An Analysis of Stress and Strain for Orthotropic Ring Shells

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Abstract

In this paper an analysis of stress and strain for orthotropic ring shells on the basis of the linear theory of thin elastic shell is presented. The asymptotic solution has been obtained.

The results are suitable for $\lambda = E_1/E_2 > 0.3$, where E_1, E_2 are reduced modulus of elasticity in the direction of the meridian and the parallel circle respectively.

1. Introduction

The vacuum vessel of Tokamak device consists of several rigid rings and bellows which are alternately welded into a torus. It is a very complicated complex structure. In order to make a preliminary analysis of its stress and strain state, it is necessary in the firststep to approximately simplify it as a corrugated ring shell, and further simplification can be made by regarding the corrugated ring shell as an orthotropic ring shell, the size of the bellow being much smaller than the corrugated ring shell. On the basis of the analysis of the entire ring shell, analysis of each bellow can also be made.

In this paper, a second-order differential equation of a complex variable is derived for circular ring shell under axisymmetrical loading by means of complex transformation on the basis of the linear theory of thin elastic shell. Equation includes a large parameter and two turning points at $\theta = 0$, $\theta = \pi$, so asymptotic solution is searched. To solve this equation, for sake of convenience, it is changed to a non-homogenous Airy equation. Besides fundamental equation and its solution, we calculate the interal forces and displacements of orthotropic ring shell for different values of λ under uniformly distributed load. A general discussion about various results is given.

2. Fundamental formulae

Complex variable displacement equations[1]

$$\tilde{\varepsilon}_2 + k a_{22}(\tilde{\mathscr{X}}_1 + \tilde{\mathscr{X}}_2) = a_{22} T^* - \frac{1}{k} M_1^*, \quad \frac{1}{a_{11}}\tilde{\varepsilon}_1 - \frac{1}{a_{22}}\tilde{\varepsilon}_2 = -\frac{1}{k}\left(\frac{1}{a_{22}} M_1^* - \frac{1}{a_{11}} M_2^*\right);$$
$$\tilde{\omega} = -\frac{2}{k} H^*, \tag{1}$$

Professor Zhang Wei directed this work.

where[2]

$$\widetilde{\mathscr{X}}_2 = -\frac{1}{k}(\widetilde{T}_1 - T_1^*), \qquad \widetilde{\varepsilon}_2 = a_{22}\widetilde{T} - \frac{1}{k}M_1^*,$$

$$\widetilde{\mathscr{X}}_1 = -\frac{1}{k}(\widetilde{T}_2 - T_2^*), \qquad \widetilde{\varepsilon}_1 = a_{11}\widetilde{T} - \frac{1}{k}M_2^*,$$

$$\widetilde{\tau} = \frac{1}{k}(\widetilde{S} - S^*), \qquad \frac{\widetilde{\omega}}{2} = \frac{H^*}{k},$$

$$\widetilde{T}_1 = T_1 - k\mathscr{X}_2, \qquad \widetilde{T}_2 = T_2 - k\mathscr{X}_1, \qquad \widetilde{T} = \widetilde{T}_1 + \widetilde{T}_2,$$

$$\widetilde{M}_1 = M_1 + k\varepsilon_2, \qquad \widetilde{M}_2 = M_2 + k\varepsilon_1,$$

$$\widetilde{S} = S + k\tau, \qquad \widetilde{H} = H - k\frac{\omega}{2},$$

$$c_1 = h\sqrt{\frac{E_1}{E_2}\,\frac{1}{12(1-\nu_1\nu_2)}}, \quad c_2 = h\sqrt{\frac{E_2}{E_1}\,\frac{1}{12(1-\nu_1\nu_2)}}, \quad k = ih^2\sqrt{\frac{E_1E_2}{12(1-\nu_1\nu_2)}},$$

$$a_{11} = \frac{1}{E_1h}, \qquad a_{22} = \frac{1}{E_2h},$$

E_1、E_2 are the modulus of elasticity and ν_1、ν_2 are the Poisson's ratios in two principal directions α and β respectively, G is the shear modulus.

T_1^*、T_2^*、S^*、H^*、M_1^*、M_2^* are a set of particular solutions which satisfy the equilibrium equations.

