ABSTRACT

ZAJACZKOWSKI, CLAIRE CHRISTINE. Surgery Obstructions for Seifert Fibered Homology Spheres. (Under the direction of Tye Lidman.)

We examine surgeries on knots in S^3 to find surgery obstructions to Seifert fibered integral homology spheres. Specifically, we find classes of Seifert fibered integral homology spheres which cannot result from surgery on knots of a particular genus. Our obstructions come from utilizing the toroidal structure of our manifolds, which for Seifert fibered integral homology spheres, is the number of singular fibers.

This dissertation begins by examining graded roots: objects that naturally encode Heegaard Floer homology of Seifert fibered integral homology spheres. We prove a necessary and sufficient condition for graded roots such that the U-action of the reduced Heegaard Floer homology is non-zero. We use this to make a general statement about how the associated U-action behaves on the reduced Heegaard Floer homology based on the number of singular fibers.

Next we examine Seifert fibered integral homology spheres with 5 singular fibers. We use our graded root result to show that the U-action of the reduced Heegaard Floer homology is non-zero.

To conclude the dissertation, we study the mapping cone which provides an easy way to compute the Heegaard Floer homology of surgery on a knot in S^3. We show that for $1/n$-surgery a genus 1 knot, the U-action on the reduced Heegaard Floer homology is always 0. Furthermore, we show that for ± 2-surgery on a genus 2 knot, the action of U^2 on the reduced Heegaard Floer homology is also 0.

Surgery Obstructions for Seifert Fibered Homology Spheres

by
Claire Christine Zajaczkowski

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2019

APPROVED BY:

Radmila Sazdanovic

Andrew Cooper

Adam Levine

Tye Lidman
Chair of Advisory Committee

DEDICATION

To Sam.

BIOGRAPHY

Claire grew up in Media, Pennsylvania before attending Gettysburg College planning to be a history major. She switched to a mathematics major after her randomly assigned adviser, Darren Glass, convinced her that math was more up her alley, and all that history reading wasn't for her. She then began graduate school at North Carolina State University, much to the glee of that same adviser, and is now poised to graduate. While not doing mathematics, Claire enjoys ballroom dancing, bar trivia, and playing every board game she can get her hands on.

ACKNOWLEDGEMENTS

The author thanks her advisor, Tye Lidman, for his guidance in research and his editorial help with writing this dissertation.

The author also thanks Jen Hom for her editorial eye, and Duncan McCoy for his help in streamlining the graded root results. The author was partially supported by DMS-1709702 while working on this paper.

Special thanks also goes to Samuel Flynn, for his regular help with coding and writing, and his general encouragement and support throughout graduate school.

Finally, the author thanks her officemates, friends, and family for their ongoing encouragement, support, and company.

TABLE OF CONTENTS

LIST OF TABLES . . . **vi**

LIST OF FIGURES . . . **vii**

Chapter 1 Introduction . . . **1**
- 1.1 Background . . . 1
- 1.2 Results . . . 2
  - 1.2.1 Overview . . . 2
  - 1.2.2 Obstructions . . . 3
- 1.3 Dehn Surgery . . . 5
- 1.4 Heegaard Floer Homology . . . 9
  - 1.4.1 Knot Floer Homology . . . 14

Chapter 2 Graded Roots . . . **17**
- 2.1 Introduction . . . 17
  - 2.1.1 Setup . . . 17
  - 2.1.2 Previous Work . . . 18
- 2.2 Results . . . 20

Chapter 3 Five Singular Fibers . . . **23**
- 3.1 Introduction . . . 23
- 3.2 $\Sigma(2,3,5,p,q)$ . . . 25
  - 3.2.1 Preliminaries . . . 25
  - 3.2.2 Results . . . 26
- 3.3 $\Sigma(2,3,7,p,q)$ . . . 39
  - 3.3.1 Preliminaries . . . 39
  - 3.3.2 Results . . . 40
- 3.4 $\Sigma(2,3,11,13,p)$ . . . 44
  - 3.4.1 Preliminaries . . . 44
  - 3.4.2 Results . . . 44

Chapter 4 Mapping Cone . . . **47**
- 4.1 Introduction . . . 47
- 4.2 Background . . . 47
  - 4.2.1 Setup . . . 47
  - 4.2.2 Previous Work . . . 49
- 4.3 Genus 1 Results . . . 50
  - 4.3.1 Preliminary Proofs . . . 51
  - 4.3.2 Proof of Theorem 1.2.5 . . . 54
  - 4.3.3 Preliminaries . . . 55
  - 4.3.4 Results . . . 57

References . . . **59**

LIST OF TABLES

Table 3.1 Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ . . . 27
Table 3.2 General Cases for $Y = \Sigma(2, 3, 5, p, q)$, with p and q sufficiently large . . . 28
Table 3.3 $pq \equiv 1 \mod 30$, $e_0 = -3$, , for p and q sufficiently large . . . 29
Table 3.4 $pq \equiv 13 \mod 30$, $e_0 = -3$, for p and q sufficiently large . . . 30
Table 3.5 $pq \equiv 17 \mod 30$, $e_0 = -2$, for p and q sufficiently large . . . 31
Table 3.6 $pq \equiv 29 \mod 30$, $e_0 = -2$, for p and q sufficiently large . . . 32
Table 3.7 $pq \equiv 13 \mod 30$, $p = 7$, $e_0 = -3$, for q sufficiently large . . . 35
Table 3.8 $pq \equiv 13 \mod 30$, $p = 7$, $e_0 = -3$, for q sufficiently large . . . 36
Table 3.9 $pq \equiv 17 \mod 30$, $e_0 = -2$, $p = 7$ . . . 37
Table 3.10 $pq \equiv 17 \mod 30$, $e_0 = -2$, $p = 7$ . . . 37
Table 3.11 Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for $\Sigma(2, 3, 7, p, q)$ and $a = 1$ . . . 40
Table 3.12 Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for $\Sigma(2, 3, 7, p, q)$ and $a = 2$ . . . 41
Table 3.13 General Cases for $Y = \Sigma(2, 3, 7, p, q)$ . . . 43
Table 3.14 $pq \equiv 1 \mod 42$, $e_0 = -2$ . . . 43
Table 3.15 General Cases for $Y = \Sigma(2, 3, 11, 13, p)$ . . . 46

LIST OF FIGURES

Figure 1.1 A regular projection for the left handed trefoil. 6
Figure 1.2 Solid torus and knot exterior 7
Figure 1.3 Induced orientation on $\partial(K)$ 7
Figure 1.4 Lens space surgery description 8
Figure 1.5 Plumbing graph for a lens Space 8
Figure 1.6 Surgery description of a Seifert fibered manifold. 9
Figure 1.7 Plumbing graph for Seifert fibered manifolds 9
Figure 1.8 Heegaard Diagram for S^3. 10
Figure 1.9 Heegaard diagram for $S^1 \times S^2$. 11
Figure 1.10 A whitney disk from x to y 12
Figure 1.11 Alternate Heegaard diagram for S^3. 14
Figure 1.12 Doubly pointed Heegaard diagram for the left handed trefoil. 14
Figure 1.13 Full knot Floer complex for the left-handed trefoil. 15

Figure 2.1 Construction of Γ_τ 19
Figure 2.2 $\tau_\Delta(i) > \tau_\Delta(j+1)$ 21
Figure 2.3 $\tau_\Delta(i) < \tau_\Delta(j+1)$ 21

Figure 4.1 The Mapping cone for $2/3$ surgery on a knot $K \subset S^3$. The v_k maps are red, h_k maps are blue. 49
Figure 4.2 The truncated mapping cone for $2/3$ surgery on a genus 2 knot $K \subset S^3$. The v_k maps are red, h_k maps are blue. 49
Figure 4.3 Levels of $\mathbf{A}_0^+$ 52
Figure 4.4 -1 Surgery Mapping Cone 52
Figure 4.5 $\frac{1}{n}$ surgery mapping cone for a genus 1 knot 54
Figure 4.6 +2 surgery mapping cone for a genus 2 knot 57

Chapter 1

Introduction

1.1 Background

Consider a knot k in a closed oriented 3-manifold M. We can perform Dehn surgery on k in M by cutting M open along the neighborhood of our knot and gluing back in a solid torus [28]. The neighborhood of our knot is a solid torus, which we denote $D^2 \times S^1$, where D^2 is the disk, and S^1 the circle. The space resulting from this construction is a closed oriented manifold, generally different from M. For this thesis we will only be considering surgery on knots in S^3. Lickorish and Wallace showed that every closed orientable 3-manifold can be obtained by integral surgery on a link in S^3 [16, 32]. Due to this result, Dehn surgery has become a fundamental method of representing 3-manifolds [1]. It is then natural to ask which manifolds can be represented by surgery on a knot in S^3, as opposed to a link.

Knots can be partitioned into three categories: torus, satellite and hyperbolic. Many surgery problems are understood for torus and satellite knots. For example, Moser completely classified surgery on torus knots in S^3 [18], and Gabai was able to first prove the Property P conjecture for satellite knots in [7]. (The Property P conjecture has since been completely proved by Kronheimer and Mrowka [15].) Since surgeries on torus and satellite knots are well understood, the most interesting surgeries to consider are those on hyperbolic knots. Surgeries on hyperbolic knots that do not yield hyperbolic manifolds are called exceptional. Exceptional surgeries are reducible, toroidal, or Seifert fibered. Thurston's hyperbolic Dehn surgery theorem says that there are only finitely many exceptional surgery slopes on a hyperbolic knot [31]. We call Dehn surgery Seifert fibered, toroidal or reducible if it yields a Seifert fibered, toroidal or reducible manifold respectively [12].

Since exceptional surgeries have remained the most elusive, there has been much work put into their study. Dean introduced a condition on knots in S^3 that guarantees a hyperbolic surgery [5], while Eudave-Muñoz extended this to include surgeries producing Seifert fibered manifolds with a projective plane orbit surface and two exceptional fibers [6]. In this same paper Eudave-Muñoz finds a collection of hyperbolic knots that yield toroidal Seifert fibered manifolds [6, Proposition 4.5]. It can be checked that these knots will never yield Seifert fibered integral homology 3 spheres. Teragaito showed that any positive integers can arise as the toroidal surgery slope of a hyperbolic knot in [30], and Wu classified toroidal surgeries on length 3 Montesinos knots [35]. For further general discussions of exceptional surgeries we suggest [3, 13, 19].

In particular we are interested in toroidal, Seifert fibered, integral homology 3-sphere surgeries, so let us explore these specifics further. Gordon and Luecke showed that the denominator ofa toroidal surgery slope is at most 2 for hyperbolic knots in [9], and Miyazaki and Motegi built on this result to show that if K is a hyperbolic, periodic knot with period 2, only an integer coefficient can yield a toroidal surgery [17]. Boyer and Zhang proved that toroidal Seifert fibered spaces cannot arise by non-integer surgery on a hyperbolic knot in S^3, a result also implied by the work of Gordon and Luecke [2, 10]. Ichihara and Jong showed there is no toroidal, Seifert fibered surgery on pretzel knots except for the trefoil in [12]. Ozsváth and Szabó showed that the family of Kinoshita-Terasaka knots $KR_{r,n}$ with $|r| \geq 2$ and $n \neq 0$ cannot yield integral Seifert fibered homology 3-spheres [24]. Wu found the only three large arborescent knots that yield exceptional toroidal surgeries in [34]. Wu also found the Montesinos knots that yield toroidal Seifert fibered surgeries in [33].

1.2 Results

1.2.1 Overview

In this dissertation our goal is to find obstructions to which knots in S^3 yield integral Seifert fibered homology spheres. Seifert fibered integral homology spheres are a result of exceptional surgery [28]. One of the main tools used is Heegaard Floer homology, and the knot invariant: knot Floer homology. Ozsváth and Szabó and many others have used Heegaard Floer homology to find such obstructions [24, 33, 34]. Here, however we use the number of singular fibers of a Seifert fibered integral homology sphere to find obstructions. This is using the toroidal structure of the manifold, as Seifert fibered homology spheres are toroidal if and only if they have 4 or more singular fibers.

The other tools used in this dissertation are Némethi's graded root [21], and Ozsváth and Szabó's mapping cone formula [26]. The graded root is a combinatorial object that allows us to compute Heegaard Floer homology more easily. This always exists when looking at Heegaard Floer homology of Seifert fibered homology spheres, making it a very useful tool for our purposes. The mapping cone formula allows us to more easily compute the Heegaard Floer homology of manifolds resulting from surgery on knots in S^3 [8].

Our general strategy is to examine the reduced Heegaard Floer homologies of both integral homology spheres and surgeries on knots. Specifically we observe how each of these behave under the U-action. We use Némethi's graded root to analyze this for Seifert fibered integral homology spheres and the mapping cone formula to analyze this for surgeries on knots. Then we compare these results in order to establish obstructions to surgeries on knots yielding Seifert fibered integral homology spheres.

1.2.2 Obstructions

Before we give our obstructions, we must introduce some notation. $HF^+(Y)$ is the Heegaard Floer homology of a 3-manifold Y; $HF_{red}(Y)$ is the reduced Heegaard Floer homology of Y. Note that $HF_{red}(Y) \cong HF_{red}(-Y)$ up to grading shift, so our results about HF_{red} will apply to Seifert fibered integral homology spheres of any orientation. Our most general result is the following.

Theorem 1.2.1. *No surgery on a knot K in S^3 of genus g can yield a Seifert fibered integral homology sphere with l or more singular fibers if $g \leq \frac{24l-103}{90}$.*

The proof of this is easy to follow given the following general result, which is proved in Section 2.

Lemma 1.2.2. *Let $Y = \Sigma(p_1, p_2, \ldots, p_\ell)$ be a Seifert fibered integral homology sphere with ℓ singular fibers. Then if*

$$0 \leq k < \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right),$$

$$U^k \cdot HF_{red}(Y) \neq 0.$$

Proof of Theorem 1.2.1. Gainullin's Theorem 3 in [8] tells us that for any knot $K \subset S^3$, $U^{g(K)+\left\lceil \frac{g_4(K)}{2} \right\rceil} \cdot HF_{red}\left(S^3_{\frac{p}{q}}(K)\right) = 0$ where $g(K)$ is the genus of the knot K, and $g_4(K)$ is the four ball genus of K. Using that $g_4(K) \leq g(K)$ we have the following:

$$\begin{aligned} g(K) + \left\lceil \frac{g_4(K)}{2} \right\rceil &\leq g(K) + \frac{g_4(K)+1}{2} \\ &\leq g(K) + \frac{g(K)+1}{2} \\ &= \frac{3g(K)+1}{2}. \end{aligned}$$

By Lemma 1.2.2 we have that $U^k \cdot HF_{red}(Y) \neq 0$ for

$$0 \leq k < \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right),$$

thus if

$$0 \leq \frac{3g(K)+1}{2} < \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right),$$

we will have a contradiction, and K will not yield a Seifert fibered integral homology sphere with ℓ or more singular fibers in this case. Now we use that

$$\sum_{i=1}^{l} \frac{1}{p_i} < \sum_{i=1}^{l} \frac{1}{q_i} < \frac{1}{2} + \frac{1}{3} + \frac{\ell - 2}{5},$$

where q_i is the i^{th} prime to simplify the following.

$$\begin{aligned}
g &< \frac{24\ell - 103}{90} \\
\frac{1}{2} + \frac{1}{3} + \frac{l-2}{5} &< \ell - 3 - 3g(K) \\
\sum_{i=1}^{l} \frac{1}{p_i} &< \ell - 2 - 3g - 1 \\
\sum_{i=1}^{l} \frac{1}{p_i} + 3g(K) + 1 &< \ell - 2 \\
3g(K) + 1 &< \ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i} \\
\frac{3g(K)+1}{2} &< \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right).
\end{aligned}$$

Thus our proof is complete. □

We also have the following result for genus 1 knots.

Theorem 1.2.3. *No surgery on a genus 1 knot in S^3 can yield a Seifert fibered integral homology sphere with 5 or more singular fibers.*

To prove this we need the following results: the first is proved in Chapter 3, the second is proved in Chapter 4.

Theorem 1.2.4. *Let $Y = \Sigma(p_1, p_2, p_3, p_4, p_5)$. Then*

$$U \cdot HF_{red}(Y) \neq 0.$$

Theorem 1.2.5. *For a genus 1 knot K and $n \in \mathbb{Z}$ we have that $U \cdot HF_{red}(S^3_{1/n}(K)) = 0$.*

Proof of Theorem 1.2.3. By Lemma 1.2.2 we have that $U \cdot HF_{red}(Y) \neq 0$ when

$$1 < \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right),$$

and Y is a Seifert fibered integral homology sphere with ℓ of more singular fibers. We now show this condition is satisfied for $\ell \geq 6$.

$$\sum_{i=1}^{6} \frac{1}{p_i} < \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13}$$
$$\sum_{i=1}^{6} \frac{1}{p_i} < 2$$
$$0 < 2 - \sum_{i=1}^{6} \frac{1}{p_i}$$
$$2 < 4 - \sum_{i=1}^{6} \frac{1}{p_i}$$
$$1 < \frac{1}{2}\left(6 - 2 - \sum_{i=1}^{6} \frac{1}{p_i}\right).$$

Now by Theorem 1.2.5 we have that $U \cdot HF_{red}(S^3_{1/n}(K)) = 0$ for a genus 1 knot K. It follows that surgery on a genus 1 knot cannot yield a Seifert fibered integral homology sphere with 6 or more singular fibers. Then by applying Theorem 1.2.4 our proof is complete.

□

We were also able to improve on Gainullin's Theorem 3 in [8] for ± 2 surgery on genus 2 knots.

Theorem 1.2.6. *Let K be a genus 2 knot in S^3, then the following holds for any Spin^c-structure i.*

$$U^2 \cdot HF_{red}\left(S^3_{\pm 2}(K), i\right) = 0.$$

1.3 Dehn Surgery

A **link** is a finite collection of smoothly embedded disjoint closed curves in a closed orientable 3-manifold. A one component link is called a knot. Two links, $\mathcal{L}$ and $\mathcal{L}'$, in a manifold M, are equivalent if there is a smooth orientation preserving automorphism $h : M \to M$ that satisfies $h(\mathcal{L}) = \mathcal{L}'$.

Every link $\mathcal{L} \subset M$ can be thickened to get its tubular neighborhood, $N(\mathcal{L})$. This neighborhood will be a collection of smoothly embedded disjoint solid tori, $D^2 \times S^1$, one for each component. The core of each of these tori, $\{0\} \times S^1$, form the link $\mathcal{L}$.

In this thesis we will predominantly be dealing with knots in S^3. Links in S^3 can be thought of as links in $\mathbb{R}^3$. Given a link $\mathcal{L}$ in $\mathbb{R}^3$, we can project our link $\mathcal{L}$ onto a plane $P \subset \mathbb{R}^3$. This is given by the orthogonal projection map $p\colon \mathbb{R}^3 \to P$. We say that p is a regular projection

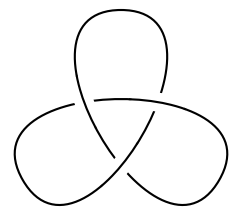


Figure 1.1: A regular projection for the left handed trefoil.

