that each row is weakly decreasing left to right, each column is strictly increasing top to bottom, and if $\alpha_{k-i} > \beta_{\ell-i}$ for some $i \geq 0$ then there is no $j > i$ such that $\beta_{\ell-j} > \beta_{\ell-i}$. A skew-II reverse shin-tableau is *standard* if it contains the numbers 1 through $|\alpha| - |\beta|$ each exactly once.

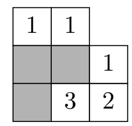
Intuitively, the third condition states that there should never be any boxes directly below a box that has been skewed-out.

Example 4.4.23. The three leftmost diagrams below are examples of skew-II reverse shin-tableaux while the rightmost diagram is not.

skew-II reverse shin:



Not:



The right-most diagram is not a skew-II reverse shin tableau because $\alpha_3 > \beta_2$ but $\beta_1 > \beta_2$. In other words, there is a skewed-out box directly above the normal box containing the 3, which violates our conditions.

We can extend the bijection $flip$ to be

$$flip : \{\text{skew standard shin-tableaux of shape } \alpha/\beta\} \to \{\text{skew-II SRST of shape } \alpha^r /\!/ \beta^r\}$$

where, given a skew standard shin-tableau T, the skew-II standard reverse shin-tableau $flip(T)$ is obtained by flipping T horizontally (reversing the order of the rows) and replacing each entry i with $|T| + 1 - i$. Again, by definition of the map, the descent composition of T is the reverse of the descent composition of $flip(T)$. This is because there is a descent at $|T| + 1 - i$ in $flip(T)$ whenever there is a descent i in T. This allows us to connect the skew extended Schur functions with the skew-II reverse extended Schur functions.

Proposition 4.4.24. *For compositions α and β where $\beta^r \subseteq \alpha^r$,*

$$\mathfrak{a}^*_{\alpha /\!/ \beta} = \sum_T x^T,$$

where the sum runs over skew-II reverse shin-tableaux T of shape $\alpha /\!/ \beta$.

Proof. The skew shin functions can be expressed as

$$\mathfrak{v}^*_{\alpha/\beta} = \sum_S F_{co_{\mathfrak{v}}(S)},$$

using the same logic as the usual shin functions (also similar to Proposition 4.4.11). Thus,

$$\mathfrak{m}^*_{\alpha/\!\!/\beta} = \omega(\mathfrak{w}^*_{\alpha^r/\beta^r}) = \sum_S F_{co_{\mathfrak{w}}(S)^r} = \sum_S F_{co_{\mathfrak{m}}(flip(S))} = \sum_U F_{co_{\mathfrak{m}}(U)},$$

where the sums run over skew standard shin-tableaux S of shape α^r/β^r and skew-II standard reverse shin-tableaux U of shape $\alpha/\!\!/\beta$. From here it is simple to expand the summation into our claim. □

With the notion of skew-II reverse shin-tableaux we now define a reverse analogue to Theorem 4.1.10.

Proposition 4.4.25. *Left Ribbon Multiplication.*

$$R_\beta \mathfrak{m}_\alpha = \sum_{\gamma \models |\alpha|+|\beta|} \sum_S \mathfrak{m}_\gamma,$$

where the sum runs over all skew-II standard reverse shin-tableaux S of shape γ/α with $co_{\mathfrak{m}}(U) = \beta$.

Proof. For compositions α and β, Theorem 4.1.10 yields

$$\mathfrak{w}_{\alpha^r} R_{\beta^r} = \sum_{\gamma \models |\alpha|+|\beta|} \sum_U \mathfrak{w}_{\gamma^r},$$

where the sum runs over skew standard shin-tableaux U of shape γ^r/α^r with $co_{\mathfrak{w}}(U) = \beta^r$. Then applying ρ gives

$$R_\beta \mathfrak{m}_\alpha = \sum_{\gamma \models |\alpha|+|\beta|} \sum_U \mathfrak{m}_\gamma,$$

where the sum runs over skew standard shin-tableaux U of shape γ^r/α^r with $co_{\mathfrak{w}}(U) = \beta^r$. Using the $flip$ bijection, we associate each tableau U with $flip(U)$ which is a skew-II standard reverse shin-tableau of shape $\alpha/\!\!/\beta$ and descent composition β. This allows us to rewrite our sum as it is stated in the claim. □

4.4.2 Row-strict reverse extended Schur and shin functions

Let α be a composition and β be a weak composition. A *row-strict reverse shin-tableau* (RSRST) of shape α and type β is a filling of the diagram of α with positive integers such that the entries in each row are strictly decreasing from left to right and the entries in each column are weakly increasing from top to bottom where each integer i appears β_i times. These are essentially a row-strict version of the reverse shin-tableaux. A row-strict reverse shin-tableau of shape $\alpha \models n$ is *standard* (SRSRST) if it includes the entries 1 through n each exactly once.

Definition 4.4.26. For $\alpha \models n$, the *row-strict reverse extended Schur function* is defined as

$$\mathfrak{R}\mathfrak{m}^*_\alpha = \sum_T x^T,$$

where the sum runs over all row-strict reverse shin-tableaux T of shape α.

Example 4.4.27. A few row-strict reverse shin-tableaux of shape $(2,3)$ and $\mathfrak{Ra}^*_{(2,3)}$ are

$$\begin{array}{|c|c|c} \hline 2 & 1 & \\ \hline 3 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 3 & 2 & \\ \hline 3 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 4 & 2 & \\ \hline 3 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 4 & 1 & \\ \hline 3 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 4 & 2 & \\ \hline 4 & 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 4 & 2 & \\ \hline 4 & 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

$$\mathfrak{Ra}^*_{(2,3)} = x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1x_2^2x_3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_4^2 + x_1x_2x_3x_4^2 + \cdots$$

For a SRSRST S, the *descent set* is defined as $Des_{\mathfrak{Ra}}(S) = \{i : \ i+1 \text{ is weakly above } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{Ra}}(S)$ is called a *descent* of S. The *descent composition* of S is defined to be $co_{\mathfrak{Ra}}(S) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{Ra}}(S) = \{i_1, \ldots, i_d\}$. Note that the set of standard row-strict reverse shin-tableaux is exactly the same as the set of standard reverse shin-tableaux.

The *row-strict reverse shin reading word* of a shin-tableau T, denoted $rw_{\mathfrak{Ra}}(T)$, is the word obtained by reading the rows of T from right to left starting with the top row and moving down. We *standardize* a standard row-strict reverse shin-tableau as follows. Given a RSRST T, its *standardization* is the SRSRST obtained by replacing the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{Ra}}(T)$, then the 2's continuing with our consecutive integers from before, then 3's, etc.

As in Proposition 4.4.11, one can show that the flat type of any row-strict reverse shin-tableau T that standardizes to a SRSRST S is a refinement of $co_{\mathfrak{Ra}}(S)$. From there, we group tableaux together based on their types and standardizations to expand a row-strict reverse extended Schur function into the fundamental basis as follows.

Proposition 4.4.28. *For a composition α,*

$$\mathfrak{Ra}^*_\alpha = \sum_S F_{co_{\mathfrak{Ra}}(S)},$$

where the sum runs over standard row-strict reverse shin-tableaux S.

Example 4.4.29. The F-expansion of the row-strict reverse extended Schur function $\mathfrak{Ra}^*_{(3,2)}$ and standard row-strict reverse shin-tableaux of shape $(3,2)$ are:

$$\mathfrak{Ra}^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)} \qquad \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 5 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}$$

Let $\mathcal{K}^{\mathfrak{Ra}}_{\alpha,\beta}$ be the number of row-strict reverse shin-tableaux of shape α and type β, and let $\mathcal{L}^{\mathfrak{Ra}}_{\alpha,\beta}$ be the number of standard row-strict reverse shin-tableaux with shape α and descent composition β. The expansions of the row-strict reverse extended Schur functions into the monomial and fundamental bases follow from Proposition 4.4.28. For a composition α,

$$\mathfrak{Ra}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{Ra}}_{\alpha,\beta} M_\beta \quad \text{and} \quad \mathfrak{Ra}^*_\alpha = \sum_\beta \mathcal{L}^{\mathfrak{Ra}}_{\alpha,\beta} F_\beta. \tag{4.16}$$

Studying the descent compositions of standard row-strict reverse tableaux allows us to relate the row-strict reverse extended Schur functions to our other bases of $QSym$.

Theorem 4.4.30. *For a composition* α*,*

$$\omega(\mathfrak{w}^*_\alpha) = \mathfrak{R}\mathfrak{m}^*_{\alpha^r} \quad \textit{and} \quad \rho(\mathfrak{R}\mathfrak{w}^*_\alpha) = \mathfrak{R}\mathfrak{m}^*_{\alpha^r} \quad \textit{and} \quad \psi(\mathfrak{m}^*_\alpha) = \mathfrak{R}\mathfrak{m}^*_\alpha.$$

Moreover, $\{\mathfrak{R}\mathfrak{m}^*_\alpha\}_\alpha$ *is a basis of* $QSym$*.*

Proof. Recall that $\sum_U F_{co_{\mathfrak{w}}(U)^r} = \sum_S F_{co_{\mathfrak{m}}(S)}$, and that the set of standard reverse shin-tableaux is equivalent to the set of standard row-strict reverse shin-tableaux (of shape α). By the complementary definitions of reverse descents and row-strict reverse descents, $Des_{\mathfrak{m}}(S)$ is the set complement of $Des_{\mathfrak{R}\mathfrak{m}}(S)$ meaning that $co_{\mathfrak{m}}(S)^c = co_{\mathfrak{R}\mathfrak{m}}(S)$. Combining these two statements, we have shown that

$$\omega(\mathfrak{w}^*_\alpha) = \sum_U \omega(F_{co_{\mathfrak{w}}(U)}) = \sum_U F_{co_{\mathfrak{w}}(U)^t} = \sum_U F_{((co_{\mathfrak{w}}(U))^r)^c} = \sum_S F_{co_{\mathfrak{R}\mathfrak{m}}(S)},$$

where the sums run over SST U of shape α and standard row-strict reverse shin-tableaux S of shape α^r. The rest follows from Proposition 4.1.14 and Theorem 4.4.13, and the fact that $\psi \circ \rho = \omega$. □

Remark 4.4.31. The row-strict reverse extended Schur basis is not equivalent to the extended Schur basis, the row-strict extended Schur basis, or the reverse extended Schur basis. $\mathfrak{R}\mathfrak{m}^*_{2,1} = F_{1,2}$ and it follows from properties of extended Schur functions that $\mathfrak{R}\mathfrak{w}^*_\beta = F_\beta$ if and only if $\beta = (m, 1^k)$ for integers $m \geq 1$ and $k \geq 0$. Thus, there is no β such that $\mathfrak{R}\mathfrak{w}^*_\beta = \mathfrak{R}\mathfrak{m}^*_{2,1}$. Next, $\mathfrak{R}\mathfrak{m}^*_{(1,2,1)} = F_{(2,2)} + F_{(1,3)}$ and the only β such that there exist standard shin-tableaux with descent composition $(2,2)$ and $(1,3)$ is $\beta = (3,1)$. However, $\mathfrak{w}^*_{(3,1)} = F_{(2,2)} + F_{(1,3)} + F_{(3,1)}$ and so there is no β such that $\mathfrak{w}^*_\beta = \mathfrak{R}\mathfrak{m}^*_{(1,2,1)}$. Further, the only β such that there exist standard reverse shin-tableaux with descent composition $(2,2)$ and $(1,3)$ is also $\beta = (3,1)$. However, $\mathfrak{m}^*_{(3,1)} = F_{(2,2)} + F_{(1,3)} + F_{(3,1)} \neq \mathfrak{R}\mathfrak{m}^*_{(1,2,1)}$. There is no β such that $\mathfrak{m}^*_\beta = \mathfrak{R}\mathfrak{m}^*_{(1,2,1)}$.

Next, we define the basis of $NSym$ that is dual to the row-strict reverse extended Schur functions and its relationship with the shin functions.

Definition 4.4.32. Define the *row-strict reverse shin basis* $\{\mathfrak{R}\mathfrak{m}_\alpha\}_\alpha$ as the unique basis of $NSym$ that is dual to the row-strict reverse extended Schur basis. Equivalently, $\langle \mathfrak{R}\mathfrak{m}_\alpha, \mathfrak{R}\mathfrak{m}^*_\beta \rangle = \delta_{\alpha,\beta}$ for all compositions α and β.

The expansions of the complete homogeneous functions and the ribbon functions of $NSym$ into the row-strict reverse shin basis are dual to those in Equation (4.16). For a composition β,

$$H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{R}\mathfrak{m}}_{\alpha,\beta} \mathfrak{R}\mathfrak{m}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{R}\mathfrak{m}}_{\alpha,\beta} \mathfrak{R}\mathfrak{m}_\alpha. \tag{4.17}$$

Like in the dual case, the row-strict reverse shin functions are related to the shin functions via the involution ω.

Proposition 4.4.33. *For a composition α, we have*

$$\mathfrak{R}\mathfrak{m}_\alpha = \omega(\mathfrak{w}_{\alpha^r}) \quad \textit{and} \quad \mathfrak{R}\mathfrak{m}_\alpha = \rho(\mathfrak{R}\mathfrak{w}_{\alpha^r}) \quad \textit{and} \quad \mathfrak{R}\mathfrak{m}_\alpha = \psi(\mathfrak{m}_\alpha).$$

Proof. Observe that

$$\langle R_\alpha, F_\beta \rangle = \langle R_{\alpha^t}, F_{\beta^t} \rangle = \langle \omega(R_\alpha), \omega(F_\beta) \rangle.$$

This property expands to other bases by linearity. For instance,

$$\langle \mathfrak{w}_{\alpha^r}, \mathfrak{w}^*_{\beta^r} \rangle = \langle \omega(\mathfrak{w}_{\alpha^r}), \omega(\mathfrak{w}^*_{\beta^r}) \rangle = \langle \omega(\mathfrak{w}_{\alpha^r}), \mathfrak{R}\mathfrak{m}^*_\beta \rangle,$$

for any compositions α and β. It follows by definition then that $\mathfrak{R}\mathfrak{m}_\beta = \omega(\mathfrak{w}_{\beta^r})$. The rest follows similarly from the invariance of ρ and ψ under duality. □

Now, we apply ω to the various results on the shin and extended Schur bases to find analogous results on the row-strict reverse shin and row-strict reverse extended Schur bases.

Theorem 4.4.34. *Let α, β be compositions, λ a partition, and m a positive integer.*

1. *(Left Pieri Rule)*
$$E_m \mathfrak{R}\mathfrak{m}_\alpha = \sum_{\alpha^r \subset^{\mathfrak{w}}_m \beta^r} \mathfrak{R}\mathfrak{m}_\beta.$$

2. $E_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \mathfrak{R}\mathfrak{m}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^t} \mathfrak{R}\mathfrak{m}_\alpha$.

3. $\mathfrak{R}\mathfrak{m}^*_\lambda = s_{\lambda'}$. *Also,* $\chi(\mathfrak{R}\mathfrak{m}_\lambda) = s_{\lambda'}$ *and* $\chi(\mathfrak{R}\mathfrak{m}_\alpha) = 0$ *when α is not a partition.*

4. *Let γ be a composition such that $\gamma_i > \gamma_{i+1}$. Then,*
$$\mathfrak{R}\mathfrak{m}_\gamma = \sum (-1)^\sigma E_{\gamma_{\sigma(1)}} E_{\gamma_{\sigma(2)}} \cdots E_{\gamma_{\sigma(\ell(\gamma))}},$$
where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \geq i-1$ for all $i \in [\ell(\gamma)]$.

5. $E_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{R}\mathfrak{m}}_{\alpha^r,\beta^r} \mathfrak{w}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{R}\mathfrak{m}}_{\alpha^r,\beta^t} \mathfrak{w}_\alpha$.

Proof. For a composition α, note that $\omega(H_\alpha) = E_{\alpha^r}$.

1. Apply $\omega = \rho \circ \psi$ to Definition 4.1.5.

2. Apply $\omega = \rho \circ \psi$ to Equation (4.3).

3. Since ω restricts to classic ω map on the Schur functions, we have $\mathfrak{R}\mathfrak{m}^*_{\lambda^r} = \omega(\mathfrak{w}^*_\lambda) = \omega(s_\lambda) = s_{\lambda'}$. Because χ is dual to the inclusion map from Sym to $QSym$ [19], we have $\chi(\mathfrak{R}\mathfrak{m}_\alpha) = \sum_\lambda($ coefficient of $\mathfrak{R}\mathfrak{m}^*_\alpha$ in $s_\lambda)s_\lambda$. Our claim follows.

4. Apply $\omega = \rho \circ \psi$ to Theorem 4.2.7.

5. Apply $\omega = \rho \circ \psi$ to Equation (4.17). □

Combining Theorem 4.4.30 and Proposition 4.4.33 allows us to express the antipode on these bases in terms of the row-strict reverse extended Schur basis and the row-strict reverse shin basis.

Corollary 4.4.35. *For a composition α,*

$$S(\mathfrak{w}_\alpha) = (-1)^{|\alpha|}\mathfrak{R}\mathfrak{m}_{\alpha^r} \quad \textit{and} \quad S^*(\mathfrak{w}^*_\alpha) = (-1)^{|\alpha|}\mathfrak{R}\mathfrak{m}^*_{\alpha^r}.$$

This result reduces the problem to studying the expansion of the row-strict reverse extended Schur functions into the extended Schur functions and vice versa, which may be interesting to approach using tableaux combinatorics.

Finally, we introduce and study skew and skew-II row-strict reverse extended Schur functions.

Definition 4.4.36. For compositions $\beta \subseteq \alpha$, the *skew row-strict reverse extended Schur functions* are defined by

$$\mathfrak{R}\mathfrak{m}^*_{\alpha/\beta} = \mathfrak{R}\mathfrak{m}^{\perp}_{\beta}(\mathfrak{R}\mathfrak{m}^*_\alpha).$$

$\mathfrak{R}\mathfrak{m}^*_{\alpha/\beta}$ expands into various bases according to Equation (3.4). For compositions $\beta \subseteq \alpha$,

$$\mathfrak{R}\mathfrak{m}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{R}\mathfrak{m}_\beta H_\gamma, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle M_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{m}_\beta R_\gamma, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle F_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{m}_\beta \mathfrak{R}\mathfrak{m}_\gamma, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle \mathfrak{R}\mathfrak{m}^*_\gamma. \tag{4.18}$$

The coefficients $\langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle$ also appear when multiplying row-strict reverse shin functions,

$$\mathfrak{R}\mathfrak{m}_\beta \mathfrak{R}\mathfrak{m}_\gamma = \sum_\alpha \langle \mathfrak{R}\mathfrak{m}_\beta \mathfrak{R}\mathfrak{m}_\gamma, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle \mathfrak{R}\mathfrak{m}_\alpha.$$

Because ω is an anti-automorphism in *NSym*, it does not map the skew extended Schur functions to the skew row-strict reverse extended Schur functions. Again, we use the right-perp operator to define functions that we show are the image of the skew extended Schur functions under ω.