For shells of revolution, if we apply the equation of Codazzi and neglect small quantities in comparison with unity, the complex variable equilibrium equations can be simplified as follows:

$$\begin{aligned}
&-ka_{22}G(\widetilde{\mu}) + \widetilde{\mu} + \left[1 - ka_{11}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2} + \frac{ka_{22}(\lambda-1)}{R_1R_2\sin\theta}\left[\frac{\partial}{\partial\theta}(R_2\cos\theta\,\widetilde{\mu}) +\right.\\
&\left.+\left(-R_1\sin\theta + R_2\cos\theta\frac{\partial}{\partial\theta}\right)\frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2}\right] = -\frac{1}{k}M_1^* - \frac{1}{R_2\sin\theta}\frac{\partial\widetilde{v}^*}{\partial\varphi} + a_{22}T^*;\\
&-G(\widetilde{\Phi}) + \left(\frac{1}{R_1} - \frac{\lambda}{R_2}\right)\frac{\widetilde{\mu}}{\sin^2\theta} + \frac{(1-\lambda)}{R_2\sin^2\theta}\frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2} = \frac{\lambda}{R_2^2\sin^3\theta}\frac{\partial\widetilde{v}^*}{\partial\varphi} +\\
&+\frac{1}{R_2\sin^2\theta}\left[\frac{1}{k}(\lambda M_1^* - M_2^*)\right] - \frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\widetilde{u}^*}{\sin\theta}\right),
\end{aligned} \tag{2}$$

where

$$G(Y) = \frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial Y}{\partial\theta}\right) + \frac{1}{R_2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2};$$

$$\widetilde{\mu} = \frac{\widetilde{u}_\rho}{R_2\sin\theta} = \frac{\widetilde{u}\cos\theta + \widetilde{w}\sin\theta}{R_2\sin\theta}, \quad \widetilde{u} = \widetilde{u}^* - \frac{(R_2\sin\theta)^2}{R_1}\frac{\partial\widetilde{\Phi}}{\partial\theta}, \quad \widetilde{v} = \widetilde{v}^* + R_2\sin\theta\frac{\partial\widetilde{\Phi}}{\partial\varphi},$$

$$\lambda = \frac{E_2}{E_1},$$

R_1、R_2 -Principal radii of curvatures.

$\widetilde{u}^*$、$\widetilde{v}^*$ are a set of particular solutions of equations$(1)_3$.

In the first symmetrical case, equations$(2)_1$ and$(2)_2$ may be resolved respectively. equation(2), may be reduced as follows:

$$-\frac{ka_{22}}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d\tilde{\mu}}{d\theta}\right)+\tilde{\mu}+\frac{ka_{22}(\lambda-1)}{R_1R_2\sin\theta}\frac{d}{d\theta}(R_2\cos\theta\cdot\tilde{\mu})=$$
$$=-\frac{1}{k}M_1^*+a_{22}T^*.$$

Introducing the complex variable of rotation,

$$\tilde{\vartheta}=-\frac{R_2}{R_1}\frac{d\tilde{\mu}}{d\theta}+(\lambda-1)\mathrm{ctg}\theta\cdot\tilde{\mu}+\mathrm{ctg}\theta\cdot\frac{1}{k}(\lambda M_1^*-M_2^*),$$

we obtain

$$\frac{d^2\tilde{\vartheta}}{d\theta^2}+\left[(2-\lambda)\frac{R_1}{R_2}\mathrm{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\tilde{\vartheta}}{d\theta}-\left[\lambda\left(\frac{R_1}{R_2}\mathrm{ctg}\theta\right)^2+\frac{R_1^2}{ka_{22}R_2}\right]\tilde{\vartheta}=$$
$$=-\frac{R_1^2}{ka_{22}}\left(\frac{Q_{1n}^*}{k}-a_{22}T_{1n}^*\right), \tag{3}$$

where

$$Q^*_{1n}=\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta M_1^*}{d\theta}-R_1\cos\theta M_2^*\right),\ T_{1n}^*=\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta T^*}{d\theta}-\lambda R_1\cos\theta T^*\right).$$

For the case of a ring shell under normal pressure P,

$$R_1=b,\qquad R_2=a\frac{1+\alpha\sin\theta}{\sin\theta},\qquad \alpha=\frac{b}{a}.$$

