Non-Reflecting Boundary for Finite Element Method

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1. Introduction

The non-reflecting boundary for the numerical analysis of the wave propagation in an infinite medium is a persistent problem of the foundation dynamics. Since Lysmer and Kuhlemeyer(2) presented the standard viscous boundary approach in 1969, various methods were proposed, as transmitting boundary by Waas(4), superposition boundary by Smith(3), and absorbing boundary by Clayton and Engquist(1).

These artificial boundaries can be grouped into two classes. The one group is based upon the general solution of the wave equation, and the other is related with the peculiarity of the complete solution. Although Lysmer and Kuhlemeyer do not state the fundamental principle of their approach, the viscous boundary falls within the former. And the superposition boundary falls within the latter.

The present study is developed in the context of the first group. The study has two distinctive points as follows. (1) The FE equations are derived from the Hamilton's Principle. The Lagrangian Function for the domain Γ with surface Ω_1 and inner imaginary boundary Ω_2 contains unknown forces in the potential energy on the imaginary boundary. The unknowns are related with the space derivatives of displacements through the stationary condition. Manipulating the general solution of the wave equation, the space derivatives can be transformed into the time derivatives. In this way the Neuman condition on the boundary Ω_2 changed to the Cauchy condition. Therefore the mathematical treatment for the non-reflecting boundary in this study is rather logical and rigorous than in literature 2 and its descendants. (2) The usual algorithm of the non-reflecting boundary by the FEM incorporates the motion of the near field domain with that of the far field domain. In this paper the far field domain is not considered, since the problem is solved for a free body of the near field under the mixed Cauchy condition.

Next section deals with the variational principle and the conversion of Neuman boundary condition into the Cauchy condition. In section 3 the results of numerical analysis for two dimensional Lamb's problem are presented. In section 4 discussions and conclusions are mentioned.

2. Boundary Condition on Inner Imaginary Boundary

Consider the finite domain Γ in an elastic half space shown in Fig.1. T_i are tractions on the surface Ω_1, and $\alpha_i + q_{ij} u_j$ are forces which act on the inner boundary Ω_2 . Potential energies V_1 and V_2 of the forces are given by

$$V_1 = -\int_{\Omega_1} T_i\, u_i\, ds$$
$$V_2 = -\int_{\Omega_2} (\alpha_i\, u_i + \frac{1}{2} q_{ij}\, u_j\, u_i)\, ds \qquad (1)$$

while the total strain energy and the kinetic energy are given by

$$U = \int_\Gamma W\, dv = \frac{1}{2}\int_\Gamma \sigma_{ij}\, \varepsilon_{ij}\, dv$$
$$K = \frac{1}{2}\int_\Gamma \rho\, \dot{u}_i\, \dot{u}_i\, dv \qquad (2)$$

The potential energy of body force is not considered. In accordance with Hamilton's principle the integral of $(U-K+V_1+V_2)$ within a time interval $[t_0, t_1]$ for given initial and final configurations must be stationary. The stationary condition yields the Euler equations and the natural boundary conditions. In the following we concentrate to consider the natural boundary condition on the inner imaginary boundary Ω_2 .

Since $\delta u_i \neq 0$ on Ω_2 ,the natural boundary conditions become

$$\{\lambda\, u_{k,k}\, \delta_{ij} + \mu \frac{1}{2}(u_{i,j} + u_{j,i})\}\ \nu_j + \alpha_i + q_{ij} u_j = 0 \qquad (3)$$

The similar boundary conditions are obtained on Ω_1 . Therefore the variational problem gives a Neuman boundary-value problem, so that we get nonunique solutions. It is required to transform the boundary conditions in order to obtain the unique solution.

To this end we use the general solution of the wave equation. Assuming the plane strain problem that the semiinfinite space is subjected to the dynamic line load through the origin O in Fig.1, we obtain the natural boundary condition on Ω_2 as

$$(\lambda+\mu)\nu_x\, \Delta + \mu(\nu_x\, u_x + \nu_z\, u_z) + \mu(-\nu_x\, w_z + \nu_z\, w_x) + \alpha + q_1 u + q_2 w = 0$$

$$(\lambda+\mu)\nu_z\, \Delta + \mu(\nu_x\, w_x + \nu_z\, w_z) + \mu(\nu_x\, u_z - \nu_z\, u_x) + \beta + r_1 u + r_2 w = 0 \qquad (4)$$

where $\Delta = u_x + w_z$

Subscripts to the displacements denote the partial differentiations. α and β are used in place of α_i , and q_1 , q_2 , r_1 and r_2 are used in place of q_{ij}.

