# A FINITE ELEMENT ANALYSIS FOR THIN SHELLS BASED ON DIRECT SOLUTION OF SHELL EQUATIONS

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## SUMMARY

### *Introduction*

Stress analysis using finite element methods is commonly explicitly based on some extremum energy principle. However, historically and conceptually, the finite element method has been formulated directly in terms of known nodal stiffnesses of the chosen elements.

For thin shell elements, such nodal stiffnesses are embodied in various systems of well tried shell equations. Finite element solutions for a number of shell structures have been obtained by satisfying such shell equations at the nodes.

### *Description of Method*

A ring type thin shell element has been used. The meridian is specified by the radii of curvature at the nodes. The displacements were assumed to be in the form of a power series. Various shell equations, such as those of Flügge, have been used to obtain solutions.

Each variable of force or displacement for each element can be represented by a linear array of the coefficients of the constants of the assumed displacement functions.

At the nodes, compatibility of displacements and equilibrium conditions give a number of equations, which can, in turn, be represented in terms of the linear arrays. The number of constants being made equal to the number of equations, solution for the constants is then possible. Back substitution will then give any required stress or displacement at any position on the shell.

### *Results Obtained*

Cylinders, spheres and toroids have been analysed with loads, which have included edge and pressure forces and both symmetrical and unsymmetrical cases. Using comparatively small numbers of elements, comparisons with analytical solutions and other published results show good agreement.

As it is not necessary to reduce pressure loads to loads at the nodes and as direct calculation of all displacements and stresses is possible, some problems, which can arise with some other approaches, are avoided.

1. Displacement, Force and Direct Stiffness Methods

Stress analysis of thin shells using Finite Elements is most commonly of the type described as the Displacement method. A displacement field is assumed and matrix stiffnesses are found by minimizing total potential energy. On the other hand, the so-called Force method basically assumes forces as the fundamental unknowns. Mixed or Hybrid approaches involve both displacements and forces as basic unknowns.

For all these methods, practical implementation to solve thin shell problems normally involves the minimization of some energy functional.

An alternative approach is to utilize the nodal stiffnesses implicit in any system of thin shell equations. This does not require any integration, to find total energies, nor any minimization of those energies. Insofar as a displacement field is assumed, this direct use of shell equations could be said to be a displacement method but insofar as this assumption is made to imply explicit force assumptions the distinction is not clear. In that both displacement and force compatibilities are enforced it has some characteristics of a mixed method.

2. A Direct Stiffness Method

In order to demonstrate this use of a system of thin shell equations, the application to cases of loading which are symmetrical with respect to a shell axis may be first considered. Insofar as thin shell equations are generally available for shells of revolution this limitation applies and the basic finite element considered is a thin shell ring element.

2.1 Element Geometry

Fig. 1 shows this ring element. The co-ordinates of a point on the middle surface are $\phi$ and/or S, in the meridional direction, and $\theta$ in the circumferential direction in a circle of radius r. The meridional radius of curvature is $r_1$ and the shell thickness is h.

In principle, $\phi$, h and $r_1$ may be taken as different at the nodal circles $n_1$ and $n_2$. In practice, it is simple to assume these different values but it may cause some unacceptable complications to include the effect of the variations of these quantities at the nodes.

2.2 Shell Equations

This approach is not dependent on the use of any particular system of shell equations. The use of more complicated equations may give the possibility of increased accuracy at the cost of a greater chance of error in the algebraic manipulations required to set up the equations in a computer program.

A number of shells have been analysed using equilibrium equations and force displacement relationships as given by Flügge [1].