Definition 4.4.37. For compositions α and β where $\beta^r \subseteq \alpha^r$, the *skew-II row-strict reverse extended Schur functions* are defined by

$$\mathfrak{R}\mathfrak{m}^*_{\alpha/\!\!/\beta} = \mathfrak{R}\mathfrak{m}^{\pm}_{\beta}(\mathfrak{R}\mathfrak{m}^*_\alpha).$$

Using Definition 3.2.1, $\mathfrak{R}\mathfrak{m}^*_{\alpha/\!\!/\beta}$ expands into various bases as follows. For compositions α, β with $\beta^r \subseteq \alpha^r$,

$$\mathfrak{R}\mathfrak{m}^*_{\alpha/\!\!/\beta} = \sum_\gamma \langle H_\gamma \mathfrak{R}\mathfrak{m}_\beta, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle M_\gamma = \sum_\gamma \langle R_\gamma \mathfrak{R}\mathfrak{m}_\beta, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle F_\gamma = \sum_\gamma \langle \mathfrak{R}\mathfrak{m}_\gamma \mathfrak{R}\mathfrak{m}_\beta, \mathfrak{R}\mathfrak{m}^*_\alpha\rangle \mathfrak{R}\mathfrak{m}^*_\gamma. \tag{4.19}$$

Theorem 4.4.38. *For compositions α and β where $\beta \subseteq \alpha$,*

$$\omega(\mathfrak{w}^*_{\alpha/\beta}) = \mathfrak{R}\mathfrak{m}^*_{\alpha^r/\!\!/\beta^r}.$$

Proof. First, observe that

$$\langle \omega(\boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma), \omega(\boldsymbol{\mathfrak{w}}^*_\alpha)\rangle = \langle \omega(\boldsymbol{\mathfrak{w}}_\gamma)\omega(\boldsymbol{\mathfrak{w}}_\beta), \omega(\boldsymbol{\mathfrak{w}}^*_\alpha)\rangle = \langle \mathfrak{R}\boldsymbol{\mathfrak{a}}_{\gamma^r}\mathfrak{R}\boldsymbol{\mathfrak{a}}_{\beta^r}, \mathfrak{R}\boldsymbol{\mathfrak{a}}_{\alpha^r}\rangle,$$

because ω is invariant under duality and an anti-isomorphism in *NSym*. Then, by Equation (4.19)

$$\omega(\boldsymbol{\mathfrak{w}}^*_{\alpha/\beta}) = \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha\rangle \omega(\boldsymbol{\mathfrak{w}}^*_\gamma) = \sum_\gamma \langle \boldsymbol{\mathfrak{w}}_\beta \boldsymbol{\mathfrak{w}}_\gamma, \boldsymbol{\mathfrak{w}}^*_\alpha\rangle \mathfrak{R}\boldsymbol{\mathfrak{a}}^*_{\gamma^r} = \sum_\gamma \langle \mathfrak{R}\boldsymbol{\mathfrak{a}}_{\gamma^r}\mathfrak{R}\boldsymbol{\mathfrak{a}}_{\beta^r}, \mathfrak{R}\boldsymbol{\mathfrak{a}}_{\alpha^r}\rangle \mathfrak{R}\boldsymbol{\mathfrak{a}}^*_{\gamma^r} = \mathfrak{R}\boldsymbol{\mathfrak{a}}^*_{\alpha^r /\!\!/ \beta^r}. \quad \square$$

The first line of the proof above shows the following equality.

Corollary 4.4.39. *For compositions α, β and γ,*

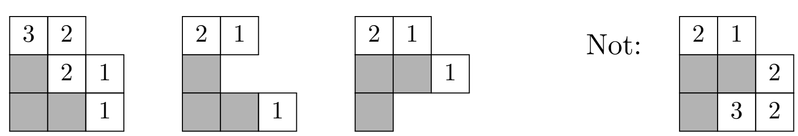
$$\mathfrak{R}\boldsymbol{\mathfrak{a}}_\gamma \mathfrak{R}\boldsymbol{\mathfrak{a}}_\beta = \sum_\alpha C^{\alpha^r}_{\beta^r, \gamma^r} \mathfrak{R}\boldsymbol{\mathfrak{a}}_\alpha.$$

The skew-II row-strict reverse extended Schur functions are again defined combinatorially via tableaux using skew-II shapes.

Definition 4.4.40. For compositions $\beta = (\beta_1, \ldots, b_\ell) \subseteq \alpha = (\alpha_1, \ldots, \alpha_k)$ such that $\beta^r \subseteq \alpha^r$, a *row-strict reverse skew-II shin-tableau* of skew shape $\alpha /\!\!/ \beta$ is a skew-II diagram $\alpha /\!\!/ \beta$ filled with integers such that each row is strictly decreasing left to right, each column is weakly increasing top to bottom, and if $\alpha_{k-i} > \beta_{\ell-i}$ for some i then there is no $j > i$ such that $\beta_{\ell-j} > \beta_{\ell-i}$. A skew-II RSRST is *standard* if it contains the numbers 1 through $|\alpha| - |\beta|$ each exactly once.

Example 4.4.41. The three leftmost diagrams below are examples of skew-II row-strict reverse shin-tableaux while the rightmost diagram is not.

skew-II row-strict reverse shin:



Not:

The expansion of skew-II row-strict reverse shin functions in terms of tableaux closely follows that of Proposition 4.4.7, but by instead applying ψ to the skew-II reverse extended Schur functions.

Proposition 4.4.42. *For compositions α and β where $\beta^r \subseteq \alpha^r$,*

$$\mathfrak{R}\boldsymbol{\mathfrak{a}}^*_{\alpha /\!\!/ \beta} = \sum_T x^T,$$

where the sum runs over skew-II reverse shin-tableaux T of shape $\alpha /\!\!/ \beta$.

We also use skew-II row-strict reverse tableaux to give a combinatorial formula for the product of a ribbon function and a row-strict reverse shin function. The proof exactly follows that of Proposition 4.4.25 but by applying ρ to Theorem 4.4.1 (2).

Proposition 4.4.43. *Left Ribbon Multiplication.*

$$R_\beta \mathfrak{Rm}_\alpha = \sum_{\gamma \models |\alpha|+|\beta|} \sum_S \mathfrak{Rm}_\gamma,$$

where the sum runs over skew-II standard row-strict reverse shin-tableaux S of shape γ/α with $co_{\mathfrak{Rm}}(U) = \beta$.

Note that the content of this chapter, up to this point, also appears in our paper [26].

4.5 Colored generalizations of the shin and extended Schur functions

To generalize the shin and extended Schur functions to $QSym_A$ and $NSym_A$, we again use combinatorial objects defined on colored composition diagrams. We proceed by first defining the colored extended Schur functions in terms of tableaux and then defining the colored shin functions via a Pieri rule. We show that these two bases are dual based on their expansions to and from other bases.

4.5.1 Colored extended Schur functions in $QSym_A$

Definition 4.5.1. For a sentence I, a *colored shin-tableau* T of shape I is a filling of the colored composition diagram I with positive integers such that each row is weakly increasing from left to right and each column is strictly increasing from top to bottom.

Colored shin-tableaux are a subset of colored immaculate tableaux, so we can use the same definitions for type, standard, standardization, descents, and colored descent compositions. Specifically, recall that the *type* of T is the weak sentence $B = (v_1, \dots, v_h)$ where v_i is the word given by reading the colors of all the boxes filled with i in the order they appear in $rw_{\mathfrak{w}}(T)$ (left to right, starting from the bottom row and moving up). We can also restrict or apply many of our results on colored immaculate tableaux to colored shin-tableaux.

Definition 4.5.2. For a sentence I, define the *colored exteneded Schur function* as

$$\boldsymbol{\mathfrak{w}}^*_I = \sum_T x_T,$$

where the sum runs over all colored shin-tableaux T of shape I.

Example 4.5.3. The colored extended Schur function for (ac, ba) is given by

$$\boldsymbol{\mathfrak{w}}^*_{(ac,ba)} = x_{ac,1}x_{ba,2} + x_{ac,1}x_{b,2}x_{a,3} + x_{a,1}x_{bc,2}x_{a,3} + x_{a,1}x_{c,2}x_{b,3}x_{a,4} + x_{a,1}x_{b,2}x_{c,3}x_{a,4} + \cdots$$

$a,1$	$c,1$
$b,2$	$a,2$

$a,1$	$c,1$
$b,2$	$a,3$

$a,1$	$c,2$
$b,2$	$a,3$

$a,1$	$c,2$
$b,3$	$a,4$

$a,1$	$c,3$
$b,2$	$a,4$

$\cdots$

Let $\mathcal{K}_{I,J}$ denote the number of colored shin-tableau of shape I with type J, and let $\mathcal{L}_{I,J}$ denote the number of standard colored shin-tableau of shape I and colored descent composition J.

Proposition 4.5.4. *For a sentence I,*

$$\boldsymbol{\mathfrak{w}}_I^* = \sum_J \mathcal{K}_{I,J} M_J \qquad \text{and} \qquad \boldsymbol{\mathfrak{w}}_I^* = \sum_J \mathcal{L}_{I,J} F_J,$$

where the sums run over all sentences J such that $|I| = |J|$.

Proof. First, we prove the monomial expansion using tools from the colored dual immaculate functions, because colored shin-tableaux are in fact colored immaculate tableaux. By Proposition 3.3.17, we have $\boldsymbol{\mathfrak{w}}_I^* = \sum_{T=\tilde{T}} M_{type(T)}$ where the sum runs over colored shin-tableaux T of shape I such that $T = \tilde{T}$ or in other words, flat shin-tableaux of shape I. Each of these colored tableau has a type B where B is a flat sentence, thus each M_B appears as many times are there are colored shin-tableaux of shape I and type B, giving us the stated colored monomial expansion.

Next, we show the fundamental expansion by first showing that $\mathcal{K}_{I,J} = \sum_{K \preceq J} \mathcal{L}_{I,K}$ for sentences I and J. Let $\mathcal{K}$ be the set of colored shin-tableaux of shape I with type J and let $\mathcal{L}$ be the set of standard shin-tableaux of shape I with descent composition K where $K \preceq J$. We prove that the standardization function $std : \mathcal{K} \to \mathcal{L}$ is a bijection to prove that $|\mathcal{K}| = |\mathcal{L}|$.

Let $std^{-1} : \mathcal{L} \to \mathcal{K}$ be defined as follows. Given a standard colored shin-tableau $S \in \mathcal{L}$ with colored descent composition $K = (u_1, \dots, u_\ell)$, we construct a colored shin-tableau $T \in \mathcal{K}$ where $J = (v_1, \dots, v_h)$ for $std^{-1}(S) = T$ as follows. Starting with an empty diagram of I, fill the boxes of T corresponding to the first v_1 boxes of S with 1's. Then, fill the boxes of T corresponding to the next v_2 boxes of S with 2's and so on until T is full. By construction, we have created a tableau T of type J that standardizes to S, and we need to check that T is indeed a colored shin-tableau. Because the colored descent composition of S is K such that $K \preceq J$, we know that when following the boxes of S in the order they are numbered, we only move to a lower row after passing a box numbered v_j for some j. Therefore, when filling our boxes to create T, every time we move to a lower row we also move to a higher number. This, combined with the fact that we fill left to right, maintains both the column strict increasing and row weak increasing conditions. Then,

$$\boldsymbol{\mathfrak{w}}_I^* = \sum_J \mathcal{K}_{I,J} M_J = \sum_J \sum_{K \preceq J} \mathcal{L}_{I,K} M_J = \sum_K \mathcal{L}_{I,K} \sum_{K \preceq J} M_J = \sum_K \mathcal{L}_{I,K} F_K. \qquad \square$$

Proposition 4.5.5. *Let $A = \{a\}$, and let I be a sentence. Then, $v(\boldsymbol{\mathfrak{w}}_I^*) = \boldsymbol{\mathfrak{w}}_{w\ell(I)}^*$. Moreover, $\{\boldsymbol{\mathfrak{w}}_I^*\}_I$ in $QSym_A$ is analogous to $\{\boldsymbol{\mathfrak{w}}_\alpha^*\}_\alpha$ in $QSym$.*

Proof. Observe that v acts on a monomial x_T, where T is a colored shin-tableau of shape I, by mapping it to the monomial $x^{T'}$ where T' is the shin-tableau of shape $w\ell(I)$ with the same integer entries as T. Thus, $v(\boldsymbol{\mathfrak{w}}_I^*) = \boldsymbol{\mathfrak{w}}_{w\ell(I)}^*$ for all alphabets A and more specifically for alphabets A containing only one color. $\square$

4.5.2 Colored shin functions in $\boldsymbol{NSym_A}$

Now we define a basis of $NSym_A$ analogous to the shin functions in $NSym$. Let $I = (w_1, \dots, w_k)$ and $J = (v_1, \dots, v_h)$ be sentences. We say I differs from J by the addition of a *colored shin-horizontal strip* with word u, denoted $J \subset_u^{\mathfrak{w}} I$, if for all $i \in [k]$:

1. $v_i \subseteq_L w_i$ (containment axiom)

2. $w((I/_L J)^r) = u$, and

3. for all indices $i \in [k]$, if $w_i > v_i$ then for all $j > i$, we have $w_j \leq v_i$ (overhang axiom).

Example 4.5.6. The sentence $(abac, cabcb, b, bca)$ differs from $(abac, cab, b, bc)$ by a colored shin-horizontal strip with word acb

a	b	a	c
c	a	b	
b			
b	c		

a	b	a	c	
c	a	b	c	b
b				
b	c	a		

Definition 4.5.7. The shin functions $\{\mathfrak{w}_I\}_I \in NSym_A$ are the unique set of functions satisfying

$$\mathfrak{w}_I H_w = \sum_J \mathfrak{w}_J,$$

where the sum runs over all sentences J which differ from I by a colored shin-horizontal strip with word w.

Theorem 4.5.8. *The colored complete homogeneous function has a positive expansion in terms of the shin functions. Specifically,*

$$H_J = \sum_I \mathcal{K}_{I,J} \mathfrak{w}_I,$$

where the sum runs over all sentences I such that $|I| = |J|$.

Proof. We proceed by induction on the length of J. First, observe that if $J = (v_1)$ we have $H_{v_1} = \mathfrak{w}_{v_1}$ by our Pieri rule. Next, assume that our rule holds for $\ell(J) = h - 1$. Now, let $J = (v_1, \dots, v_h)$ and $J^\natural = (v_1, \dots, v_{h-1})$. By our assumption, $H_{J^\natural} = \sum_T \mathfrak{w}_{sh(T)}$ and thus $H_J = H_{J^\natural} H_{v_h} = \sum_T \mathfrak{w}_{sh(T)} H_{v_h}$ where the sum runs over colored shin-tableaux T with content $J^\natural$. By Definition 4.5.7, this yields

$$H_J = \sum_T \sum_K \mathfrak{w}_K,$$

where the sums run over all colored shin-tableaux T of type $J^\natural$ and all sentences K that differ from $sh(T)$ by the addition of a shin-horizontal strip of word v_h.

Consider a colored shin-tableau T of type $J^\natural$ with shape $sh(T)$, and the diagrams obtained by adding a colored shin-horizontal strip of word v_h. From each of these diagrams we create a new colored shin-tableau T' where the boxes added by the shin-horizontal strip are filled with h's and the other boxes are filled as they are in T. It is clear that the increasing rows condition is satisfied. The overhang axiom guarantees that there are no boxes in the same column below any of the newly added h boxes, thus the strictly increasing column condition is met. As a result, T' is a colored shin-tableau of type J. Our sum is rewritten as

$$H_J = \sum_{T'} \boldsymbol{\mathfrak{w}}_{sh(T')} = \sum_I \mathcal{K}_{I,J} \boldsymbol{\mathfrak{w}}_J,$$

where the sums runs over all colored shin-tableau of type J and all sentences I such that $|I| = |J|$. □

The duality of the colored shin and colored extended Schur now follows from Proposition 2.2.7.

Corollary 4.5.9. *$\{\boldsymbol{\mathfrak{w}}_I\}_I$ is the unique basis of $NSym_A$ that is dual to $\{\boldsymbol{\mathfrak{w}}^*_I\}_I$.*

Now the expansion of the ribbon basis into the shin functions also follows from duality by applying Proposition 2.2.7 to Proposition 4.5.4.

Corollary 4.5.10. *For a sentence I,*

$$R_J = \sum_I \mathcal{L}_{I,J} \boldsymbol{\mathfrak{w}}_J,$$

where the sum runs over all sentences I such that $|I| = |J|$.

Finally, we verify that the colored shin basis in $NSym_A$ is analogous to the shin basis in $NSym$.

Proposition 4.5.11. *Let $A = \{a\}$ be an alphabet and I be a sentence. Then $\upsilon(\boldsymbol{\mathfrak{w}}_I) = \boldsymbol{\mathfrak{w}}_{w\ell(I)}$. Moreover, $\{\boldsymbol{\mathfrak{w}}_I\}_I$ in $NSym_A$ is analogous to $\{\boldsymbol{\mathfrak{w}}_\alpha\}_\alpha$ in $NSym$.*

Proof. By Proposition 3.3.47, for all sentences I and J we have

$$\langle \boldsymbol{\mathfrak{w}}_I, \boldsymbol{\mathfrak{w}}^*_J \rangle = \langle \upsilon(\boldsymbol{\mathfrak{w}}_I), \upsilon(\boldsymbol{\mathfrak{w}}^*_J) \rangle = \langle \upsilon(\boldsymbol{\mathfrak{w}}_I), \boldsymbol{\mathfrak{w}}^*_{w\ell(J)} \rangle.$$

Because A is an alphabet of size one, $w\ell$ is a bijection from sentences in A to compositions meaning that if $w\ell(I) = w\ell(J)$ then $I = J$. Thus, we need $\upsilon(\boldsymbol{\mathfrak{w}}_I) = \boldsymbol{\mathfrak{w}}_{w\ell(I)}$ whenever $I = J$ for any J. It follows that for any sentence I, we have $\upsilon(\boldsymbol{\mathfrak{w}}_I) = \boldsymbol{\mathfrak{w}}_{w\ell(I)}$. □

4.5.3 Skew and skew-II colored extended Schur functions

We define colored skew and skew-II extended Schur functions in $QSym_A$ using colored versions of the perp and right-perp operators.

Definition 4.5.12. For $H \in NSym_A$, define the action of the linear operator $H^\perp$ on $M \in QSym_A$ as $\langle G, H^\perp M \rangle = \langle HG, M \rangle$ and the action of the linear operator $H^\pm$ on $M \in QSym_A$ as $\langle G, H^\pm M \rangle = \langle GH, M \rangle$ for all $G \in NSym_A$. Thus, for dual bases $\{A_I\}_I$ of $QSym_A$ and $\{B_I\}_I$ of $NSym_A$, we have

$$H^\perp(M) = \sum_I \langle HB_I, M \rangle A_I, \qquad H^\pm(M) = \sum_I \langle B_I H, M \rangle A_I.$$

Definition 4.5.13. For sentences I and J with $J \subseteq_L I$, the *colored skew extended Schur functions* are defined as

$$\mathfrak{w}_J^{\perp}(\mathfrak{w}_I^*) = \mathfrak{w}_{I/J}^*.$$

The colored skew extended Schur function $\mathfrak{w}_{I/J}^*$ expands into various bases as follows:

$$\mathfrak{w}_{I/J}^* = \sum_K \langle \mathfrak{w}_J H_K, \mathfrak{w}_I^* \rangle M_K = \sum_K \langle \mathfrak{w}_J R_K, \mathfrak{w}_I^* \rangle F_K = \sum_K \langle \mathfrak{w}_J \mathfrak{w}_K, \mathfrak{w}_I^* \rangle \mathfrak{w}_K^*. \tag{4.20}$$

The coefficients $\mathcal{C}_{J,K}^I = \langle \mathfrak{w}_J \mathfrak{w}_K, \mathfrak{w}_I^* \rangle$ are also the coefficients that appear when multiplying colored shin functions

$$\mathfrak{w}_J \mathfrak{w}_K = \sum_I \langle \mathfrak{w}_J \mathfrak{w}_K, \mathfrak{w}_I^* \rangle \mathfrak{w}_I.$$

Like the non-colored case, these functions can be expressed in terms of a skew version of our earlier colored tableaux.

Definition 4.5.14. For sentences $J = (v_1, \ldots, v_h) \subseteq_L I = (w_1, \ldots, w_k)$, a colored shin tableaux of shape I/J is a colored skew shape I/J filled with integers such that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. Additionally, I and J must be such that for any $w_i > v_i$, there exists no j such that $j > i$ and $v_j > v_i$.

Proposition 4.5.15. *For sentences I and J such that $J \subseteq_L I$, the coefficient $\langle \mathfrak{w}_J H_K, \mathfrak{w}_I^* \rangle$ is equal to the number of colored skew shin-tableaux of shape I/J and type K. Moreover,*

$$\mathfrak{w}_{I/J}^* = \sum_T x_T,$$

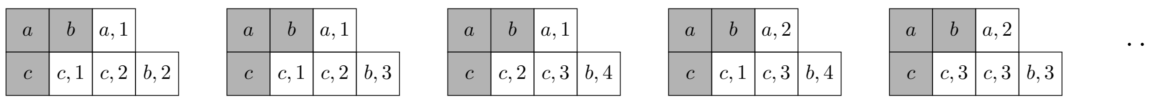
where the sum runs over colored skew shin-tableaux T of shape I/J.