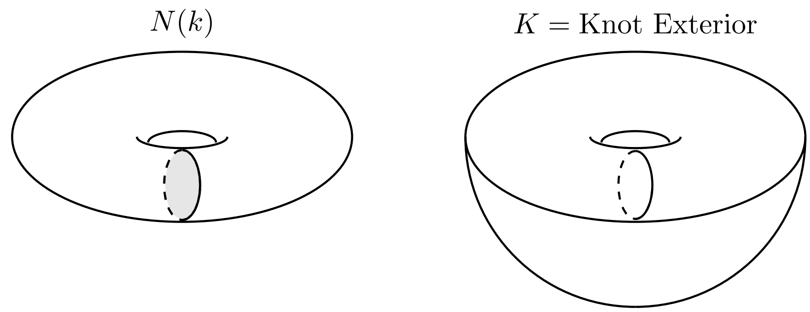


Figure 1.2: Solid torus and knot exterior

for $\mathcal{L}$ if for every link $p^{-1}(x)$, $x \in P$, intersects $\mathcal{L}$ in 2 or fewer points, and the Jacobian has rank one at every intersection point $y \in p^{-1}(x)$. Every link admits a regular projection, and so knots and links are often drawn as smooth curves in the plane with under or over crossings at each double point, see Figure 1.1.

Now we let us explore the specifics of Dehn surgery. The idea behind this is taking a link $\mathcal{L} \subset M$, cutting out a tubular neighborhood of $\mathcal{L}$, and gluing a solid torus back in its place under some mapping. This gives us a new manifold M'. The space resulting from this construction is a closed oriented manifold, generally different from M. For this dissertation we will only be considering surgery on knots in S^3.

Next we give the formal construction for **Dehn surgery**. Since we will be dealing with surgery on knots in S^3, we will give our construction for knots instead of links, but it can easily be extended to links. Let K be a knot in a closed orientable 3-manifold M, and let $N(K)$ be its tubular neighborhood. Then we cut open M along the 2-torus $\partial N(K)$. This results in two manifolds; the knot exterior $E(K) = M \setminus \text{int}N(K)$, and the solid torus $N(K)$, see Figure 1.2. We will identify $N(K)$ as the standard solid torus $D^2 \times S^1$. It follows that $E(K)$ is a manifold with torus boundary, and $M = E(K) \cup (D^2 \times S^1)$. We now can use a homeomorphism $h \colon \partial D^2 \times S^1 \to \partial E(K)$, to glue the solid torus back in to $E(K)$. The new manifold we get from this construction is $M' = E(K) \cup_h (D^2 \times S^1)$. We say that M' is obtained from M by surgery along the knot K.

This new manifold M' depends on the homeomorphism h. This homeomorphism, h, is completely determined by where h sends the meridian of the solid torus. We can think of

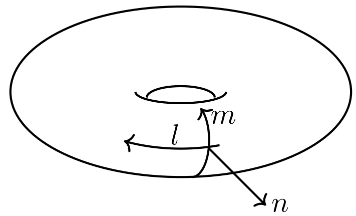


Figure 1.3: Induced orientation on $\partial(K)$

this meridian as $\partial D^2 \times \{\star\}$, and the image of this as some curve $c = h(\partial D^2 \times \{\star\})$ on the boundary of K. To see why this completely determines h, we can think of attaching the solid torus $D^2 \times S^1 = (D^2 \times I) \cup D^3$ in two steps. First we glue in $D^2 \times I$, for I some segment of S^1. Then we attach the 3-ball, D^3, along its boundary S^2. Since all orientation preserving homeomorphisms of S^2 are isotopic to the identity this second gluing is unimportant. Thus the image of the meridian determines the homeomorphism h.

When we consider $M = S^3$, a simple closed curve on $\partial E(K)$ is given by a pair of relatively prime integers, up to isotopy. A meridian of $N(K)$ is a generator of $H_1(E(K))$; we will call this curve m on ∂K. The canonical longitude is a curve l on $\partial E(K)$, that is homologically trivial in $E(K)$. This curve is unique up to isotopy. The two curves m and l form a basis for $H_1(\partial E(K))$ which is unique up to isotopy and orientation reversals of m and l. We fix an orientation by choosing the standard orientation on S^3, this induces an orientation on $E(K)$. Then we orient m and l, such that the triple $\langle m, l, n\rangle$ is positively oriented, where n is a normal vector to $\partial E(K)$ pointing inside $E(K)$.

Now we have that any simple closed curve c on ∂K is isotopic to a curve of the form $pm+ql$. Since the orientation of c does not matter for determining h, the pairs (p, q) and $(-p, -q)$ define the same curve c. We will think of the pair (p, q) as the reduced fraction p/q. Then there is a correspondence between the isotopy classes of non-trivial simple closed curves on ∂K and the set of reduced fractions p/q. We need to add in the fraction $1/0 = \infty$, which gives us the curve m. We call p/q surgery a **rational surgery**. If $q = \pm 1$ we call p/q **integral surgery**.

Example 1.3.1. Let k be the unknot in S^3. (The unknot is equivalent to the circle.) Performing $1/0$ surgery on k gives us S^3 again. In fact, performing $1/0$ surgery on any knot in S^3, will give us S^3 again. Now consider $0/1$ surgery on k, this will give us $S^1 \times S^2$.

One key type of manifold is lens spaces. These manifolds are more easily understood via surgery. For $p \geq 2$, the lens space $L(p, q)$ can be obtained by gluing together two solid tori. The gluing homeomorphism sends the meridian of the first torus, μ_1, to the curve $-q\mu_2 + p \cdot \lambda_2$ on the second torus.

Alternatively, we can think of $L(p, q)$ as surgery on the unknot. If you think of the second torus as the trivial knot exterior, we then send the meridian μ_1 to the curve $ql - pm$. Thus $L(p, q)$ is given by $-p/q$ surgery on the unknot in S^3.

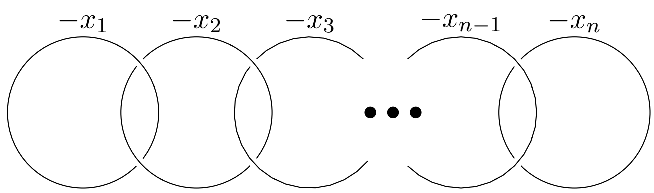


Figure 1.4: Lens space surgery description

Figure 1.5: Plumbing graph for a lens Space

$L(p, q)$ can also be represented as surgery on a link. Let $p/q = [x_1, \ldots, d_n]$ denote a continued fraction decomposition of p/q. Then $L(p, q)$ is surgery on a chain of linked unknots with surgery coefficients x_i. This can be seen in Figure 1.4

We can also represent surgeries as plumbing graphs. These are graphs where each vertex represents an unknot, and two vertices are connected by an edge if those unknots are linked as in Figure 1.4. Thus $L(p, q)$ can be described as the plumbing graph in Figure 1.5

Another important class of manifolds are Seifert fibered manifolds. A **Seifert fibered manifold** is a manifold together with a decomposition into a disjoint union of circles, called fibers, such that each fiber has a tubular neighborhood, that forms a standard fibered torus. A standard fibered torus is the surface bundle of the automorphism of a disk given by rotation by $2\pi b/a$, where $a > 0$, and a and b are relatively prime. The central fiber of a standard fibered torus is called a singular fiber. A Seifert manifold, written $M((a_1, b_1), \ldots, (a_n, b_n))$ has n singular fibers. This manifold has a surgery description shown in Figure 1.6, if the base orbifold is S^2. It can also be described with plumbing graph shown in Figure 1.7, where $a_i/b_i = [x_{i1}, \ldots, x_{im}]$.

A **Siefert fibered integral homology 3-sphere** is a Seifert fibered manifold with the same homolgy as S^3 where we take our coefficients in $\mathbb{Z}$. The surgery description presented in Figure 1.6 yields a Seifert fibered homology sphere if each pair p_i and p_j relatively prime and we chose the q_i's such that

$$\sum \frac{q_i}{p_i} = \frac{1}{p_1 \cdots p_\ell}.$$

1.4 Heegaard Floer Homology

Heegaard Floer homology is an invariant of closed 3 manifolds, introduced by Ozsváth and Szabó. This invariant is isomorphic to Seiberg-Witten Floer Homology [25]. Heegaard Floer homology assigns an $\mathbb{F}[U]$ module to each Spin^c-structure of a 3-manifold. We will take $\mathbb{F} = \mathbb{Z}_2$ for the remainder of this dissertation.

In order to construct Heegaard Floer homology we must first introduce some preliminaries. A genus g **handlebody** is an orientable 3-manifold given by the union of the 3-ball, B^3, with g

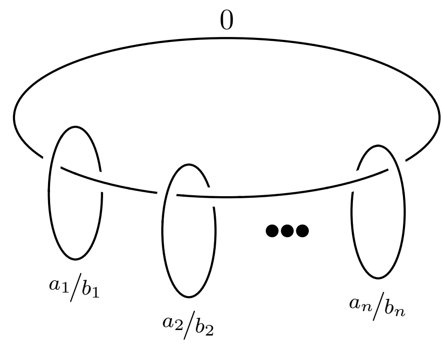


Figure 1.6: Surgery description of a Seifert fibered manifold.

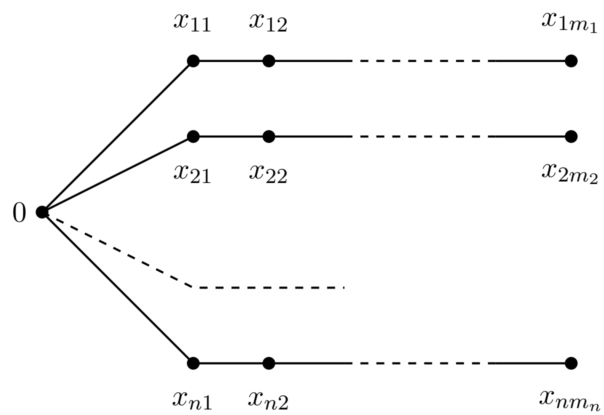


Figure 1.7: Plumbing graph for Seifert fibered manifolds

many 1-handles, $D^2 \times [-1, 1]$. These one handles are solid cyllinders, and are glued to B^3 via a map that sends the $2g$ ends of the g many cyllinders, to $2g$ disjoint disks in ∂B^3. The resulting manifold is orientable. The boundary of a genus g handlebody is homeomorphic to a Riemann surface of genus g.

Any closed orientable 3-manifold can be obtained by gluing two handlebodies together. These handlebodies must have the same genus, g. We call the decomposition of the manifold M into genus g handlebodies the genus g **Heegaard splitting**, or Heegaard decomposition, of M. It is easy to convince yourself of the existence of these Heegaard splittings by considering a triangulation of M. The union of the vertices and edges gives some graph Y. Thickening this graph gives us a handlebody U_0. It is clear that $M - U_0$ is also a handlebody, thus we have a Heegaard splitting of M.

It should be clear that a manifold M has many different Heegaard splittings, thus we want a way to get between different splittings of M. If we have a genus g splitting of M, denoted $M = U_0 \cup_\Sigma U_1$, we can get a genus $g + 1$ splitting of M. This is constructed by taking two points in Σ and connecting them by some unknotted arc γ in U_1. Then U_0' is U_0 and a tubular neighborhood of γ, called N. Similarly $U_1' = U_1 - N$. We have a new decomposition,

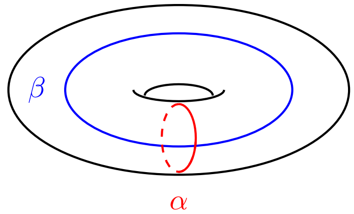


Figure 1.8: Heegaard Diagram for S^3.

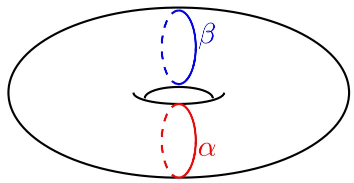


Figure 1.9: Heegaard diagram for $S^1 \times S^2$.

$M = U_0' \cup_{\Sigma'} U_1'$, which is called the **stabilization** of $M = U_0 \cup_\Sigma U_1$. A result by Singer [29] gives us that any two Heegaard decompositions of M can be connected by a series of stabilizations, or destabilizations, up to diffeomorphism.

For a manifold M with genus g Heegaard splitting (Σ_g, U_0, U_1), a compatible **Heegaard diagram** is given by Σ_g with a collection of curves $\alpha = (\alpha_1, \dots, \alpha_g)$ and $\beta = \beta_1, \dots, \beta_g$. We require that α is a set of attaching circles for U_0, and β is a set of attaching circles for U_1. A set of attaching circles, $(\gamma_1, \dots, \gamma_g)$ for U is a collection of closed embedded curves in Σ_g such that the γ_i's are all disjoint, $\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected, the γ_i's bound disjoint embedded disks in U [25].

Example 1.4.1. The Heegaard diagrams for $1/0 = \infty$ and $0/1$ surgery on the unknot in S^3 can be seen in Figures 1.8 and 1.9 respectively.

Since one Heegaard splitting can emit many compatible Heegaard diagrams, we need a way to move between various diagrams. This comes in the form of the Heegaard moves: **isotopy**, **handle slide** and **stabilization**. These moves on Heegaard diagrams do not change the underlying three manifold. Consider a set of attaching circles, $(\gamma_1, \dots, \gamma_g)$, for a handlebody U. An isotopy moves the attaching circles in a one parameter family such that the curves remain disjoint. A handle slide takes two curves, say γ_1 and γ_2, and replaces γ_1 with γ_1' where γ_1', γ_1 and γ_2 bound an embedded pair of pants in $\Sigma - \gamma_3 - \cdots - \gamma_g$.

Stabilization comes from making $\Sigma' = \Sigma \# E$, where E is a genus 1 surface. Then we replace $(\alpha_1, \dots, \alpha_g)$ with $(\alpha_1, \dots, \alpha_g, \alpha_{g+1})$, and similarly with the β_i's. The curves α_{g+1} and β_{g+1} are supported in E and meet transversally at a single point.

Now we must introduce the concept of **pointed Heegaard diagrams**, $(\Sigma_g, \alpha, \beta, z)$. Here $z \in \Sigma_g$ is our basepoint, and must be disjoint from the α and β curves. There are pointed Heegaard moves defined similarly to those above. We now allow for isotopy of the basepoint

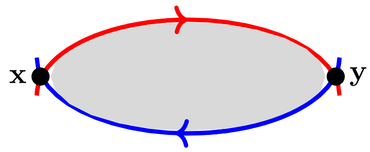


Figure 1.10: A whitney disk from x to y

and require that z is disjoint from the curves we isotopy. For a handle slide we require that z is not in the pair of pants region defined in the handle slide.

Our generators for Heegaard Floer homology live in the ambient space $\text{Sym}^g(\Sigma_g)$, so let us explore those specifics further. Inside $\text{Sym}^g(\Sigma_g)$, the attaching circles for our Heegaard diagram induce a pair of smoothly embedded, g-dimensional tori,

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g, \text{ and } \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g.$$

A Whitney disk is a way to get a relationship between points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and let e_1 and e_2 be the arcs in $\partial\mathbb{D}$ with $\text{Re}(z) \geq 0$, and $\text{Re}(z) \leq 0$ respectively. Given a pair of intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a **Whitney disk** connecting $\mathbf{x}$ and $\mathbf{y}$ is a continuous map

$$u\colon \mathbb{D} \to \text{Sym}^g(\Sigma_g).$$

We require $u(-i) = \mathbf{x}$, $u(i) = \mathbf{y}$, $u(e_1) \subset \mathbb{T}_\alpha$, and $u(e_2) \subset \mathbb{T}_\beta$. Let $\pi_2(\mathbf{x}, \mathbf{y})$ be the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ and $\mathbf{y}$. We also have a multiplicative structure, which we can think of as a way to glue disks from $\mathbf{x}$ to $\mathbf{y}$ and from $\mathbf{y}$ to $\mathbf{z}$ to get a disk from $\mathbf{x}$ to $\mathbf{z}$. Thus the multiplication is as follows:

$$\star\colon \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{z}) \to \pi_2(\mathbf{x}, \mathbf{z}).$$

Since it is hard to picture disks in $\text{Sym}^g(\Sigma_g)$, it is helpful to think of their "shadow" in Σ_g. For any two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and some point $w \in \Sigma$ in the complement of the α and β curves, let n_w denote the algebraic intersection number

$$n_w(\phi) = \#\phi^{-1}(\{w\} \times \text{Sym}^{g-1}(\Sigma_g)),$$

for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Now we are ready to define domains, the shadow of disks in Σ_g. Let $D_1, \ldots D_m$ denote the closures of the components of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. Then for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ the domain associated to ϕ is given by:

$$\mathcal{D}(\phi) = \sum_{i=1}^{m} n_{z_i}(\phi) D_i,$$

where z_i are points in the interior of D_i.

We must introduce the concept of **Spin^c-structures**. We know that every closed oriented

3-manifold Y admits a nowhere vanishing vector field. Two nowhere vanishing vector fields, v_1 and v_2, on Y are homologous if there is a ball B in Y such that $v_1|_{Y-B}$ is homotopic to $v_2|_{Y-B}$. Then we define Spin^c structures as equivalence classes of nowhere vanishing vector fields modulo this equivalence relation. As stated earlier, Heegaard Floer homology assigns an $\mathbb{F}[U]$-module to each Spin^c-structure of a manifold. Seifert fibered homology spheres only have one Spin^c-structure, and thus this concept will only become important in discussions in Chapter 4.

We know that a suitable complex structure on Σ induces a complex structure on $\mathrm{Sym}^g(\Sigma_g)$. For a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ let $\mathcal{M}(\phi)$ denote the space of holomorphic representatives of ϕ. The space $\mathcal{M}(\phi)$ admits an $\mathbb{R}$ action that corresponds to complex automorphisms of the unit disk that preserve i and $-i$. We can divide $\mathcal{M}(\phi)$ by this $\mathbb{R}$ action and define the unparametrized moduli space

$$\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\phi)}{\mathbb{R}}.$$

The expected dimension of the moduli space $\mathcal{M}(\phi)$ is called the **Maslov index** and is denoted $\mu(\phi)$. The Maslov index is additive, $\mu(\phi_1 \star \phi_2) = \mu(\phi_1) + \mu(\phi_2)$, and for the homotopy class of the constant map $\mu = 0$.