These can be put in the form:

Equilibrium equations

$$Q_\phi = \frac{dM_\phi}{ds} + (M_\phi - M_\theta)\frac{\cos\phi}{r}$$

$$p = \frac{dQ_\phi}{ds} + \frac{1}{r}\left(Q_\phi\cos\phi + N_\theta\sin\phi + \frac{r}{r_1}N_\phi\right) \qquad (1a)$$

$$0 = \frac{dN_\phi}{ds} + N_\phi\frac{\cos\phi}{r} - N_\theta\frac{\cos\phi}{r} - Q_\phi\frac{1}{r_1}$$

Force displacement relationships

$$N_\phi = D(\frac{dv}{ds} + \frac{w}{r_1} + \mu\frac{\delta}{r})$$

$$N_\theta = D(\frac{\delta}{r} + \mu(\frac{dv}{ds} + \frac{w}{r_1}))$$

$$M_\phi = K(\frac{d\beta}{ds} + \mu\frac{\cos\phi}{r}\beta) \qquad (1b)$$

$$M_\theta = K(\frac{\cos\phi}{r}.\beta + \mu\frac{d\beta}{ds})$$

where v and w are the meridional and normal components of displacement of the middle surface and $N_\phi$, $N_\theta$, $M_\phi$, $M_\theta$ and $Q_\phi$ are the stress resultants as defined in detail by Flügge [1].

$D = Eh/(1-\mu^2)$ $K = Eh^3/12(1-\mu^2)$ where

h is shell thickness, E Young's modulus and $\mu$ Poisson's Ratio.

$\beta$, the rotation of a tangent to the meridian and $\delta$, the displacement perpendicular to the axis are given by:

$$\beta = \frac{dw}{ds} - \frac{v}{r_1}$$

$$\delta = w\sin\phi + v\cos\phi \qquad (2)$$

and V, the resultant axial force per unit length and H, the resultant force component per unit length perpendicular to the axis are:

$$V = N_\phi\sin\phi + Q_\phi\cos\phi$$

$$H = Q_\phi\sin\phi - N_\phi\cos\phi \qquad (3)$$

2.3 Assumed Displacement Relations

S, the co-ordinate of distance along the meridian, can conveniently be taken to be zero at one node. The variations of displacements with S are assumed to be power series of the type used in normal Displacement methods:

$$v = k_1 + k_2S + K_3S^2 + k_4S^3 + k_5S^4$$

$$w = k_6 + k_7S + k_8S^2 + k_9S^3 + k_{10}S^4 + k_{11}S^5 \qquad (4)$$

$k_1 - k_{11}$ are unknown constants, whose values are determined from the boundary conditions at the nodes. These boundary conditions are made to include the satisfaction of the shell eqs. (1a) and (1b).

2.4 Rigid Body Motion

For the assumed axisymmetric loading conditions, the only possible rigid body motion is an axial translation of the complete shell. When necessary, this can be eliminated by putting the axial deflection, $w\cos\phi - v\sin\phi$, equal to zero at one node. For simplicity this can be at the end where S = 0.

2.5 Boundary Conditions at the Nodes

If the rigid body motion has been eliminated the conditions to give equations for the determination of the constants $k_1 - k_{11}$ in the displacement functions are:

At each of the two nodes:

(i) The value of the pressure, p.

(ii) The value of the axial load per unit length, V.

(iii) The condition for $\frac{dN_\phi}{ds}$ as defined by the third equation (1a).

(i) and (ii) are statically determined by the pressure and axial loads applied to the shell. From (i), (ii) and (iii) six equations result, for each element, when the two nodes are considered.

For any determinate case of boundary conditions at the nodes, four further equations must be defined by the values of meridional moment $M_\phi$, radial force H, radial deflection $\delta$ and rotation of the tangent $\beta$, at the two nodes.

Now if the 10 equations resulting from these conditions, together with the equation for rigid body motion elimination, are obtained for each element, solution is possible for the 11 constants $k_1 - k_{11}$ for each element.

Such a solution, for a complete shell structure, will include all requirements for compatibility between elements and also all the external boundary conditions of the complete structure.