From the general solution for the surface line source the displacements are writen as

$$u = l_x A_P f(c_1 t - l_x x - l_z z) + l_z A_{SV}\, g(c_2 t - l_x x - l_z z)$$
$$w = l_z A_P f(c_1 t - l_x x - l_z z) - l_x A_{SV}\, g(c_2 t - l_x x - l_z z) \qquad (5)$$

f and g are displacement functions, l_x and l_z are directional cosines of wave propagation direction, A_P and A_{SV} are amplitudes of p and sv waves, and c_1 and c_2 are their velocities.

From eqs.(5) we obtain the relations between space and time derivatives of displacements as

$$u_x = -\frac{l_x^2}{c_1} A_1 - \frac{l_x l_z}{c_2} A_2$$

$$u_z = -\frac{l_x l_z}{c_1} A_1 - \frac{l_z^2}{c_2} A_2$$

$$w_x = -\frac{l_x l_z}{c_1} A_1 + \frac{l_x^2}{c_2} A_2 \qquad (6)$$

$$w_z = -\frac{l_z^2}{c_1} A_1 + \frac{l_x l_z}{c_2} A_2$$

where $A_1 = l_x \dot{u} + l_z \dot{w}$, $A_2 = l_z \dot{u} - l_x \dot{w}$

Substituting eqs.(6) into eqs.(5), we obtain the expressions for the forces acting on the boundary Ω_2 by particle velocities as

$$\alpha + q_1 u + q_2 w = \frac{(\lambda+\mu)\nu_x}{c_1} A_1 + \mu \left\{ \frac{l_x \gamma + l_z \omega}{c_1} A_1 + \frac{l_z \gamma - l_x \omega}{c_2} A_2 \right\}$$

$$\beta + r_1 u + r_2 w = \frac{(\lambda+\mu)\nu_z}{c_2} A_1 + \mu \left\{ \frac{l_z \gamma - l_x \omega}{c_1} A_1 - \frac{l_x \gamma + l_z \omega}{c_2} A_2 \right\} \qquad (7)$$

where $\gamma = l_x \nu_x + l_z \nu_z$ $\quad \omega = l_x \nu_z - l_z \nu_x$

Thus, the Neuman condition problem for the domain Γ is converted into the mixed Cauchy condition problem, so that the unique solutions can be obtained from the Hamilton's principle by using the direct method as the FEM.

Comparing eqs.(7) to the formulas of the standard viscous boundary, it is understood that the standard viscous approximation is correct for one dimensional problem.

3. Numerical Computation by FEM

The proposed method has been applied to one and two dimensional (plane strain) problems to investigate the accuracy.

The result for one dimensional problem agreed completely with the theoretical solution. No reflections did occur on the imaginary boundary.

In this paper the two dimensional results are presented to demonstrate the validity of the proposal. The global form and the FE discretization of the analyzed region, and the surface traction are shown in Fig.2. The load varies sinusoidaly with time, and acts by one cycle. Regarding the stepwise integration the Houbolt's method is used. And the Simpson's formula is used for the boundary integration.

Fig.3 is the computational result for the input of longer period (0.4 sec/cycle). It shows the displacements vs. time in vertical direction of some points. There are no disturbances in the displacements caused by the imaginary boundary.

Fig.4 shows the results for two different periods (0.1 sec/cycle and 0.02 sec/cycle)for comparison. Although the presented curves are the displacements of only some surface points, the proposed imaginary boundary seems to have a sufficient function as a non-reflecting boundary without reference to the periodic time of the incident wave. Arrows in the figure of horizontal displacement show the arrival of the Rayleigh wave. The numerical result yields 1889 m/sec for the wave velocity. The theoretical velocity is 0.9554 times c_2 ,that is 1911 m/sec.

4. Discussion and Conclusion

(1)In order to investigate the effect of the critical angle the cutoff domain was used in the numerical examples. The results were independent to the incident angle to the cutoff line.

(2)In the numerical results the rigid displacement and rotation were not found. The mixed Cauchy condition was valid.

(3)The demonstrative examples were very few. Nevertheless the proposed boundary seems to be efficient.

(4)With respect to the application of this method to engineering problems several conclusions will be suggested. Among of them it is comprehensive and most important to provide the method with treatment for the synchronous arrival of a number of waves with the different angle of direction to the imaginary boundary.