Introducing the above mentioned parameters in to (3), we obtain the fundamental equations of ring shells:

$$\frac{d^2\tilde{\vartheta}}{d\theta^2}+\frac{(2-\lambda)\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\tilde{\vartheta}}{d\theta}+\left[-\lambda\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\right)^2+i2\beta^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]\tilde{\vartheta}=2i\beta^2a\left(\frac{Q_{1n}^*}{k}-a_{22}T_{1n}^*\right), \tag{4}$$

where

$$2\beta^2=\frac{b^2}{ah}\sqrt{12(1-\nu_1\nu_2)\frac{E_2}{E_1}}.$$

After we introduce

$$p(\theta)=\frac{(2-\lambda)\alpha\cos\theta}{1+\alpha\sin\theta},\qquad q(\theta)=\frac{\sin\theta}{1+\alpha\sin\theta},\qquad r(\theta)=-\lambda\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\right)^2,$$

and neglect the small quantities in comparison with unity, equation(4) can be written as

$$\frac{d^2\tilde{\vartheta}}{d\theta^2}+p\frac{d\tilde{\vartheta}}{d\theta}+(-i\eta)^3q\tilde{\vartheta}=f_1(\theta)+f_2(\theta), \tag{5}$$

where

$$f_1(\theta)=(-i\eta)^3A_0f_{10}(\theta),\qquad f_2(\theta)=(-i\eta)^3B_0f_{20}(\theta),$$

$$f_{10}(\theta)=\frac{i}{2}\frac{\cos\theta}{1+\alpha\sin\theta},\qquad f_{20}(\theta)=\frac{\cos\theta}{(1+\alpha\sin\theta)^2};$$

$$A_0=\frac{\overline{\Omega_z^1}}{\pi}+i\frac{2(1-\lambda)Pb^2a}{k_1\eta^3},\qquad B_0=\frac{\lambda Pb^2a}{2k_1\eta^3},\qquad k_1=-\frac{k}{i},\qquad \eta=\sqrt[3]{2\beta^2}.$$

$\overline{\Omega_z^1}$ is a term, which relates to the boundary forces and can be determined from the displacement condition.

3. The solution of equations

(1) The general solution of equation(5)[2]:

Let the general solution be

$$\tilde{\vartheta}_0 = \varphi(\theta)\cdot H_0[\psi(\theta)].$$

First, set $H_0 = U(\psi)$, which satisfies the Airy equation

$$\frac{d^2U}{d\psi^2} + \psi U = 0 .$$

Further, the following conditions

$$\psi(\theta) = -i\eta\omega(\theta);$$

$$\omega(\theta) = Sign\theta\left(\frac{3}{2}\int_0^{|\theta|}\sqrt{\left|\frac{\sin\theta}{1+a\sin\theta}\right|}\right)^{2/3}, \qquad \varphi(\theta) = \sqrt[4]{\frac{\omega(\theta)}{\sin\theta(1+a\sin\theta)^{3-2\lambda}}};$$

$$\omega'(\theta) = \sqrt{\frac{\sin\theta}{\omega(\theta)\cdot(1+a\sin\theta)}}$$

are satified and neglect the small quantities, then homogeneous equation(5) is satisfied approximately.

Since the solution of Airy equation is known, the general solution of equation (5) may be written as:

$$\tilde{\vartheta}_0 = \varphi(\theta)[\tilde{C}_1U_1(iy) + \tilde{C}_2U_2(iy)], \tag{6}$$

where

$$iy = \psi .$$

U_1, U_2 are two linearly independent solutions of Airy equation. In general, we take them as an Airy's function.

$\tilde{C}_1$, $\tilde{C}_2$, are complex coefficients which must be determined from the boundary conditions.

(2) The particular solution of equation (5) can be formed by the particular solution of the following two equations:

$$\frac{d^2\tilde{\vartheta}}{d\theta^2} + p\frac{d\tilde{\vartheta}}{d\theta} + (-i\eta)^3 q\tilde{\vartheta} = f_1(\theta); \qquad \frac{d^2\tilde{\vartheta}}{d\theta^2} + p\frac{d\tilde{\vartheta}}{d\theta} + (-i\eta)^3 q\tilde{\vartheta} = f_2(\theta). \tag{7}$$

Let the particular solution of equation $(7)_1$:

$$\tilde{\vartheta}_1 = \varphi(\theta)\cdot H_1[\psi(\theta)],$$

in which

$$\varphi(\theta) = \sqrt[4]{\frac{\omega(\theta)}{\sin\theta(1+a\sin\theta)^{3-2\lambda}}}, \qquad \psi(\theta) = \omega(\theta).$$

and the inhomogeneous differential equation is changed into:

$$\frac{d^2H_1}{d\omega^2} + i\eta^3\omega H_1 = i\eta^3A_0[g_1(\omega) - g_1(0)] + i\eta^3A_0g_1(0), \tag{8}$$

where

$$g_1(\omega) = \sqrt{\frac{(1+a\sin\theta)^{2-\lambda}}{[\omega'(\theta)]^{\frac{3}{2}}}}\, f_{10}(\theta) = \sqrt[4]{\frac{\omega^3(\theta)(1+a\sin\theta)^{7-2\lambda}}{\sin^3\theta}}\, f_{10}(\theta). \tag{9}$$