2.6 Calculation of all Variables in Terms of the Constants in the Displacement Functions

From eq. (4) $\frac{dv}{ds}$, $\frac{d^2v}{ds^2}$, $\frac{dw}{ds}$, $\frac{d^2w}{ds^2}$, $\frac{d^3w}{ds^3}$, and $\frac{d^4w}{ds^4}$ can be readily found by successive differentiation. Hence $\beta$, $\frac{d\beta}{ds}$ and $\delta$ can also be found in terms of S and the constants $k_1 - k_{11}$.

Now as all variables of force are given in terms of displacements or differentials of displacements (eq. (1b)), these forces may likewise be found in the same terms. Finally eq. (1a) can be written in these same terms.

In practice, apart from the differentiations, all the calculations for these successive derivations can be easily and conveniently performed by computer.

For each end of each element the linear arrays for v, w, $\frac{dw}{ds}$, $\beta$, $\frac{d\beta}{ds}$, $M_\phi$, $M_\theta$, $\delta$, $N_\phi$, $N_\theta$, $\frac{dM_\phi}{ds}$, $Q_\phi$, $\frac{dQ_\phi}{ds}$, p, $\frac{dN_\phi}{ds}$, V and H are successively evaluated and stored in the computer.

2.7 The System of Equations for the Solution of a Complete Shell

The eleven dimensioned linear arrays, for all the variables involved in the equations for the boundary conditions at the two nodes of each element, can be used to form 11n equations for a complete shell divided into n elements.

(a) For each element 6n equations arise from the satisfaction of eq. (1a).

(b) n equations are given by the elimination of rigid body motions.

(c) At each of the n-1 internal nodes 4 equations are given by the equalities of $\delta$, $\beta$, $M_\phi$ and H between adjacent elements.

(d) Finally 4 equations for the boundary conditions at the two outside edges of the complete shell must be obtained. Typically these might be $\delta = \beta = 0$, for a rigid edge constraint, or $M_\phi$ and $H = 0$, for a free edge condition.

The total number of equations from (a) + (b) + (c) + (d) is $6n + 4(n-1) + n + 4 = 11n$ i.e. the appropriate requirement to solve for the 11 constants for each of the n elements.

For a non-branched shell the corresponding matrix of the coefficients of the constants in these equations can be written in a banded form, of band width 22.

In any case, inversion of this matrix will enable the system of equations to be solved for the numerical values of all the constants $k_1 - k_{11}$, for all the elements.

Back substitution in the appropriate linear arrays for all the variables of force and deflection, will then enable the computer to evaluate all variables at the nodes.

## 3. Results

### 3.1 Cylinder with Edge Shear Loading

Fig. 2 shows an example quoted by Zienkiewicz [2] from the original paper of Grafton and Strome [3], in which results for various numbers of conical elements were given.

The results of the present Finite element method, using 7 elements, are seen to compare favourably with theoretical values calculated on the assumption that the cylinder was long, e.g. the error in deflection at the loading point was 0.7%, compared to the reported values of the above reference of 31.7% error, using 9 elements, and 3.1% error, using 24 elements.

### 3.2 Hemispherical Tank Containing Liquid

This test case involves varying pressure load and a high radius/thickness ratio. Nevertheless Fig. 3 shows that,even with the small number of elements used,the stress concentration at the lower fixed edge has been obtained with some precision. As shown here, it is useful to use smaller elements near to points of rapidly varying stress. The size of elements was in fact chosen by judgment relative to the 'critical', or boundary layer, length determined by a common guidance formula.

### 3.3 Boundary Conditions at the Pole of a Closed Shell

In the case of the top of the spherical shell shown in Fig. 3, or any similar case where $r = 0$, calculations for the linear arrays for the coefficients of the constants $k_1 - k_{11}$ would involve terms using $\frac{1}{r}$ and are therefore not feasible.

It has been found possible, however, to use the displacement functions of eqs. (4) in the normal way for the cap element 7 (Fig. 3). The lower edge, common with element 6, being dealt with in the usual way. 6 boundary conditions are then required for the equations at the pole 0 (including rigid body motion elimination).