We are finally ready to define the Floer chain complexes. Let Y be a rational homology 3-sphere with compatible pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$. Choose a Spin^c structure $t \in \mathrm{Spin}^c(Y)$. Let $\widehat{\mathrm{CF}}(\alpha, \beta, t)$ denote the free Abelian group generated by $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in Spin^c structure t. Then we can define the boundary map

$$\partial \colon \widehat{CF}(\alpha, \beta, t) \to \widehat{CF}(\alpha, \beta, t),$$

as

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi)=1 | n_z(\phi)=0} c(\phi) \cdot \mathbf{y}.$$

We define $c(\phi)$ to be the signed number of points in $\widehat{\mathcal{M}}$ if $\mu(\phi) = 1$. The Heegaard Floer homology groups $\widehat{HF}(Y, t)$ are the homology groups of $(\widehat{CF}(\alpha, \beta, t), \partial)$.

Let $CF^\infty(\alpha, \beta, t)$ be the free abelian group generated by pairs $[\mathbf{x}, i]$, where $\mathbf{x}$ is in $t \in \mathrm{Spin}^c$, and $i \in \mathbb{Z}$. We define the boundary map as follows:

$$\partial[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} c(\phi) \cdot [\mathbf{y}, i - n_z(\phi)].$$

Additionally, we have an isomorphism U on $CF^\infty(\alpha, \beta, t)$ given by $U([\mathbf{x}, i]) = [\mathbf{x}, i-1]$, that decreases the grading by 2. We can define $CF^-(\alpha, \beta, t)$, as the subgroup of $CF^\infty(\alpha, \beta, t)$ generated by the pairs $[\mathbf{x}, i]$ where $i < 0$. Then

$$CF^+(\alpha, \beta, t) = {}^{CF^\infty(\alpha,\beta,t)}/_{CF^-(\alpha,\beta,t)}.$$

Then the Floer homology groups $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$ are the homology groups of $(CF^+(\alpha,\beta,t),\partial)$, $(CF^-(\alpha,\beta,t),\partial)$ and $(CF^\infty(\alpha,\beta,t),\partial)$ respectively. We also define the primary object of this thesis,

$$HF_{red} = {}^{HF^+}/_{U^d HF^+}, \text{ for } d >> 0.$$

The Heegaard Floer chain complexes are equivalent under Heegaard moves. Thus these chain complexes, and therefore the Heegaard Floer homologies, are invariants for 3-manifolds.

In practice, computing Heegaard Floer homology is not too difficult to compute. To see this process we will compute $HF^\infty(S^3)$ using two different Heegaard diagrams in Example 1.4.2.

Example 1.4.2. Figure 1.8 is a Heegaard diagram for S^3. We know Heegaard Floer homology is generated by the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Since we only have one such intersection point, called x, we see that $HF^+(S^3) = \mathbb{Z}_2[U^{-1}]\langle x\rangle$. Alternatively, we could use the Heegaard diagram for S^3 pictured in Figure 1.11. Here we have three intersection points, x, y and z. Thus our generators are $[x,i]$, $[y,i]$ and $[z,i]$. There is a disk from y to x that crosses the basepoint w and there is a disk from y to z, which gives us a differential. Thus we have the following differentials:

$$\partial[x,i] = 0, \quad \partial[y,i] = [z,i] + [x,i-1], \quad \partial[z,i] = 0.$$

Thus we have that $HF^+(S^3) = \mathbb{Z}_2[U^{-1}]$ as expected.

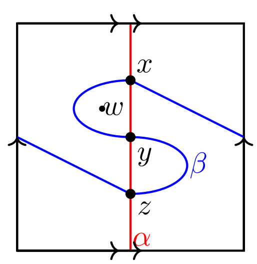


Figure 1.11: Alternate Heegaard diagram for S^3.

1.4.1 Knot Floer Homology

There is a variant of Heegaard Floer homology specifically for knots in 3-manifolds, called knot Floer homology, [25, 27]. We will specifically be discussing knots in S^3. We need a variant of Heegaard diagrams for knots. For knots $K \subset S^3$ this will be a Heegaard diagram for S^3, $(\Sigma_g, \alpha, \beta, w, z)$ with two basepoints, w and z. We then connect w and z by a curve a in $\Sigma_g - \alpha_1 - \cdots - \alpha_g$, and by a curve b in $\Sigma_g - \beta_1 - \cdots - \beta_g$. By pushing a into our first handlebody, and b into our second, we obtain a knot $K \subset S^3$. We say that this diagram is a **doubly-pointed Heegaard diagram compatible with** K.

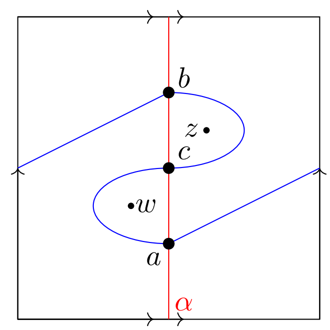


Figure 1.12: Doubly pointed Heegaard diagram for the left handed trefoil.

The chain complex to compute one flavor of knot Floer homology is CFK^∞, also referred to as the **full knot complex**. This is a doubly-filtered chain complex. We think of CFK^∞ as freely generated over $\mathbb{Z}$ by triples $[\mathbf{x}, i, j]$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i \in \mathbb{Z}$, and $j \in \mathbb{Z}$ is given by $A(x) = j - i$, where $A(x)$ is the Alexander grading of x. The triple $[\mathbf{x}, i, j]$ corresponds to the generator $U^{-i}\mathbf{x}$, as the U-action on $[\mathbf{x}, i, j]$ yields $[\mathbf{x}, i-1, j-1]$ or $U^{-i+1}\mathbf{x}$.

We often represent CFK^∞ graphically in the xy - plane. Here we represent $[\mathbf{x}, i, j]$ as a dot at (i, j). We draw our differentials in as arrows between these generators. If a disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ gives us a differential, the change in x-coordinate is $-n_w(\phi)$, and the change in vertical coordinate is $-n_z(\phi)$.

Example 1.4.3. Consider Figure 1.12 as our doubly pointed Heegaard diagram for the left handed trefoil. We know that $[a, i, j]$, $[b, i, j]$ and $[c, i, j]$ are the generators. We have the following differentials,

$$\partial[a, i, j] = [b, i, j-1], \quad \partial[c, i, j] = [b, i-1, j].$$

Thus the full knot floer complex can be drawn as in Figure 1.13.

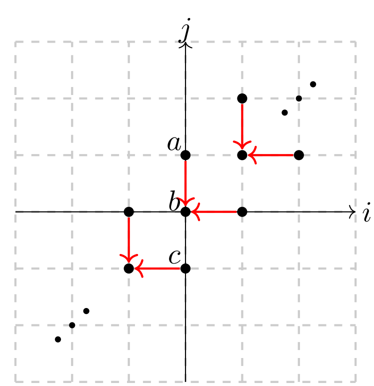


Figure 1.13: Full knot Floer complex for the left-handed trefoil.

From CFK^∞ we can define several auxilary complexes. The subcomplex of CFK^∞ where

$i = 0$ is exactly $\widehat{CF}(S^3)$ where the knot filtration is given by the j-coordinate. The chain complex CFK^- corresponds to the subcomplex of CFK^∞ where $i \leq 0$.

Structure This dissertation is laid out as follows. In Chapter 2 we layout the necessary background and construction for graded roots. We prove a preliminary Lemma 2.2.1, and then use this to prove Lemma 1.2.2. In Chapter 3 we prove Theorem 1.2.4. This is first done by using a corollary to Lemma 1.2.2 to distinguish between finite and infinite cases to prove. Then the three infinite cases are each proven separately. In Chapter 4 we introduce the mapping cone for computing Heegaard Floer homology of surgery on a knot. We then improve on Gainullin's work in [8] for genus 1 and genus 2 knots.

Chapter 2

Graded Roots

2.1 Introduction

In this section we introduce Némethi's graded root [21] and setup the preliminary results necessary for our obstructions. Originally this object was constructed from the plumbing graph for a plumbed 3-manifold. We will predominantly be following the layout of Can and Karakurt [4].

2.1.1 Setup

First, a graded root is defined as follows:

Definition 2.1.1. (Némethi, [21, Definition 3.1.2]) Let R be an infinite tree with vertices V and edges E. We denote by $[u, v]$ the edge with end-points u and v. We say that R is a **graded root** with grading $\chi\colon V \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in E$,

(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v], [u, w] \in E$, and $v \neq w$,

(c) χ is bounded below, and $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$ and $|\chi^{-1}(k)| = 1$ for k sufficiently large.

Now we can give the basics for the construction of a graded root for a given Seifert fibered homology sphere, $Y = \Sigma(p_1, \ldots, p_\ell)$, where $p_1 < p_2 < \cdots < p_\ell$. We define a function $\Delta\colon \mathbb{N} \to \mathbb{Z}$ by

$$\Delta(n) = 1 + |e_0|n - \sum_{i=1}^{l} \left\lceil \frac{np_i'}{p_i} \right\rceil, \tag{2.1}$$

where $(e_0, p_1', \ldots, p_\ell')$ is the unique solution to

$$e_0 p_1 \cdots p_\ell + p_1' p_2 \cdots p_\ell + p_1 p_2' \cdots p_\ell + \cdots + p_1 p_2 \cdots p_\ell' = -1,$$

and

$$0 < p_i' \leq p_i - 1 \text{ for } i = 1, 2, \ldots, \ell.$$

Note the above equation requires $e_0 < 0$. By [4, Theorem 4.1, (2)] we have that $\Delta(n)$ is always positive for $n > N_0$ and

$$N_0 = p_1 p_2 \cdots p_\ell \left((ell - 2) - \sum_{i=1}^{l} \frac{1}{p_i} \right).$$

The general definition follows.

Definition 2.1.2. (Karakurt-Lidman, [14, Definition 3.1]) A **Δ-sequence** is a pair (X, δ) where X is a well ordered finite set, and $\delta\colon X \to \mathbb{Z} \setminus \{0\}$ with $\delta(x_0) > 0$ where x_0 is the minimum of X.

For our purposes, we will have the Δ-sequence (X, Δ), where $\Delta(n)$ is given by (2.1) and $X = \{x \in \mathbb{N} \mid \Delta(x) \neq 0 \text{ and } x \leq N_0\}$. We also have that Δ will be symmetric, as $\Delta(n) = -\Delta(N_0 - n)$ [4, Theorem 4.1, (2)]. For the remainder of this dissertation when we refer to a Δ-sequence we will be referring to the sequence $(\Delta(x_0), \Delta(x_1), \ldots \Delta(x_k))$, where x_k is the last integer that satisfies $\Delta(x_k) < 0$, as we will only be using the function defined in (2.1).

Once we have our Δ-sequence, we can define a τ function $\tau_\Delta\colon \{0, 1, \ldots, k\} \to \mathbb{Z}$ using the recurrence relation $\tau_\Delta(n+1) - \tau_\Delta(n) = \Delta(x_n)$ with the initial condition $\tau_\Delta(0) = 0$. Let $\{\tau_\Delta(0), \ldots, \tau_\Delta(k)\}$, be called the τ-sequence.

Now that we know how to get a τ-sequence for a Seifert fibered homology sphere, we can use this to construct our graded root, following [21, Example 3.1.3]. For every $n \in \mathbb{Z}_{\geq 0}$, let R_n be the infinite graph with vertex set $\mathbb{Z} \cap [\tau(n), \infty)$ and edge set $\{[k, k+1]\colon k \in \mathbb{Z} \cap [\tau(n), \infty)\}$. We then identify all common vertices and edges of R_n and R_{n+1}. This gets us an infinite tree Γ_τ, and we assign a function $\chi(v)$ that gives the unique integer corresponding to the grading of the vertex v.

Example 2.1.3. The construction of Γ_τ for the τ-sequence $\{0, 1, 0, -1, -2, -3, -4, -3\}$, is seen in Figure 2.1.

2.1.2 Previous Work

Now that we can construct our graded root, we will introduce some properties and terminology. To any graded root Γ, we associate a $\mathbb{Z}$-graded, $\mathbb{F}[U]$-module. We let $\mathbb{H}^+(\Gamma)$ be the free $\mathbb{F}$ vector space generated by the vertex set of Γ, and the degree of the generator corresponding to the vertex, v has degree $2\chi(v)$. The U-action on $\mathbb{H}^+(\Gamma)$ is a degree -2 endomorphism that sends each vertex v to the sum of all vertices w connected to v by an edge where $\chi(w) < \chi(x)$. If no vertices w satisfy this, then U sends v to zero. We can now define two finitely generated $\mathbb{F}[U]$-modules:

$$\mathbb{H}_{\text{red}}(\Gamma) = \text{Coker}(U^n), \text{ for large } n,$$

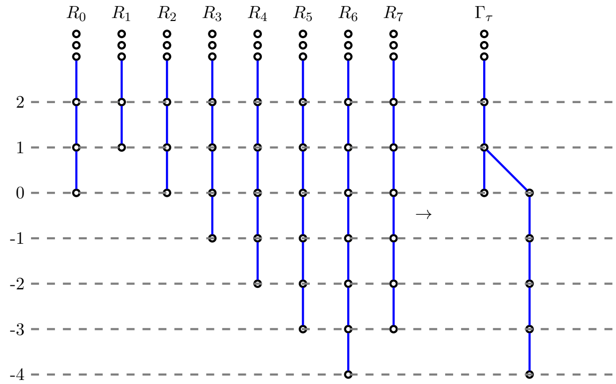


Figure 2.1: Construction of Γ_τ

$$\widehat{\mathbb{H}}(\Gamma) = \text{Ker}(U) \oplus \text{Coker}(U)[-1].$$

We will also refer to the image of U^n for large n as the tower, denoted $\mathcal{T}^+$, which will eventually stabilize and become constant. We can then view $\mathbb{H}_{\text{red}}(\Gamma)$, which is the primary focus of this dissertation, as ${}^{\mathbb{H}^+(\Gamma)}/_{\mathcal{T}^+}$.

These groups encode Heegaard Floer homology as follows:

Theorem 2.1.4. *(Némethi, [20, Section 11.3]) Let Y be a positively oriented integral homology sphere, then we have the following isomorphisms of $\mathbb{F}[U]$-modules up to an overall degree shift:*

(1) $HF^+(-Y) \cong \mathbb{H}^+(\Gamma)$,

(2) $HF_{red}(-Y) \cong \mathbb{H}_{red}(\Gamma)$,

(3) $\widehat{HF}(-Y) \cong \widehat{\mathbb{H}}(\Gamma)$.

We must introduce the concept of refinements and merges before proceeding. In general, refinements and merges allow us to get between different Δ-sequences that produce the same graded roots. This will become particularly important in our discussion of 5 singular fibers in Chapter 3.

Definition 2.1.5. (Karakurt-Lidman, [14, Section 3]) Let (X, Δ) be a Δ-sequence. Let t be a positive integer and $x \in X$ with $|\Delta(x)| \geq t$. From this we construct a new Δ-sequence (X', Δ') as follows. The set X' is obtained by removing x from X and putting t consecutive elements $x_1, \ldots, x_t$ in its place. Now, choose nonzero integers $n_1, \ldots, n_t$ each with the same sign as $\Delta(x)$ such that $n_1 + \cdots + n_t = \Delta(x)$. The new Δ' agrees with Δ on $X \setminus \{x\}$ and satisfies $\Delta'(x_i) = n_i$ for $i = 1, \ldots, t$. (X', Δ') is called a **refinement** of (X, Δ) at x. Conversely (X, Δ) is called a **merge** of (X', Δ')

Proposition 2.1.6. *(Karakurt-Lidman, [14, Proposition 3.6]) Refinements and merges do not change $\mathbb{H}^+(\Gamma)$ or $\mathbb{H}_{red}(\Gamma)$.*

2.2 Results

Now we will prove the necessary results about graded roots, which will later be used in our surgery obstructions.

Lemma 2.2.1. *If a Δ-sequence for a graded root Γ contains x such that $\Delta(x) = k+1$ and $x' > x$ such that $\Delta(x') = -(k+1)$ then*

$$U^k \cdot \mathbb{H}_{red}(\Gamma) \neq 0.$$

Proof. Since $\Delta(x) = k+1$ we have

$$\tau_\Delta(i+1) - \tau_\Delta(i) = k+1$$

where x is in the i^{th} position in our Δ-sequence. Similarly since $\Delta(x') = -(k+1)$ we have

$$\tau_\Delta(j) - \tau_\Delta(j+1) = k+1$$

where x' is in the j^{th} position in our Δ-sequence. Since we assumed $x' > x$, we know that $j > i$. It is also clear that

$$\tau_\Delta(i+1) > \tau_\Delta(i) \quad \text{and} \quad \tau_\Delta(j) > \tau_\Delta(j+1).$$

We now have two cases. First suppose $\tau_\Delta(i) \geq \tau_\Delta(j+1)$. This is shown in Figure 2.2. We have that

$$U^k \cdot a = b$$

in this case, and since a and c are both at grading $n+k$, a is not in the image of U^z for large z. Thus $a \in \mathbb{H}_{red}(\Gamma)$ and $U^k \cdot a \neq 0$ so

$$U^k \cdot \mathbb{H}_{red}(\Gamma) \neq 0.$$

Now consider when $\tau_\Delta(i) > \tau_\Delta(j+1)$, which is shown in Figure 2.3. We can proceed exactly as above to find that

$$U^k \cdot \mathbb{H}_{red}(\Gamma) \neq 0.$$

Thus our proof is complete.

□

The $\Delta(x) = k+1$ condition in Lemma 2.2.1 is equivalent to requiring a sequence of values $(x_1, \ldots, x_j)$, that give us $k+1$ many consecutive $+1$'s in our Δ-sequence after the appropriate refinements and merges. Similarly, $\Delta(x') = -(k+1)$ gives us $k+1$ consecutive -1's. Thus after

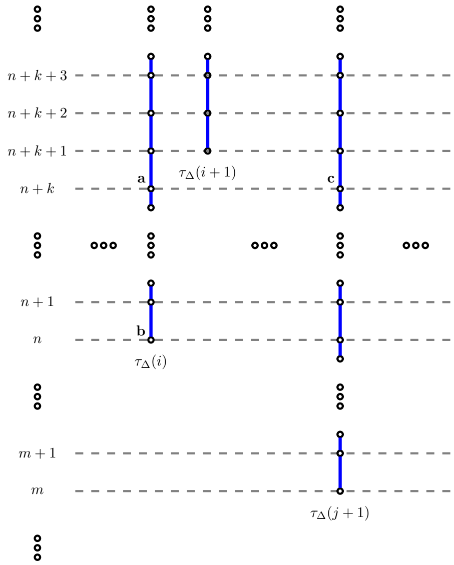


Figure 2.2: $\tau_\Delta(i) > \tau_\Delta(j+1)$

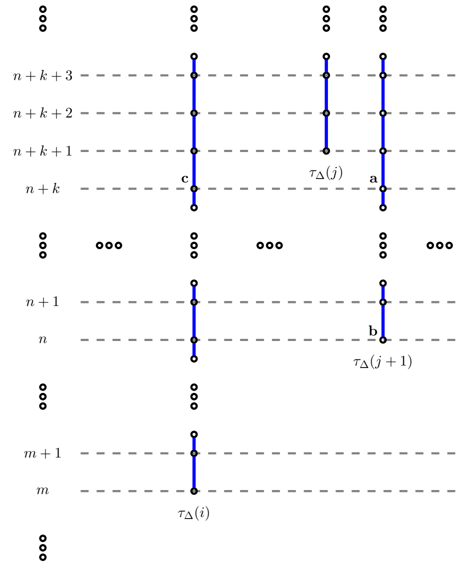


Figure 2.3: $\tau_\Delta(i) < \tau_\Delta(j+1)$

all appropriate refinements and merges we have

$$\{\dots, k+1, \dots, -(k+1), \dots\}$$

in our Δ-sequence. There is no need to prove this separately as we know refinements and merges do not change $\mathbb{H}_{red}$.