Proof. Let $I = (w_1, \ldots, w_k)$, $J = (v_1, \ldots, v_h)$, and $K = (u_1, \ldots, u_j)$. Observe that

$$\mathfrak{w}_J H_K = \sum_{J = J^0 \subset^{\mathfrak{w}}_{u_1} J^1 \subset^{\mathfrak{w}}_{u_2} J^2 \cdots \subset^{\mathfrak{w}}_{u_j} J^j = I} \mathfrak{w}_I^*$$

via repeated applications of the right Pieri rule. Then $\langle \mathfrak{w}_J H_K, \mathfrak{w}_I^* \rangle$ counts the number of unique sequences $(J, J^1, J^2, \ldots, J^j)$ such that $J = J^0 \subset^{\mathfrak{w}}_{u_1} J^1 \subset^{\mathfrak{w}}_{u_2} J^2 \cdots \subset^{\mathfrak{w}}_{u_j} J^j = I$. These sequences are in bijection with colored skew shin-tableaux of shape I/J and type K as follows. Associate the sequence $(J, J^1, J^2, \ldots, J^j = I)$ with the colored skew shin-tableau of shape I/J obtained by filling the boxes in J^i/J^{i-1} with i's for each $i \in [j]$. Thus when reading the type of this tableau, the entries filled with i's are given by the word u_i, and in fact the type is K. The containment condition of the Pieri rule ensures that rows are weakly increasing and the overhang axiom ensures that columns are strictly increasing. The overhang axiom also ensures the extra condition on the relative lengths of rows in J. Thus, each sequence is associated to a colored skew shin-tableau in this way, and it is simple to see that the reverse is true as well. It follows that $\langle \mathfrak{w}_J H_K, \mathfrak{w}_I^* \rangle$ is equal to the number of colored skew shin-tableaux of shape I/J and type K. Now it is simple to expand Equation (4.20) into a sum over colored skew shin-tableaux. $\square$

Example 4.5.16. The colored skew extended Schur function for $(aba, cccb)/(ab, c)$ is given by



$$\mathfrak{w}^*_{(aba,cccb)/(ab,c)} = x_{ca,1}x_{cb,2} + x_{ca,1}x_{c,2}x_{b,3} + x_{a,1}x_{c,2}x_{c,3}x_{b,4} + x_{c,1}x_{a,2}x_{c,3}x_{b,4} + x_{a,2}x_{ccb,3} + \cdots$$

The rule for multiplication of a shin function by a ribbon function also generalizes to the colored case using colored skew shin-tableaux.

Theorem 4.5.17. *For sentences I and J,*

$$\mathfrak{w}_I R_J = \sum_K \sum_S \mathfrak{w}_K,$$

where the sums run over sentences K and skew standard colored shin-tableau S of shape K/I with $co_A(S) = J$.

Proof. We proceed by induction on the length of J. If $\ell(J) = 0$ then $\mathfrak{w}_I R_\emptyset = \mathfrak{w}_I$ and $\sum_K \sum_S \mathfrak{w}_K = \mathfrak{w}_I$ because the only possible skew colored standard shin tableau S with descent composition $\emptyset$ is the one of shape I/I. Next, assume that our statement holds for all J such that $|J| \leq h-1$. Now, consider the sentences $I = (w_1, \ldots, w_k)$ and $J = (v_1, \ldots, v_h)$. By the definition of multiplication on $\{R_I\}_I$ (Equation 2.9), observe that

$$\mathfrak{w}_I R_J = \mathfrak{w}_I R_{J^\natural} R_{v_h} - \mathfrak{w}_I R_{(v_1,\ldots,v_{h-2},v_{h-1}\cdot v_h)},$$

where $J^\natural = (v_1, \ldots, v_{h-1})$. Then, by induction, and substituting $R_{v_h} = H_{v_h}$, we have

$$\mathfrak{w}_I R_J = \left(\sum_{K_1} \sum_{S_1} \mathfrak{w}_{K_1} \right) H_{v_h} - \sum_{K_2} \sum_{S_2} \mathfrak{w}_{K_2},$$

where the sums run over sentences K_1 and skew standard colored shin-tableaux S_1 of shape K_1/I with $co_A(S_1) = J^\natural$, sentences K_2 and skew standard colored shin-tableaux S_2 of shape K_2/I with $co_A(S_1) = (|v_1|, \ldots, |v_{h-2}|, |v_{h-1}| + |v_h|)$. Notice in the first portion of the sum, we can apply the right Pieri rule. When the colored shin-horizontal strip of word v_h is added to K_1, fill in each of the new boxes from left to right, bottom to top starting with the number $|K_1| + 1$ and increasing by 1 each time. For each tableau S_1 we create a set of new skew standard colored shin-tableaux U_1 of shape L_1 where $co_A(U_1) = J$ or $co_A(U_1) = (|v_1|, \ldots, |v_{h-2}|, |v_{h-1}| + |v_h|)$. This is because there are no new descents created within our added strip, so we only need to consider whether the first block is placed below the highest numbered block in S_1, creating a descent, or not. Splitting into these two cases, we write

$$\left(\sum_{K_1}\sum_{S_1}\mathfrak{w}_{K_1}\right)H_{v_h}=\sum_{L_1}\sum_{U_1}\mathfrak{w}_{L_1}+\sum_{L_2}\sum_{U_2}\mathfrak{w}_{L_2},$$

where the sums run over all sentences L_1, skew SCST U_1 of shape L_1/I where $co^{\mathfrak{w}}_A(U_1)=J$, sentences L_2, and skew SCST U_2 of shape L_2/I where $co^{\mathfrak{w}}_A(U_2)=(v_1,\ldots,v_{h-2},v_{h-1}\cdot v_h)$.

Thus our overall sum becomes

$$\mathfrak{w}_I R_J=\sum_{L_1}\sum_{U_1}\mathfrak{w}_{L_1}+\sum_{L_2}\sum_{U_2}\mathfrak{w}_{L_2}-\mathfrak{w}_I R_{(v_1,\ldots,v_{h-1},v_{h-1}\cdot v_h)}.$$

Notice that the second and third portions of our sums are equal up to a sign. Therefore, the overall sum simplifies to

$$\mathfrak{w}_I R_J=\sum_{L_1}\sum_{U_1}\mathfrak{w}_{L_1},$$

where the sum runs over sentences L_1 and skew SCST U_1 of shape L_1/I with $co^{\mathfrak{w}}_A(U_1)=J$. □

Additionally, the comultiplication of the colored extended Schur functions can be expressed in terms of skew extended Schur functions.

Proposition 4.5.18. *For a sentence I,*

$$\Delta\mathfrak{w}^*_I=\sum_{J\subseteq_L I}\mathfrak{w}^*_J\otimes\mathfrak{w}^*_{I/J}.$$

Proof. The product of the colored shin functions uniquely defines the coproduct of the colored extended Schur functions due to Hopf algebra properties [37]. Specifically,

$$\Delta(\mathfrak{w}^*_I)=\sum_{J,K}\langle\mathfrak{w}_J\mathfrak{w}_K,\mathfrak{w}^*_I\rangle\,(\mathfrak{w}^*_J\otimes\mathfrak{w}^*_K)=\sum_{J\subseteq_L I}\mathfrak{w}^*_J\otimes\left(\sum_K\langle\mathfrak{w}_J\mathfrak{w}_K,\mathfrak{w}^*_I\rangle\mathfrak{w}^*_K\right)=\sum_{J\subseteq_L I}\mathfrak{w}^*_J\otimes\mathfrak{w}^*_{I/J}$$ □

Now, we define a colored skew-II extended Schur function using the colored right-perp operator.

Definition 4.5.19. For sentences I and J, the *skew-II colored extended Schur function* is defined as

$$\mathfrak{w}^*_{I/\!\!/J}=\mathfrak{w}^{\perp}_J(\mathfrak{w}^*_I),$$

where $\mathfrak{w}_{I/\!\!/J}=0$ if $J^r\not\subseteq_L I^r$.

By Definition 4.5.12, $\mathfrak{w}^*_{I/\!\!/J}$ expands into various bases as follows. For sentences I and J,

$$\mathfrak{w}^*_{I/\!\!/J}=\sum_K\langle H_K\mathfrak{w}_J,\mathfrak{w}^*_I\rangle M_K=\sum_K\langle R_K\mathfrak{w}_J,\mathfrak{w}^*_I\rangle F_K=\sum_K\langle\mathfrak{w}_K\mathfrak{w}_J,\mathfrak{w}^*_I\rangle\mathfrak{w}^*_K. \tag{4.21}$$

As in Remark 4.3.11, these functions do not always expand positively into the monomial basis and thus cannot be expressed as positive sums over skew-II colored shin-tableaux. We can still express the comultiplication of the colored extended Schur basis in terms of the skew-II colored extended Schur functions like in Proposition 4.3.12

Proposition 4.5.20. *For a composition I,*

$$\Delta(\boldsymbol{\mathfrak{w}}_I^*) = \sum_J \boldsymbol{\mathfrak{w}}_{I/\!\!/J}^* \otimes \boldsymbol{\mathfrak{w}}_J^*,$$

where the sum runs over compositions J where $J^r \subseteq_L I^r$.

Proof. For a sentence I,

$$\Delta(\boldsymbol{\mathfrak{w}}_I^*) = \sum_{J,K} \langle \boldsymbol{\mathfrak{w}}_K \boldsymbol{\mathfrak{w}}_J, \boldsymbol{\mathfrak{w}}_I^* \rangle \boldsymbol{\mathfrak{w}}_K^* \otimes \boldsymbol{\mathfrak{w}}_J^* = \sum_J \left(\sum_K \langle \boldsymbol{\mathfrak{w}}_K \boldsymbol{\mathfrak{w}}_J, \boldsymbol{\mathfrak{w}}_I^* \rangle \boldsymbol{\mathfrak{w}}_K^* \right) \otimes \boldsymbol{\mathfrak{w}}_J^* = \sum_J \boldsymbol{\mathfrak{w}}_{I/\!\!/J}^* \otimes \boldsymbol{\mathfrak{w}}_J^*. \qquad \square$$

where the sum runs over sentences J.

CHAPTER

5

THE YOUNG QUASISYMMETRIC AND YOUNG NONCOMMUTATIVE SCHUR FUNCTIONS

We define the colored Young quasisymmetric basis of $QSym_A$ and the colored Young noncommutative Schur basis of $NSym_A$. These generalize the Young quasisymmetric Schur functions and Young noncommutative Schur functions respectively. We also introduce skew colored Young quasisymmetric Schur functions in $QSym_A$.

5.1 Background

The quasisymmetric Schur functions were introduced by Haglund, Luoto, Mason, and van Willigenburg. These specialize the later developed quasisymmetric Macdonald polynomials of Corteel, Haglund, Mandelshtam, Mason, and Williams [24]. Tewari and van Willigenburg defined a class of 0-Hecke modules whose characteristics are the quasisymmetric Schur functions in [81], and König proved the indecomposability of certain submodules in [46]. Many other authors have studied the quasisymmetric Schur functions and their duals in the context of 0-Hecke algebras, the quasisymmetric Hall-Littlewood polynomials, dual equivalence, and more, for example in [15, 44, 45, 54, 73, 75, 84, 85]. We begin, however, with the related Young quasisymmetric Schur functions because they are more compatible with Schur functions. The Young quasisymmetric Schur functions were introduced by Luoto, Mykytiuk, and van Willigenburg in [56] as the image of the quasisymmetric

Schur functions under ρ.

5.1.1 Young quasisymmetric Schur functions

Given a tableau T, let (i, j) denote the box in row i and column j, and let $T(i, j)$ denote the value in the box (i, j).

Definition 5.1.1. For a composition α, a *semistandard Young composition tableau* (SSYCT) T of shape α is defined to be a filling of the diagram of α with positive integers such that

1. the entries in each row are weakly increasing from left to right,

2. the entries in the first column are strictly increasing from top to bottom, and

3. (triple rule) let $T(i, j) = 0$ if $(i, j) \notin \alpha$ for $i, j \in \mathbb{Z}_{>0}$. If $i > j$ and $T(i, k) \leq T(j, k+1)$, then $T(i, k+1) < T(j, k+1)$.

The *type* of T is the weak composition $(\beta_1, \beta_2, \ldots)$ where β_i is the number of times i appears in T.

A *SSYCT* T of type $\beta = (\beta_1, \ldots, \beta_h)$ is associated with the monomial $x^T = x_1^{\beta_1} \cdots x_h^{\beta_h}$ which may also be written as x^β.

Example 5.1.2. The semistandard Young composition tableaux of shape $(2, 2)$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} \quad \cdots$$

Definition 5.1.3. For a composition α, the *Young quasisymmetric Schur function* is given by

$$\hat{\mathbf{s}}^*_\alpha = \sum_T x^T,$$

where the sum runs over all SSYCTs T of shape α.

Example 5.1.4. According to the tableaux in Example 5.1.2, the Young quasisymmetric Schur function $\hat{\mathbf{s}}^*_{(2,2)}$ is given by

$$\hat{\mathbf{s}}^*_{(2,2)} = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + 2x_1 x_2 x_3 x_4 + \cdots$$

A *standard Young composition tableau* (SYCT) of shape α is a Young composition tableau in which the integers 1 through $|\alpha|$ each appear exactly once. A SYCT U has a *descent* in position i if $(i + 1)$ is in a column weakly left of i in U. We denote the set of descents in U as $\hat{Des}(U)$, called the *descent set* of U. The *descent composition* of U is defined as $\hat{co}(U) = comp(\hat{Des}(U))$. Each semistandard Young composition tableau T of size n can be associated with a standard Young composition tableau called the *standardization* of T. Given a semistandard Young composition tableau T, form a standard Young composition tableau $std(T) = U$ by relabeling the boxes of T

with integers 1 through n in the following way. Starting with the leftmost lowest box filled with a 1, relabel all the boxes filled with 1's moving from left to right, bottom to top, filling them with consecutive integers $1, 2, 3, \ldots$. Continue relabeling all the boxes originally filled with 2's with the next consecutive integer, again going from left to right, bottom to top. Continue this process for boxes originally filled with 3's, 4's, and so on until the entire tableau has been relabelled.

Example 5.1.5. The two semistandard Young composition tableaux below both have shape (3,3) and type (2,2,1,1) but have different standardizations.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \qquad\qquad T_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array}$$

$$std(T_1) = U_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 5 \\ \hline \end{array} \qquad\qquad std(T_2) = U_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$$

$$D\hat{e}s(U_1) = \{2,5\},\ \hat{co}(U_1) = (2,3,1) \qquad\qquad D\hat{e}s(U_2) = \{2,4\},\ \hat{co}(U_2) = (2,2,2)$$

Let $\hat{K}_{\alpha,\beta}$ be the number of SSYCT of shape α and type β, and let $\hat{L}_{\alpha,\beta}$ be the number of SYCT of shape α and descent composition β.

Proposition 5.1.6. *[56] The Young quasisymmetric Schur functions expand positively into the monomial and fundamental bases as*

$$\hat{\mathbf{s}}^*_\alpha = \sum_\beta \hat{K}_{\alpha,\beta} M_\beta \qquad \textit{and} \qquad \hat{\mathbf{s}}^*_\alpha = \sum_\beta \hat{L}_{\alpha,\beta} F_\beta.$$

Schur functions have a positive expansion in terms of Young quasisymmetric Schur functions.

Proposition 5.1.7. *[56] For a partition λ,*

$$s_\lambda = \sum_{sort(\alpha)=\lambda} \hat{\mathbf{s}}^*_\alpha,$$

where the sum runs over compositions α such that $sort(\alpha) = \lambda$.

Example 5.1.8. The Schur function $s_{(4,2,1)}$ expands into Young quasisymmetric Schur functions as

$$s_{(4,2,1)} = \hat{\mathbf{s}}^*_{(1,2,4)} + \hat{\mathbf{s}}^*_{(1,4,2)} + \hat{\mathbf{s}}^*_{(2,1,4)} + \hat{\mathbf{s}}^*_{(2,4,1)} + \hat{\mathbf{s}}^*_{(4,1,2)} + \hat{\mathbf{s}}^*_{(4,2,1)}.$$

There exist two left Pieri rules for the Young quasisymmetric Schur functions. These rules make use of the following operators on skew diagrams. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition with the maximum entry m and let $1 \leq s \leq m$. If there is an i such that $s = \alpha_i$ and $s \neq \alpha_j$ for $j < i$, then

$$\hat{\mathfrak{d}}_s(\alpha) = (\alpha_1, \ldots, \alpha_{i-1}, (s-1), \alpha_{i+1}, \ldots, \alpha_k),$$

otherwise $\hat{\mathfrak{d}}_s(\alpha) = \emptyset$. Then for $S = \{s_1 < \cdots < s_j\}$ and $M = \{m_1 \leq \cdots \leq m_j\}$,

$$\hat{\mathfrak{h}}_S(\alpha) = \hat{\mathfrak{d}}_{s_1}(\ldots(\hat{\mathfrak{d}}_{s_{j-1}}(\hat{\mathfrak{d}}_{s_j}(\alpha)))\ldots) \quad \text{and} \quad \hat{\mathfrak{v}}_M(\alpha) = \hat{\mathfrak{d}}_{m_j}(\ldots(\hat{\mathfrak{d}}_{m_2}(\hat{\mathfrak{d}}_{m_1}(\alpha)))\ldots).$$

If there are any zeros in $\hat{\mathfrak{h}}_S(\alpha)$ or $\hat{\mathfrak{v}}_M(\alpha)$, remove them to create a composition.

Example 5.1.9. Let $\alpha = (1, 2, 3, 2)$. Then,

$$\hat{\mathfrak{h}}_{2,3}(\alpha) = \hat{\mathfrak{d}}_2(\hat{\mathfrak{d}}_3((1, 2, 3, 2))) = \hat{\mathfrak{d}}_2((1, 2, 2, 2)) = (1, 1, 2, 2),$$

$$\hat{\mathfrak{v}}_{2,2}(\alpha) = \hat{\mathfrak{d}}_2(\hat{\mathfrak{d}}_2((1, 2, 3, 2))) = \hat{\mathfrak{d}}_2((1, 1, 3, 2)) = (1, 1, 3, 1).$$

Additionally, let a horizontal strip be a skew shape that has no more than one box in each column and a vertical strip be a skew shape with no more than one box in each row. For a horizontal strip δ, let $S(\delta)$ denote the set of columns occupied by its skew diagram, and for a vertical strip ε, let $M(\varepsilon)$ denote the multiset of columns occupied by its skew diagram. In both, multiplicities are given by the number of boxes in each column and elements of the set should be listed in weakly increasing order.