These are that: $k_1 = k_2 = k_3 = k_6 = k_7 = k_9 = 0$ and correspond to the equivalent boundary conditions at a normal edge where $S = 0$.

## 4. Unsymmetrical Loading

It is common practice, when analysing shells of revolution with unsymmetrical loadings, to decompose the loads into the sum of a number of terms of a Fourier series. Of these terms, the most important is that giving a resultant lateral force. For this 'n = 1' case (n referring here to the number of the term in the Fourier series) all variables are assumed to vary sinusoidally with $\theta$, e.g. $v = \bar{v}\cos\theta$ and $M_\phi = \bar{M}_\phi\cos\theta$, where $\bar{v}$ and $\bar{M}_\phi$ represent the maximum values of these variables, which occur at $\theta = 0^\circ$.

For unsymmetrical loading it is necessary to introduce the appropriate extra variables of displacement and force. These must include, in the notation of Flügge, a lateral displacement u, for which the displacement relation assumed is (for the 'n = 1' case):

$$u = (k_{12} + k_{13}S + k_{14}S^2 + k_{15}S^3 + k_{16}S^4)\ \bar{u}\sin\theta$$

where $\bar{u}$ is the maximum value of u, which occurs at $\theta = 90^{\circ}$.

This introduces a requirement for 5 more equations for the extra boundary conditions, which exists for unsymmetrical loading.

Appropriate extra variables of force, $N_{\phi\theta}$ and $M_{\phi\theta}$, must be included and for the formulation of the boundary conditions the resultant 'Kirchoff' shear forces $N_1$ and $Q_1$ must be used, where for the 'n = 1' case:

$$N_1 = N_{\phi\theta} - M_{\phi\theta}\sin\phi/r$$

and $$Q_1 = Q_{\phi} + M_{\phi\theta}/r$$

A detailed treatment of these points with the appropriate notation and sign conventions has been given in References [4] and [5]. Equations of the form of eqs. (1a), (1b), (2) and (3) can be obtained, including all the extra variables required for unsymmetrical loading.

From these, in the same manner as for symmetrical loading, all variables of displacement and force can be found in the form of linear arrays of the, now 16, constants for each element.

With the addition of 5 extra boundary conditions for each element the solution can proceed in the same way as for the symmetrical case.

The five additional boundary conditions required are:

(i) Lateral rigid body motion elimination.

(ii) The statically determinate value of the resultant lateral shear force at each node.

(iii) The satisfaction of an equilibrium condition, at each node, of the form of the third equation of eqs. (1a) but for $\frac{dN_{\phi\theta}}{ds}$ instead of $\frac{dN_{\phi}}{ds}$

Thus for n elements 16 n equations can be found and solved for the 16n constants. Back substitution will then enable the values of any variables at the nodes to be calculated.

## 5. Results for an Unsymmetrically Loaded Shell

Fig. 4 shows a laterally loaded shell with the geometry of a corrugated pipe. The results of an analysis of this case, using a numerical integration solution, were given in References [4] and [5]. An experimental loading test showed good confirmation of the theoretical analysis. This case was therefore used as a test example for unsymmetrical loading. Accordingly, the lower quarter of this shell was analysed by the finite element approach of this present paper.

Using 2, 3, 6 and 9 elements successively, convergence of the solutions for the stress resultants was demonstrated and the distributions of $N_{\phi}$, $N_{\theta}$, $M_{\phi}$ and $M_{\theta}$ were in good agreement with the results given in References [4] and [5].

The values of the stress resultants were calculated at the nodes and also at two equally spaced points between the nodes.

Fig. 4 shows that for the moments $M_{\phi}$ and $M_{\theta}$, even for 2 elements, good results are obtained. For the forces $N_{\theta}$, again results were good for low numbers of elements. The distributions of $N_{\phi}$ were good for 3 elements but for 2 elements, although at the nodes the values were without significant error, between the nodes, very marked oscillations of the distribution curves away from the true values occurred.