The next Lemma follows easily.

Lemma 1.2.2. *Let $Y = \Sigma(p_1, p_2, \dots, p_\ell)$ be a Seifert fibered integral homology sphere with ℓ singular fibers. Then if*

$$0 \leq k < \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right),$$

$$U^k \cdot HF_{red}(Y) \neq 0.$$

Proof. Consider $n = kp_1p_2\cdots p_\ell$. We have that

$$\begin{aligned}
\Delta(n) &= 1 + |e_0|n - \sum_{i=1}^{l} \frac{np_i'}{p_i} \\
&= 1 + |e_0|n - k \cdot \left(p_1'p_2\cdots p_\ell + p_1p_2'\cdots p_\ell + \cdot + p_1p_2\cdots p_\ell'\right) \\
&= 1 + |e_0|n - k\left(-1 - e_0p_1\cdots p_\ell\right) \\
&= 1 + k + |e_0|kp_1\cdots p_\ell + ke_0p_1\cdots p_\ell \\
&= k+1,
\end{aligned}$$

using substitution from the Diophantine equation. By using that $\Delta(x) = -\Delta(N_0 - x)$ we have that

$$\Delta(N_0 - n) = -(k+1).$$

Now using our assumptions on k we have the following,

$$\begin{aligned} n &= kp_1 \cdots p_\ell \\ n &< \frac{1}{2}\left(\ell - 2 - \sum_{i=1}^{l} \frac{1}{p_i}\right) p_1 \cdots p_\ell \\ n &< \frac{1}{2} N_0 \\ 2n &< N_0 \\ n &< N_0 - n. \end{aligned}$$

Thus $\Delta(n) = k + 1$, $\Delta(N_0 - n) = -(k + 1)$ and $N_0 - n > n$ and so the conditions of Lemma 2.2.1 are satisfied and $U^k \cdot HF_{red}(Y) \neq 0$. □

Chapter 3

Five Singular Fibers

3.1 Introduction

The goal of this section is to prove Theorem 1.2.4. Our strategy is to find values such that the conditions of Lemma 2.2.1 are met. First, this will involve finding exact values $x' > x$ such that $\Delta(x) = 2$ and $\Delta(x') = -2$. But this process will not prove all of the necessary cases, so we will then have to find a sequence of values $(x, x+1, x+2, \ldots, x+j)$, where $x + j < N_0/2$ such that

$$(\Delta(x), \Delta(x+1), \ldots, \Delta(x+j-1), \Delta(x+j)) = (1, 0, \ldots, 0, 1).$$

Then by (2.2.1) we have that

$$(\Delta(N_0 - x - j), \Delta(N_0 - x - j + 1), \ldots, \Delta(N_0 - x - 1), \Delta(N_0 - x)) = (-1, 0, \ldots, 0, -1).$$

By performing the appropriate merges we have a Δ-sequence as follows

$$\{\ldots, 2, \ldots, -2\}.$$

Then the proof of Theorem 1.2.4 follows from Lemma 2.2.1.

First consider the following corollary to Lemma 1.2.2.

Corollary 3.1.1. *Let* $Y = \Sigma(p_1, p_2, p_3, p_4, p_5)$. *Then*

$$U \cdot HF_{red}(Y) \neq 0,$$

when one of the following holds,

1. $p_1 \geq 4$,
2. $p_1 = 3$ *and* $p_5 \geq 17$,
3. $p_1 = 2$, $p_2 = 3$, *and* $p_3 \geq 17$,

4. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$ *and* $p_4 \geq 83$,

5. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $p_4 = 43$, $p_5 \geq 1811$,

6. $p_1 = 2$, $p_2 = 3$, $p_3 = 11$, $p_4 \geq 15$ *and* $p_5 \geq 101$.

Proof. This simply involves testing that $p_1 p_2 \cdots p_5 \leq {}^{N_0}/2$ in the given cases. Then as

$$\Delta(p_1 p_2 \cdots p_5) = 2$$

and

$$\Delta(N_0 - p_1 p_2 \cdots p_5) = -2$$

the proposition follows easily. □

This does not prove Theorem 1.2.4 entirely as not all cases are covered by Corollary 3.1.1. We are left with three infinite cases: $\Sigma(2,3,5,p,q)$, $\Sigma(2,3,7,p,q)$, $\Sigma(2,3,11,13,p)$, and series of finite cases. The finite cases can be easily checked to satisfy $U \cdot HF_{red}(Y) \neq 0$. Thus to prove Theorem 1.2.4 we only need to address the following infinite cases: $\Sigma(2,3,5,p,q)$, $\Sigma(2,3,7,p,q)$, $\Sigma(2,3,11,13,p)$.

We will address each of these infinite cases separately in the following lemmas, proved later in this section.

Lemma 3.1.2. *Let* $Y = \Sigma(2,3,5,p,q)$, *for* $p \neq 7$. *Then* $U \cdot HF_{red}(Y) \neq 0$.

Lemma 3.1.3. *Let* $Y = \Sigma(2,3,5,7,q)$. *Then* $U \cdot HF_{red}(Y) \neq 0$.

Lemma 3.1.4. *Let* $Y = \Sigma(2,3,7,p,q)$. *Then* $U \cdot HF_{red}(Y) \neq 0$.

Lemma 3.1.5. *Let* $Y = \Sigma(2,3,11,13,q)$. *Then* $U \cdot HF_{red}(Y) \neq 0$.

With these lemmas, we are ready to prove Theorem 1.2.4.

Theorem 1.2.4. *Let* $Y = \Sigma(p_1, p_2, p_3, p_4, p_5)$. *Then*

$$U \cdot HF_{red}(Y) \neq 0.$$

Proof. This follows immediately from Corollary 3.1.1 and Lemmas 3.1.2, 3.1.3, 3.1.4 and 3.4.1. □

3.2 $\Sigma(2,3,5,p,q)$

3.2.1 Preliminaries

Before we proceed with our proof of Lemmas 3.1.2 and 3.1.3 we must first complete some algebraic preliminaries. Consider the Seifert fibered homology sphere

$$Y = \Sigma(2,3,5,p,q),$$

with Diophantine solutions

$$D = (e_0, 1, a, b, p', q').$$

Then the Diophantine equation simplifies as follows

$$-1 = 30pqe_0 + 15pq + 10apq + 6bpq + 30p'q + 30pq'.$$

Reducing this mod 30 we get the following,

$$\begin{aligned} 29 &\equiv 15pq + 10apq + 6bpq \mod 30 \\ 29 &\equiv pq(15 + 10a + 6b) \mod 30 \end{aligned}$$

By the requirements of the Diophantine equation we know that $a \in \{1,2\}$ and $b \in \{1,2,3,4\}$. Using this and reducing $pq \mod 30$ we can solve for a and b. That gives us the following,

$$\begin{aligned} pq &\equiv 1 \mod 30 \implies D = (e_0,1,2,4,p',q'), & pq &\equiv 17 \mod 30 \implies D = (e_0,1,1,2,p',q'), \\ pq &\equiv 7 \mod 30 \implies D = (e_0,1,2,2,p',q'), & pq &\equiv 19 \mod 30 \implies D = (e_0,1,2,1,p',q'), \\ pq &\equiv 11 \mod 30 \implies D = (e_0,1,1,4,p',q'), & pq &\equiv 23 \mod 30 \implies D = (e_0,1,1,3,p',q'), \\ pq &\equiv 13 \mod 30 \implies D = (e_0,1,2,3,p',q'), & pq &\equiv 29 \mod 30 \implies D = (e_0,1,1,1,p',q'). \end{aligned}$$

Note in all the above cases, $apq \equiv 2 \mod 3$ and $bpq \equiv 4 \mod 5$.

We can also come up with a bound on e_0 using the Diophantine equation:

$$\begin{aligned} -1 &= 30pqe_0 + 15pq + 10apq + 6bpq + 30p'q + 30pq' \\ -30pqe_0 &= 15pq + 10apq + 6bpq + 30p'q + 30pq' + 1 \\ |e_0| &= \frac{15pq + 10apq + 6bpq + 30p'q + 30pq' + 1}{30pq} \\ |e_0| &= \frac{1}{2} + \frac{a}{3} + \frac{b}{5} + \frac{p'}{p} + \frac{q'}{q} + \frac{1}{30pq} \\ |e_0| &< \frac{1}{2} + \frac{2}{3} + \frac{4}{5} + \frac{p}{p} + \frac{q}{q} + \frac{1}{30 \cdot 7 \cdot 11} = \frac{4582}{1155} \approx 3.96. \end{aligned}$$

Therefore $e_0 = -1, -2, -3$.

As stated above, the goal is to find a consecutive sequence of values

$$(\Delta(x), \Delta(x+1), \ldots, \Delta(x+j-1), \Delta(x+j)) = (1, 0, \ldots, 0, 1), \text{ such that } x + j < N_0/2.$$

In the majority of cases for $\Sigma(2,3,5,p,q)$ this will be a pair of elements of the form $xpq, xpq \pm 1$. With this in mind, consider the following simplification of $\Delta(xpq+k)$, for $x \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$.

$$\begin{aligned}
\Delta(xpq+k) = 1 &+ |e_0|(xpq+k) - \left\lceil \frac{xpq+k}{2} \right\rceil - \left\lceil \frac{a(xpq+k)}{3} \right\rceil - \left\lceil \frac{b(xpq+k)}{5} \right\rceil \\
&- \left\lceil \frac{p'(xpq+k)}{p} \right\rceil - \left\lceil \frac{q'(xpq+k)}{q} \right\rceil \\
&= 1 + |e_0|xpq + k|e_0| - \left\lceil \frac{xpq+k}{2} \right\rceil - \left\lceil \frac{a(xpq+k)}{3} \right\rceil - \left\lceil \frac{b(xpq+k)}{5} \right\rceil \\
&- xp'q - \left\lceil \frac{kp'}{p} \right\rceil - xpq' - \left\lceil \frac{kq'}{q} \right\rceil \\
&= 1 + |e_0|xpq + k|e_0| - \left\lceil \frac{xpq+k}{2} \right\rceil - \left\lceil \frac{a(xpq+k)}{3} \right\rceil - \left\lceil \frac{b(xpq+k)}{5} \right\rceil \\
&- \left\lceil \frac{kp'}{p} \right\rceil - \left\lceil \frac{kq'}{q} \right\rceil - x\left(-pqe_0 - \frac{pq}{2} - \frac{apq}{3} - \frac{bpq}{5} - \frac{1}{30}\right) \\
&= 1 + k|e_0| + \frac{xpq}{2} - \left\lceil \frac{xpq+k}{2} \right\rceil + \frac{axpq}{3} - \left\lceil \frac{a(xpq+k)}{3} \right\rceil + \frac{bxpq}{5} - \left\lceil \frac{b(xpq+k)}{5} \right\rceil \\
&- \left\lceil \frac{kp'}{p} \right\rceil - \left\lceil \frac{kq'}{q} \right\rceil + \frac{x}{30}.
\end{aligned}$$

Using this simplification above we are able to compute $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for the x values that satisfy the majority of our $\Sigma(2,3,5,p,q)$ cases. These values are presented in Table 3.1.

3.2.2 Results

Now we are ready to prove Lemma 3.1.2.

Lemma 3.1.2. *Let $Y = \Sigma(2,3,5,p,q)$, for $p \neq 7$. Then $U \cdot HF_{red}(Y) \neq 0$.*

As stated above, we wish to find values that provide us with 2 consecutive $+1's$ in our Δ-sequence that occur prior to $N_0/2$. Then by Lemma 2.2.1 our proof is complete. The values that satisfy this condition for sufficiently large p and q are found in Tables 3.2, 3.3, 3.4, 3.5 and 3.6. We separate the values in these tables into two cases. Type A are those where the values are of the form $xpq \pm k$. Type B cases are those that follow a different pattern and are color coded green.

Table 3.1: Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$

	$a=1$ $b=1$	$a=1$ $b=2$	$a=1$ $b=3$	$a=1$ $b=4$	$a=2$ $b=1$	$a=2$ $b=2$	$a=2$ $b=3$	$a=2$ $b=4$
$\Delta(25pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$
$\Delta(25pq)$	1	1	1	1	1	1	1	1
$\Delta(25pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$
$\Delta(26pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$
$\Delta(26pq)$	1	1	1	1	1	1	1	1
$\Delta(26pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$
$\Delta(27pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$
$\Delta(27pq)$	1	1	1	1	1	1	1	1
$\Delta(27pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$

Proof. Given the values in Tables 3.2, 3.3, 3.4, 3.5 and 3.6, our proof involves checking the value of Δ of these numbers, and confirming that they occur before $N_0/2$. For Type A cases, the Δ function of these numbers can either be found in Table 3.1 or computed using the formula for $\Delta(xpq+k)$. Type B cases we will address later.

For the Type A cases it remains to show that these values occur before $N_0/2$. Since it is clear that

$$25pq \leq 26pq \leq 27pq \leq 28pq \leq 28pq+6$$

we need only to check that $28pq+6 \leq N_0/2$. It can be easily checked that this holds when

$$p > 10, \text{ and } q \geq \frac{10p+4}{p-10},$$

which are satisfied in the following cases,

- $p = 11, \quad q > 114$
- $p = 13, \quad q > 43$
- $p = 17, \quad q > 24$
- $p = 19, \quad q > 21$
- $p > 20$

The finite cases that do not satisfy these inequalities may be easily checked to satisfy

$$U \cdot HF_{red} \neq 0.$$

Thus it remains to address the following.

1. $pq \equiv 1 \mod 30$, $e_0 = -3$, $1/3 < p'/p < 2/3$ and $0 < q'/q < 1/3$, and where p'/p and q'/q are switched.

Table 3.2: General Cases for $Y = \Sigma(2,3,5,p,q)$, with p and q sufficiently large

	$e_0 = -1$	$e_1 = -2$	$e_0 = -3$
$pq \equiv 1 \mod 30$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	See Table 3.3
$pq \equiv 7 \mod 30$	$26pq - 1$ $26pq$	$25pq - 1$ $25pq$	$27pq$ $27pq + 1$
$pq \equiv 11 \mod 30$	$25pq - 1$ $25pq$	$26pq - 1$ $26pq$	$25pq$ $25pq + 1$
$pq \equiv 13 \mod 30$	$26pq - 1$ $26pq$	$25pq - 1$ $25pq$	See Table 3.4
$pq \equiv 17 \mod 30$	$25pq - 1$ $25pq$	See Table 3.5	$25pq$ $25pq + 1$
$pq \equiv 19 \mod 30$	$26pq - 1$ $26pq$	$25pq - 1$ $25pq$	$26pq + 1$ $26pq + 1$
$pq \equiv 23 \mod 30$	$25pq - 1$ $25pq$	$27pq - 1$ $27pq$	$25pq$ $25pq + 1$
$pq \equiv 29 \mod 30$	$25pq - 1$ $25pq$	See Table 3.6	$25pq$ $25pq + 1$

2. $pq \equiv 13 \mod 30$, $e_0 = -3$, $3/5 < p'/p < 4/5$ and $3/5 < q'/q < 4/5$

3. $pq \equiv 17 \mod 30$, $e_0 = -2$, $1/5 < p'/p \leq 2/5$ and $1/5 < q'/q \leq 2/5$

4. $pq \equiv 29 \mod 30$, $e_0 = -2$, $1/2 < q'/q < 3/5$ and $0 < p'/p < 1/2$ and where p'/p and q'/q are switched.

5. $pq \equiv 29 \mod 30$, $e_0 = -2$, $3/5 < p'/p < 1$ and $0 < q'/q < 1/2$, and where p'/p and q'/q are switched.

Case 1: Assume that $1/3 < p'/p < 2/3$ and $0 < q'/q < 1/3$. First we check that $\Delta(15pq+30pq') = 1$ and $\Delta(15pq + 30pq' + 1) = 1$.

$$\begin{aligned}
\Delta(15pq + 30pq') &= 1 + |e_0|(15pq + 30pq') - \left\lceil \frac{15pq + 30pq'}{2} \right\rceil - \left\lceil \frac{2(15pq + 30pq')}{3} \right\rceil \\
&\quad - \left\lceil \frac{4(15pq + 30pq')}{5} \right\rceil - \left\lceil \frac{p'(15pq + 30pq'}{p} \right\rceil - \left\lceil \frac{q'(15pq + 30pq'}{q} \right\rceil \\
&= 1 + 15pq|e_0| - 30pq'|e_0| - \frac{15pq + 30pq' + 1}{2} - \frac{30pq + 60pq'}{3} \\
&\quad - \frac{60pq + 120pq'}{5} - 15p'q - 30p'q' - 15pq' - \left\lceil \frac{30q'}{q}(pq') \right\rceil \\
&= 1 + 15pq|e_0| + 30pq'|e_0| - \frac{15pq + 30pq' + 1}{2} - \frac{30pq + 60pq'}{3} \\
&\quad - \frac{60pq - 120pq'}{5} - 30p'q' - 15\left(-pqe_0 - \frac{pq}{2} - \frac{2pq}{3} - \frac{4pq}{5} - \frac{1}{30}\right)
\end{aligned}$$

Table 3.3: $pq \equiv 1 \mod 30$, $e_0 = -3$, , for p and q sufficiently large

	$p'/p \in (0, \frac{1}{3})$	$p'/p \in (\frac{1}{3}, \frac{1}{2})$	$p'/p \in (\frac{1}{2}, \frac{2}{3})$	$p'/p \in (\frac{2}{3}, 1)$
$q'/q \in (0, \frac{1}{3})$	$20pq$ $20pq+1$ $20pq+2$	$15pq+30pq'$ $15pq+30pq'+1$	$15pq+30pq'$ $15pq+30pq'+1$	$20pq-3$ $\vdots$ $22pq$
$q'/q \in (\frac{1}{3}, \frac{1}{2})$	$15pq+30p'q$ $15pq+30p'q+1$	$20pq$ $20pq+1$ $20pq+2$	$22pq-3$ $\vdots$ $22pq$	$22pq-3$ $\vdots$ $22pq$
$q'/q \in (\frac{1}{2}, \frac{2}{3})$	$15pq+30p'q$ $15pq+30p'q+1$	$22pq-3$ $\vdots$ $20pq+2$	$22pq-2$ $22pq-1$ $22pq$	$22pq-3$ $\vdots$ $22pq$
$q'/q \in (\frac{2}{3}, 1)$	$22pq-3$ $\vdots$ $22pq$	$22pq-3$ $\vdots$ $22pq$	$22pq-3$ $\vdots$ $22pq$	$22pq-2$ $\vdots$ $22pq$

$$- \left\lceil \frac{30q'}{q} \left(-pqe_0 - \frac{pq}{2} - \frac{2pq}{3} - \frac{4pq}{5} - \frac{1}{30} - p'q \right) \right\rceil$$
$$= 1$$

It can be similarly checked that $\Delta(15pq + 30pq' + 1) = 1$. Now we must check that

$$15pq + 30pq' + 1 < N_0/2.$$

It is easily checked that this is satisfied when

$$q' < \frac{29pq - 30q - 3p - 2}{60p}.$$

Since $q' < q/3$ by assumption, we check when

$$\frac{q}{3} < \frac{29pq - 30q - 3p - 2}{60p},$$

which is satisfied when $q > 6$. Thus $15pq + 30pq' + 1 < N_0/2$ is satisfied for $q > 6$, which is trivially true. The case where $1/3 < q'/q < 2/3$ and $0 < p'/p < 1/3$ can be checked the same way, and so this case is complete.