Theorem 5.1.10. *[56] Let α be a composition and r a positive integer. Then,*

$$\hat{\mathbf{s}}^*_{(r)}\hat{\mathbf{s}}^*_\alpha = \sum_\beta \hat{\mathbf{s}}^*_\beta,$$

where the sum runs over all compositions β such that

1. *$\delta = sort(\beta)/sort(\alpha)$ is a horizontal strip,*
2. *$\beta \models |\alpha| + r$, and*
3. *$\hat{\mathfrak{h}}_{S(\delta)}(\beta) = \alpha$.*

Additionally, we have that

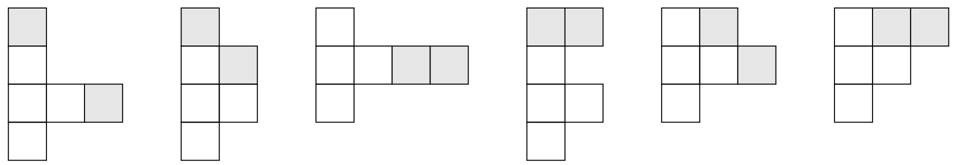
$$\hat{\mathbf{s}}^*_{(1^r)}\hat{\mathbf{s}}^*_\alpha = \sum_\beta \hat{\mathbf{s}}^*_\beta,$$

where the sum runs over all compositions β such that

1. *$\varepsilon = sort(\beta)/sort(\alpha)$ is a vertical strip,*
2. *$\beta \models |\alpha| + r$, and*
3. *$\hat{\mathfrak{v}}_{M(\varepsilon)}(\beta) = \alpha$.*

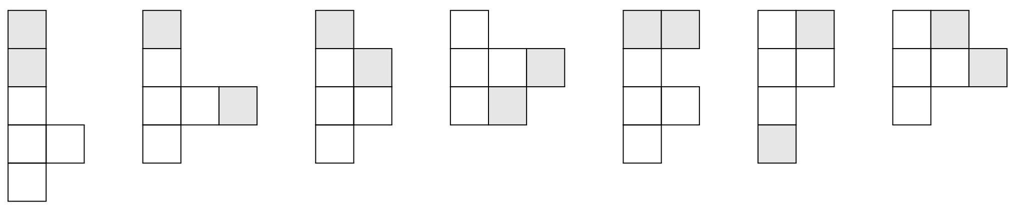
Example 5.1.11. For the first rule, we have

$$\hat{\mathbf{s}}^*_{(2)}\hat{\mathbf{s}}^*_{(1,2,1)} = \hat{\mathbf{s}}^*_{(1,1,3,1)} + \hat{\mathbf{s}}^*_{(1,2,2,1)} + \hat{\mathbf{s}}^*_{(1,4,1)} + \hat{\mathbf{s}}^*_{(2,1,2,1)} + \hat{\mathbf{s}}^*_{(2,3,1)} + \hat{\mathbf{s}}^*_{(3,2,1)}.$$



For the second rule, we have

$$\hat{\mathbf{s}}^*_{(1,1)}\hat{\mathbf{s}}^*_{(1,2,1)} = \hat{\mathbf{s}}^*_{(1,1,1,2,1)} + \hat{\mathbf{s}}^*_{(1,1,3,1)} + \hat{\mathbf{s}}^*_{(1,2,2,1)} + \hat{\mathbf{s}}^*_{(1,3,2)} + \hat{\mathbf{s}}^*_{(2,1,2,1)} + \hat{\mathbf{s}}^*_{(2,2,1,1)} + \hat{\mathbf{s}}^*_{(2,3,1)}.$$



Another especially interesting property of these functions is that dual immaculate functions expand positively into the Young quasisymmetric Schur basis. This was proved by Allen, Hallam, and Mason in [4] using an analogue of Schensted Insertion. The combinatorial formula for the coefficients in this expansion counts objects they define called dual immaculate recording tableaux (DIRTs), which we do not define here.

Theorem 5.1.12. *[4] The dual immaculate quasisymmetric functions decompose into Young quasisymmetric Schur functions in the following way:*

$$\mathfrak{S}^*_\alpha = \sum_\beta d_{\alpha,\beta}\hat{\mathbf{s}}^*_\beta,$$

where $d_{\alpha,\beta}$ is the number of DIRTs of shape β with row strip shape α^r.

In [58], Marcum and Niese also give positive expansions of certain extended Schur functions into the Young quasisymmetric Schur basis using *DIRTs*. In this case, however, not all extended Schur functions expand positively into the Young quasisymmetric Schur basis.

Theorem 5.1.13. *[58] Let α be a shuffle of (1^k) and a partition λ. Then*

$$\boldsymbol{\mathfrak{w}}^*_\alpha = \sum_\beta d^*_{\alpha,\beta}\hat{\mathbf{s}}^*_\beta,$$

where $d^*_{\alpha,\beta}$ counts the number of a special class of DIRTs with row strip shape α^r, shape β, and one additional condition.

Corollary 5.1.14. *[58] Let α be a shuffle of (1^k) and a partition λ where $k > 0$ and all parts of λ are greater than 2, and let $\alpha_{i_1} = \cdots = \alpha_{i_k} = 1$. Let $A(\alpha)$ denote the set of compositions β such that* *sort*$(\alpha) = $ *sort*(β) *and* $\beta_j > 1$ *if* $j > i_k$. *Then,*

$$\boldsymbol{\mathfrak{w}}^*_\alpha = \sum_{\beta \in A(\alpha)} d^*_{\alpha,\beta} \hat{\mathbf{s}}^*_\beta,$$

where $d^*_{\alpha,\beta} \in \{0,1\}$.

Like the other Schur-like bases of $QSym$, there are skew Young quasisymmetric functions that can be defined in terms of a skew version of the SSYCT. These were also introduced in [56]. The notions of type, standard, standardization, descents, descent compositions, and associated monomials all exactly follow those of the usual SSYCT.

Definition 5.1.15. Let α and β be compositions such that $\beta \subseteq \alpha$ and if $(i,j) \in \alpha$ but $(i,j) \notin \beta$ then there is no row α_t of length $j-1$ such that $t > i$. A *skew semistandard Young composition tableau* T of shape α/β is defined as a filling of the skew shape α/β with positive integers such that

1. the entries in each row are weakly increasing from left to right,
2. the entries in the first column are strictly increasing from top to bottom,
3. (triple rule) let $T(i,j) = \infty$ when $(i,j) \notin \alpha$ and let $T(i,j) = 0$ when $(i,j) \in \beta$. If $i > j$ and $(j, k+1) \in \alpha/\beta$ and $T(i,k) \leq T(j,k+1)$, then $T(i,k+1) < T(j,k+1)$.

Skew semistandard Young composition tableaux are not defined for skew shapes that do not meet the conditions described on α and β in the definition above. Intuitively, this means that there will never be a box B in α/β such that there exists a row below B whose length is exactly the number of boxes to the left of B. Note that in [56], an alternate notation is used to denote only the shapes α/β that meet the criteria described.

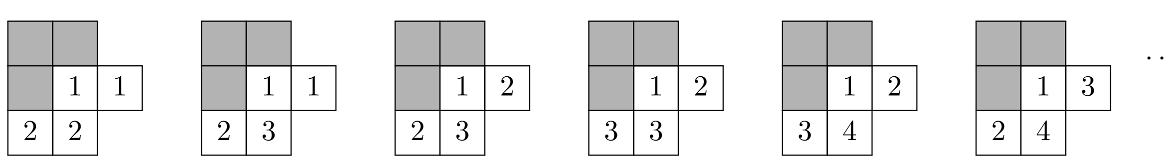
Definition 5.1.16. For compositions $\beta \subseteq \alpha$, the *skew Young quasisymmetric function* is defined as

$$\hat{\mathbf{s}}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over skew SSYCT of shape α/β. If there exists a box $(i,j) \in \alpha$ but $(i,j) \notin \beta$ such that there is a row α_t of length $j-1$ where $t > i$, then $\hat{\mathbf{s}}^*_{\alpha/\beta} = 0$.

Example 5.1.17. The skew Young quasisymmetric Schur function $\hat{\mathbf{s}}^*_{(2,3,2)/(2,1)}$ is given by

$$\hat{\mathbf{s}}^*_{(2,3,2)/(2,1)} = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + 2x_1x_2x_3x_4 + \cdots$$



We conclude this subsection by noting that there exists a bijection $\hat{\rho}_\alpha$ between skew SSYCT of inner shape α and skew SSYT of inner shape $sort(\alpha)$. This bijection is first defined in [56], but is based on an analogous bijection developed by Mason in [59]. Many of the properties of these bases and tableaux stem from this bijection.

5.1.2 Young noncommutative Schur functions

The Young noncommutative Schur functions were also introduced in [56].

Definition 5.1.18. For a composition α, the *Young noncommutative Schur function* $\hat{\mathbf{s}}_\alpha$ is defined by $\langle \hat{\mathbf{s}}_\alpha, \hat{\mathbf{s}}^*_\beta \rangle = \delta_{\alpha,\beta}$ for every composition β.

By definition, $\{\hat{\mathbf{s}}_\alpha\}_\alpha$ is the basis of *NSym* that is dual to $\{\hat{\mathbf{s}}^*_\alpha\}_\alpha$ in *QSym*. Therefore, complete homogeneous functions and ribbon functions expand into the Young noncommutative Schur basis as

$$H_\beta = \sum_\alpha \hat{K}_{\alpha,\beta} \hat{\mathbf{s}}_\alpha \qquad\qquad R_\beta = \sum_\alpha \hat{L}_{\alpha,\beta} \hat{\mathbf{s}}_\alpha. \tag{5.1}$$

Perhaps the most exciting property of the Young noncommutative Schur functions is that they have a Littlewood-Richardson type rule. We state that result here without defining all the necessary background. Most important is that the Young noncommutative Schur structure coefficients are positive and they count certain skew standard Young composition tableaux.

Theorem 5.1.19. *[56] Let α and β be compositions. Then,*

$$\hat{\mathbf{s}}_\alpha \hat{\mathbf{s}}_\beta = \sum_{\gamma \models |\alpha|+|\beta|} \hat{C}^\gamma_{\alpha,\beta} \hat{\mathbf{s}}_\gamma,$$

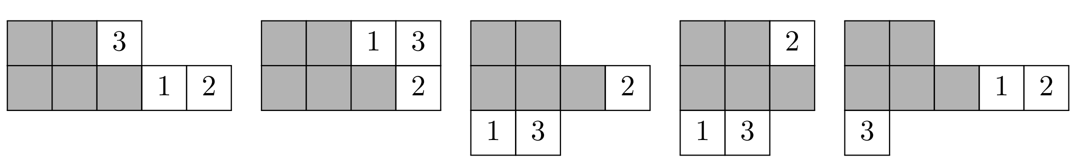
where $\hat{C}^\gamma_{\alpha,\beta}$ counts the number of skew SYCTs T of shape γ/α such that using using Schensted insertion $\hat{\rho}^{-1}(P(w_{col}(T))) = U_\beta$.

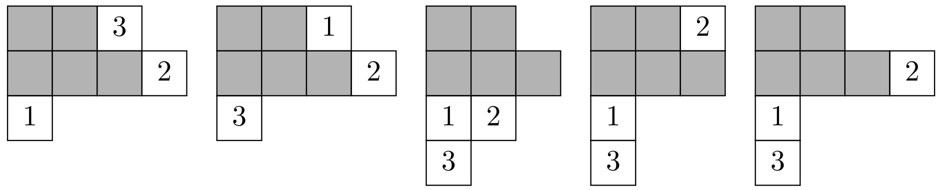
These coefficients are also those that appear in the skew Young quasisymmetric Schur functions,

$$\hat{\mathbf{s}}^*_{\gamma/\alpha} = \sum_\beta \hat{C}^\gamma_{\alpha,\beta} \hat{\mathbf{s}}^*_\beta.$$

Example 5.1.20. [56] The noncommutative Littlewood-Richardson coefficients of the product below count the following tableaux:

$$\hat{\mathbf{s}}_{(2,3)} \hat{\mathbf{s}}_{(2,1)} = \hat{\mathbf{s}}_{(3,5)} + \hat{\mathbf{s}}_{(4,4)} + \hat{\mathbf{s}}_{(2,4,2)} + \hat{\mathbf{s}}_{(3,3,2)} + \hat{\mathbf{s}}_{(2,5,1)} + 2\hat{\mathbf{s}}_{(3,4,1)} + \hat{\mathbf{s}}_{(2,3,2,1)} + \hat{\mathbf{s}}_{(3,3,1,1)} + \hat{\mathbf{s}}_{(2,4,1,1)}$$





These noncommutative Littlewood-Richardson coefficients have the following relationship to the classical Littlewood-Richardson coefficients.

Corollary 5.1.21. *[56] Let λ and μ be partitions and let α and β be compositions such that $sort(\alpha) = \lambda$ and $sort(\beta) = \mu$. Then,*

$$c^{\nu}_{\lambda,\mu} = \sum_{sort(\gamma)=\nu} \hat{C}^{\gamma}_{\alpha,\beta}.$$

The Young noncommutative Schur functions also have Pieri rules that follow from their Littlewood-Richardson rules.

Corollary 5.1.22. *[14] Let α be a composition and r a positive integer. Then,*

$$\hat{\mathbf{s}}_{\alpha}\hat{\mathbf{s}}_{(r)} = \sum_{\gamma} \hat{\mathbf{s}}_{\gamma},$$

where the sum runs over all compositions $\gamma \supseteq \alpha$ obtained by adding r boxes to α such that

1. *$\ell(\gamma) \leq \ell(\alpha) + 1$,*
2. *no two boxes are in the same column, and*
3. *a box can only be added to that row if there is no row below it with the same length.*

Additionally,

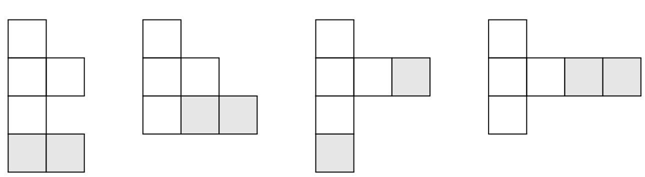
$$\hat{\mathbf{s}}_{\alpha}\hat{\mathbf{s}}_{(1^r)} = \sum_{\gamma} \hat{\mathbf{s}}_{\gamma},$$

where the sum runs over all compositions $\gamma \subseteq \alpha$ obtained by adding r boxes to α such that

1. *$\ell(\gamma) \leq \ell(\alpha) + 1$,*
2. *no two boxes are added in the same row, and*
3. *a box can only be added to that row if there is no row below it with the same length.*

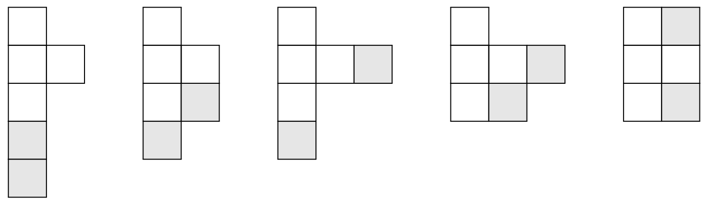
Example 5.1.23. The first rule yields

$$\hat{\mathbf{s}}_{(1,2,1)}\hat{\mathbf{s}}_{(2)} = \hat{\mathbf{s}}_{(1,2,1,2)} + \hat{\mathbf{s}}_{(1,2,3)} + \hat{\mathbf{s}}_{(1,3,1,1)} + \hat{\mathbf{s}}_{(1,4,1)}.$$



The second rule yields

$$\hat{\mathbf{s}}_{(1,2,1)}\hat{\mathbf{s}}_{(1,1)} = \hat{\mathbf{s}}_{(1,2,1,1,1)} + \hat{\mathbf{s}}_{(1,2,2,1)} + \hat{\mathbf{s}}_{(1,3,1,1)} + \hat{\mathbf{s}}_{(1,3,2)} + \hat{\mathbf{s}}_{(2,2,2)}.$$



Interestingly, Allen, Hallam, and Mason showed that Young noncommutative Schur functions and immaculate functions are equal when indexed by the same partition. Notably, this is not true for the shin functions.

Theorem 5.1.24. *[4] Let α be a composition. Then $\hat{\mathbf{s}}_\alpha = \mathfrak{S}_\alpha$ if and only is α is a partition.*

For compositions in general, the expansions are as follows.

Proposition 5.1.25. *[4] The Young noncommutative Schur functions decompose into the dual immaculate functions in the following way:*

$$\hat{\mathbf{s}}_\alpha = \sum_\beta d_{\beta,\alpha}\mathfrak{S}_\beta,$$

where $d_{\beta,\alpha}$ is the number of DIRTs of shape α and row strip shape β^r. Moreover, for the hook shape $(1^k, n-k)$ we have

$$\hat{\mathbf{s}}_{(1^k,n-k)} = \sum_{\ell(\beta)=k+1} \mathfrak{S}_\beta.$$

Allen, Hallam, and Mason also provide a Remmel-Whitney-Style algorithm for decomposing $\mathfrak{S}^*_\alpha$ in terms of $DIRT$s. In [67], Niese gives a Remmel-Whitney rule for the decomposition of a product of a Young quasisymmetric Schur function and a Schur function into the Young quasisymmetric Schur basis.

5.1.3 Quasisymmetric Schur functions and other bases related by involutions

The images of the Young quasisymmetric Schur functions under ψ, ρ, and ω are row-strict Young quasisymmetric Schur functions, quasisymmetric Schur functions, and row-strict quasisymmetric functions, respectively. The quasisymmetric Schur functions were introduced first by Haglund, Luoto, Mason, and van Willigenburg in [39]. In this section, we define skew tableaux and skew functions only because the definitions for non-skew cases follow by letting the inner shape be $\emptyset$. The skew quasisymmetric Schur functions were introduced by Bessenrodt, Luoto, and van Willigenburg in [14]. Note that for consistency we refer to certain functions as 'skew-II' that appear in the literature as 'skew'.

Definition 5.1.26. Let α and β be compositions such that $\beta^r \subseteq \alpha^r$ and if $(i,j) \in \alpha$ but $(i,j) \notin \beta$ then there is no row α_t of length $j-1$ such that $t < i$. A *skew semistandard reverse composition tableau* (skew SSRCT) T of shape $\alpha /\!\!/ \beta$ is defined to be a filling of the skew-II shape $\alpha /\!\!/ \beta$ with positive integers such that

1. the entries in each row are weakly decreasing from left to right,
2. the entries in the first column are strictly increasing from top to bottom,
3. (triple rule) let $T(i,j) = \infty$ if $(i,j) \in \beta$ and $T(i,j) = 0$ if $(i,j) \notin \alpha$ for $i, j \in \mathbb{Z}_{>0}$. If $i < j$ and $(j, k+1) \in \alpha /\!\!/ \beta$ and $T(i,k) \geq T(j,k+1)$, then $T(i,k+1) > T(j,k+1)$.

Definition 5.1.27. Let α and β be compositions such that $\beta^r \subseteq \alpha^r$. Then the *skew quasisymmetric Schur function* is defined as

$$\check{\mathbf{s}}^*_{\alpha /\!\!/ \beta} = \sum_T x^T,$$

where the sum runs over skew semistandard reverse composition tableaux T of shape $\alpha /\!\!/ \beta$. If $\beta = \emptyset$ then we write $\check{\mathbf{s}}^*_\alpha$ and call it the *quasisymmetric Schur function*.

The quasisymmetric Schur functions $\{\check{\mathbf{s}}^*_\alpha\}_\alpha$ form a basis of $QSym$, [39].

Example 5.1.28. The following are semistandard reverse composition tableaux of shape $(3,2)$:

1	1	1
2	2	

3	2	1
4	1	

2	2	2
3	1	

3	2	1
5	4	

4	3	1
5	2	

4	3	2
5	1	

Definition 5.1.29. For a composition α, the *noncommutative Schur function* $\check{\mathbf{s}}_\alpha$ is defined by

$$\langle \check{\mathbf{s}}_\alpha, \check{\mathbf{s}}^*_\beta \rangle = \delta_{\alpha,\beta},$$

for all compositions β.

The noncommutative Schur functions $\{\check{\mathbf{s}}_\alpha\}_\alpha$ are thus the basis of *NSym* that is dual to the quasisymmetric Schur functions. These bases relate to the Young quasisymmetric and noncommutative Schur functions by the involution ρ.

Proposition 5.1.30. *[56] For a composition* α*,*

$$\rho(\hat{\mathbf{s}}^*_\alpha) = \check{\mathbf{s}}^*_{\alpha^r} \quad \textit{and} \quad \rho(\hat{\mathbf{s}}_\alpha) = \check{\mathbf{s}}_{\alpha^r}.$$

Mason and Searles describe when Young quasisymmetric Schur and quasisymmetric Schur *polynomials* are equal, which also gives insight into the behavior of Young quasisymmetric Schur and quasisymmetric Schur functions.

Theorem 5.1.31. *[64] The equality*

$$\hat{\mathbf{s}}^*_\alpha(x_1, \ldots, x_n) = \check{\mathbf{s}}^*_\beta(x_1, \ldots, x_n),$$

holds if and only if $\alpha = \beta$ and either α has all parts the same, or all parts of α are 1 or 2, or $n = \ell(\alpha)$ and consecutive parts of α differ by at most 1.

The quasisymmetric Schur and noncommutative Schur functions also have properties analogous to those in Section 5.1.1. We will not state these analogues, but they can be obtained by applying ρ to the original results. One additional result is given in [80], where Tewari proves a Murnaghan-Nakayam rule for the noncommutative Schur functions.

Theorem 5.1.32. *[80] Given a composition α and a positive integer n,*

$$\psi_n \check{\mathbf{s}}_\alpha = \sum_\beta (-1)^{ht(\beta /\!\!/ \alpha)} \check{\mathbf{s}}_\alpha,$$

where the sum runs over compositions β such that $\beta /\!\!/ \alpha$ is a border-strip that meets certain conditions.

The row-strict quasisymmetric Schur functions were introduced by Mason and Remmel in [62]. These are defined on a variant of reverse composition tableaux with strictly decreasing rows.