These results were obtained for the simplified equations given by Flügge [1]. Solutions with other more complex sets of equations were obtained but for reasons unknown, although results were good with the larger numbers of elements, errors with smaller numbers of elements were greater. A possible explanation is that there was some undiscovered error in the algebraic manipulation of the formulae.

## 6. Conclusions

The curved axisymmetric finite element proposed has been shown to give good results with a fairly small number of elements when the analysis has been by the described direct use of shell equations. This arises from the comparatively high order of the displacement functions used and, insofar as the order or 'degrees of freedom' used corresponds exactly to all the conditions of equilibrium and force-displacement relationships of normal thin shell theory, good conformity is to be expected at the nodes.

As the equations include these complete equalities of displacements and stress resultants at the nodes, there is no difficulty in calculating values of displacements or stresses at the nodes, including points of stress concentration.

Integration difficulties, when calculating the internal strain energies of elements, do not arise and transformations from local to general co-ordinates systems are not required.

There is no need to approximate distributed pressure loadings by forces concentrated at the nodes, although, if this is not done, many of the advantages of having symmetrical matrices may be lost. However 'in core' solutions have been found quite feasible owing to the small number of elements used.

The possibility of only satisfying some ,and not all, of the relevant shell equations would allow more elements to be used for a given computer storage requirement.

## 7. Acknowledgments

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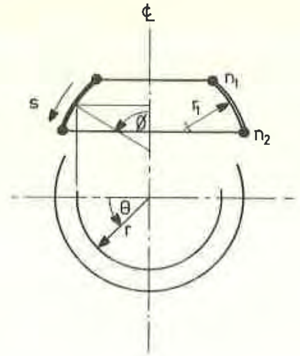