The remaining cases can be checked using the same process.

Case 2: Let $pq \equiv 13 \mod 30$, $e_0 = -3$, $3/5 < p'/p, q'/q < 4/5$. It is easily checked that

$$\Delta(60pq' - 9pq) = 1 = \Delta(60pq' - 9pq + 3).$$

Table 3.4: $pq \equiv 13 \mod 30$, $e_0 = -3$, for p and q sufficiently large

	$p'/p \in \left(0, \frac{1}{5}\right)$	$p'/p \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$p'/p \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$p'/p \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$p'/p \in \left(\frac{4}{5}, 1\right)$
$q'/q \in \left(0, \frac{1}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$28pq$ $\vdots$ $28pq+6$	$28pq$ $\vdots$ $28pq+6$	$28pq-4$ $\vdots$ $28pq$
$q'/q \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$28pq$ $\vdots$ $28pq+6$	$90pq'-3pq$ $\vdots$ $90pq'-3pq+3$	$28pq-4$ $\vdots$ $28pq$
$q'/q \in \left(\frac{4}{5}, 1\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$28pq-4$ $\vdots$ $28pq$	$28pq-4$ $\vdots$ $28pq$	$28pq-4$ $\vdots$ $28pq$

$$\Delta(60pq' - 9pq + 1) = 0 = \Delta(60pq' - 9pq + 2).$$

Note that if $\left\lceil \frac{3p'}{p} \right\rceil = 3$, then $q > \frac{3}{5}$, by the Diophantine equation. Since we assumed $3/5 < q'/q < 4/5$, we must have $\left\lceil \frac{3p'}{p} \right\rceil = 2$. Also using the Diophantine equation we have that $p'/p > 3/5$, implies $q'/q < 19/30$. We check that

$$60pq' - 9pq + 3 < N_0/2,$$

when

$$q' < \frac{77pq - 30p - 30q - 6}{120p}.$$

Since $q' < \frac{19q}{30}$, we check when

$$\frac{19q}{30} < \frac{77pq - 30p - 30q - 6}{120p},$$

which is satisfied when

$$p > 30, \text{ and } q > \frac{30p + 6}{p - 30}.$$

The above inequalities are satisfied in the following cases,

- $p > 60$
- $p = 31, \quad q > 936$
- $p = 37, \quad q > 159$
- $p = 41, \quad q > 112$

Table 3.5: $pq \equiv 17 \mod 30$, $e_0 = -2$, for p and q sufficiently large

	$p'/p \in \left(0, \frac{1}{5}\right)$	$p'/p \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$p'/p \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$p'/p \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$p'/p \in \left(\frac{4}{5}, 1\right)$
$q'/q \in \left(0, \frac{1}{5}\right)$	$28pq$ $\vdots$ $28pq+3$	$28pq$ $\vdots$ $28pq+4$	$28pq$ $\vdots$ $28pq+4$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$
$q'/q \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$28pq$ $\vdots$ $28pq+4$	$51pq-60pq'-3$ $\vdots$ $51pq-60pq'$	$28pq-6$ $\vdots$ $28pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$
$q'/q \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$28pq$ $\vdots$ $28pq$	$28pq-6$ $\vdots$ $28pq$	$28pq-6$ $\vdots$ $28pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$
$q'/q \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$
$q'/q \in \left(\frac{4}{5}, 1\right)$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$	$26pq-2$ $26pq-1$ $26pq$

- $p = 43, \quad q > 99$
- $p = 47, \quad q > 83$
- $p = 53, \quad q > 69$
- $p = 59, \quad q > 61$

We still must address this for $p < 30$. We check each of these cases individually, using our restrictions on p'/p, $pq \mod 30$ and the Diophantine equation. In each case we are trying to show that

$$q' < \frac{77pq - 30q - 30p - 3}{130p}. \tag{3.1}$$

- $p = 11$: Then $p' \in \{7, 8\}$. By the Diophantine equation we have $q' \leq \frac{197q-1}{330}$, and this satisfies (3.1) when $q > \frac{329}{11}$.
- $p = 13$: Then $p' \in \{8, 9, 10\}$. By the Diophantine equation we have $q' \leq \frac{241q-1}{390}$, and this satisfies (3.1) when $q > 11$.
- $p = 17$: Then $p' \in \{11, 12, 13\}$. By the Diophantine equation we have $q' \leq \frac{299q-1}{510}$, and this satisfies (3.1) when $q > 55$.
- $p = 19$: Then $p' \in \{12, 13, 14, 15\}$. By the Diophantine equation we have $q' \leq \frac{343q-1}{570}$, and this satisfies (3.1) when $q > 9$.
- $p = 23$: Then $p' \in \{14, 15, 16, 17, 18\}$. By the Diophantine equation we have $q' \leq \frac{431q-1}{690}$, and this satisfies (3.1) when $q > 40$.

Table 3.6: $pq \equiv 29 \mod 30$, $e_0 = -2$, for p and q sufficiently large

	$p'/p \in (0, \frac{1}{2})$	$p'/p \in (\frac{1}{2}, \frac{3}{5})$	$p'/p \in (\frac{3}{5}, 1)$
$q'/q \in (0, \frac{1}{2})$	$26pq$ $26pq + 1$ $26pq + 2$	$60p'q - 8pq$ $60p'q - 8pq + 1$ $60q' - 8q + 2$	$45pq - 30p'q - 1$ $45pq - 30p'q$
$q'/q \in (\frac{1}{2}, \frac{3}{5})$	$60pq' - 8pq$ $60pq' - 8pq + 1$ $60pq' - 8pq + 2$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$
$q'/q \in (\frac{3}{5}, 1)$	$45pq - 30pq' - 1$ $45pq - 30pq'$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$

- $p = 29$: Then $p' \in \{18, 19, 20, 21, 22, 23\}$. By the Diophantine equation we have $q' \leq \frac{533q-1}{870}$, and this satisfies (3.1) when $q > 12$.

The remaining finite cases can be checked by hand, and thus this case is complete.

Case 3: $pq \equiv 17 \mod 30$, $e_0 = -2$, $1/5 < p'/p, q'/q < 2/5$.
It is easily checked that

$$\Delta(51pq - 60pq' - 3) = 1 = \Delta(51pq - 60pq').$$

$$\Delta(51pq - 60pq' - 2) = 0 = \Delta(51pq - 60pq' - 1).$$

Using our assumptions on p' and q' and the Diophantine equation we have that

$$51pq - 60pq' < N_0/2,$$

when $p > 30$, and $q > \frac{30p}{p-30}$. The above inequalities are satisfied in the following cases,

- $p > 60$
- $p = 31, \quad q > 930$
- $p = 37, \quad q > 158$
- $p = 41, \quad q > 111$
- $p = 43, \quad q > 99$
- $p = 47, \quad q > 82$
- $p = 53, \quad q > 69$
- $p = 59, \quad q > 61$

We still must address this for $p < 30$. We check each of these cases individually, using our restrictions on p'/p, $pq \mod 30$ and the Diophantine equation. In each case we are trying to show that

$$q' > \frac{43pq + 30p + 30q}{120p}. \tag{3.2}$$

- $p = 11$: Then $p' \in \{3, 4\}$. By the Diophantine equation we have $q' \geq \frac{133q-1}{210}$, and this satisfies (3.2) when $q > 1$.
- $p = 13$: Then $p' \in \{3, 4, 5\}$. By the Diophantine equation we have $q' \geq \frac{149q-1}{390}$, and this satisfies (3.1) when $q > 1$.
- $p = 17$: Then $p' \in \{4, 5, 6\}$. By the Diophantine equation we have $q' \geq \frac{211q-1}{510}$, and this satisfies (3.2) when $q > 6$.
- $p = 19$: Then $p' \in \{12, 13, 14, 15\}$. By the Diophantine equation we have $q' \geq \frac{227q-1}{570}$, and this satisfies (3.2) when $q > 9$.
- $p = 23$: Then $p' \in \{14, 15, 16, 17, 18\}$. By the Diophantine equation we have $q' \geq \frac{259q-1}{690}$, and this satisfies (3.2) when $q > 40$.
- $p = 29$: Then $p' \in \{18, 19, 20, 21, 22, 23\}$. By the Diophantine equation we have $q' \geq \frac{337q-1}{870}$, and this satisfies (3.2) when $q > 12$.

The remaining finite cases can be checked by hand, and thus this case is complete.

Case 4: $pq \equiv 29 \mod 30$, $e_0 = -2$, $1/2 < q'/q < 3/5$ and $0 < p'/p < 1/2$.
It is easily checked that

$$\Delta(60pq' - 8pq) = 1 = \Delta(60pq' - 8pq + 2),$$

$$\Delta(60pq' - 8pq + 1) = 0.$$

Using our assumptions of p' and q' and the Diophantine equation we have that

$$60pq' - 8pq + 2 < N_0/2,$$

when $p > 10$ and $q > \frac{30p+4}{3p-30}$. These are satisfied in the following cases,

- $p > 20$
- $p = 11, \quad q > 111$
- $p = 13, \quad q > 43$
- $p = 17, \quad q > 25$
- $p = 19, \quad q > 21$

The case where $1/2 < p'/p < 3/5$ and $0 < q'/q < 1/2$ can be checked similarly, completing this case.

Case 5: $pq \equiv 29 \mod 30$, $e_0 = -2$, $0 < q'/q < 1/2$, $3/5 < p'/p < 1$.
It is easily checked that

$$\Delta(45pq - 30p'q - 1) = 1 = \Delta(45pq - 30p'q).$$

Using our assumptions on p' and q' and the Diophantine equation we have that

$$45pq - 30p'q < N_0/2,$$

when $p > 6$ and $q > \frac{6p}{p-6}$. These inequalities are satisfied in the following cases,

- $p > 12$
- $p = 11, \quad q > 13$

The case where $0 < p'/p < 1/2$, $3/5 < q'/q < 1$, can be checked similarly. Thus our proof is complete. □

Now we prove the other case of $\Sigma(2,3,5,p,q)$.

Lemma 3.1.3. *Let $Y = \Sigma(2,3,5,7,q)$. Then $U \cdot HF_{red}(Y) \neq 0$.*

Table 3.7: $pq \equiv 13 \mod 30$, $p = 7$, $e_0 = -3$, for q sufficiently large

	$p'/p \in \left(0, \frac{1}{5}\right)$	$p'/p \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$p'/p \in \left(\frac{2}{5}, \frac{3}{5}\right)$
$q'/q \in \left(0, \frac{1}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	
$q'/q \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$1420q - 1890q' - 7$ $\vdots$ $1420q - 1890q'$
$q'/q \in \left(\frac{4}{5}, 1\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$	$520q - 420q' - 2$ $\vdots$ $520q - 420q'$

Table 3.8: $pq \equiv 13 \mod 30$, $p = 7$, $e_0 = -3$, for q sufficiently large

	$p'/p \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$p'/p \in \left(\frac{4}{5}, 1\right)$
$q'/q \in \left(0, \frac{1}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$26pq$ $26pq+1$ $26pq+2$	$26pq$ $26pq+1$ $26pq+2$
$q'/q \in \left(\frac{2}{5}, \frac{3}{5}\right)$	$280q - 210q' - 7$ $\vdots$ $280q - 210q'$	
$q'/q \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$60pq' - 9pq$ $\vdots$ $60pq' - 9pq + 3$	
$q'/q \in \left(\frac{4}{5}, 1\right)$		

Proof. Consider Table 3.2. The largest value in this table is $27pq$, it is easily checked that for $p = 7$, $27pq < N_0/2$ holds when $q > 42$. The remaining finite cases can be easily checked and then we have that $Y = \Sigma(2, 3, 5, 7, q)$ satisfies $U \cdot HF_{red}(Y)$ except for the following cases,

1. $pq \equiv 1 \mod 30$, $e_0 = -3$,
2. $pq \equiv 29 \mod 30$, $e_0 = -2$,
3. $pq \equiv 13 \mod 30$, $e_0 = -3$,
4. $pq \equiv 17 \mod 30$, $e_0 = -2$.

We will now address each of these cases separately.

Case 1: Consider Table 3.3. Excluding Type B cases, the largest value here is $22pq$, and this satisfies $22pq < N_0/2$ for $p = 7$ when $q > \frac{14}{5}$. Now it remains to check the Type B cases, but we have already shown that $15pq + 30pq' + 1 < N_0/2$ for $q > 6$. Thus this case is complete.

Case 2: Consider Table 3.6, the largest value here is $26pq + 2$, excluding Type B cases. It can be checked that for $p = 7$, $26pq + 2 < N_0/2$ is satisfied when $q > 214/19$. Now it remains to check the Type B cases. Consider when $q'/q < 1/2$ and $3/5 < p'/p$. We have already shown that $45pq - 30p'q < N_0/2$ when $p > 6$ and $q > \frac{6p}{p-6}$. Thus this is satisfied when $q > 42$ for $p = 7$. The case where $p'/p < 1/2$ and $3/5 < q'/q$ can be checked similarly. Now consider when $1/2 < q'/q < 3/5$

Table 3.9: $pq \equiv 17 \mod 30$, $e_0 = -2$, $p = 7$

	$p'/p \in \left(0, \frac{1}{5}\right)$	$p'/p \in \left(\frac{1}{5}, \frac{2}{5}\right)$	$p'/p \in \left(\frac{2}{5}, \frac{3}{5}\right)$
$q'/q \in \left(0, \frac{1}{5}\right)$			$182q - 2$ $\vdots$ $182q$
$q'/q \in \left(\frac{1}{5}, \frac{2}{5}\right)$		$51pq - 60pq' - 3$ $\vdots$ $51pq - 60pq'$	$240q - 210q' - 10$ $\vdots$ $240q - 210q' - 10$
$q'/q \in \left(\frac{2}{5}, \frac{3}{5}\right)$		$777q - 1260q' - 7$ $\vdots$ $777q - 1260q'$	
$q'/q \in \left(\frac{3}{5}, \frac{4}{5}\right)$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$
$q'/q \in \left(\frac{4}{5}, 1\right)$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$	$26pq - 2$ $26pq - 1$ $26pq$

Table 3.10: $pq \equiv 17 \mod 30$, $e_0 = -2$, $p = 7$

	$p'/p \in \left(\frac{3}{5}, 1\right)$
$q'/q \in (0, 1)$	$26pq - 2$ $26pq - 1$ $26pq$

and $p'/p < 1/2$. Since $p = 7$, we must have $p' \in \{1, 2, 3\}$ to satisfy $p'/p < 1/2$. When $p' = 1$ we can solve the Diophantine equation to find that

$$q' = \frac{173q - 1}{210}$$

but then $q'/q < 173/210$ and thus $q'/q > 3/5$, so this case cannot occur for $p = 7$ and $p' = 1$. For $p' = 2$ we follow the same argument to find that

$$q' = \frac{143q - 1}{210},$$

but again this only satisfies $1/2 < q'/q < 3/5$ for $1/38 < q < 1/17$. Therefore this case cannot occur for $p = 7$ and $p' = 2$. For $p' = 3$ we check that $60pq' - 8pq + 2 < N_0/2$ is satisfied for $q > \frac{210}{43}$ and thus this case is complete. The case where $q'/q < 1/2, 1/2 < p'/p < 3/5$ can be checked to satisfy $60p'q - 8pq + 2 < N_0/2$ when $q > 214/15$. Thus this case is complete.

Case 3: We proceed as in Case 2 and use the Diophantine equation to limit the possible values for p' in each case. Then we check to make sure that q'/q satisfies the desired condition; if not that cell in the table is left blank and highlighted in red, as this case is not possible for $p = 7$. For the cases that are possible, we have found values that give us the desired 2 consecutive positive ones in our Δ-sequence, presented in Tables 3.7 and 3.8. It remains to check that these occur before $N_0/2$ in our Δ-sequence:

- $3/5 < p'/p < 4/5, 2/5 < q'/q < 3/5$
  It follows that $p' = 5$, and thus $280q - 210q' < N_0/2$ for $q > \frac{-208}{41}$.
- $2/5 < p'/p < 3/5, 3/5 < q'/q < 4/5$
  It follows that $p' = 4$, and thus $1420q - 1890q' - 7 < N_0/2$ for $q > \frac{214}{45}$.
- $3/5 < p'/p < 4/5, 3/5 < q'/q < 4/5$
  It follows that $p' = 5$, and thus $60pq' - 9pq + 3 < N_0/2$ for $q > \frac{209}{73}$.
- $2/5 < p'/p < 3/5, 4/5 < q'/q < 1$
  It follows that $p' = 3$, and thus $520q - 420q' < N_0/2$ for $q > \frac{214}{19}$.

Case 4: We proceed exactly as in Case 3 and the values that give us the desired 2 consecutive positive ones in our Δ-sequence are presented in Tables 3.9 and 3.10. It remains to check that these occur before $N_0/2$ in our Δ-sequence:

- $2/5 < p'/p < 3/5, 0 < q'/q < 1/5$
  It is easily checked that $182q < N_0/2$ for $q > \frac{210}{19}$.
- $1/5 < p'/p < 2/5, 1/5 < q'/q < 2/5$
  It follows that $p' = 2$, and thus $51pq - 60pq' < N_0/2$ for $q > \frac{2354}{2331}$.
- $2/5 < p'/p < 3/5, 1/5 < q'/q < 2/5$
  It follows that $p' = 3$, and thus $240q - 210q' - 10 < N_0/2$ for $q > \frac{64}{15}$.
- $1/5 < p'/p < 2/5, 2/5 < q'/q < 3/5$
  It follows that $p' = 2$, and thus $777q - 1260q' < N_0/2$ for $q > \frac{222}{41}$.