Definition 5.1.33. Let α and β be compositions such that $\beta^r \subseteq \alpha^r$ and if $(i,j) \in \alpha$ but $(i,j) \notin \beta$ then there is no row α_t of length $j-1$ such that $t < i$. A *skew row-strict semistandard reverse composition tableau* T of shape $\alpha /\!\!/ \beta$ is defined to be a filling of the skew-II shape $\alpha /\!\!/ \beta$ with positive integers such that

1. the entries in each row are weakly decreasing from left to right,
2. the entries in the first column are strictly increasing from top to bottom,
3. (triple rule) let $T(i,j) = 0$ if $(i,j) \notin \alpha$ and $T(i,j) = \infty$ if $(i,j) \in \beta$ for $i, j \in \mathbb{Z}_{>0}$. If $i < j$ and $T(j, k+1) < T(i,k)$, then $T(j,k+1) \leq T(i,k+1)$.

Definition 5.1.34. Let α and β be compositions such that $\beta^r \subseteq \alpha^r$. Then the *skew row-strict quasisymmetric Schur function* is defined as

$$\check{\mathbf{rs}}^*_{\alpha /\!\!/ \beta} = \sum_T x^T,$$

where the sum runs over skew semistandard row-strict reverse composition tableaux T of shape $\alpha /\!\!/ \beta$. If $\beta = \emptyset$ then we write $\check{\mathbf{rs}}^*_\alpha$ for the *row-strict quasisymmetric Schur function.*

The row-strict quasisymmetric Schur functions $\{\check{\mathbf{rs}}^*_\alpha\}_\alpha$ form a basis of $QSym$, [62].

Example 5.1.35. The following are semistandard row-strict reverse composition tableaux of shape $(3, 2)$:

$$\begin{array}{|c|c|c|}\hline 3&2&1\\\hline 4&3\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 3&2\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 3&1\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 5&4\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&1\\\hline 5&2\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 4&3&2\\\hline 5&1\\\cline{1-2}\end{array}$$

Definition 5.1.36. For a composition α, the *row-strict noncommutative Schur function* $\check{\mathbf{rs}}_\alpha$ is defined by

$$\langle \check{\mathbf{rs}}_\alpha, \check{\mathbf{rs}}^*_\beta \rangle = \delta_{\alpha,\beta},$$

for all compositions β.

The row-strict noncommutative Schur functions $\{\check{\mathbf{rs}}_\alpha\}_\alpha$ are thus the basis of *NSym* that is dual to the row-strict quasisymmetric Schur functions. These bases relate to the Young quasisymmetric and noncommutative Schur functions by the involution ω.

Proposition 5.1.37. *[62] For a composition α,*

$$\omega(\hat{\mathbf{s}}^*_\alpha) = \check{\mathbf{rs}}^*_{\alpha^r}, \quad \psi(\check{\mathbf{s}}^*_\alpha) = \check{\mathbf{rs}}^*_\alpha, \quad \omega(\hat{\mathbf{s}}_\alpha) = \check{\mathbf{rs}}_{\alpha^r}, \quad \textit{and} \quad \psi(\check{\mathbf{s}}_\alpha) = \check{\mathbf{rs}}_\alpha.$$

Lastly, the row-strict Young quasisymmetric functions were introduced by Mason and Niese in [61]. They are defined on a variant of Young composition tableaux with strictly increasing rows.

Definition 5.1.38. Let α and β be compositions such that $\beta \subseteq \alpha$ and if $(i, j) \in \alpha$ but $(i, j) \notin \beta$ then there is no row α_t of length $j - 1$ such that $t > i$. A *skew semistandard row-strict Young composition tableau* T of shape α/β is defined to be a filling of the skew shape α/β with positive integers such that

1. the entries in each row are strictly decreasing from left to right,
2. the entries in the first column are weakly increasing from top to bottom,
3. (triple rule) let $T(i, j) = 0$ if $(i, j) \in \beta$ and $T(i, j) = \infty$ if $(i, j) \notin \alpha$ for $i, j \in \mathbb{Z}_{>0}$. If $i > j$ and $T(j, k) < T(i, k + 1)$, then $T(j, k + 1) \leq T(i, k + 1)$.

Definition 5.1.39. Let α and β be compositions such that $\beta \subseteq \alpha$. Then the *skew row-strict Young quasisymmetric Schur function* is defined as

$$\hat{\mathbf{rs}}^*_{\alpha/\beta} = \sum_T x^T,$$

where the sum runs over skew semistandard row-strict Young composition tableaux T of shape α/β. If $\beta = \emptyset$ then this becomes the *row-strict Young quasisymmetric Schur function* $\hat{\mathbf{rs}}^*_\alpha$.

The row-strict Young quasisymmetric Schur functions $\{\hat{\mathbf{rs}}^*_\alpha\}_\alpha$ form a basis of *QSym*, [61].

Example 5.1.40. The following are semistandard row-strict Young composition tableaux of shape $(2,3)$:

1	2	
2	3	4

1	2	
1	2	3

1	3	
1	2	3

1	2	
3	4	5

1	4	
2	3	5

1	5	
2	3	4

Definition 5.1.41. For a composition α, the *row-strict Young noncommutative Schur function* $\hat{\mathbf{rs}}_\alpha$ is defined by

$$\langle \hat{\mathbf{rs}}_\alpha, \hat{\mathbf{rs}}^*_\beta \rangle = \delta_{\alpha,\beta},$$

for all compositions β.

The row-strict Young noncommutative Schur functions $\{\hat{\mathbf{rs}}_\alpha\}_\alpha$ are the basis of *NSym* that is dual to the row-strict quasisymmetric functions by definition. These bases relate to the Young quasisymmetric and noncommutative Schur functions by the involution ψ.

Proposition 5.1.42. *[61] For a composition α,*

$$\psi(\hat{\mathbf{s}}^*_\alpha) = \hat{\mathbf{rs}}^*_\alpha, \quad \rho(\check{\mathbf{rs}}^*_\alpha) = \hat{\mathbf{rs}}^*_{\alpha^r}, \quad \omega(\check{\mathbf{s}}^*_\alpha) = \hat{\mathbf{rs}}^*_{\alpha^r},$$

$$\psi(\hat{\mathbf{s}}_\alpha) = \hat{\mathbf{rs}}_\alpha, \quad \rho(\check{\mathbf{rs}}_\alpha) = \hat{\mathbf{rs}}_{\alpha^r}, \quad \omega(\check{\mathbf{s}}_\alpha) = \hat{\mathbf{rs}}_{\alpha^r}.$$

Most results for the quasisymmetric and Young quasisymmetric bases (or their duals) have analogs for the row-strict quasisymmetric and row-strict Young quasisymmetric bases (or their duals) that can be found by simply applying the appropriate involution.

5.2 Colored Generalizations of the Young quasisymmetric Schur functions and their duals

We generalize the last of our primary Schur-like bases, the Young quasisymmetric Schur functions and the Young noncommutative Schur functions, to $QSym_A$ and $NSym_A$. We also prove expansions to and from other bases, a right Pieri rule, and properties of skew colored Young quasisymmetric Schur functions.

5.2.1 The colored Young quasisymmetric Schur functions in $QSym_A$

We define the Young quasisymmetric and noncommutative Schur functions in $QSym_A$ and $NSym_A$ with yet another type of colored tableaux. For a colored tableau T, let (i,j) denote the box in row i and column j and let $T(i,j)$ denote the integer filling box (i,j).

Definition 5.2.1. For a sentence I, a *semistandard colored Young composition tableau* (SSCYCT) T of shape α is defined to be a filling of the diagram of I with positive integers such that:

1. the entries in each row are weakly increasing from left to right,

2. the entries in the first column are strictly increasing from top to bottom, and

3. (triple rule) let $T(i,j) = \infty$ when $(i,j) \notin I$. If $i > j$ and $T(i,k) \leq T(j,k+1)$, then $T(i,k+1) < T(j,k+1)$.

The *type* of T is a weak sentence $J = (v_1, v_2, \ldots)$ where v_i is the word obtained by reading the colors of each box in T filled with an i, starting at the lowest leftmost box and moving left to right, bottom to top.

A SSCYCT T of type $J = (v_1, \ldots, v_h)$ is associated with the monomial $x_T = x_{v_1,1} \cdots x_{v_h,h}$ which may also be written as x_J.

Example 5.2.2. Some of the semistandard colored Young composition tableaux of shape (ab, cb) and their types are

$a,1$	$b,1$
$c,2$	$b,2$

(ab, cb)

$a,1$	$b,1$
$c,2$	$b,3$

(ab, c, b)

$a,1$	$b,3$
$c,2$	$b,2$

(a, cb, b)

$a,1$	$b,2$
$c,3$	$b,3$

(a, b, cb)

$a,1$	$b,2$
$c,3$	$b,4$

(a, b, c, b)

$a,1$	$b,4$
$c,2$	$b,3$

(a, c, b, b)

$\cdots$

Definition 5.2.3. For a sentence I, the *colored Young quasisymmetric Schur function* is given by

$$\hat{\mathbf{s}}_I^* = \sum_T x_T,$$

where the sum runs over all SSCYCTs T of shape I.

Example 5.2.4. The colored Young quasisymmetric Schur function $\hat{\mathbf{s}}^*_{(ab,cb)}$ is given by

$$\hat{\mathbf{s}}^*_{(ab,cb)} = x_{ab,1}x_{cb,2} + x_{ab,1}x_{c,2}x_{b,3} + x_{a,1}x_{cb,2}x_{b,3} + x_{a,1}x_{b,2}x_{cb,3} + x_{a,1}x_{b,2}x_{c,3}x_{b,4} + x_{a,1}x_{c,2}x_{b,3}x_{b,4} + \cdots$$

Proposition 5.2.5. *Let $A = \{a\}$, and let I be a sentence. Then, $v(\hat{\mathbf{s}}_I^*) = \hat{\mathbf{s}}^*_{w\ell(I)}$. Moreover, $\{\hat{\mathbf{s}}_I^*\}_I$ in $QSym_A$ is analogous to $\{\hat{\mathbf{s}}_\alpha^*\}_\alpha$ in $QSym$.*

Proof. Observe that v acts on a monomial x_T where T is a colored SSYCT of shape I by mapping it to the monomial $x^{T'}$ where T' is the SSYCT of shape $w\ell(I)$ with the same integer entries as T. Thus, $v(\hat{\mathbf{s}}_I^*) = \hat{\mathbf{s}}^*_{w\ell(I)}$ for all alphabets A and including those containing only one color. □

A *standard colored Young composition tableau* (SCYCT) of shape I is a colored Young composition tableau in which the integers 1 through $|I|$ each appear exactly once. A SCYCT U has a *descent* in position i if $(i+1)$ is in a row weakly left of i in U. We denote the set of descents in U as $\hat{Des}_A(U)$, called the *descent set* of U. The *colored descent composition* of U is defined as the sentence $J = (v_1, \ldots, v_h)$ obtained by reading the colors filling each box of U in the order they are numbered and splitting into a new word after every descent.

Each semistandard colored Young composition tableau T of size n can be associated with a standard colored Young composition tableau called the *standardization* of T. Given a semistandard colored Young composition tableau T, form a standard colored Young composition tableau $std(T) = U$ by relabeling the boxes of T with integers 1 through n in the following way. Starting with the leftmost lowest box filled with a 1, relabel all the boxes filled with 1's moving from left to right, bottom to top, filling them with consecutive integers $1, 2, 3, \ldots$. Continue relabeling all the boxes originally filled with 2's with the next consecutive integers, again going from left to right, bottom to top, and increasing by 1 with each move. Continue this process for boxes originally filled with 3's, 4's, and so on until the entire tableau has been relabelled. Additionally, recall from Section 3.2 that the flattening of a tableau T with entries $\{i_1 < i_2 < \cdots, < i_k\}$ is the tableau $\tilde{T}$ obtained by replacing each i_j with j so that the tableau has entries $\{1, 2, \ldots, k\}$.

Example 5.2.6. The two semistandard colored Young composition tableaux below both have shape (abc,acc) and type (ab,ac,c,c) but have different standardizations.

$$T_1 = \begin{array}{|c|c|c|} \hline a,1 & b,1 & c,4 \\ \hline a,2 & c,2 & c,3 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|} \hline a,1 & b,1 & c,2 \\ \hline a,2 & c,3 & c,4 \\ \hline \end{array}$$

$$std(T_1) = U_1 = \begin{array}{|c|c|c|} \hline a,1 & b,2 & c,6 \\ \hline a,3 & c,4 & c,5 \\ \hline \end{array} \qquad std(T_2) = U_2 = \begin{array}{|c|c|c|} \hline a,1 & b,2 & c,4 \\ \hline a,3 & c,5 & c,6 \\ \hline \end{array}$$

$D\hat{e}s(U_1) = \{2, 5\}, \; \hat{co}_A(U_1) = (ab, acc, c)$ $\qquad$ $D\hat{e}s(U_1) = \{2, 4\}, \; \hat{co}_A(U_1) = (ab, ac, cc)$

Let $\hat{K}_{I,J}$ be the number of SSCYCT of shape I and type J, and let $\hat{L}_{I,J}$ be the number of SCYCT of shape I and colored descent composition J.

Proposition 5.2.7. *The colored Young quasisymmetric Schur functions expand positively into the colored monomial and colored fundamental bases as*

$$\hat{\mathbf{s}}^*_I = \sum_J \hat{K}_{I,J} M_J \qquad \textit{and} \qquad \hat{\mathbf{s}}^*_I = \sum_J \hat{L}_{I,J} F_J,$$

where the sum runs over sentences J such that $|I| = |J|$.

Proof. Every semistandard colored Young composition tableau is also a colored immaculate tableau, so by Proposition 3.3.17 we can write

$$\hat{\mathbf{s}}^*_I = \sum_{T = \tilde{T}} M_{type(T)},$$

where the sum runs over flat semistandard colored Young composition tableaux T of shape I. From here, it is simple to obtain the first statement in the result. To obtain the second, we first show that given a standard colored Young composition tableau S,

$$F_{\hat{co}(S)} = \sum_{B \preceq \hat{co}(S)} M_B = \sum_{\substack{T=\tilde{T} \\ std(T)=S}} M_{type(T)}. \tag{5.2}$$

The method for standardization here is the same as the one for colored immaculate tableaux so by Proposition 3.3.15 we know that there is a unique tableaux T with a given type and standardization. Thus, each $M_{type(T)}$ only appears once in our sum. It remains to show that for any $B \preceq \hat{co}(S)$, there is a flat SSCYCT T of type B such that $std(T) = S$. Let $B = (b_1, \ldots, b_t)$ for words b_1, b_2, $\ldots$ b_t. We construct our tableau T (which has the same shape as S) by considering the boxes in the order they are numbered in S. We fill the first $|b_1|$ boxes with 1's, the next $|b_2|$ boxes with 2's, and so on. It follows easily that the rows of T are weakly increasing and the first column is strictly increasing, so we just need to verify that the triple rule holds.

Let $i > j$ and we will consider the triple (i, k), $(j, k+1)$ and $(i, k+1)$. Because S is a SCYCT, we know that if $S(i, k) \leq S(j, k+1)$ then $S(i, k+1) < S(j, k+1)$ by the triple rule. Now, assume that $T(i, k) \leq T(j, k+1)$. This implies that $S(i, k) < S(j, k+1)$ by construction, and thus $S(i, k+1) < S(j, k+1)$. This means that when we labelled T, we must have filled the box $(i, k+1)$ before $(j, k+1)$ so $T(i, k+1) \leq T(j, k+1)$. Further, since $(j, k+1)$ is weakly to the left of $(i, k+1)$ there must be a descent between the two in S. Since the type B we used to create T was a refinement of $\hat{co}(S)$, the boxes $(j, k+1)$ and $(i, k+1)$ were associated with different words b_r in B. Thus, $T(i, k+1) < T(j, k+1)$ so the triple rule holds for T and T is a SSCYCT.

Thus,

$$\hat{s}^*_I = \sum_{T=\tilde{T}} M_{type(T)} = \sum_{S} \sum_{\substack{T=\tilde{T} \\ std(T)=S}} M_{type(T)} = \sum_{S} F_{\hat{co}(S)},$$

where the sums run over standard colored Young composition tableaux of shape I and flat SSCYCT T that standardize to S. The expression above simplifies to our second statement. ☐

5.2.2 The colored Young noncommutative Schur functions in $\boldsymbol{NSym}_A$

Definition 5.2.8. For a sentence I, define the *colored Young noncommutative Schur function* $\hat{\mathbf{s}}_I$ by $\langle \hat{\mathbf{s}}_I, \hat{\mathbf{s}}^*_J \rangle = \delta_{I,J}$ for every composition J.

By definition, $\{\hat{\mathbf{s}}_I\}_I$ is the basis of $NSym_A$ that is dual to $\{\hat{\mathbf{s}}^*_I\}_I$ in $QSym_A$.

Proposition 5.2.9. *Let $A = \{a\}$ be an alphabet and I be a sentence. Then $\upsilon(\hat{\mathbf{s}}_I) = \hat{\mathbf{s}}_{w\ell(I)}$. Moreover, $\{\hat{\mathbf{s}}_I\}_I$ in $NSym_A$ is analogous to $\{\hat{\mathbf{s}}_\alpha\}_\alpha$ in NSym.*

Proof. Let A be an alphabet of size one. By Proposition 3.3.47, for all sentences I and J we have

$$\langle \hat{\mathbf{s}}_I, \hat{\mathbf{s}}^*_J \rangle = \langle \upsilon(\hat{\mathbf{s}}_I), \upsilon(\hat{\mathbf{s}}^*_J) \rangle = \langle \upsilon(\hat{\mathbf{s}}_I), \hat{\mathbf{s}}^*_{w\ell(J)} \rangle.$$

Because A is an alphabet of size one, $w\ell$ is a bijection from sentences in A to compositions meaning that if $w\ell(I) = w\ell(J)$ then $I = J$. Thus, we have

$$\langle \hat{\mathbf{s}}_I, \hat{\mathbf{s}}_J^* \rangle = \delta_{w\ell(I),w\ell(J)} \quad \text{so} \quad \langle \upsilon(\hat{\mathbf{s}}_I), \hat{\mathbf{s}}_{w\ell(J)}^* \rangle = \delta_{w\ell(I),w\ell(J)} = \langle \hat{\mathbf{s}}_{w\ell(I)}, \hat{\mathbf{s}}_{w\ell(J)}^* \rangle,$$

for all sentences I, J. It follows that for any sentence I, we have $\upsilon(\hat{\mathbf{s}}_I) = \hat{\mathbf{s}}_{w\ell(I)}$. □

The colored complete homogeneous functions and colored ribbon functions expand positively into the Young noncommutative Schur basis following Propositions 2.2.7 and 5.2.7.

Corollary 5.2.10. *For a sentence J,*

$$H_J = \sum_I \hat{K}_{I,J} \hat{\mathbf{s}}_I \qquad\qquad R_J = \sum_I \hat{L}_{I,J} \hat{\mathbf{s}}_I,$$

where the sum runs over sentences I such that $|I| = |J|$.

Like in the non-colored case, the colored noncommutative Schur functions have a right Pieri rule. Here, however, it must be proved directly since we do not have a colored Littlewood-Richardson rule from which it may follow.