Fig. 1 - Geometry of a Ring Finite Element

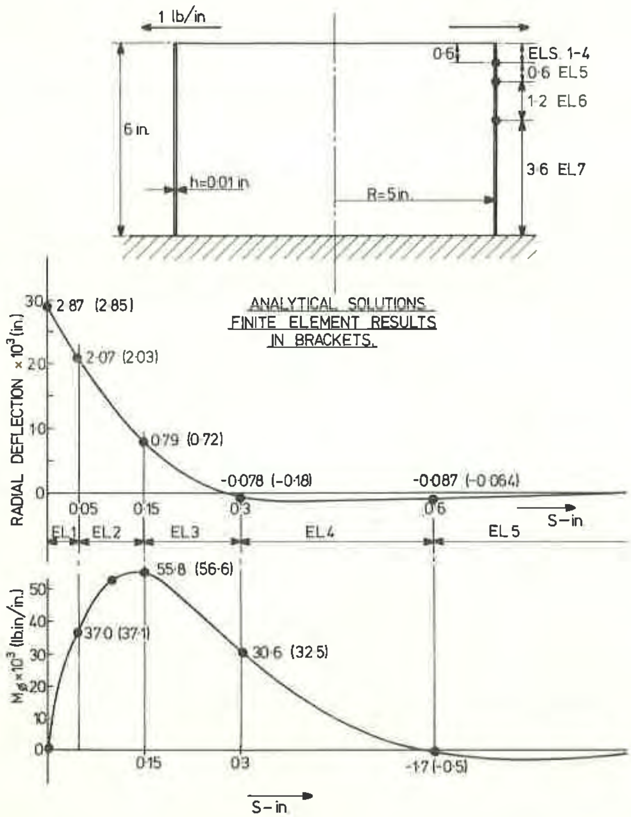


Fig. 2 - Cylindrical Shell - Comparison of Results with Theory

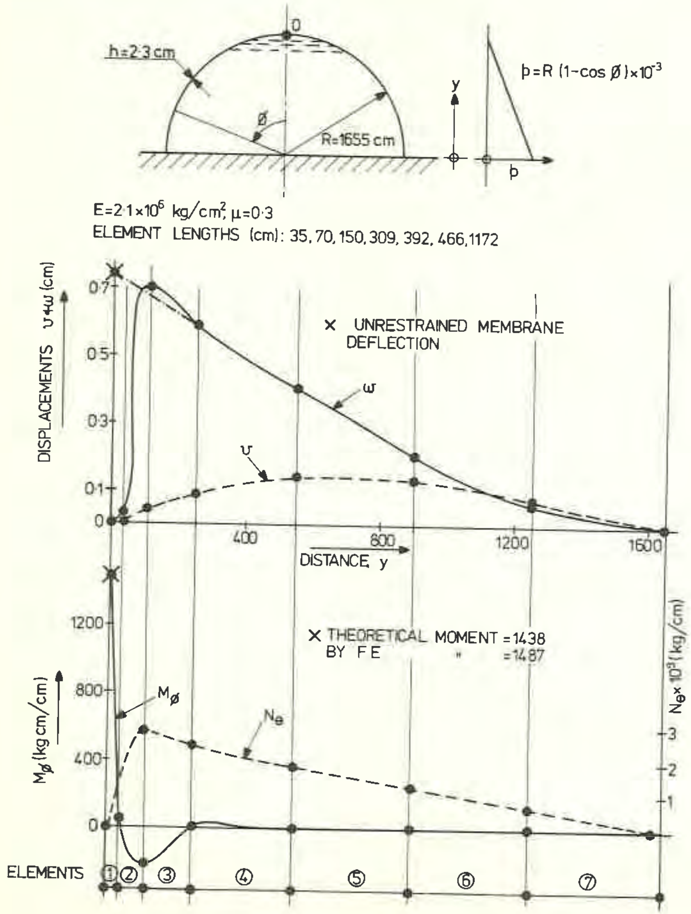


Fig. 3 - Hemispherical Tank - Distributions of Displacements and Stress Resultants

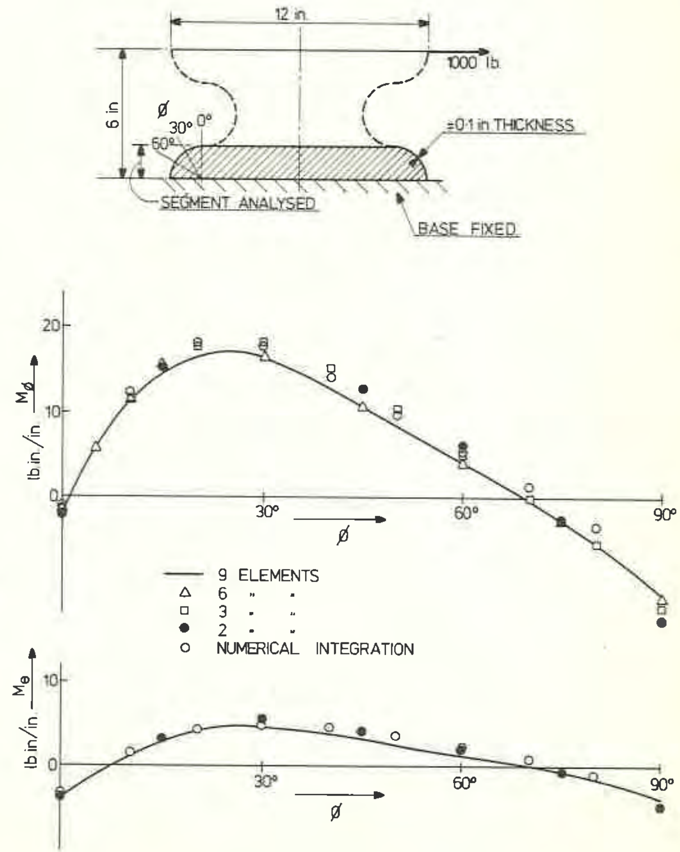


Fig. 4 - Section of Laterally Loaded Shell - Distributions of $M_\phi$ and $M_\theta$.