□

3.3 $\Sigma(2,3,7,p,q)$

3.3.1 Preliminaries

We proceed exactly as in the $\Sigma(2,3,5,p,q)$ case. Consider the Diophantine equation for the Seifert fibered homology sphere

$$Y = \Sigma(2,3,7,p,q)$$

with solution

$$D = (e_0, 1, a, b, p', q'),$$

by reducing this modulo 42 we are able to solve for the Diophantine solutions based on $pq \mod 42$.

$$\begin{aligned}
pq &\equiv 1 \mod 42 \implies D = (e_0,1,1,1,p',q'), & pq &\equiv 23 \mod 42 \implies D = (e_0,1,2,4,p',q'),\\
pq &\equiv 5 \mod 42 \implies D = (e_0,1,2,3,p',q'), & pq &\equiv 25 \mod 42 \implies D = (e_0,1,1,2,p',q'),\\
pq &\equiv 11 \mod 42 \implies D = (e_0,1,2,2,p',q'), & pq &\equiv 29 \mod 42 \implies D = (e_0,1,2,1,p',q'),\\
pq &\equiv 13 \mod 42 \implies D = (e_0,1,1,6,p',q'), & pq &\equiv 31 \mod 42 \implies D = (e_0,1,1,5,p',q'),\\
pq &\equiv 17 \mod 42 \implies D = (e_0,1,2,5,p',q'), & pq &\equiv 37 \mod 42 \implies D = (e_0,1,1,4,p',q'),\\
pq &\equiv 19 \mod 42 \implies D = (e_0,1,1,3,p',q'), & pq &\equiv 41 \mod 42 \implies D = (e_0,1,2,6,p',q').
\end{aligned}$$

Note in all the above cases, $apq \equiv 1 \mod 3$ and $bpq \equiv 1 \mod 7$.

We can also come up with a bound on e_0 using the Diophantine equation:

$$\begin{aligned}
-1 &= 42pqe_0 + 21pq + 14apq + 6bpq + 42p'q + 42pq'\\
-42pqe_0 &= 21pq + 14apq + 6bpq + 42p'q + 42pq' + 1\\
|e_0| &= \frac{21pq + 14apq + 6bpq + 42p'q + 42pq' + 1}{42pq}\\
|e_0| &= \frac{1}{2} + \frac{a}{3} + \frac{b}{7} + \frac{p'}{p} + \frac{q'}{q} + \frac{1}{42pq}\\
|e_0| &< \frac{1}{2} + \frac{2}{3} + \frac{6}{7} + \frac{p}{p} + \frac{q}{q} + \frac{1}{42*11*13} = \frac{4028}{1001} \approx 4.023.
\end{aligned}$$

Therefore $e_0 = -1, -2, -3, -4$.

Using a similar simplification as we did for $\Sigma(2,3,5,p,q)$ we are able to compute $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for the x values that satisfy the majority of our $\Sigma(2,3,7,p,q)$ cases. These values are presented in Tables 3.11 and 3.12.

Table 3.11: Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for $\Sigma(2,3,7,p,q)$ and $a = 1$

	$b = 1,2$	$b = 3$	$b = 4$	$b = 5,6$
$\Delta(26pq-1)$	$1-\lvert e_0\rvert$	$1-\lvert e_0\rvert$	$1-\lvert e_0\rvert$	$2-\lvert e_0\rvert$
$\Delta(26pq)$	1	1	1	1
$\Delta(26pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$
$\Delta(35pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$
$\Delta(35pq)$	1	1	1	1
$\Delta(35pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$
$\Delta(36pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$
$\Delta(36pq)$	1	1	1	1
$\Delta(36pq+1)$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$
$\Delta(39pq-1)$	$2-\lvert e_0\rvert$	$2-\lvert e_0\rvert$	$3-\lvert e_0\rvert$	$3-\lvert e_0\rvert$
$\Delta(39pq)$	1	1	1	1
$\Delta(39pq+1)$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-2$	$\lvert e_0\rvert-3$	$\lvert e_0\rvert-3$

3.3.2 Results

Now we are ready to prove Lemma 3.1.4.

Lemma 3.1.4. *Let $Y = \Sigma(2,3,7,p,q)$. Then $U \cdot HF_{red}(Y) \neq 0$.*

As with Lemma 3.1.2, we wish to find values that provide us with 2 consecutive $+1's$ in our Δ-sequence that occur prior to $N_0/2$. The values that satisfy this condition are found in Tables 3.13 and 3.14.

Proof. We must now show that the values in Tables 3.13 and 3.14 occur before $N_0/2$. Since it is clear that

$$26pq \leq 35pq \leq 36pq \leq 39pq < 39pq + 1 < 39pq + 2$$

we need only to check that $39pq + 1 \leq N_0/2$. It can be easily checked that this holds when

$$p > 6, \text{ and } q \geq \frac{42p + 4}{7p - 42},$$

which are satisfied in the following cases,

- $p = 11, \quad q > 25,$
- $p > 12.$

The finite cases that do not satisfy these inequalities may be easily checked to satisfy

$$U \cdot HF_{red} \neq 0.$$

Table 3.12: Key values of $\Delta(xpq)$ and $\Delta(xpq \pm 1)$ for $\Sigma(2,3,7,p,q)$ and $a = 2$

	$b = 1,2$	$b = 3$	$b = 4$	$b = 5,6$
$\Delta(26pq - 1)$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$
$\Delta(26pq)$	1	1	1	1
$\Delta(26pq + 1)$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 4$	$\lvert e_0\rvert - 4$	$\lvert e_0\rvert - 4$
$\Delta(35pq - 1)$	$3 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$
$\Delta(35pq)$	1	1	1	1
$\Delta(35pq + 1)$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$
$\Delta(36pq - 1)$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$
$\Delta(36pq)$	1	1	1	1
$\Delta(36pq + 1)$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$
$\Delta(39pq - 1)$	$2 - \lvert e_0\rvert$	$2 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$	$3 - \lvert e_0\rvert$
$\Delta(39pq)$	1	1	1	1
$\Delta(39pq + 1)$	$\lvert e_0\rvert - 2$	$\lvert e_0\rvert - 2$	$\lvert e_0\rvert - 3$	$\lvert e_0\rvert - 3$

Thus it remains to address the following when $pq \equiv 1 \mod 42$ and $e_0 = -2$. First let $1/2 < p'/p$, and $q'/q < 1/2$. It can be easily checked that

$$\Delta(63pq - 42pq' - 1) = 1 = \Delta(63pq - 42pq').$$

We must show that $63pq - 42pq' < N_0/2$. This is true when

$$p' > \frac{41pq + 42p + 42q}{84q}.$$

Using that $1/2 < p'/p$, the above is satisfied when $p > 42$ and $q > \frac{42p}{p-42}$. These inequalities are satisfied in the following cases,

- $p > 84$
- $p = 43, \quad q > 1806$
- $p = 47, \quad q > 394$
- $p = 53, \quad q > 202$
- $p = 55, \quad q > 177$
- $p = 59, \quad q > 145$
- $p = 61, \quad q > 134$
- $p = 65, \quad q > 118$
- $p = 67, \quad q > 112$
- $p = 71, \quad q > 102$
- $p = 73, \quad q > 98$
- $p = 79, \quad q > 89$

The case where $p'/p < 1/2$ and $q'/q > 1/2$ can be checked similarly. Thus our proof is complete. $\square$

Table 3.13: General Cases for $Y = \Sigma(2,3,7,p,q)$

	$e_0 = -1$	$e_1 = -2$	$e_0 = -3$	$e_0 = -4$
$pq \equiv 1 \mod 42$	$26pq - 1$ $26pq$	See Table 3.14	$26pq$ $26pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 5 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$39pq$ $39pq + 1$	$35pq$ $35pq + 1$
$pq \equiv 11 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$36pq$ $\vdots$ $36pq + 3$	$26pq$ $26pq + 1$
$pq \equiv 13 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$35pq$ $35pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 17 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$35pq$ $35pq + 1$
$pq \equiv 19 \mod 42$	$26pq - 1$ $26pq$	$36pq$ $36pq + 1$ $36pq + 2$	$35pq$ $35pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 23 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$36pq$ $\vdots$ $36pq + 3$	$35pq$ $35pq + 1$
$pq \equiv 25 \mod 42$	$26pq - 1$ $26pq$	$36pq$ $36pq + 1$ $36pq + 1$	$26pq$ $26pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 29 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq - 1$	$36pq$ $36pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 31 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$35pq$ $35pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 37 \mod 42$	$26pq - 1$ $26pq$	$36pq$ $\vdots$ $36pq + 3$	$35pq$ $35pq + 1$	$26pq$ $26pq + 1$
$pq \equiv 41 \mod 42$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$26pq - 1$ $26pq$	$35pq$ $35pq + 1$

Table 3.14: $pq \equiv 1 \mod 42$, $e_0 = -2$

	$p'/p \in \left(0, \frac{1}{2}\right)$	$p'/p \in \left(\frac{1}{2}, 1\right)$
$q'/q \in \left(0, \frac{1}{2}\right)$	$39pq$ $39pq + 1$ $39pq + 2$	$63pq - 42p'q - 1$ $63pq - 42p'q$
$q'/q \in \left(\frac{1}{2}, 1\right)$	$63pq - 42pq' - 1$ $63pq - 42pq'$	$63pq - 42pq' - 1$ $63pq - 42pq'$

3.4 $\Sigma(2, 3, 11, 13, p)$

3.4.1 Preliminaries

Before we proceed, we must first complete some algebraic preliminaries. This will follow the general idea of the previous two sections, but we take a slightly different approach. Consider the Seifert fibered homology sphere

$$Y = \Sigma(2, 3, 11, 13, p),$$

with Diophantine solutions

$$D = (e_0, 1, a, b, c, p').$$

Then the Diophantine equation simplifies as follows

$$-1 = 858pe_0 + 429p + 286ap + 78bp + 66cp + 858p'.$$

Reducing this modulo 3, 11 and 13 we get that

$$qp \equiv 2 \mod 3, \quad bp \equiv 10 \mod 11, \quad cp \equiv 12 \mod 13.$$

We can also come up with a bound on e_0 using the Diophantine equation:

$$\begin{aligned}
-1 &= 858pe_0 + 429p + 286ap + 78bp + 66cp + 858p' \\
|e_0| &= \frac{1}{2} + \frac{a}{3} + \frac{b}{11} + \frac{c}{13} + \frac{p'}{p} + \frac{1}{858p} \\
|e_0| &< \frac{1}{2} + \frac{2}{3} + \frac{10}{11} + \frac{12}{13} + \frac{p}{p} \approx 3.99
\end{aligned}$$

Therefore $e_0 = -1, -2, -3$.

3.4.2 Results

We are now ready to prove the following.

Lemma 3.4.1. *Let $Y = \Sigma(2, 3, 11, 13, q)$. Then $U \cdot HF_{red}(Y) \neq 0$.*

Proof. Consider the following,

$$\Delta(781p) = 1 + |e_0|(781p) - \left\lceil \frac{781p}{2} \right\rceil - \left\lceil \frac{781ap}{3} \right\rceil - \left\lceil \frac{781bp}{11} \right\rceil - \left\lceil \frac{781cp}{13} \right\rceil - \left\lceil \frac{781pp'}{p} \right\rceil$$

Using that

$$qp \equiv 2 \mod 3, \quad bp \equiv 10 \mod 11, \quad cp \equiv 12 \mod 13,$$

and that

$$781 \equiv 1 \mod 3, \quad 781 \equiv 0 \mod 11, \quad 781 \equiv 1 \mod 13,$$

we know the value of the ceiling functions. Now we solve the Diophantine equation for p',

$$p' = \frac{-1}{858} - pe_0 - \frac{p}{2} - \frac{ap}{3} - \frac{bp}{11} - \frac{cp}{13}.$$

Now we are able to simplify $\Delta(781p)$ fully.

$$\begin{aligned}
\Delta(781p) &= 1 + |e_0|(781p) - \frac{781p+1}{2} - \frac{781ap+1}{3} - 71bp - \frac{781cp+1}{13} - 781p' \\
&= 1 + |e_0|(781p) - \frac{781p+1}{2} - \frac{781ap+1}{3} - 71bp - \frac{781cp+1}{13} \\
&\quad - 781\left(\frac{-1}{858} - pe_0 - \frac{p}{2} - \frac{ap}{3} - \frac{bp}{11} - \frac{cp}{13}\right) \\
&= 1 + |e_0|(781p) - \frac{781p+1}{2} - \frac{781ap+1}{3} - 71bp - \frac{781cp+1}{13} \\
&\quad + \frac{781}{858} + 781pe_0 + \frac{781p}{2} + \frac{781qp}{3} + \frac{781bp}{11} + \frac{781cp}{13} \\
&= 1 + \frac{781}{858} - \frac{1}{2} - \frac{1}{3} - \frac{1}{13} \\
&= 1.
\end{aligned}$$

Using this same process we have that

$$\Delta(781p+1) = \begin{cases} |e_0| - 1, & a = 1, c = 1 \\ |e_0| - 2, & a = 1 \text{ and } c \neq 1, \text{ or } a = 2, c = 1 \\ |e_0| - 3, & a = 2, c \neq 1 \end{cases}$$

$$\Delta(781p-1) = \begin{cases} 2 - |e_0|, & a = 1, c \neq 12 \\ 3 - |e_0|, & a = 1 \text{ and } c = 12, \text{ or } a = 2, c \neq 12 \\ 4 - |e_0|, & a = 2, c = 12 \end{cases}$$

We can see in Table 3.15 the values that give us two consecutive ones, given conditions on $|e_0|$, a and c. Since $781p - 1 < 781p < 781p + 1$, we need to show that $781p + 1 < N_0/2$. This is easily checked to be true when $p \geq 6$, which is always true for $Y = \Sigma(2, 3, 11, 13, p)$. It remains to check the following two cases,

1. $|e_0| = 2$, $a = 1$, $c \neq 1, 12$

2. $|e_0| = 3$, $a = 1$, $c \neq 1, 12$

Case 1: It can be checked that $\Delta(1287p - 858p') = 1$ and $\Delta(2387p - 858p') = 3 - |e_0| = 1$. Now it remains to check that $1287p - 858p' < N_0//2$. We use the Diophantine equation to show that

$$p' \geq \frac{505p - 1}{858},$$

Table 3.15: General Cases for $Y = \Sigma(2,3,11,13,p)$

	$e_0 = -1$	$e_1 = -2$	$e_0 = -3$
$a = 1, c = 1$	$781p - 1$ $781p$	$781p$ $781p + 1$	$781p$ $781p + 1$
$a = 1, c = 12$	$781p - 1$ $781p$	$781p - 1$ $781p$	$781p$ $781p + 1$
$a = 1, c \neq 1, 12$	$781p - 1$ $781p$		$781p$ $781p + 1$
$a = 2, c = 1$	$781p - 1$ $781p$	$781p - 1$ $781p$	$781p$ $781p + 1$
$a = 2, c = 12$	$781p - 1$ $781p$	$781p - 1$ $781p$	$781p - 1$ $781p$
$a = 2, c \neq 1, 12$	$781p - 1$ $781p$	$781p - 1$ $781p$	

and then plug this into our above inequality. We see that $1287p - 858p' < N_0//2$ is satisfied when $p > \frac{860}{151} \approx 5.6$, which is trivially true, and so this case is complete.

Case 2: It is easily checked that $\Delta(1573p - 858p' - 1) = 1$ and $\Delta(1573p - 858p') = 1$. We find that $1573p - 858p' < N_0/2$ when $p \geq 6$.

□

Chapter 4

Mapping Cone

4.1 Introduction

In this section we give the proper notation and theorems used to set up the mapping cone for rational surgery on a knot in S^3. The mapping cone formula was introduced by Ozsváth and Szabó in [26], but we follow Gainullin's general structure in [8].

4.2 Background

4.2.1 Setup

Once we have the full knot complex for a knot, $C = CFK^\infty(K)$, we use the mapping cone to compute Heegaard Floer homology of surgery on K. We have that C is homotopy equivalent as a filtered complex to a complex where all filtration preserving differentials are trivial [27]. Therefore we can replace the group, viewed as a chain complex, with its homology at each filtration level. We will work with this object, called the reduced complex.

This complex C has an associated U-action, which corresponds to translation by the vector $(-1, -1)$. C is invariant under this shift, so the group at filtration level (i, j) is the same as the one at $(i-1, j-1)$, and we can view U as the identity map between these. We know that U is a chain map, and U is invertible, so C is an $\mathbb{F}[U, U^{-1}]$ - module. It follows that C is generated by elements at filtration level $i = 0$, and we will refer to this complex at filtration level $(0, j)$ as $\widehat{HFK}(K, j)$.

We can now define the following quotient complexes of C.

$$A_k^+(K) = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z},$$

and

$$B^+ = C\{i \geq 0\} \cong CF^+(S^3).$$

We also have two chain maps $v_k \colon A_k^+(K) \to B^+$, and $h_k \colon A_k^+(K) \to B^+$. The map v_k is simply projection, sending all generators with $i > k$ to 0, and the identity elsewhere. The map h_k first projects onto $C\{j \geq k\}$, then multiplies by U^k, and then identifies $C\{j \geq 0\}$ with $C\{i \geq 0\}$ via a chain homotopy equivalence. This homotopy equivalence exists because both complexes represent $CF^+(S^3)$. It is known that v_k is an isomorphism if $k \geq g(K)$ and h_k is an isomorphism if $k \leq -g(K)$, where $g(K)$ is the genus of the knot [26].

Now we can define chain complexes

$$\mathcal{A}^+_{i,p/q}(K) = \bigoplus_{n\in\mathbb{Z}}(n, A^+_{\left\lfloor \frac{i+pn}{q}\right\rfloor}(K)),$$

and

$$\mathcal{B}^+ = \bigoplus_{n\in\mathbb{Z}}(n, B^+).$$

We use the index n just to distinguish between different copies of the same group, and the index i represents the Spin^c-structure. Then we have a chain map

$$D^+_{i,p/q} \colon \mathcal{A}^+_{i,p/q} \to \mathcal{B}^+,$$

where $D^+_{i,p/q}$ is given by an appropriate sums of v_k and h_k, which we now define. We require $v_k \colon (n, A^+_k(K) \to (n, B^+)$, and $h_k \colon (n, A^+_k(K) \to (n+1, B^+)$. We can define this map explicitly as follows,

$$D^+_{i,p/q}(\{k, a_k\})_{k\in\mathbb{Z}}) = \{(k, b_k)\}_{k\in\mathbb{Z}}, \text{ where } \quad b_k = v_{\left\lfloor \frac{i+pk}{q}\right\rfloor}(a_k) + h_{\left\lfloor \frac{i+p(k-1)}{q}\right\rfloor}(a_{k-1}).$$

The complexes $A^+_k(K)$ and B^+ inherit a relative $\mathbb{Z}$-grading from C. Now let $\mathbb{X}^+_{i,p/q}$ denote the mapping cone of $D^+_{i,p/q}$. This has a relative $\mathbb{Z}$-grading by requiring $D^+_{i,p/q}$ decreases the grading by 1.