Theorem 5.2.11. *Let I be a sentence and v a word. Then,*

$$\check{\mathbf{s}}_I H_v = \sum_J \check{\mathbf{s}}_J,$$

where the sum runs over all sentences $I \subseteq_L J$ obtained by adding $|v|$ boxes to I such that

1. *$\ell(J) \le \ell(I) + 1$,*
2. *no two boxes are added in the same column,*
3. *$(J/_L I)^r = v$, and*
4. *a box can only be added to a row if there is no lower row of the same length.*

Proof. Let $\{G_I\}_I$ be a set of noncommutative symmetric functions defined recursively by the Pieri rule above with G_I in place of $\hat{\mathbf{s}}_I$. We show first that $H_K = \sum_J \hat{K}_{J,K} G_K$ for all sentences K, using induction on the length of K. When $\ell(K) = 1$, it follows from the Pieri rule that $H_K = G_K$, which agrees with our statement. Now assume the statement is true for $\ell(K) = k-1$. Let $K = (w_1, \dots, w_k)$ and let $K^\natural = (w_1, \dots, w_{k-1})$. By our inductive assumption,

$$H_{K^\natural} = \sum_J \hat{K}_{J,K^\natural} G_J, \tag{5.3}$$

where the sum runs over sentences J. Note that because $\hat{K}_{J,K^\natural}$ counts SSCYCT tableaux of shape J and type $K^\natural$, we can rewrite $\sum_J \hat{K}_{J,K^\natural} G_J$ as $\sum_Y G_{shape(Y)}$ where the sum runs over semistandard colored Young composition tableaux of type $K^\natural$. We multiply Equation (5.3) on the right by H_{w_k} to obtain

$$H_{K^\natural}H_{w_k} = H_K = \sum_J \hat{K}_{J,K^\natural}G_J H_{w_k} = \sum_Y G_{shape(Y)}H_{w_k},$$

where the last sum runs over semistandard colored Young composition tableaux of type $K^\natural$. Given a semistandard colored Young composition tableau Y of type $K^\natural$, we claim that

$$G_{shape(Y)}H_{w_k} = \sum_T G_{shape(T)}$$

where the sum runs over semistandard colored Young composition tableaux T of type K such that $shape(Y) \subseteq_L shape(T)$ and each box in T that corresponds to a box in Y is filled with the same number as the box in Y, in which case we say Y is a *subtableau* of T. Observe that the multiset of sentences obtained through the Pieri rule for $\{G_\alpha\}_\alpha$ is equivalent to the multiset of shapes of the tableaux obtained by adding boxes in the same way to Y and filling those new boxes with k. Adding boxes in this way creates a new colored tableau T of type K that is in fact a colored Young composition tableau. Note that while the shapes of these tableaux are not unique, the tableaux themselves are by construction. Since boxes are added on the right or bottom, the condition that rows are weakly increasing and the first column is strictly increasing is maintained, so we only need to verify that the triple rule is satisfied. Let $i > j$ and consider any pair of blocks (i, c) and $(j, c+1)$ such that $T(i, c) \leq T(j, c+1)$. If $T(i, c)$ and $T(j, c+1)$ are both less than k, then their triple including $(i, c+1)$ satisfies the triple rule since it was present in the original tableau Y. If $T(j, c+1) = k$, then $T(i, c+1) < k$ because two boxes cannot be added in the same column using the Pieri rule, so both cannot be new boxes. Thus, this triple satisfies the triple rule. The triple rule does not apply to any other triples, meaning we have shown it is satisfied in all cases. Therefore, our new tableaux T is a SSCYCT of type K. As stated earlier, each tableau created this way is unique so it remains to show that the set of tableaux that is produced does include every SSCYCT of type K.

Given a SSCYCT T of type K, it is easy to see that removing every box filled with a k produces a SSCYCT of type $K^\natural$. This is because the remaining tableau must meet the requirements of SSCYCT since it was a subtableau of one already. Due to the triple rule and the other conditions, the boxes must each come from a different column, and they must come from the right or bottom of the tableau. Additionally, the triple rule asserts that we cannot have $T(j, c+1) = k$ and $T(i, c) \neq k$ with $(i, c) \in T$ and $(i, c+1) \notin T$, because then $T(i, c+1) = \infty$ which violates the triple rule. Thus, any boxes filled with k must not leave a row that would become the same length as a lower row. Therefore, the configuration of boxes removed is also one that would be added by the Pieri rule meaning there is indeed a SSCYCT with our given parameters that corresponds to each diagram generated by applying the Pieri rule. We have shown that

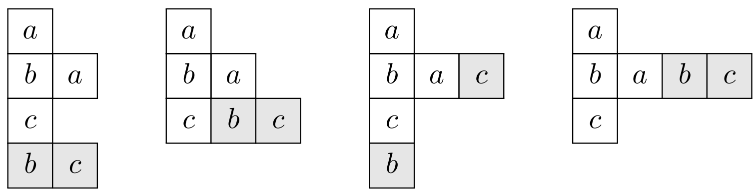
$$H_K = \sum_Y G_{shape(Y)}H_{w_k} = \sum_Y \sum_T G_{shape(T)} = \sum_Z G_{shape(Z)} = \sum_I \hat{K}_{I,K}G_I,$$

where the sums run over colored Young composition tableaux Y of type $K^\natural$, colored Young composition tableaux T of type K such that Y is a subtableau of T, semistandard colored Young composition tableaux of type K, and sentences I such that $|I| = |K|$, respectively.

Since the transition matrix of $\hat{K}$ coefficients between $\{H_I\}_I$ and $\{G_I\}_I$ is the same as the transition matrix between $\{H_I\}_I$ and $\{\hat{\mathbf{s}}_I\}_I$, we must have $\hat{\mathbf{s}}_I = G_I$ for every sentence I. Thus, the colored Young noncommutative functions are the unique set of functions that satisfy our Pieri rule. □

Example 5.2.12. Multiplying $\hat{\mathbf{s}}_{(a,ba,c)}$ by $H_{(bc)}$ yields the following:

$$\hat{\mathbf{s}}_{(a,ba,c)}H_{(bc)} = \hat{\mathbf{s}}_{(a,ba,c,bc)} + \hat{\mathbf{s}}_{(a,ba,cbc)} + \hat{\mathbf{s}}_{(a,bac,c,b)} + \hat{\mathbf{s}}_{(a,babc,b)}.$$



With this Pieri rule in hand, we return to $QSym_A$ to define skew versions of our functions.

Definition 5.2.13. For sentences I, J, the *skew colored Young quasisymmetric function* is defined

$$\hat{\mathbf{s}}^*_{I/J} = \hat{\mathbf{s}}^\perp_J(\hat{\mathbf{s}}^*_I).$$

By Definition 4.5.12, the skew colored Young quasisymmetric Schur function $\hat{\mathbf{s}}^*_{I/J}$ expands into various bases as follows:

$$\hat{\mathbf{s}}^*_{I/J} = \sum_K \langle \hat{\mathbf{s}}_J H_K, \hat{\mathbf{s}}^*_I \rangle M_K = \sum_K \langle \hat{\mathbf{s}}_J R_K, \hat{\mathbf{s}}^*_I \rangle F_K = \sum_K \langle \hat{\mathbf{s}}_J \hat{\mathbf{s}}_K, \hat{\mathbf{s}}^*_I \rangle \hat{\mathbf{s}}^*_K. \quad (5.4)$$

Definition 5.2.14. Let I and J be compositions such that $J \subseteq_L I$ and if $(i,j) \in I$ but $(i,j) \notin J$ then there is no row I_t of length $j-1$ such that $t > i$. A *skew semistandard colored Young composition tableau* T of shape I/J is defined to be a filling of the skew colored shape I/J with positive integers such that:

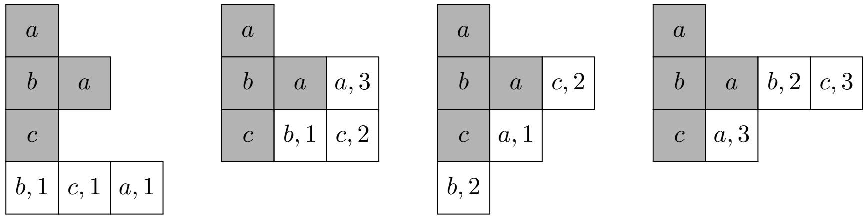
1. the entries in each row are weakly increasing from left to right,

2. the entries in the first column are strictly increasing from top to bottom,

3. (triple rule) let $T(i,j) = \infty$ when $(i,j) \notin I$ and $T(i,j) = 0$ when $(i,j) \in J$. If $i > j$ and $(j, k+1) \in I/J$ and $T(i,k) \leq T(j,k+1)$, then $T(i,k+1) < T(j,k+1)$.

Skew semistandard colored Young composition tableaux are not defined for colored skew shapes that do not meet the conditions described on I and J in the definition above. The notions of type, standard, standardization, descents, colored descent compositions, and associated monomials all exactly follow those of the usual SSCYCT.

Example 5.2.15. The following are skew semistandard colored Young composition tableaux:



Proposition 5.2.16. *Let I and J be sentences. Then,*

$$\hat{\mathbf{s}}_{I/J} = \sum_T x_T,$$

where the sum runs over all skew semistandard colored Young composition tableaux of shape I/J.

Proof. Using the same logic in the proof of Theorem 5.2.11 with repeated application of the Pieri rule, we have

$$\hat{\mathbf{s}}_J H_K = \sum_I \sum_T x_T,$$

where the sums run over sentences I and skew colored Young composition tableaux of shape I/J and type K. We only need a few additional observations since we are now dealing with skew colored tableaux. The fourth condition for the colored Pieri rule (Theorem 5.2.11) ensures that in every shape I/J we obtain, if $(i, j) \in I$ but $(i, j) \notin J$ then there is no row I_t of length $j-1$ such that $t > i$. Additionally, because we have set the boxes inside the inner shape J to be 0, the existence of that inner shape does not introduce any conflicts with triples. Therefore, given specific sentences I and J, the coefficient $\langle \hat{\mathbf{s}}_J H_K, \hat{\mathbf{s}}_I^* \rangle$ counts the number of skew colored Young composition tableaux of shape I/J and type K. By expanding the sum in Equation (5.4) using Proposition 3.3.17, we have

$$\hat{\mathbf{s}}^*_{I/J} = \sum_K \langle \hat{\mathbf{s}}_J H_K, \hat{\mathbf{s}}_I^* \rangle M_K = \sum_T x_T,$$

where the sum runs over skew colored Young composition tableaux T of shape I/J. $\square$

CHAPTER

6

COLORED GENERALIZATIONS OF THE SYMMETRIC FUNCTIONS

In the classical case, the algebra Sym is a subalgebra of $QSym$ and the commutative image of $NSym$. It is natural to ask if there is a similar subalgebra of $QSym_A$ and to look at the commutative image of $NSym_A$. We define a pair of dual Hopf algebras, $PSym_A$ and Sym_A, where $PSym_A$ is the commutative image of $NSym_A$ and Sym_A is a subalgebra of $QSym_A$.

6.1 The algebra of p-sentences $PSym_A$

Let A be an alphabet with a total order. Define the *graded lexicographic order* on words by $v \preceq_{g\ell} w$ if $\ell(v) < \ell(w)$ or if both $\ell(v) = \ell(w)$ and $v \preceq_\ell w$. A *p-sentence* is a sentence $P = (w_1, \ldots, w_k)$ such that $w_1 \succeq_{g\ell} \cdots \succeq_{g\ell} w_k$. Given a sentence $I = (v_1, \ldots, v_j)$, let $sort(I)$ be the p-sentence obtained by sorting the words in I by graded lexicographic order. Given a weak sentence $K = (u_1, \ldots, u_m)$, let $sort(K) = sort(\tilde{K})$ and let $\sigma(K)$ be $(u_{\sigma(1)}, \ldots, u_{\sigma(m)})$ for $\sigma \in S_m$. Let $PSent_A$ denote the set of p-sentences for the alphabet A. Let $WSent_A$ denote the set of weak sentences and $Sent_A$ denote the set of sentences. See Appendix B to review earlier notation.

Example 6.1.1. The sentence $P = (abb, cab, ba, cc, a, b)$ is a p-sentence. Given the sentence $I = (c, aba, bc)$, the associated p-sentence is $sort(I) = (aba, bc, c)$. Given the permutation $\sigma = 231$, we have $\sigma(I) = (bc, c, aba)$.

Definition 6.1.2. Define the algebra $PSym_A$ as the algebra generated by $\{h_w\}_w$ for all words w

in the alphabet A, where generators commute, that is $h_w h_v = h_v h_w$. We write

$$PSym_A = \mathbb{Q}[h_w : \text{words } w].$$

Definition 6.1.3. For a p-sentence $P = (w_1, \ldots, w_k)$, we define the *colored complete homogeneous p-symmetric function* as

$$h_P = h_{w_1} h_{w_2} \ldots h_{w_k}.$$

We have constructed this algebra so that the set $\{h_P : P \in PSent_A\}$ of colored complete homogeneous p-symmetric functions is a basis of $PSym_A$, which we also call the *h-basis*. Using this basis, we see that $PSym_A$ admits a Hopf algebra structure.

Proposition 6.1.4. *$PSym_A$ is a graded Hopf algebra with multiplication given by*

$$h_P h_Q = h_{sort(P \cdot Q)},$$

the natural unity map $u(k) = k \cdot 1$, the comultiplication given by

$$\Delta(h_Q) = \sum_{J \subseteq_R Q} h_{sort(Q/_R J)} \otimes h_{sort(J)},$$

and the counit

$$\epsilon(h_P) = \begin{cases} 1 & \text{if } P = \emptyset, \\ 0 & \text{otherwise.} \end{cases}.$$

Proof. We have already defined $PSym_A$ as an algebra, so it remains to show that $(PSym_A, \Delta, \epsilon)$ is a coalgebra. Observe that

$$\begin{aligned}(id \otimes \Delta) \circ \Delta(h_Q) &= (id \otimes \Delta)\left(\sum_{J \subseteq_R Q} h_{sort(Q/_R J)} \otimes h_{sort(J)}\right) \\ &= \sum_{J \subseteq_R Q} h_{sort(Q/_R J)} \otimes \left[\sum_{K \subseteq_R sort(J)} h_{sort(sort(J)/_R K)} \otimes h_{sort(K)}\right] \\ &= \sum_{J \subseteq_R Q,\, K \subseteq_R J} h_{sort(Q/_R J)} \otimes h_{sort(J/_R K)} \otimes h_{sort(K)} \end{aligned} \tag{6.1}$$

To prove co-associativity, we show that $(\Delta \otimes id) \circ \Delta(h_P)$ is equal to (6.1).

$$\begin{aligned}(\Delta \otimes id) \circ \Delta(h_Q) &= (\Delta \otimes id)\left(\sum_{Y \subseteq_R Q} h_{sort(Q/_R Y)} \otimes h_{sort(Y)}\right) \\ &= \sum_{Y \subseteq_R Q,\, Z \subseteq_R sort(Q/_R Y)} h_{sort(sort(Q/_R Y)/_R Z)} \otimes h_{sort(Z)} \otimes h_{sort(Y)} \end{aligned} \tag{6.2}$$

We now reorganize this double sum so that the sum over Y is on the inside and we have a new sum on the outside. Each term in Equation (6.2) sum is associated with a unique pair of weak sentences Y and Z such that $Y \subseteq_R Q$ and $Z \subseteq_R \mathit{sort}(Q/_R Y)$. Each pair Y and Z can be associated with a unique pair of weak sentences X and Y as follows. For each $Y \subseteq_R Q$, pick a permutation σ_Y such that $\sigma_Y(Q/_R Y) = \mathit{sort}(Q/_R Y)$, and note that $\sigma_Y(Q/_R Y) = \sigma_Y(Q)/_R \sigma_Y(Y)$. Then for a pair (Y, Z) we can define a weak sentence X such that $\mathit{sort}(Q/_R Y)/_R Z = (\sigma_Y(Q)/_R \sigma_Y(Y))/_R Z = \sigma_Y(Q)/_R \sigma_Y(X)$. In other words, $\sigma_Y(X)/_R \sigma_Y(Y) = Z$. By this definition the pair (X, Y) is uniquely defined by (Y, Z). Observe that every weak sentence $X \subseteq_R Q$ appears because, in the cases that $Y = \emptyset$, we have $\sigma_Y(X) = Z$ and $\sigma_Y = id$, so there is an $X = Z$ for all $Z \subseteq_R \sigma_Y(Q) = Q$. Further, for each X, there is a pair (X, Y) for each $Y \subseteq_R X$. By the way we have defined X, we now have $\mathit{sort}(Q/_R Y)/_R Z = \sigma_Y(Q)/_R \sigma_Y(X)$ which implies that $\mathit{sort}(\mathit{sort}(Q/_R Y)/_R Z) = \mathit{sort}(\sigma_Y(Q)/_R \sigma_Y(X)) = \mathit{sort}(Q/_R X)$. Further, $\mathit{sort}(Z) = \mathit{sort}(\sigma_Y(X)/_R \sigma_Y(Y)) = \mathit{sort}(X/_R Y)$. Thus, when we uniquely associate a pair (Y, Z) with a pair (X, Y) we also uniquely associate equivalent summands $h_{\mathit{sort}(\mathit{sort}(Q/_R Y)/_R Z)} \otimes h_{\mathit{sort}(Z)} \otimes h_{\mathit{sort}(Y)}$ and $h_{\mathit{sort}(Q/_R X)} \otimes h_{\mathit{sort}(X/_R Y)} \otimes h_{\mathit{sort}(Y)}$. Using this equality, we rewrite (6.2) as

$$(\Delta \otimes id) \circ \Delta(h_Q) = \sum_{X \subseteq_R Q} \sum_{Y \subseteq_R X} h_{\mathit{sort}(Q/_R X)} \otimes h_{\mathit{sort}(X/_R Y)} \otimes h_{\mathit{sort}(Y)}.$$

This is equivalent to the sum in Equation (6.1), so we have shown that $(PSym_A, \Delta, \epsilon)$ is coassociative. To verify the axioms for the counit, we check that the two expressions below are equivalent to $id(h_Q) = h_Q$ and to each other. Note that $\mathbb{Q} \otimes PSym_A \cong PSym_A \cong PSym_A \otimes \mathbb{Q}$ so the elements $1 \otimes h_Q$ and $h_Q \otimes 1$ are equivalent to h_Q.

$$(id \otimes \epsilon) \circ \Delta(h_Q) = (id \otimes \epsilon) \left(\sum_{J \subseteq_R Q} h_{\mathit{sort}(Q/_R J)} \otimes h_{\mathit{sort}(J)} \right) = h_Q \otimes 1,$$
$$(\epsilon \otimes id) \circ \Delta(h_Q) = (id \otimes \epsilon) \left(\sum_{J \subseteq_R Q} h_{\mathit{sort}(Q/_R J)} \otimes h_{\mathit{sort}(J)} \right) = 1 \otimes h_Q.$$

Thus, $(PSym_A, \Delta, \epsilon)$ is a coalgebra.