Due to slow change of $g_1(\omega)$, we get the first partial solution $H_1^{(1)}$ as an approximate solution of equation (8)

$$H_1^{(1)} = [g_1(\omega) - g_1(0)]\frac{A_0}{\omega(\theta)},$$

with some small errors near $\omega = 0$.

As to the second partial solution $H_1^{(2)}$, it must sitisfy the following equation

$$\frac{d^2 H_1^{(2)}}{d\psi^2} + \psi H_1^{(2)} = -i\eta g_1(0)\cdot A_0.$$

Using the method of indeterminate coefficient, we obtain

$$H_1^{(2)} = -i\eta g_1(0) A_0 e_0(\psi),$$

where $e_0(\psi)$ is a convergent series in the whole complex plane. we take as given by Tumarkin [3] the complex function e_0 of the pure imaginary variable iy as follows:

$$e_0(iy) = 1.288\left[1 - \frac{1}{3!}(iy)^3 + \frac{4}{6!}(iy)^6 - \frac{4\cdot 7}{9!}(iy)^9 + \frac{4\cdot 7\cdot 10}{12!}(iy)^{12} - \cdots\right] -$$

$$- 0.9389\left[1 - \frac{2}{4!}(iy)^3 + \frac{2\cdot 5}{7!}(iy)^6 - \frac{2\cdot 5\cdot 8}{10!}(iy)^9 + \cdots\cdots\right] +$$

$$+ (iy)^2\left[\frac{1}{2!} - \frac{3}{5!}(iy)^3 + \frac{3\cdot 6}{8!}(iy)^6 - \frac{3\cdot 6\cdot 9}{11!}(iy)^9 + \cdots\cdots\right].$$

According to equation (9),

$$g_1(0) = \frac{i}{2},$$

then we obtain

$$H_1^{(2)} = -\frac{\eta}{2} A_0 E[-\eta\omega(\theta)],$$

$$H_1 = H_1^{(1)} + H_1^{(2)} = A_0\left\{\frac{i}{2\omega(\theta)}\left[\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)^{3-2\lambda}}{\sin^3\theta}}\cos\theta - 1\right] - \frac{1}{2}\eta E[-\eta\omega(\theta)]\right\},$$

$$\widetilde{\vartheta}_1 = \varphi(\theta)\cdot A_0\left\{\frac{i}{2\omega(\theta)}\left[\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)^{3-2\lambda}}{\sin^3\theta}}\cos\theta - 1\right] - \frac{1}{2}\eta E[-\eta\omega(\theta)]\right\}. \quad (10)$$

For the same reason we obtain the particular solution of equation $(7)_2$:

$$\widetilde{\vartheta}_2 = \varphi(\theta)\cdot B_0\left\{\frac{1}{\omega(\theta)}\left[\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)^{3-2\lambda}}{\sin^3\theta}}\frac{\cos\theta}{1+\alpha\sin\theta} - 1\right] + i\eta E[-\eta\omega(\theta)]\right\}. \quad (11)$$

Combining equation (6),(10)and(11), we obtain the complete of equation (5) as

$$\widetilde{\vartheta} = \widetilde{\vartheta}_0 + \widetilde{\vartheta}_1 + \widetilde{\vartheta}_2 = \varphi(\theta)[\widetilde{C}_1 U_1(-i\eta\omega) + \widetilde{C}_2 U_2(-i\eta\omega)] +$$

$$+ A_0\left\{iL_1(\theta) - \frac{1}{2}\eta\varphi(\theta)\cdot E[-\eta\omega(\theta)]\right\} + B_0\{L_2(\theta) + i\eta\varphi(\theta)\cdot E[-\eta\omega(\theta)]\},$$

where

$$L_1(\theta) = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{1}{\sqrt[4]{\omega^3(\theta)\sin\theta(1+\alpha\sin\theta)^{3-2\lambda}}}\right),$$

$$L_2(\theta) = \frac{\cos\theta}{\sin\theta(1+\alpha\sin\theta)} - \frac{1}{\sqrt[4]{\omega^3(\theta)\sin\theta(1+\alpha\sin\theta)^{3-2\lambda}}}.$$