Theorem 4.2.1 (Ozsváth-Szabó, [26])**.** *There is a relatively graded isomorphism of* $\mathbb{F}[U]$*-modules*

$$H_*\left(\mathbb{X}^+_{i,p/q}\right) \cong HF^+\left(S^3_{p/q}(K), i\right).$$

The picture you should have in your head for the mapping cone is depicted in Figure 4.1 for $2/3$ surgery on some knot in S^3. We have 3 copies of each A^+_i and each B^+_i, which the subscripts denote. We also have that the h_k maps will move 2 groups horizontally. For general p/q surgery we will have q copies of each group and the h_k maps will move p groups horizontally, moving right for positive p and left for negative p.

Since we know that v_k is an isomorphism if $k \geq g(K)$ and h_k is an isomorphism if $k \leq -g(K)$, when we are working with a specific knot, we know much of the mapping cone is acyclic. We can delete this acyclic part, which we will refer to as the truncated mapping cone. This truncated mapping cone is depicted for $2/3$ surgery of some genus 2 knot in S^3 in Figure 4.2

Now let $\mathbf{A}^+_{i,p/q}(K) = H_*(A^+_k(K))$ and $\mathbf{B}^+ = H_*(B^+)$, $\mathbb{A}^+_{i,p/q}(K) = H_*(\mathcal{A}^+_{i,p/q}(K))$, $\mathbb{B}^+ =$

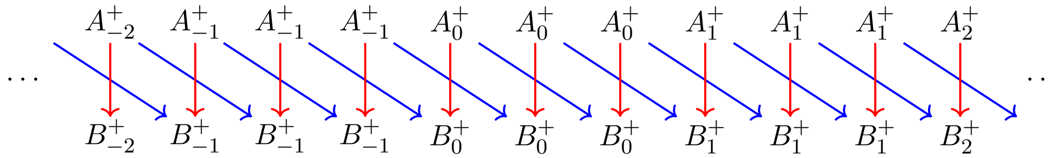


Figure 4.1: The Mapping cone for $2/3$ surgery on a knot $K \subset S^3$. The v_k maps are red, h_k maps are blue.

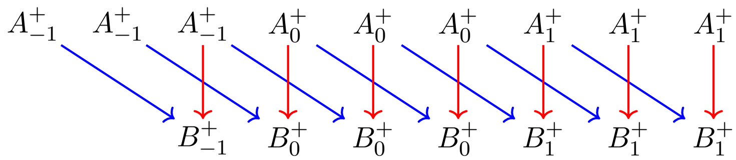


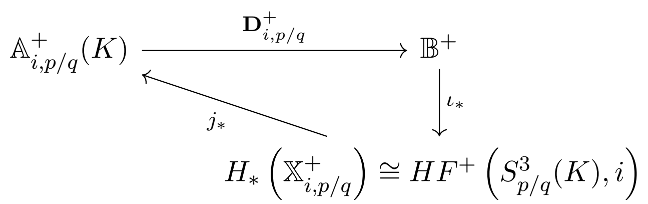
Figure 4.2: The truncated mapping cone for $2/3$ surgery on a genus 2 knot $K \subset S^3$. The v_k maps are red, h_k maps are blue.

$H_*(\mathcal{B}^+)$ and let $\mathbf{v}_k, \mathbf{h}_k$ and $\mathbf{D}^+_{i,p/q}$ denote the maps induced by v_k, h_k and $D^+_{i,p/q}$ on homology respectively.

The short exact sequence

$$0 \longrightarrow \mathcal{B}^+ \xrightarrow{\iota} \mathbb{X}^+_{i,p/q} \xrightarrow{j} \mathcal{A}^+_{i,p/q}(K) \longrightarrow 0$$

induces the exact triangle



All maps above are U-equivariant.

4.2.2 Previous Work

From the exact triangle we have the following,

Corollary 4.2.2. *(Gainullin, [8, Section 2]) If the surgery slope p/q is positive, then the map $\mathbf{D}^+_{i,p/q}$ will be surjective, so $HF^+\left(S^3_{p/q}(K), i\right) \cong \ker\left(\mathbf{D}^+_{i,p/q}\right)$.*

We need to establish some important decompositions of the maps given earlier. First we have that we can decompose $\mathbf{A}^+_k(K)$ as $\mathbf{A}^+_k(K) \cong \mathbf{A}^T_k(K) \oplus \mathbf{A}^{red}_k(K)$, where $\mathbf{A}^{red}_k(K)$ is a finite-dimensional vector space in the kernel of some power of U, and $\mathbf{A}^T_k(K) \cong \mathcal{T}^+$. We also have an isomorphism $\phi\colon \mathbf{A}^+_k(K) \to \mathbf{A}^+_{-k}(K)$ that is U-equivariant, grading preserving, and satisfies $\mathbf{v}_k \circ \phi = \mathbf{h}_k$, where we view $\mathbf{v}_k$ and $\mathbf{h}_k$ as maps into $CF^+(S^3)$, and ϕ as an isomorphism from

$CF^+(S^3)$ to itself [23]. Note, we will often drop K from the above notation and refer to $\mathbf{A}_k^+(K)$ as simply $\mathbf{A}_k^+$.

We have that

$$\mathbf{D}_{i,p/q}^+ = \mathbf{D}_{i,p/q}^T \oplus \mathbf{D}_{i,p/q}^{red}$$

where the first map is the restriction of $\mathbf{D}_{i,p/q}^+$ to $\mathbb{A}_{i,p/q}^T(K) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}^+$ and the second one is the restriction to $\mathbb{A}_{i,p/q}^{red}(K)$. Similarly we have restrictions of $\mathbf{v}_k$ and $\mathbf{h}_k$ to $\mathcal{T}^+$ which will just be multiplication by some non-negative powers of U. These powers of U will be denoted by V_k and H_k. We will need the following properties of these integers, (see [22, Section 2], [27, Section 7], [11, Lemma 2.5])

$$V_k \geq V_{k+1} \text{ and } H_k \leq H_{k+1}, \quad \forall k \in \mathbb{Z}, \tag{4.1}$$

$$V_k = H_{-k}, \quad \forall k \in \mathbb{Z}, \tag{4.2}$$

$$V_k \to +\infty \text{ as } k \to -\infty \quad \text{ and } \quad H_k \to +\infty \text{ as } k \to +\infty \tag{4.3}$$

$$V_k = 0 \text{ for } k \geq g(K) \quad \text{ and } \quad H_k = 0 \text{ for } k \leq -g(K), \tag{4.4}$$

$$V_k - 1 \leq V_{k+1} \leq V_k, \forall k \in \mathbb{Z}, \tag{4.5}$$

$$H_k = V_k + k, \forall k \in \mathbb{Z}. \tag{4.6}$$

Finally, we state a Corollary of Gainullin that we will reference frequently in the next section.

Corollary 4.2.3. *([8, Corollary 30])* $U^g \cdot \mathbb{A}_{i,p/q}^{red} = 0$.

4.3 Genus 1 Results

The goal of this section is to prove Theorem 1.2.3. Since we are working with Seifert fibered integral homology spheres in this section, our spaces will only have one Spin^c-structure, thus we will often drop the index i as it is not important for our purposes. We will also often drop the p/q subscript if it is clear what surgery we are discussing.

4.3.1 Preliminary Proofs

Before proving Theorem 1.2.5, we must prove some preliminary results about the $\mathbf{v_k}$ and $\mathbf{h_k}$ maps, and subsequently, V_k and H_k.

Lemma 4.3.1. *Let K be a knot in S^3 with genus g. Then $H_i \leq g$ when $i \leq g$ and $V_i \leq g$ when $i \geq 0$.*

Proof. By (4.6) $H_g = V_g + g$ and by (4.4) $V_g = 0$. Therefore $H_g = g$. By (4.1) we have that $H_i \leq g$ for $i \leq g$. Since $V_0 = H_0$ by (4.6), and $H_0 \leq g$, we know that $V_0 \leq g$. Then by (4.1) we have that $V_i \leq g$ when $i \geq 0$. ∎

Lemma 4.3.2. *For a genus 1 knot K we have that $U \cdot HF_{red}(S^3_{\pm 1}(K)) = 0$.*

Proof. By the mapping cone formula $HF^+(S^3_1(K)) = H_*(A_0^+)$ and by Corollary 4.2.3 we have

$$U^{g(K)} \cdot H_*(A_0^{red}) = 0,$$

where $g(K)$ is the genus of K, thus

$$U \cdot HF_{red}(S^3_1(K)) = 0.$$

We know that $S^3_{p/q}(K) = -S^3_{-p/q}(\overline{K})$, where $\overline{K}$ denotes the the reflection of K, [24]. Thus it follows that

$$S^3_{+1}(K) = -S^3_{-1}(\overline{K}).$$

Since K and $\overline{K}$ are both genus 1 knots it follows that

$$U \cdot HF_{red}(S^3_{-1}(K)) = 0,$$

by the arguments for $+1$ surgery. □

Before we proceed, we must introduce the concept of levels, a way to refer to the relative grading of $\mathbf{A}_k^+$ and $\mathbf{B}_k^+$. We define level 0 to contain the bottom of $\mathcal{T}^+$, and those elements that have this same grading, call this grading g. Then level 1 will consist of those elements in grading $g+2$, level 2 will consist of those elements in grading $g+4$. Thus in general elements in grading $g+2k$ will be in level k. Elements in different parity of grading will be in half levels. So an element in grading $g+2k+1$ will be in level $\frac{2k+1}{2}$.

Example 4.3.3. Consider $\mathbf{A}_0^+$ in Figure 4.3. The a_i's are the vertices in $\mathbf{A}_0^+$, while the other elements are in $\mathbf{A}_0^{red}$. We see that a_0 is the bottom of the tower, and thus is in level 0. The only generators in different parity of grading are w and z. We can then say all elements are in the following levels,

Level 0: a_0, v, y	Level 2: a_2	Level $\frac{3}{2}$: w
Level 1: a_1, x	Level 3: a_3, u	Level $\frac{5}{2}$: z

Remark 4.3.4. *Note that linear combinations of elements in half levels will always be sent to 0 under the U-action by Corollary 4.2.3. Thus elements in half levels can never contribute to $U \cdot HF_{red}$ being nonzero, so we need not discuss them to prove Theorem 1.2.3.*

Now we use Lemma 4.3.2 to restrict the allowable maps for $\mathbf{v}_0$ and $\mathbf{h}_0$. Consider Figure 4.4, which depicts the truncated mapping cone for -1 surgery on a genus 1 knot. This figure shows the labeling used to refer to elements of $\mathbf{A}_0^T$, $\mathbf{B}_0^T$ and $\mathbf{B}_{-1}^T$, where the subscript of the element refers to the level it is in.

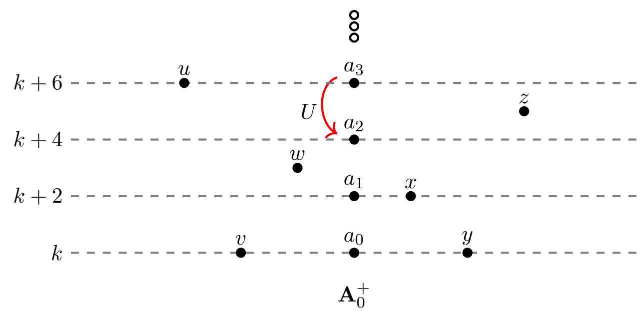


Figure 4.3: Levels of $\mathbf{A}_0^+$

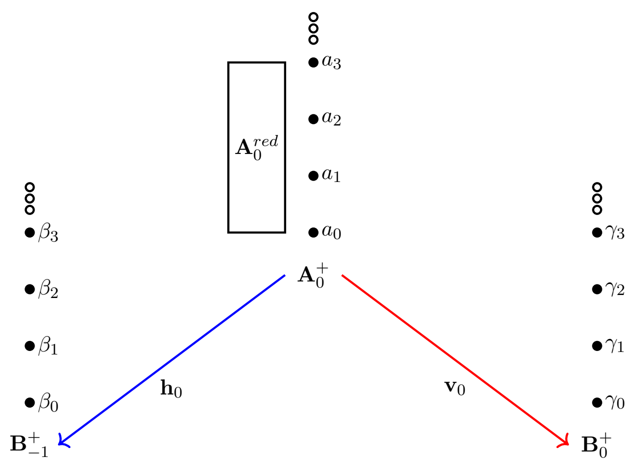


Figure 4.4: -1 Surgery Mapping Cone

Remark 4.3.5. *Note we have only drawn a box to represent $\mathbf{A}_0^{red}$, and neither of the $\mathbf{B}^+$ complexes have a reduced part as they represent $CF^+(S^3)$.*

Lemma 4.3.6. *Let x_i be in level i of $\mathbf{A}_0^{red}$. Then $\mathbf{v_0}(x_i) = 0 = \mathbf{h_0}(x_i)$ for $i \geq V_0 + 1$, or $i < 0$.*

Proof. We know that $\mathbf{v_0}$ must send x_i to an element in level $i - V_0$ of $\mathbf{B}_0^+$, due to the grading restriction on $\mathbf{v}_0$. Therefore if $i < 0$, $\mathbf{v}_0(x_i) = 0$ as there are no elements in B_0^+ at level i for negative i. Now consider $i \geq V_0 + 1$. Assume indirectly that $\mathbf{v_0}(x_i) = \gamma_{i-(V_0)}$. Then we have that

$$U(\mathbf{v_0}(x_i)) = U \cdot \gamma_{(i-V_0)} = \gamma_{(i-V_0-1)} \neq 0, \text{ since } i \geq V_0 + 1.$$

But by the U-equivariance of $\mathbf{v}_0$ we also have

$$U(\mathbf{v}_0(x_i)) = \mathbf{v}_0(U \cdot x_i) = \mathbf{v}_0(0) = 0.$$

Thus we have a contradiction so $\mathbf{v}_0(x_i) = 0$. It can be shown that $\mathbf{h}_0(x_i) = 0$ using the same arguments. □

Lemma 4.3.7. *Let $V_0 = H_0 = 1$ and x be in level 1 of $\mathbf{A}_0^{red}$. Then $\mathbf{v_0}(x) = 0 = \mathbf{h_0}(x)$.*

Proof. Due to the grading restrictions on $\mathbf{v}_0$ and $\mathbf{h}_0$ we know that $\mathbf{v}_0(x) = 0$ or $\mathbf{v}_0(x) = \gamma_0$ and similarly $\mathbf{h}_0(x) = 0$ or $\mathbf{h}_0(x) = \beta_0$. Thus the following three cases are the only possible maps for $\mathbf{v}_0$ and $\mathbf{h}_0$ besides $\mathbf{v_0}(x) = 0 = \mathbf{h_0}(x)$. We will show that in each case we get a contradiction, and thus $\mathbf{v_0}(x) = 0 = \mathbf{h_0}(x)$ must hold.

Case 1: Let $\mathbf{h}_0(x) = \beta_0$, $\mathbf{v}_0(x) = \gamma_0$. Here we have

$$\mathbf{D}^+(x + a_1) = 2\beta_0 + 2\gamma_0 = 0.$$

Since we are working with -1-surgery we have that $x + a_1 \in \ker(\mathbf{D}^+) \cong HF_{red}$. This contradicts Corollary 4.3.2, and our proof is complete.

Case 2: Let $\mathbf{h}_0(x) = \beta_0$, $\mathbf{v}_0(x) = 0$. Recall there exists an isomorphism $\phi\colon \mathbf{A}_0^+ \to \mathbf{A}_0^+$ that is U-equivariant, grading preserving, and satisfies $\mathbf{v}_0 \circ \phi = \mathbf{h}_0$. Therefore $\mathbf{v}_0(\phi(x)) = \gamma_0$, and $\mathbf{h}_0(\phi(x)) = 0$. It follows that $\phi(x) \neq x$, and $U \cdot \phi(x) = 0$ by the U-equivariance of ϕ. Assume indirectly that $\phi(x) = a_1 + r$ where $r \in A_0^{red}$. Then $U \cdot (a_1 + r) = a_0$. But this contradicts the U-equivariance of ϕ, so $\phi(x) \in A_0^{red}$. Then we have

$$\mathbf{D}^+(a_1 + x + \phi(x)) = \beta_0 + \gamma_0 + \beta_0 + \gamma_0 = 0.$$

Since we are working with -1-surgery we have that $x + a_1 \in \ker(\mathbf{D}^+) \cong HF_{red}$. This contradicts Corollary 4.3.2, and our proof is complete.

Case 3: Let $\mathbf{h}_0(x) = 0$, $\mathbf{v}_0 = \gamma_0$. This is argued similarly to case 2.

Thus we have shown that the only allowable maps for $\mathbf{v}_0$ and $\mathbf{h}_0$ must satisfy $\mathbf{v}_0(x) = 0 = \mathbf{h}_0(x)$, for any element x in level 1 of $\mathbf{A}_0^{red}$. □

Proposition 4.3.8. *Let K be a genus 1 knot K in S^3, and let x be in level i of $\mathbf{A}_0^{red}$. Then $\mathbf{v}_0(x_i) = 0 = \mathbf{h}_0(x_i)$ for $i \neq 0$.*

Proof. This follows directly from Lemmas 4.3.6 and 4.3.7. □

Given these restrictions, we are now ready to prove Theorem 1.2.5. Figure 4.5 depicts the truncated mapping cone for $\frac{1}{n}$ surgery on genus 1 knots for positive n. We consider the element $a^j_{(0,i)}$ to be in level j of the i^{th} copy of $\mathbf{A}_0^+$, denoted $\mathbf{A}^+_{(0,i)}$. We have a similar notation for $b^l_{(0,k)}$ in level l of $\mathbf{B}^+_{0,k}$.

4.3.2 Proof of Theorem 1.2.5

Now we are ready to prove our main result for genus 1 knots.