To show that $PSym_A$ is a bialgebra, we must show that the equations in Definition 2.2.2 are satisfied. Here we use the explicit notation for multiplication, $\mu : PSym_A \otimes PSym_A \to PSym_A$ where $\mu(h_P \otimes h_S) = h_{\mathit{sort}(P \cdot S)}$. Also, let 1_P denote the multiplicative identity in $PSym_A$ and T denote the map $T(x \otimes y) = y \otimes x$. First, we check that $\Delta(m(h_P \otimes h_S)) = (m \otimes m)(id \otimes T \otimes id)(\Delta \otimes \Delta)(h_P \otimes h_S)$. For the left-hand side, we have

$$\Delta(m(h_P \otimes h_S)) = \Delta(h_{\mathit{sort}(P \cdot S)}) = \sum_{J \subseteq_R \mathit{sort}(P \cdot S)} h_{\mathit{sort}(\mathit{sort}(P \cdot S)/_R J)} \otimes h_{\mathit{sort}(J)}.$$

For the righthand side, we have

$$(m \otimes m)(id \otimes T \otimes id)(\Delta \otimes \Delta)(h_P \otimes h_S) = \cdots$$

$$= (m \otimes m)(id \otimes T \otimes id) \sum_{\substack{Y \subseteq_R P \\ Z \subseteq_R S}} h_{sort(P/_R Y)} \otimes h_{sort(Y)} \otimes h_{sort(S/_R Z)} \otimes h_{sort(Z)}$$

$$= (m \otimes m) \sum_{\substack{Y \subseteq_R P \\ Z \subseteq_R S}} h_{sort(P/_R Y)} \otimes h_{sort(S/_R Z)} \otimes h_{sort(Y)} \otimes h_{sort(Z)}$$

$$= \sum_{\substack{Y \subseteq_R P \\ Z \subseteq_R S}} h_{sort(P/_R Y \cdot S/_R Z)} \otimes h_{sort(Y \cdot Z)} = \sum_{\substack{Y \subseteq_R P \\ Z \subseteq_R S}} h_{sort((P \cdot S)/_R (Y \cdot Z))} \otimes h_{sort(Y \cdot Z)}$$

where $Y \cdot Z = (y_1, \ldots, y_{\ell(P)}, z_1, \ldots, z_{\ell(S)})$ for $Y = (y_1, y_2, \ldots, y_{\ell(Y)})$ and $z = (z_1, z_2, \ldots, z_{\ell(Z)})$ and empty words are added to the ends of Y and Z so that they are of length $\ell(P)$ and $\ell(S)$ respectively. Let σ be a permutation such that $sort(P \cdot S) = \sigma(P \cdot S)$ and note that $sort(\sigma(P \cdot S)/_R \sigma(Y \cdot Z)) = sort((P \cdot S)/_R (Y \cdot Z)$. Rename $\sigma(Y \cdot Z)$ as J and note that $sort(J) = sort(Y \cdot Z)$. Then we can rewrite our sum as

$$(m \otimes m)(id \otimes T \otimes id)(\Delta \otimes \Delta)(h_P \otimes h_S) = \sum_{J \subseteq_R sort(P \cdot S)} h_{sort(sort(P \cdot S)/_R J)} \otimes h_{sort(J)},$$

so the equation holds. Next observe that $m \circ (\epsilon \otimes \epsilon)(h_P \otimes h_S) = \epsilon \circ m(h_P \otimes h_S)$ because both sides of the equation equal 1 if $h_P = h_S = 1_P$, and 0 otherwise. Third, we have for $k \in \mathbb{Q}$ that $(u \otimes u)\Delta(k) = \Delta(u(k))$ because both sides of the equation equal $1_P \otimes 1_P$. Finally, we have $k = \epsilon \circ u(k) = \epsilon(k \cdot 1_P) = k\epsilon(1_P) = k$. The four properties above confirm that $PSym_A$ is a bialgebra. It is also easily seen to be graded by the size of p-sentences and connected, so $PSym_A$ is a Hopf algebra by Proposition 2.2.4. □

Example 6.1.5. Multiplication and comultiplication on the h-basis works as follows,

$$h_{(aba,c)} h_{(bb,a)} = h_{(aba,bb,a,c)},$$

$$\Delta(h_{(ab,bc)}) = h_{(ab,bc)} \otimes 1 + h_{(bc,a)} \otimes h_{(b)} + h_{(ab,b)} \otimes h_{(c)} + \\ + h_{(a,b)} \otimes h_{(b,c)} + h_{(ab)} \otimes h_{(bc)} + h_{(a)} \otimes h_{(bc,b)} + 1 \otimes h_{(ab,bc)}.$$

Proposition 6.1.6. *The antipode of $PSym_A$ is given by*

$$S(h_P) = \sum_{J \preceq P} (-1)^{\ell(J)} h_{sort(J)},$$

where the sum runs over sentences J that refine the p-sentence P, extended linearly.

Proof. Using the recursive definition of antipode and the comultiplication Δ of Equation 2.3 where $w = a_1 \cdots a_i$ is a word of length i,

$$S(h_w) = -\sum_{j=0}^{i-1} S(h_{(a_1 \cdots a_j)}) h_{(a_{j+1} \cdots a_i)}.$$

In the case that $i = 1$, we have $S(a_1) = -a_1$ which agrees with our formula. Assume that the formula holds for words $w = a_1 \cdots a_i$ where $i = k$. Now let $w = a_1 a_2 \cdots a_k a_{k+1}$ be a word. We have

$$S(h_{a_1 a_2 \cdots a_k a_{k+1}}) = -\sum_{j=0}^{k} S(h_{(a_1 \cdots a_j)}) h_{(a_{j+1} \cdots a_{k+1})} = -h_w - \sum_{j=1}^{k} \sum_{J \preceq (a_1 \cdots a_j)} (-1)^{\ell(J)} h_{sort(J)} h_{(a_{j+1} \cdots a_k)}$$
$$= \sum_{K \preceq w} (-1)^{\ell(K)} h_{sort(K)}.$$

Then, because the antipode is an anti-endomorphism,

$$S(h_P) = S(h_{w_k}) \cdots S(h_{w_1}) = \sum_{I_k \preceq w_k} (-1)^{\ell(I_k)} h_{sort(I_k)} \cdots \sum_{I_1 \preceq w_1} (-1)^{\ell(I_1)} h_{sort(I_1)}$$
$$= \sum_{I_k \preceq w_k} \cdots \sum_{I_1 \preceq w_1} (-1)^{\ell(I_k) + \cdots + \ell(I_1)} h_{sort(I_k)} \cdots h_{sort(I_1)} = \sum_{J \preceq P} (-1)^{\ell(J)} h_{sort(J)}.$$ □

Example 6.1.7. For the p-sentence (aba, c), we have

$$S(h_{(aba,c)}) = -h_{(aba,c)} + h_{(ba,a,c)} + h_{(ab,a,c)} - h_{(a,a,b,c)}.$$

Now we can show that $PSym_A$ is in fact the commutative image of $NSym_A$. We do so by defining a morphism χ called the colored forgetful map that sends the (noncommutative) generators of $NSym_A$ to the (commutative) generators of $PSym_A$.

Theorem 6.1.8. *The algebra homomorphism $\chi : NSym_A \to PSym_A$ defined by $H_w \to h_w$ is a surjective Hopf algebra morphism.*

Proof. It is simple to see that χ is a surjective algebra homomorphism by definition. For the purposes of this proof, let Δ_N denote the comultiplication on $NSym_A$ and Δ_P denote the comultiplicaiton on $PSym_A$. We see that χ is a coalgebra homomorphism (Equation 2.2) since

$$(\chi \otimes \chi)\Delta_N(H_I) = (\chi \otimes \chi)(\sum_{J \subseteq_R I} H_{I/_R J} \otimes H_J)$$
$$= \sum_{J \subseteq_R I} h_{sort(I/_R J)} \otimes h_{sort(J)} = \Delta_P(h_{sort(I)}) = \Delta_P(\chi(H_I))$$

Additionally, $\epsilon_N(\chi(H_P)) = \epsilon_P(H_P)$ because both are 1 if $P = \emptyset$ and 0 otherwise. Thus, by Proposition 2.2.4, the map χ is a Hopf algebra morphism. □

We have established that $PSym_A$ is a Hopf algebra and the commutative image of $NSym_A$, which is analogous to the relationship of Sym and $NSym$. To show that $PSym_A$ is a colored generalization of Sym, we define a map from $PSym_A$ to Sym and show that it is a Hopf isomorphism if A is a unary alphabet.

Definition 6.1.9. Define the *uncoloring map*

$$\upsilon : PSym_A \to Sym \quad \text{by} \quad \upsilon(h_P) = h_{w\ell(P)}.$$

Proposition 6.1.10. *When A is an alphabet of size one, the map $\upsilon : PSym_A \to Sym$ is a Hopf isomorphism.*

Proof. Let $A = a$ and note that $w\ell$ is now a bijection between partitions and p-sentences, because there is only one p-sentence for each shape λ. Let λ_a denote the unique p-sentence of shape λ and n_a the unique word of length n in the alphabet $A = \{a\}$. Then the formulas for multiplication and comultiplication of the colored complete homogeneous basis simplify to

$$h_{\lambda_a} h_{\mu_a} = h_{sort(\lambda_a \cdot \mu_a)},$$

$$h_{n_a} = \sum_{0 \le i \le n} h_{i_a} \otimes h_{(n-i)_a}.$$

These are exactly the formulas for the complete homogeneous basis of the symmetric functions from Equations (2.7) and (2.8) when λ_a is replaced with λ and so on. From here it is simple to see that υ is a bialgebra isomorphism and thus, by Corollary 2.2.5, a Hopf isomorphism. □

6.2 The colored symmetric functions Sym_A

We now introduce another colored generalization of Sym that is a subalgebra of $QSym_A$ and is dual to $PSym_A$. As before, we say a monomial $x_{i_1,v_1} \cdots x_{i_j,v_j}$, where $i_1 < \cdots < i_j$, is associated with the sentence $I = (v_1, \cdots, v_j)$, and now also the p-sentence $sort(I)$.

Definition 6.2.1. Let Sym_A denote the set of *colored symmetric functions* $f \in \mathbb{Q}[x_A]$ such that

$$f(x_{A,1}, x_{A,2}, \ldots) = f(x_{A,\sigma(1)}, x_{A,\sigma(2)}, \ldots).$$

In other words, if two monomials are associated with the same p-sentence, then they have the same coefficients.

Example 6.2.2. The function

$$f = x_{a,1}x_{bc,2} + x_{bc,1}x_{a,2} + x_{a,1}x_{bc,3} + x_{bc,1}x_{a,3} + \cdots + x_{a,5}x_{bc,7} + x_{bc,5}x_{a,7} + \cdots$$

is in Sym_A because each term $x_{a,i}x_{bc,j}$ has the same coefficient as the term $x_{a,\sigma(i)}x_{bc,\sigma(j)}$ for any permutation σ of $\mathbb{N}$. The following function is not in Sym_A:

$$g = x_{a,1}x_{bc,2} + 3x_{bc,1}x_{a,2} + \cdots$$

By definition, $Sym_A \subseteq QSym_A$. It is simple to see that Sym_A is in fact a subspace of $QSym_A$.

Definition 6.2.3. For a p-sentence P, the *colored monomial symmetric function* is defined

$$m_P = \sum_{sort(I)=P} M_I,$$

where the sum runs over sentences I such that $sort(I) = P$. Equivalently, if $\ell(P) = k$ and $I = (v_1, \ldots, v_k)$,

$$m_P = \sum_{sort(I)=P} \sum_{i_1<\cdots<i_k} x_{v_1,i_1} \cdots x_{v_k,i_k} = \sum_{\substack{K\in WSent_A \\ sort(K)=P}} x_K.$$

Example 6.2.4. Consider the p-sentence (ab, c, c). Then,

$$\begin{aligned} m_{(ab,c,c)} &= M_{(ab,c,c)} + M_{(c,ab,c)} + M_{(c,c,ab)} \\ &= \sum_{i_1<i_2<i_3} (x_{ab,i_1}x_{c,i_2}x_{c,i_3} + x_{c,i_1}x_{ab,i_2}x_{c,i_3} + x_{c,i_1}x_{c,i_2}x_{ab,i_3}). \end{aligned}$$

Proposition 6.2.5. *The set $\{m_P\}_P$ is a basis of Sym_A.*

Proof. Consider a colored symmetric function $f = \sum_{K\in WSent_A} b_K x_K$ where b_K are rational coefficients. Since $f \in QSym_A$, we rewrite $f = \sum_{J\in Sent_A} b_J M_J$ because $b_K = b_{\tilde{K}}$ for any weak sentence K. By definition of Sym_A, if $P = sort(J)$ for a sentence J and p-sentence P, then $b_J = b_P$. Thus, $f = \sum_{P\in PSent_A} b_P m_P$, so the colored monomial symmetric functions span Sym_A. Additionally, for any sentence J, the monomial x_J only appears in the colored monomial symmetric function $m_{sort(\tilde{K})}$. It follows that the colored monomial symmetric functions are linearly independent. □

To show that Sym_A is a Hopf subalgebra of $QSym_A$, we must show that multiplication, comultiplication, and the antipode of $QSym_A$ restrict to Sym_A.

Lemma 6.2.6. *Let $P = (w_1, \ldots, w_k)$ and $S = (v_1, \ldots, v_j)$ be p-sentences. Then,*

$$m_P m_S = \sum_{Q\in PSent_A} r^Q_{P,S} m_Q,$$

where, for $\ell(Q) = m$, the coefficient $r^Q_{P,S}$ is the number of pairs of weak sentences $Y = (y_1, \ldots, y_m)$ and $Z = (z_1, \ldots, z_m)$ such that $sort(Y) = P$ and $sort(Z) = S$ and $Q = (y_1z_1, y_2z_2, \ldots, y_mz_m)$, and multiplication is inherited from $QSym_A$.

Proof. Let $P = (w_1, \ldots, w_k)$ and $S = (v_1, \ldots, v_j)$ be p-sentences. Let K be a weak sentence with $sort(K) = Q$ and $\ell(K) = \ell$. Using the definition of multiplication on $QSym_A$, each time the term x_K appears in the product $m_P m_S$, it is as the product of two monomials $x_I x_J$ where $I = (v_1, \ldots, v_\ell)$ and $J = (u_1, \ldots, u_\ell)$ are weak sentences such that $sort(I) = P$ and $sort(J) = S$ and $(v_1u_1, v_2u_2, \ldots, v_\ell u_\ell) = K$. Let $\sigma_K \in S_\ell$ be a permutation such that $\sigma_K(K) = Q$, in other words $(v_{\sigma_K(1)}u_{\sigma_K(1)}, \ldots, v_{\sigma_K(\ell)}u_{\sigma_K(\ell)}) = sort(K) \cdot (\emptyset^{\ell-m}) = Q \cdot (\emptyset^{\ell-m})$. For any given (I, J) as described above, let $Y = (y_1, \ldots, y_\ell) = \sigma_K(I)$ and $Z = (z_1, \ldots, z_\ell) = \sigma_K(J)$, and observe that we now have Y, Z such that $sort(Y) = P$ and $sort(Z) = S$ and $Q = (y_1z_1, y_2z_2, \ldots, y_mz_m)$ with $(y_{m+1}z_{m+1}, \ldots, y_\ell z_\ell) = (\emptyset^{\ell-m})$. In this way, each pair (I, J) can be associated with a unique pair (Y, Z) because the pair (Y, Z) is defined by (I, J) by the application of a fixed permutation. We also obtain every pair (Y, Z) such that $sort(Y) = P$ and $sort(Z) = S$ and $Q = (y_1z_1, y_2z_2, \ldots, y_mz_m)$ because we can do the exact same logic in reverse. Thus, we have shown that for any weak sentence K such that $sort(K) = Q$, the monomial x_K appears exactly $r^Q_{P,S}$ times in the product of m_Pm_S. Now, our claim follows from the definition of m_Q. □

Example 6.2.7. Using this formula the product $m_{(bc,a)}m_{(b)}$ expands as

$$m_{(bc,a)}m_{(b)} = m_{(bc,a,b)} + m_{(bcb,a)} + m_{(ab,bc)}.$$

For p-sentences $P = (w_1, \ldots, w_k)$ and $Q = (v_1, \ldots, v_j)$, write $Q \sqsubseteq P$ if $\{v_1, \ldots, v_j\}$ is a submultiset of $\{w_1, \ldots, w_k\}$. Now, assuming $Q \sqsubseteq P$, define

$$P \setminus Q = (u_1, \ldots, u_{k-j}) \in PSent_A$$

such that $\{w_1, \ldots, w_k\} = \{v_1, \ldots, v_j\} \sqcup \{u_1, \ldots, u_{k-j}\}$ where $\sqcup$ is the union of multisets. For example, $\{aaa, ab, ab\} \sqsubseteq \{aaa, ab, ab, ab, ca, ca\}$, and $\{aaa, ab, ab, ab, ca, ca\} \setminus \{aaa, ab, ab\} = \{ab, ca, ca\}$.

Lemma 6.2.8. *Let $P = (w_1, \ldots, w_k)$ be a p-sentence . Then the coproduct of m_P is given by*

$$\Delta(m_P) = \sum_{Q \sqsubseteq P} m_Q \otimes m_{P \setminus Q},$$

where Δ is inherited from $QSym_A$.

Proof. Observe that

$$\Delta(m_P) = \sum_{sort(I)=P} \Delta(M_I) = \sum_{sort(I)=P} \sum_{J \cdot K = I} M_J \otimes M_K.$$

As written, the latter equation sums over all unique rearrangements of P and then splits those rearrangements into two parts. Instead, rewrite the sum to first isolate a part of P and then sum over all unique rearrangements of that part and what remains.

$$\begin{aligned}\Delta(m_P) &= \sum_{Q \sqsubseteq P} \sum_{sort(J)=Q} \sum_{sort(K)=P\setminus Q} M_J \otimes M_K = \sum_{Q \sqsubseteq P} \left(\sum_{sort(J)=Q} M_J \right) \otimes \left(\sum_{sort(K)=P\setminus Q} M_K \right) \\ &= \sum_{Q \sqsubseteq P} m_Q \otimes m_{P\setminus Q}.\end{aligned}$$

□

Example 6.2.9. Consider the p-sentence (aba, bb, ca). Then,

$$\Delta(m_{(aba,bb,ca)}) = 1 \otimes m_{(aba,bb,ca)} + m_{(aba)} \otimes m_{(bb,ca)} + m_{(bb)} \otimes m_{(aba,ca)} + m_{(ca)} \otimes m_{(aba,bb)} + \\ + m_{(aba,bb)} \otimes m_{(ca)} + m_{(aba,ca)} \otimes m_{(bb)} + m_{(bb,ca)} \otimes m_{(aba)} + m_{(aba,bb,ca)} \otimes 1$$

Lemma 6.2.10. *Let $P = (w_1, \ldots, w_k)$ and recall S^* is the antipode of $QSym_A$. Then,*

$$S^*(m_P) \in Sym_A.$$

Proof. Observe that

$$S^*(m_P) = \sum_{sort(I)=P} S^*(M_I) = \sum_{sort(I)=P} \sum_{J^r \succeq I} (-1)^{\ell(I)} M_J.$$

Let K and L be two sentences such that $sort(K) = sort(L)$. It suffices to show that M_K and M_L appear the same number of times in the sum above because this implies the sum is a colored symmetric function. The function M_K will appear once for each unique sentence $I = (w_1, \ldots, w_k)$ such that $sort(I) = P$ and $K^r \succeq I$. In this case, $K^r = (w_1 \cdots w_{i_1}, w_{i_1+1} \cdots w_{i_2}, \cdots)$ for some $1 \leq i_1 < i_2 < \cdots < i_{\ell(K)} \leq k$. For this appearance of K, we can see that L appears once as the coarsening of the sentence I obtained by rearranging $(w_1 \cdots w_{i_1}, w_{i_1+1} \cdots w_{i_2}, \cdots)^r$ to equal L, then splitting the sentence in between each adjacent w_j and w_{j+1}. This way, we can match each M_K with a unique M_L. We can do the reverse to match each M_L with a unique M_K, so there must be the same number of both. This proves our claim. □

Lemma 6.2.6 and 6.2.8 show that Sym_A is a subalgebra and a subcoalgebra of $QSym_A$, respectively. Lemma 6.2.10 verifies that the antipode of $QSym_A$ restricts to Sym_A, therefore Sym_A is a Hopf subalgebra of $QSym_A$.

Theorem 6.2.11. *Sym_A is a Hopf algebra. Specifically, it is a Hopf subalgebra of $QSym_A$.*

Next, we verify that Sym_A is an appropriate analogue to the symmetric functions.

Definition 6.2.12. Define the *uncoloring map*

$$\upsilon : Sym_A \to Sym \quad \text{by} \quad \upsilon(m_P) = m_{w\ell(P)}.$$

Proposition 6.2.13. *When A is an alphabet of size one, the map $\upsilon : Sym_A \to Sym$ is a Hopf isomorphism.*

Proof. Let $A = a$ and note that $w\ell$ is now a bijection between partitions and p-sentences , because there is only one p-sentence for each shape λ. Let λ_a denote the unique p-sentence of shape λ and n_a the unique word of length n in the alphabet $A = \{a\}$. Then our formulas for multiplication and comultiplication of the colored monomial basis of Sym_A simplify to

$$m_{\lambda_a} m_{\mu_a} = \sum_{\nu_a} r^{\nu_a}_{\lambda_a,\mu_a} m_{\nu_a} = \sum_{\nu_a\colon\, \nu \in \lambda \overset{Q}{\sqcup\!\sqcup}\, \mu} m_{\nu_a},$$

$$m_{\lambda_a} = \sum_{\mu_a \sqsubseteq \lambda_a} m_{\mu_a} \otimes m_{\lambda_a \setminus \mu_a}.$$

These are exactly the formulas for the monomial basis of the symmetric functions from Equations (2.5) and (2.6) when λ_a is replaced with λ and so on, thus from here it is simple to see that υ is a bialgebra isomorphism and thus a Hopf isomorphism. □

We have now defined two colored generalizations of Sym, one relating to $NSym_A$ and one to $QSym_A$. These two generalizations are in fact dual Hopf algebras.