(3) Internal forces and displacements

The internal forces are

$$T_1 = R_e\widetilde{T}_1 = \frac{Pb}{2}\,\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta} + \frac{k_1\overline{\Omega}_z^1}{2\pi b}\,\frac{\alpha\sin\theta}{1+\alpha\sin\theta} + \frac{k_1}{b}\,\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\, I_m\widetilde{\vartheta},$$

$$T_2 = R_e\widetilde{T}_2 = \frac{Pb}{2} + \frac{k_1}{b} I_m\frac{d\widetilde{\vartheta}}{d\theta},$$

$$M_1 = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}(\mathscr{X}_1 + \nu_2 \mathscr{X}_2) = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}\left[\frac{1}{b} R_e \frac{d\widetilde{\vartheta}}{d\theta} + \nu_2 \frac{\cos\theta}{a(1+\alpha\sin\theta)} R_e \widetilde{\vartheta}\right],$$

$$M_2 = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}(\mathscr{X}_2 + \nu_1 \mathscr{X}_1) = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}\left[\frac{\cos\theta}{a(1+\alpha\sin\theta)} R_e \widetilde{\vartheta} + \frac{\nu_1}{b} R_e \frac{d\widetilde{\vartheta}}{d\theta}\right],$$

horizontal displacement is

$$u_\rho = \frac{a(1+\alpha\sin\theta)}{E_2 h}(T_2 - \nu_2 T_1),$$

we cosider that vertical displacement u_z parallel to the axis of rotation is zero at $\theta = -\pi/2$, and obtain:

$$u_z = -b\int_{-\frac{\pi}{2}}^{\theta}\left[\cos\theta \cdot R_e\widetilde{\vartheta} + \frac{1}{E_1 h}(T_1 - \nu_1 T_2)\sin\theta\right]d\theta.$$

(4) Determination of complex coefficients $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{\Omega}_z'$.

Since the ring shell under uniformly distributed normal pressure has a symmetrical plane through $\theta = \pm\pi/2$, we have the following conditions:

(A) $R_e\widetilde{\vartheta} = 0$ at $\theta = \pm\pi/2$. In addition, relative parpendicular displacement u_z between the points $\theta = -\pi/2$ and $\theta = \pi/2$ is zero:

$$-b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\cos\theta \cdot R_e\widetilde{\vartheta} + \frac{1}{E_1 h}(T_1 - \nu_1 T_2)\sin\theta\right]d\theta = 0.$$

(B) Transverse shear force $Q_\rho = 0$ at $\theta = \pm\pi/2$

$$Q_\rho = \frac{k_1}{a(1+\alpha\sin\theta)} I_m\widetilde{\vartheta}. \ [2]$$

From $I_m\widetilde{\vartheta} = 0$, two imaginary parts C_3 and C_4 of $\widetilde{C}_1$ and $\widetilde{C}_2$ may be determined. Now, all the coefficients are determined and therefore the problem is solved.

4. Example of calculation and conclusions

(1) We calculate the internal forces $T_1(\theta)$, $T_2(\theta)$, $M_1(\theta)$, $M_2(\theta)$, under a uniformly distributed load $P = -1\text{kg/cm}^2$, The following parameters of the orthotropic ring shell are given: $a = 100\text{cm}$, $\alpha = 0.25$, $h = 0.05\text{cm}$,
$E_1 = 1.9\times10^6\text{kg/cm}^2$, $\nu_1 = 0.3$, $\lambda = 1.0$, 0.75, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1.

(2) From these results we obtain

(A) When λ is equal to unity (i,e. isotropic), results obtained agree with the tabular values given by K.F.Chernykh.

(B) When $\lambda < 1$, as the value of λ decreases, internal forces also decrease and even their sign may change. For our calculated example, the magnitude of λ should not be smaller than 0.3 , $\lambda \not< 0.3$, otherwise deformation go beyond the scope of deformation.

5. Refernces

[1] A.P.Mukoed, Prikl Mekh, v10 n8 Aug 1974 P36-42
[2] K.F.Chernykh, Linear theory of shells, PartI, 1962; PartII,1964
[3] S.A.Tumarkin, Prikl Mate i Mekh, v23 n6 1959 P1083-1094