Theorem 1.2.5. *For a genus 1 knot K and $n \in \mathbb{Z}$ we have that $U \cdot HF_{red}(S^3_{1/n}(K)) = 0$.*

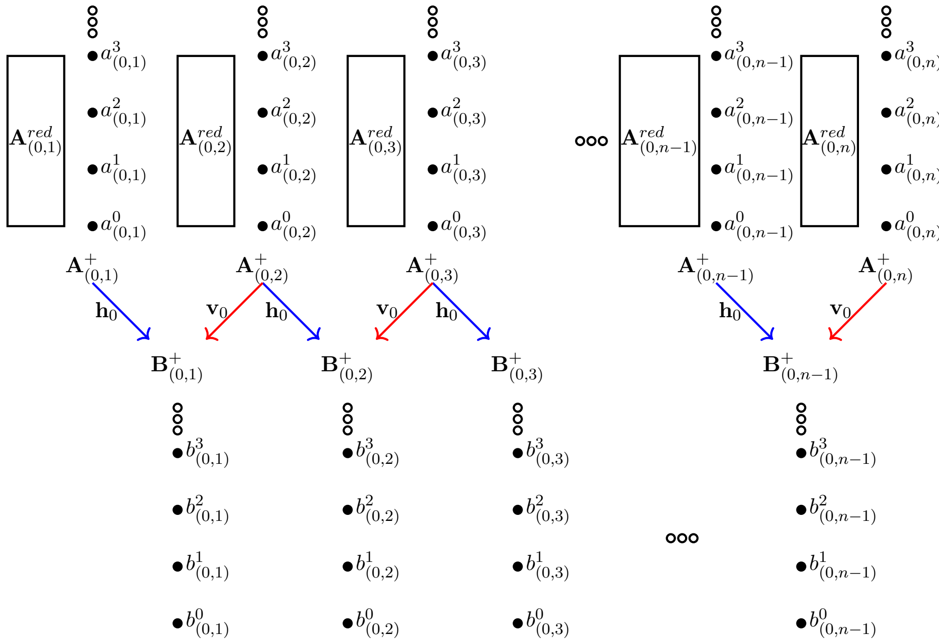


Figure 4.5: $\frac{1}{n}$ surgery mapping cone for a genus 1 knot

Proof. Assume $n > 0$. By Proposition 4.3.8, we can write any element of $\ker(\mathbf{D}^+_{1/n})$ as $A + R$ where

$$A = a^i_{(0,1)} + a^i_{(0,2)} + \cdots + a^i_{(0,n)}$$

for $i \geq 0$, and R is the sum of elements in $\mathbf{A}^{red}$. Additionally, any element at level 0 could also be in $\ker\left(\mathbf{D}^+_{1/n}\right)$. We need not consider these elements as$U \cdot y$ for y in level 0 is 0. We know that A is in the tower as by construction. Thus $HF_{red}(S^3_{1/n}(K))$ consists of elements in $\mathbb{A}^{red}_0$. Therefore $U \cdot HF_{red}(S^3_{1/n}(K)) = 0$ by Corollary 4.2.3, for positive n.

Now consider negative surgery. We know that $S^3_{1/n}(K) = -S^3_{-1/n}(\overline{K})$ where $\overline{K}$ is the mirror of K. Since K and $\overline{K}$ are both genus 1 knots it follows that

$$U \cdot HF_{red}(S^3_{-1/n}(K)) = 0$$

by the arguments for positive $\frac{1}{n}$ surgery above. □

Genus 2 Results

4.3.3 Preliminaries

In this section we will work with ± 2 surgery on genus 2 knots. For ± 2 surgery we will have two Spin^c-structures, which we will call $\mathfrak{s}$ and $\mathfrak{t}$. Specifically, we choose $\mathfrak{s}$ such that $HF^+(S^3_{+2}(K), \mathfrak{s}) \cong \ker(\mathbf{D}^+_{\mathfrak{s}})$, where $\ker(\mathbf{D}^+_{\mathfrak{s}})$ is the appropriate sum of $\mathbf{h}_{-1}$ and $\mathbf{v}_1$. Then $HF^+(S^3_{+2}(K), \mathfrak{t}) \cong \ker(\mathbf{v}_0)$.

Before we look at the mapping cone, we introduce map factorizations of v_k and h_k, [22]. The map v_k factors through v_{k+1} via the following,

$$A_k^+ \xrightarrow{f} A_{k+1}^+ \xrightarrow{v_{k+1}^+} B^+.$$

The map f sends every element $[x, i, j]$ with $i \geq 0$ or $j \geq k+1$ to itself, and sends everything else to 0. We have a similar factorization for h_{k+1} via h_k,

$$A_{k+1}^+ \xrightarrow{U} C\{i \geq -1, j \geq k\} \xrightarrow{\psi} A_k^+ \xrightarrow{h_k^+} B^+.$$

Here ψ sends everything with $i = -1$ to 0, and everything else to itself. Now we introduce an additional subquotient of CFK^∞, which we will call the strip, and write as S_k. We define this as follows,

$$S_k = \{[x, i, j] \colon j = k, i < 0\},$$

and we refer to the homology of the strip as H_k. It is from the structure CFK^∞ that $U \cdot H_k = 0$, regardless of k. We also have a clear inclusion $S_k \hookrightarrow A_k$ as modules. Now using the map $f \colon A_k \to A_{k+1}^+$ we have the following short exact sequence,

$$0 \to S_k \xrightarrow{\iota} A_k^+ \xrightarrow{f} A_{k+1}^+ \to 0,$$

which induces the following exact triangle,

$$\begin{array}{ccc} \mathbf{A}_k^+ & \xrightarrow{f_*} & \mathbf{A}_{k+1}^+ \\ & {\scriptstyle \iota_*} \nwarrow & \downarrow {\scriptstyle d} \\ & & H_k. \end{array}$$

Now we are ready to prove some preliminary results. The truncated mapping cone for +2-surgery on a genus 2 knot is shown in Figure 4.6, and we follow this same notation.

Lemma 4.3.9. *Let $x \in \mathbf{A}_1^{red}$ at level $i \geq 2$. Then $\mathbf{v}_1(x) = 0$.*

Proof. Let $x \in \mathbf{A}_1^{red}$ at level $i \geq 2$. Assume indirectly that $\mathbf{v}_1(x) \neq 0$. By the grading restrictions of $\mathbf{v}_1$, we know that $\mathbf{v}_1(x) = b_{(-1, i-V_1)}$. If $V_1 = 2$, we have that $H_1 = 3$, but this contradicts Lemma 4.3.1, so we must have $V_1 \leq 1$.

We know that $\mathbf{v}_2 \colon \mathbf{A}_2^+ \to \mathbf{B}^+$ is an isomorphism since we are working with genus 2 knots. By the map factorization above we have that $\mathbf{v}_1(x) = \mathbf{v}_2(f(x)) = b_{(-1, i-V_1)}$. Thus $f(x) = a_{(2, i-V_1)}$, following the same naming conventions seen in Figure 4.6.

Now consider $f(x + a_{(1,i)})$. By the grading restrictions this maps to $2a_{(2, i-V_1)} = 0$ as we are working over $\mathbb{Z}_2$. Thus $x + a_{(1,i)} \in \ker(f)$. By the exact triangle above it follows that $x + a_{(1,i)} \in \operatorname{Im}(\iota_*)$. Thus there exists some element $y \in H_k$, that satisfies $\iota_*(y) = x + a_{(1,i)}$. By the U equivariance of ι_* we have that

$$\iota_*(Uy) = U\iota_*(y) = U(x + a_{(1,i)}),$$

which is nonzero, because we chose $i \neq 0$. But thus Uy nonzero for $y \in H_k$. This is a contradiction as $U \cdot H_k = 0$, and thus our proof is complete. □

Lemma 4.3.10. *Let $x \in \mathbf{A}_{-1}^{red}$ at level $i \geq 2$. Then $\mathbf{h}_{-1}(x) = 0$.*

Proof. Let $x \in \mathbf{A}_{-1}^{red}$ at level $i \geq 2$. Assume indirectly that $\mathbf{h}_{-1}^{+}(x) \neq 0$. By the grading restrictions on $\mathbf{h}_{-1}$ we know that $\mathbf{h}_{-1}(x) = b_{(-1,i-H_{-1})}$.

Recall there exists an isomorphism $\phi\colon \mathbf{A}_{-1}^{+} \to \mathbf{A}_{1}^{+}$ that is U-equivariant, grading preserving, and satisfies $\mathbf{v}_1 \circ \phi = \mathbf{h}_{-1}$. Thus we know that $\mathbf{v}_1(\phi(x)) = b_{(-1,i-H_{-1})}$, and $\phi(x)$ occurs at level i, since $V_1 = H_{-1}$. Since $U^2 \cdot x = 0$ by Corollary 4.2.3, by the U-equivariance of ϕ we have that $U^2 \cdot \phi(x) = 0$. Assume indirectly that $\phi(x) = a_{(1,i)} + r$ where $r \in A_0^{red}$. Then $U^2 \cdot (a_{(1,i)} + r) = a_{(1,i-2)}$, as we assumed $i \geq 2$. But this contradicts the U-equivariance of ϕ, so $\phi(x) \in A_0^{red}$. Then we have that $\mathbf{v}_1(\phi(x)) = b_{(-1,i-H_{-1})}$, which contradicts Lemma 4.3.9 and so our proof is complete. □

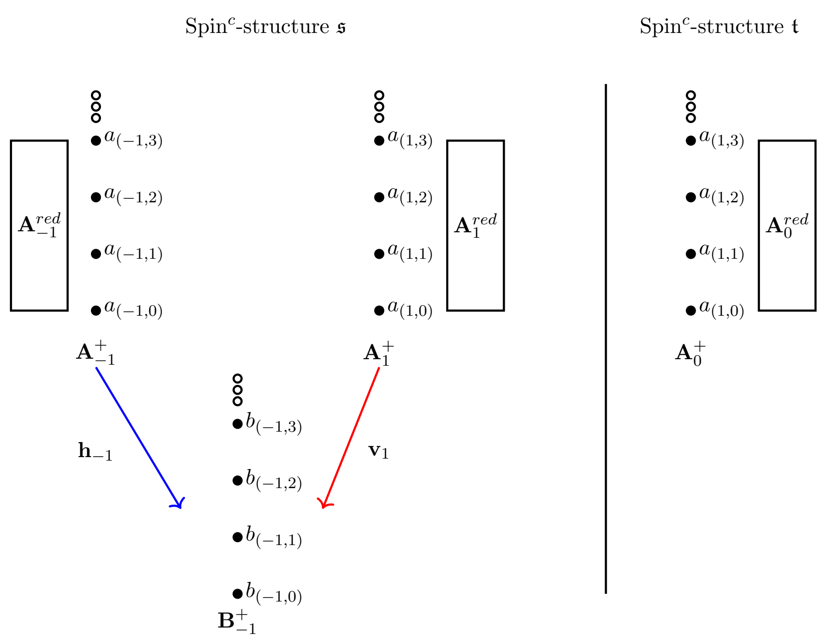


Figure 4.6: +2 surgery mapping cone for a genus 2 knot

4.3.4 Results

Lemma 4.3.11. *Let K be a genus 2 knot in S^3, then*

$$U^2 \cdot HF_{red}\left(S^3_{\pm 2}(K), \mathfrak{t}\right) = 0.$$

Proof. First we consider +2 surgery on a genus 2 knot. Here we have $HF^{+}(S^3_2(K),\mathfrak{t}) = \mathbf{A}^{+}_0$, and thus $HF_{red}(S^3_2(K)) \cong \mathbf{A}^{red}_0$. Then by Corollary 4.2.3 we have that $U^2 \cdot HF_{red}(S^3_2(K)) = 0$, as desired. Now consider -2 surgery on a genus 2 knot, let $Y = S^3_{-2}(K)$. We have that $S^3_{-2}(K) = Y = -S^3_2(\overline{K})$, where $\overline{K}$ is the mirror of K. Thus we have that

$$\begin{aligned} U^2 \cdot HF_{red}\,(Y,\mathfrak{t}) &= U^2 \cdot HF_{red}\left(S^3_{-2}(\overline{K}),\mathfrak{t}\right) \\ &\cong U^2 \cdot HF_{red}\left(S^3_2(K),\mathfrak{t}\right) \\ &= 0. \end{aligned}$$

□

Lemma 4.3.12. *Let K be a genus 2 knot in S^3, then*

$$U^2 \cdot HF_{red}\left(S^3_{\pm 2}(K),\mathfrak{s}\right) = 0.$$

Proof. Consider +2 surgery on a genus 2 knot. We know that

$$HF^{+}\left(S^3_{+2}(K),\mathfrak{s}\right) = \ker(\mathbf{D}^{+}_{i,2}),$$

for positive surgeries. By Lemmas 4.3.9 and 4.3.10, we know that elements of $A^{red}_{\pm 1}$ at level $i \geq 2$ must be sent to 0.

Consider elements in $\ker\left(\mathbf{D}^{+}_{i,2}\right)$ that come from level 2 or above in $\mathbf{A}^{+}_{\pm 1}$. These elements consist of sums $a + x$, where $a = a_{(1,i)} + a_{(-1,i)}$ and x is the sum of elements in $\mathbf{A}^{red}_{\pm 1}$. Since we took a and x to be elements that occur at or above level 2, we know $U^2 \cdot (a + x) = U^2 \cdot a$, which is zero in $HF_{red}\left(S^3_{+2}(K),\mathfrak{s}\right)$. Thus $U^2 \cdot HF_{red}\left(S^3_{+2}(K),\mathfrak{s}\right) = 0$ in this case.

Now consider elements of the form $a + x$ as above, where these are appropriate sums such that $a + x \in \ker(\mathbf{D}^{+}_{i,2})$ and all elements summed are at level 1 or below. By construction $U^2(a + x) = 0$ in this case, as anything in the tower at level 1 or 0 will be sent to 0 by U^2, and any $U^2 \cdot x = 0$ by Corollary 4.2.3. Thus

$$U^2 \cdot HF_{red}\left(S^3_2(K),\mathfrak{s}\right) = 0.$$

Now we must address -2 surgery. Let $Y = S^3_{-2}(K)$, where K is a genus 2 knot. We have that $S^3_{-2}(K) = Y = -S^3_2(\overline{K})$, where $\overline{K}$ is the mirror of K. Thus we have that

$$\begin{aligned} U^2 \cdot HF_{red}\,(Y,\mathfrak{s}) &= U^2 \cdot HF_{red}\left(S^3_{-2}(K),\mathfrak{s}\right) \\ &= U^2 \cdot HF_{red}\left(-S^3_2(\overline{K}),\mathfrak{s}\right) \\ &= 0. \end{aligned}$$

□

Theorem 1.2.6. *Let K be a genus 2 knot in S^3, then the following holds for any Spin^c-structure i.*

$$U^2 \cdot HF_{red}\left(S^3_{\pm 2}(K), i\right) = 0.$$

Proof. This follows directly from Lemmas 4.3.11 and 4.3.12. □

REFERENCES

[1] Boyer, S. (2002). Dehn surgery on knots. In *Handbook of geometric topology*, pages 165–218. North-Holland, Amsterdam.

[2] Boyer, S. and Zhang, X. (1994). Exceptional surgery on knots. *Bull. Amer. Math. Soc. (N.S.)*, 31(2):197–203.

[3] Brittenham, M. and Wu, Y.-Q. (2001). The classification of exceptional Dehn surgeries on 2-bridge knots. *Comm. Anal. Geom.*, 9(1):97–113.

[4] Can, M. B. and Karakurt, C. (2014). Calculating Heegaard-Floer homology by counting lattice points in tetrahedra. *Acta Math. Hungar.*, 144(1):43–75.

[5] Dean, J. C. (2003). Small Seifert-fibered Dehn surgery on hyperbolic knots. *Algebr. Geom. Topol.*, 3:435–472.

[6] Eudave-Muñoz, M. (2002). On hyperbolic knots with Seifert fibered Dehn surgeries. In *Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999)*, volume 121, pages 119–141.

[7] Gabai, D. (1989). Surgery on knots in solid tori. *Topology*, 28(1):1–6.

[8] Gainullin, F. (2017). The mapping cone formula in Heegaard Floer homology and Dehn surgery on knots in S^3. *Algebr. Geom. Topol.*, 17(4):1917–1951.

[9] Gordon, C. M. and Luecke, J. (1995). Dehn surgeries on knots creating essential tori. I. *Comm. Anal. Geom.*, 3(3-4):597–644.

[10] Gordon, C. M. and Luecke, J. (2004). Non-integral toroidal Dehn surgeries. *Comm. Anal. Geom.*, 12(1-2):417–485.

[11] Hom, J., Lidman, T., and Zufelt, N. (2015). Reducible surgeries and Heegaard Floer homology. *Math. Res. Lett.*, 22(3):763–788.

[12] Ichihara, K. and Jong, I. D. (2010). Toroidal Seifert fibered surgeries on Montesinos knots. *Comm. Anal. Geom.*, 18(3):579–600.

[13] Ichihara, K. and Masai, H. (2016). Exceptional surgeries on alternating knots. *Comm. Anal. Geom.*, 24(2):337–377.

[14] Karakurt, C. and Lidman, T. (2015). Rank inequalities for the Heegaard Floer homology of Seifert homology spheres. *Trans. Amer. Math. Soc.*, 367(10):7291–7322.

[15] Kronheimer, P. B. and Mrowka, T. S. (2004). Witten's conjecture and property P. *Geom. Topol.*, 8:295–310.

[16] Lickorish, W. B. R. (1962). A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540.

[17] Miyazaki, K. and Motegi, K. (2000). Toroidal surgery on periodic knots. *Pacific J. Math.*, 193(2):381–396.

[18] Moser, L. (1971). Elementary surgery along a torus knot. *Pacific J. Math.*, 38:737–745.

[19] Motegi, K. (2003). Dehn surgeries, group actions and Seifert fiber spaces. *Comm. Anal. Geom.*, 11(2):343–389.

[20] Némethi, A. (2005). On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds. *Geom. Topol.*, 9:991–1042.

[21] Némethi, A. (2007). Graded roots and singularities. In *Singularities in geometry and topology*, pages 394–463. World Sci. Publ., Hackensack, NJ.

[22] Ni, Y. and Wu, Z. (2015). Cosmetic surgeries on knots in S^3. *J. Reine Angew. Math.*, 706:1–17.

[23] Ozsváth, P. and Szabó, Z. (2004a). Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116.

[24] Ozsváth, P. and Szabó, Z. (2004b). On Heegaard Floer homology and Seifert fibered surgeries. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 181–203. Geom. Topol. Publ., Coventry.

[25] Ozsváth, P. and Szabó, Z. (2006). An introduction to Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 3–27. Amer. Math. Soc., Providence, RI.

[26] Ozsváth, P. S. and Szabó, Z. (2011). Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.*, 11(1):1–68.

[27] Rasmussen, J. A. (2003). *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Harvard University.

[28] Saveliev, N. (2002). *Invariants for homology 3-spheres*, volume 140 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin. Low-Dimensional Topology, I.

[29] Singer, J. (1933). Three-dimensional manifolds and their Heegaard diagrams. *Trans. Amer. Math. Soc.*, 35(1):88–111.

[30] Teragaito, M. (2010). Toroidal Dehn surgery on hyperbolic knots and hitting number. *Topology Appl.*, 157(1):269–273.

[31] Thurston, W. P., Kerckhoff, S., Floyd, W. J., and Milnor, J. W. (1997). *The geometry and topology of three-manifolds.* publisher not identified.

[32] Wallace, A. H. (1960). Modifications and cobounding manifolds. *Canad. J. Math.*, 12:503–528.

[33] Wu, Y.-Q. (2011a). The classification of toroidal Dehn surgeries on Montesinos knots. *Comm. Anal. Geom.*, 19(2):305–345.

[34] Wu, Y.-Q. (2011b). Exceptional Dehn surgery on large arborescent knots. *Pacific J. Math.*, 252(1):219–243.

[35] Wu, Y.-Q. (2013). Immersed surfaces and Seifert fibered surgery on Montesinos knots. *Trans. Amer. Math. Soc.*, 365(5):2469–2488.