Theorem 6.2.14. *The Hopf algebras $PSym_A$ and Sym_A are dually paired by the inner product*

$$PSym_A \times Sym_A : \langle \cdot, \cdot \rangle \to \mathbb{Q} \text{ defined by } \langle h_P, m_S \rangle = \delta_{P,S}.$$

Proof. Let $PSym_A^*$ be the graded dual algebra to $PSym_A$ paired by the inner product $\langle \cdot, \cdot \rangle : PSym_A \times PSym_A^* \to \mathbb{Q}$ defined by $\langle h_P, h_S^* \rangle = \delta_{P,S}$ where $\{h_S^*\}_S$ is the basis dual to $\{h_P\}_P$. Then by Proposition 2.2.8, the multiplication for this basis is defined as

$$h_Q^* h_S^* = \sum_P r^P_{Q,S} h_P^*,$$

where $r^P_{Q,S}$ is the number of pairs of weak sentences $Y = (y_1, \ldots, y_m)$ and $Z = (z_1, \ldots, z_m)$ such that $sort(Y) = Q$, $sort(Z) = S$ and $P = (y_1 z_1, y_2 z_2, \ldots, y_m z_m)$. Comultiplication is expressed as

$$\Delta(h_S^*) = \sum_{P \sqsubseteq S} h_P^* \otimes h_{S \setminus P}^*.$$

Notice that these formulas match those in Lemmas 6.2.6 and 6.2.8 for multiplication and comultiplicaiton of the colored monomial basis of Sym_A. It is clear from this definition that there is a bijective bialgebra morphism, and thus a Hopf isomorphism by [37, Corollary 1.4.27], between $PSym_A^*$ and Sym_A defined by $h_P^* \longleftrightarrow m_p$. □

Proposition 6.2.15. *The map $\chi : NSym_A \to PSym_A$ is adjoint to the inclusion $\iota : Sym_A \hookrightarrow QSym_A$ with respect to the duality pairing.*

Proof. For a p-sentence P and a sentence I,

$$\langle \chi(H_I), m_P \rangle = \langle h_{sort(I)}, m_P \rangle = \begin{Bmatrix} 1 & \text{if } sort(I) = P \\ 0 & \text{otherwise} \end{Bmatrix} = \left\langle H_I, \sum_{sort(J)=P} M_J \right\rangle = \langle H_I, \iota(m_P) \rangle.$$

Thus by Equation (2.4) we have shown the maps are adjoint. □

Figure 6.1 depicts the relationships between $Sym, NSym, QSym, PSym_A, Sym_A, NSym_A$, and $QSym_A$. Dual algebras are connected by dashed lines.

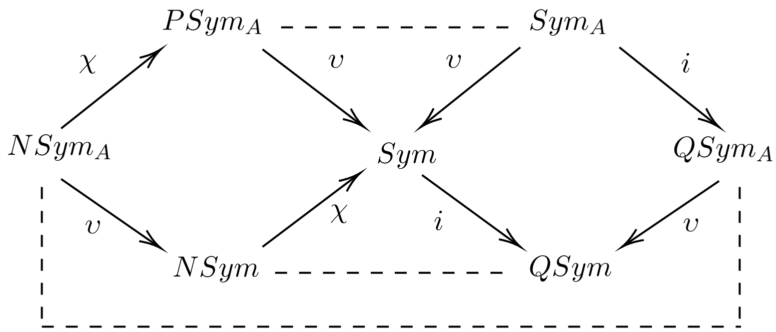


Figure 6.1: Relationships between algebras.

6.3 The colored Schur and colored dual Schur functions

One generalization of the Schur basis to $PSym_A$ and Sym_A is clear fairly immediately. A colored shin-tableau of shape P is a *colored semistandard Young tableau* (CSSYT) if P is a p-sentence. That is, a colored semistandard Young tableau is a colored diagram of shape P filled with positive integers such that each row is weakly increasing from left to right and each column is strictly increasing from top to bottom. These tableaux inherit the definitions of type, standard, standardization, descent, and other statistics from colored shin-tableaux. Additionally, $\mathcal{K}_{P,J}$ denotes the number of colored semistandard Young tableaux of shape P and type J.

Definition 6.3.1. For a p-sentence P, the *colored dual Schur function* is defined as

$$s^*_P = \sum_{Q \in PSent_A} \mathcal{K}_{P,Q} m_Q.$$

Example 6.3.2. The only CSSYT of shape (abb, ca) whose types are p-sentences are given below, so we have the following colored dual Schur function.

$$s^*_{(abb,ca)} = m_{(abb,ca)} + m_{(ab,cb,a)} + m_{(ab,ca,b)}$$

$a,1$	$b,1$	$b,1$
$c,2$	$a,2$	

$a,1$	$b,1$	$b,2$
$c,2$	$a,3$	

$a,1$	$b,1$	$b,3$
$c,2$	$a,2$	

In most cases, s^*_P can not be expanded into a sum of monomials x_T associated with all of the colored semistandard Young tableaux of shape P. For example, in the function

$$s^*_{(a,ba)} = m_{(a,ba)} + m_{(a,b,a)} = x_{a,1}x_{ba,2} + x_{ba,1}x_{a,2} + x_{a,1}x_{b,2}x_{a,3} + \cdots ,$$

there is a term associated with the sentence (ba, a) which cannot possibly be the type of a colored Young tableau of shape (a, ba).

Proposition 6.3.3. *The colored dual Schur functions $\{s^*_P\}_P$ form a basis of Sym_A.*

Proof. Consider the transition matrix between the colored dual Schur basis and the colored monomial symmetric basis where the indices are ordered first by reverse lexicographic order on partitions applied to the $w\ell$ of the sentences and second by the lexicographic order on words. For example, the sentence (ab, c) comes after $(abcb, c)$ but before (cc, b). We show that this matrix is upper unitriangular. The entries on the diagonal, $\mathcal{K}_{P,P}$, are always greater than 0 because there always exists a CSSYT of shape P and type P given by filling each box in row i with i's. It remains to show that each entry below the diagonal is 0. Observe that because every CSSYT is a colored immaculate tableau, the number of CSSYT of a certain shape and type is always less than the number of colored immaculate tableau of that shape and type. We have chosen our order so that any entry below the diagonal in this matrix corresponds to an entry below the diagonal in the matrix used in the proof of Theorem 3.3.23 for colored immaculate tableaux. We showed in the proof of Theorem 3.3.23 that the entries below the diagonal in the immaculate matrix are zero, and so the entries below the diagonal in the colored Schur transition matrix are zero as well. Therefore, the transition matrix is upper unitriangular and the colored dual Schur functions form a basis. □

Definition 6.3.4. For a p-sentence P, the *colored Schur function s_P* is defined by

$$\langle s_P, s^*_Q \rangle = \delta_{P,Q},$$

for all p-sentences Q. In other words, $\{s_P\}_P$ is defined as basis of $PSym_A$ that is dual to the colored dual Schur basis of Sym_A.

Proposition 6.3.5. *Let A be a unary alphabet. Then, for p-sentences P and Q,*

$$\upsilon(s^*_P) = s_{w\ell(P)} \qquad \text{and} \qquad \upsilon(s_Q) = s_{w\ell(Q)}.$$

Proof. First, we show that $\upsilon(s^*_P) = s_{w\ell(P)}$. Observe that when $|A| = 1$, colored semistandard Young tableaux are in clear bijection with semistandard Young tableaux and so $\mathcal{K}_{P,Q} = K_{w\ell(P),w\ell(Q)}$. Additionally, there is only one sentence P such that $w\ell(P) = \lambda$ for any partition λ. Then, if $w\ell(P) = \lambda$, we have

$$\upsilon(s^*_P) = \sum_Q \mathcal{K}_{P,Q}\upsilon(m_Q) = \sum_Q K_{w\ell(P),w\ell(Q)} m_{w\ell(Q)} = \sum_\mu K_{\lambda,\mu} m_\mu = s_\lambda.$$

Since the matrices for $\mathcal{K}_{P,Q}$ and $K_{\lambda,\mu}$ are equal, so are their inverse matrices meaning $\mathcal{K}^{-1}_{P,Q} = K^{-1}_{w\ell(P),w\ell(Q)}$. Therefore, if $w\ell(Q) = \mu$,

$$\upsilon(s_Q) = \sum_P K^{-1}_{w\ell(P),w\ell(Q)} \upsilon(h_P) = \sum_P K^{-1}_{w\ell(P),w\ell(Q)} h_{w\ell(P)} = \sum_\lambda K^{-1}_{\lambda,\mu} h_\lambda = s_\mu. \quad \square$$

Example 6.3.6. The colored Schur function for (aaa, aa) is given by

$$s^*_{(aaa,aa)} = m_{(aaa,aa)} + 2m_{(aa,aa,a)} + 3m_{(aa,a,a,a)} + 5m_{(a,a,a,a,a)}.$$

Applying the uncoloring map, we get

$$\upsilon(s^*_{(aaa,aa)}) = m_{(3,2)} + 2m_{(2,2,1)} + 3m_{(2,1,1,1)} + 5m_{(1,1,1,1,1)} = s_{(3,2)}.$$

A variety of interesting questions about Sym_A and $PSym_A$ remain, including the following:

1. Which properties of the Schur functions do the colored Schur and colored dual Schur functions generalize?

2. How do the colored Schur and colored dual Schur bases relate to the various colored Schur-like bases of $NSym_A$ and $QSym_A$?

3. What are the commutative images of the colored immaculate, colored Young noncommutative Schur, and colored shin bases?

4. Do any subsets of these images form bases of $PSym_A$ that generalize the Schur functions? Where is there overlap between these images, if any?

5. Are there generalizations of the Schur functions in Sym_A that we can express as sums of the colored dual immaculate, colored Young quasisymmetric Schur, and colored extended Schur functions in meaningful combinatorial ways?

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APPENDICES

APPENDIX

A

BASIC NOTATION

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_j)$ be a compositions, γ be a weak composition, and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. We generally use α, β, γ for compositions and λ, μ, ν for partitions.

$[k] = \{1, 2, \ldots, k-1, k\}$

$|\alpha| = \sum_i \alpha_i$ (size)

$\ell(\alpha) = k$ (length)

α^c if α is represented as blocks of stars separated by bars, then to find α^c put bars in exactly the opposite places from α (complement)

$\alpha^r = (\alpha_k, \ldots, \alpha_1)$ (reverse)

$\alpha^t = (\alpha^c)^r = (\alpha^r)^c$ (transpose)

λ' the partition obtained by flipping λ over the main diagonal (conjugate)

$sort(\alpha)$ the partition obtained be reordering the parts of α into a partition

α/β α with the first β_i boxes of row i removed or shaded out for all $i \leq j$ (skew shape)

$\alpha/\!\!/\beta$ α with the first β_i boxes of row $k+1-i$ removed or shaded for $i \leq j$ (skew-II shape)

$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_j)$ (concatenation)

$\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1, \beta_2, \ldots, \beta_j)$ (near-concatenation)

$\alpha \sqcup\!\sqcup \beta$ the sum of all compositions obtained by interweaving α and β (shuffle)

$\beta \preceq \alpha$ $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots \alpha_k\} \subseteq \{\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \cdots + \beta_j\}$(refinement order)

$\beta \leq_\ell \alpha$ $\beta_i < \alpha_i$ where i is the first positive integer such that $\alpha_i \neq \beta_i$ (lexicographic order)

$\beta \subseteq \beta$ $\beta_i \leq \alpha_i$ for $i \in [j]$ and $j \leq k$ (dominance order)

$\tilde{\gamma}$ the composition obtained by removing the parts of γ equal to zero (flattening)

Sym the algebra of symmetric functions

$\{m_\lambda\}_\lambda$ the monomial symmetric functions

$\{h_\lambda\}_\lambda$ the complete homogeneous symmetric functions

$\{e_\lambda\}_\lambda$ the elementary symmetric functions

$\{s_\lambda\}_\lambda$ the Schur symmetric functions

SSYT semistandard young tableaux

$K_{\alpha,\beta}$ the number of SSYTs of shape α and thpe β (Kostka number)

$QSym$ the algebra of quasisymmetric functions

$\{M_\alpha\}_\alpha$ the monomial quasisymmetric functions

$\{F_\alpha\}_\alpha$ the fundamental quasisymmetric functions

$NSym$ the algebra of noncommutative symmetric functions

$\{H_\alpha\}_\alpha$ the complete homogeneous noncommutative symmetric functions

$\{R_\alpha\}_\alpha$ the ribbon noncommutative symmetric functions

$\langle \cdot, \cdot \rangle : NSym \times QSym \to \mathbb{Q}$ the bilinear inner product defined $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$

$\{E_\alpha\}_\alpha$ the elementary noncommutative symmetric functions

$\{\Psi_\alpha\}_\alpha$ the noncommutative power sums of the first kind

$\chi : NSym \to Sym$ the linear morphism defined $\chi(H_\alpha) = h_{sort(\alpha)}$ (forgetful map)

$F^{\perp}(H) = \sum_{\alpha} \langle H, FB_{\alpha} \rangle A_{\alpha}$ for $\langle A_{\alpha}, B_{\beta} \rangle = \delta_{\alpha,\beta}$ and $H \in NSym$, $F \in QSym$ (perp operator)

$H^{\perp}(F) = \sum_{\alpha} \langle HA_{\alpha}, F \rangle B_{\alpha}$ for $\langle A_{\alpha}, B_{\beta} \rangle = \delta_{\alpha,\beta}$ and $H \in NSym$, $F \in QSym$ (perp operator)

$H^{\pm}(F) = \sum_{\alpha} \langle A_{\alpha} H, F \rangle B_{\alpha}$ for $\langle A_{\alpha}, B_{\beta} \rangle = \delta_{\alpha,\beta}$, $H \in NSym$, $F \in QSym$ (right-perp operator)

ψ involutions that map $F_{\alpha} \to F_{\alpha^c}$ or $R_{\alpha} \to R_{\alpha^c}$

ρ involutions that map $F_{\alpha} \to F_{\alpha^r}$ or $R_{\alpha} \to R_{\alpha^r}$

ω involutions that map $F_{\alpha} \to F_{\alpha^t}$ or $R_{\alpha} \to R_{\alpha^t}$

$Des_{\_}()$ the descent set of a standard ______ tableaux

$co_{\_}()$ the descent composition of a standard ______ tableaux

std() maps semistandard tableaux to standard tableaux (standardization)

flip() maps a standard tableaux of shape α to a standard tableaux of shape α^r where each entry i is replaced by $|\alpha| - i + 1$

immaculate tableaux weakly increasing rows, strictly increasing first column

RSIT, RIT, RSRIT reverse, row-strict reverse immaculate tableaux

$\mathfrak{K}_{\alpha,\beta}$ the number of immaculate tableaux of shape α and type β

$\alpha \subset_s^{\mathfrak{S}} \beta$ when $\beta \vdash |\alpha| + s$ and $\alpha \subseteq \beta$ and $\ell(\beta) \leq \ell(\alpha) + 1$.

$\{\mathfrak{S}_{\alpha}\}_{\alpha}$ the immaculate noncommutative symmetric functions

$\{\mathfrak{RS}_{\alpha}\}_{\alpha}$ the row-strict immaculate noncommutative symmetric functions

$\{\mathfrak{D}_{\alpha}\}_{\alpha}$ the reverse immaculate noncommutative symmetric functions

$\{\mathfrak{RD}_{\alpha}\}_{\alpha}$ the row-strict reverse immaculate noncommutative symmetric functions

shin-tableaux weakly increasing rows, strictly increasing columns

RSST, RST, RSRST row-strict, reverse, row-strict reverse shin-tableaux

$\mathcal{K}_{\alpha,\beta}$ the number of shin tableaux of shape α and type β

$\alpha \subset_r^{\text{ש}} \beta$ when $\beta \vdash |\alpha + r|$ and $\alpha \subseteq \beta$ and if $\beta_i > \alpha_i$ and $j > i$, we have $\beta_j \leq \alpha_i$

$\{\mathfrak{w}_\alpha\}_\alpha$ the shin noncommutative symmetric functions

$\{\mathfrak{R}\mathfrak{w}_\alpha\}_\alpha$ the row-strict shin noncommutative symmetric functions

$\{\reflectbox{$\mathfrak{w}$}_\alpha\}_\alpha$ the reverse shin noncommutative symmetric functions

$\{\mathfrak{R}\reflectbox{$\mathfrak{w}$}_\alpha\}_\alpha$ the row-strict reverse shin noncommutative symmetric functions

SSYCT weakly increasing rows, strictly increasing first column, triple rule

SSRCT weakly decreasing rows, strictly increasing first column, triple rule

$\hat{K}_{\alpha,\beta}$ the number of semistandard Young composition tableaux of shape α and type β

$\check{K}_{\alpha,\beta}$ the number of semistandard reverse composition tableaux of shape α and type β

$\{\hat{\mathbf{s}}^*_\alpha\}_\alpha$ the Young quasisymmetric Schur functions

$\{\hat{\mathbf{rs}}^*_\alpha\}_\alpha$ the row-strict Young quasisymmetric Schur functions

$\{\check{\mathbf{s}}^*_\alpha\}_\alpha$ the quasisymmetric Schur functions

$\{\check{\mathbf{rs}}^*_\alpha\}_\alpha$ the row-strict quasisymmetric Schur functions

APPENDIX

B

COLORED NOTATION

Let A be a finite alphabet with a total order $\leq$. Let $I = (w_1, \dots, w_k)$ and $J = (v_1, \dots, v_h)$ be sentences and $K = (u_1, \dots, u_g)$ a weak sentence. Let $w = a_1 \dots a_n$ and $v = b_1 \dots b_m$. Let U be a standard colored immaculate tableau and thus also a standard colored row-strict immaculate tableau.

$|w| = n$ (size)

$w \cdot v = a_1 \dots a_n b_1 \dots b_m$ (concatenation of words)

$w \preceq_\ell v$ if $a_i < b_i$ for the first positive integer i such that $a_i \neq b_i$ (lexicographic order)

$w \preceq_{g\ell} v$ if $|w| < |v|$ or both $|w| = |v|$ and $w \preceq_\ell v$ (graded lexicographic order)

$|I| = \sum_{i=1}^{k} |w_i|$ (size)

$\ell(I) = k$ (length)

$w(I) = w_1 \cdot w_2 \cdots w_k$ (maximal word)

$w\ell(I) = (|w_1|, |w_2|, \dots, |w_k|)$ (word lengths)

$J \preceq I$ for $w(I) = w(J)$, if J splits at each location that I splits (refinement order)

$\mu(J, I) = (-1)^{\ell(J)-\ell(I)}$ for $J \preceq I$ (Möbius function on the poset of sentences ordered by $\preceq$)

$I \cdot J = (w_1, \ldots, w_n, v_1, \ldots, v_m)$ (concatenation of sentences)

$I \odot J = (w_1, \ldots, w_{n-1}, w_n \cdot v_1, v_2, \ldots, v_m)$ (near-concatenation)

$I^r = (w_k, \ldots, w_2, w_1)$ (reversal)

I^c unique sentence with $w(I^c) = w(I)$ that splits exactly where I does not (complement)

$\tilde{K}$ the sentence obtained by removing all empty words from K (flattening)

$K \subseteq_L I$ there exists a weak sentence $I/_L K$ as defined below (left-containment)

$I/_L K = (q_1, \ldots, q_t)$ such that $w_i = u_i q_i$ for all $i \in [k]$

$K \subseteq_R I$ there exists a weak sentence $I/_R K$ as defined below (right-containment)

$I/_R K = (q_1, \ldots, q_t)$ such that $w_i = q_i u_i$ for all $i \in [k]$

$\langle \cdot, \cdot \rangle : NSym \otimes QSym \to \mathbb{Q}$ $\langle H_I, M_J \rangle = \delta_{I,J}$ (inner product)

$co_{\overline{A}}(U)$ colored descent composition of a standard ____ tableau

$std(T)$ the standardization of a tableau T

CIT colored immaculate tableaux

$J \subset_u^{\mathfrak{S}} I$ if $w_i = v_i q_i$ for $1 \leq i \leq k$ such that $q_k \cdots q_1 = u$ and $k \leq h+1$ where $v_{h+1} = \emptyset$

CST colored shin-tableaux

$J \subset_u^{\mathfrak{w}} I$ when $v_i \subseteq_L w_i$ and $w((I/_L J)^r) = u$ and if $|w_i| > |v_i|$ and $j > i$ then $|w_j| \leq |v_i|$

SSCYCT semistandard colored Young composition tableaux