References

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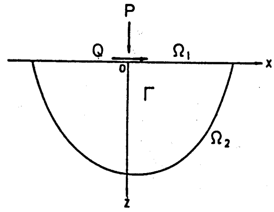

Fig.1 Imaginary Boundary in Half Space

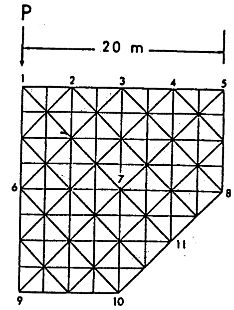

Fig.2 FE Discretization of Analyzed Domain

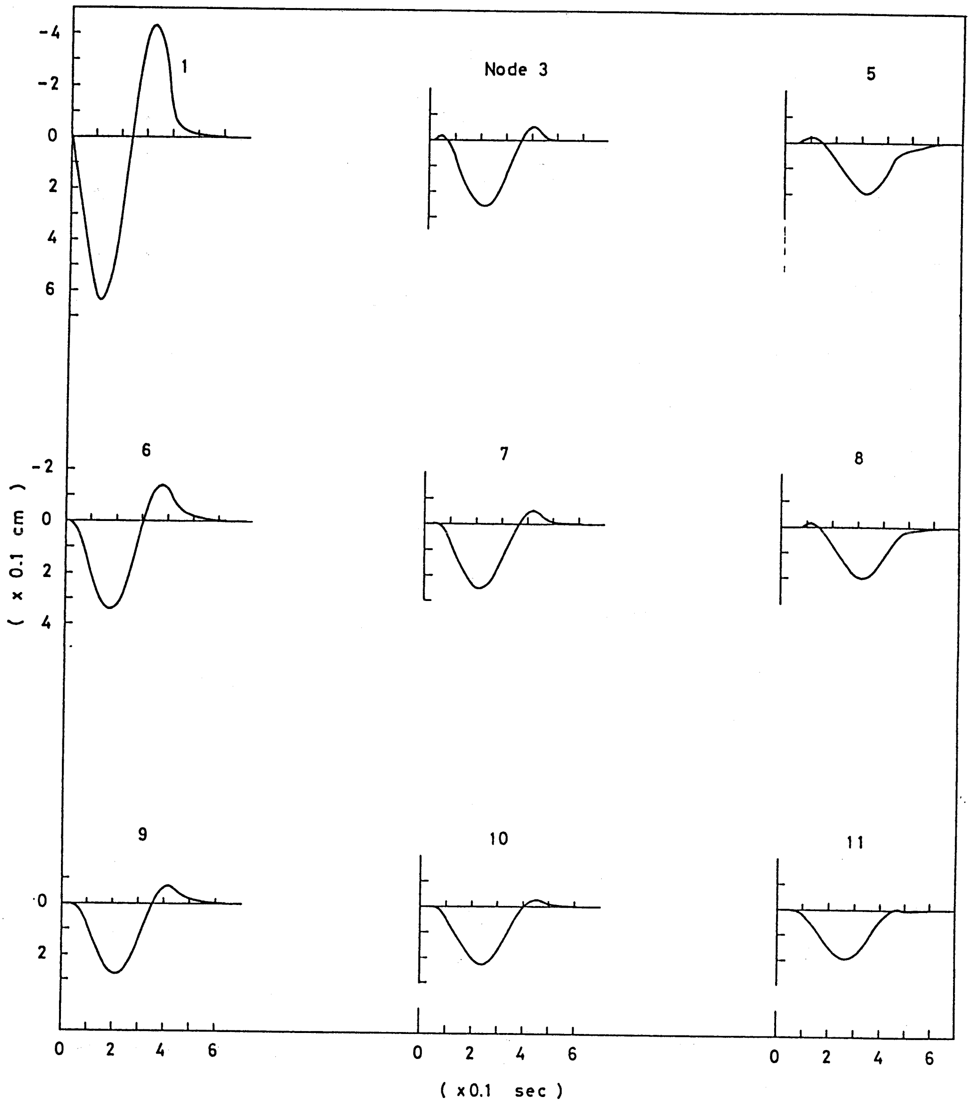

Fig. 3 Vertical Displacement Response to Line Load $P\sin\frac{2\pi}{T}t$

(λ=1000 kgf/cm² μ=250 kgf/cm² P=100 kgf T=0.4sec)

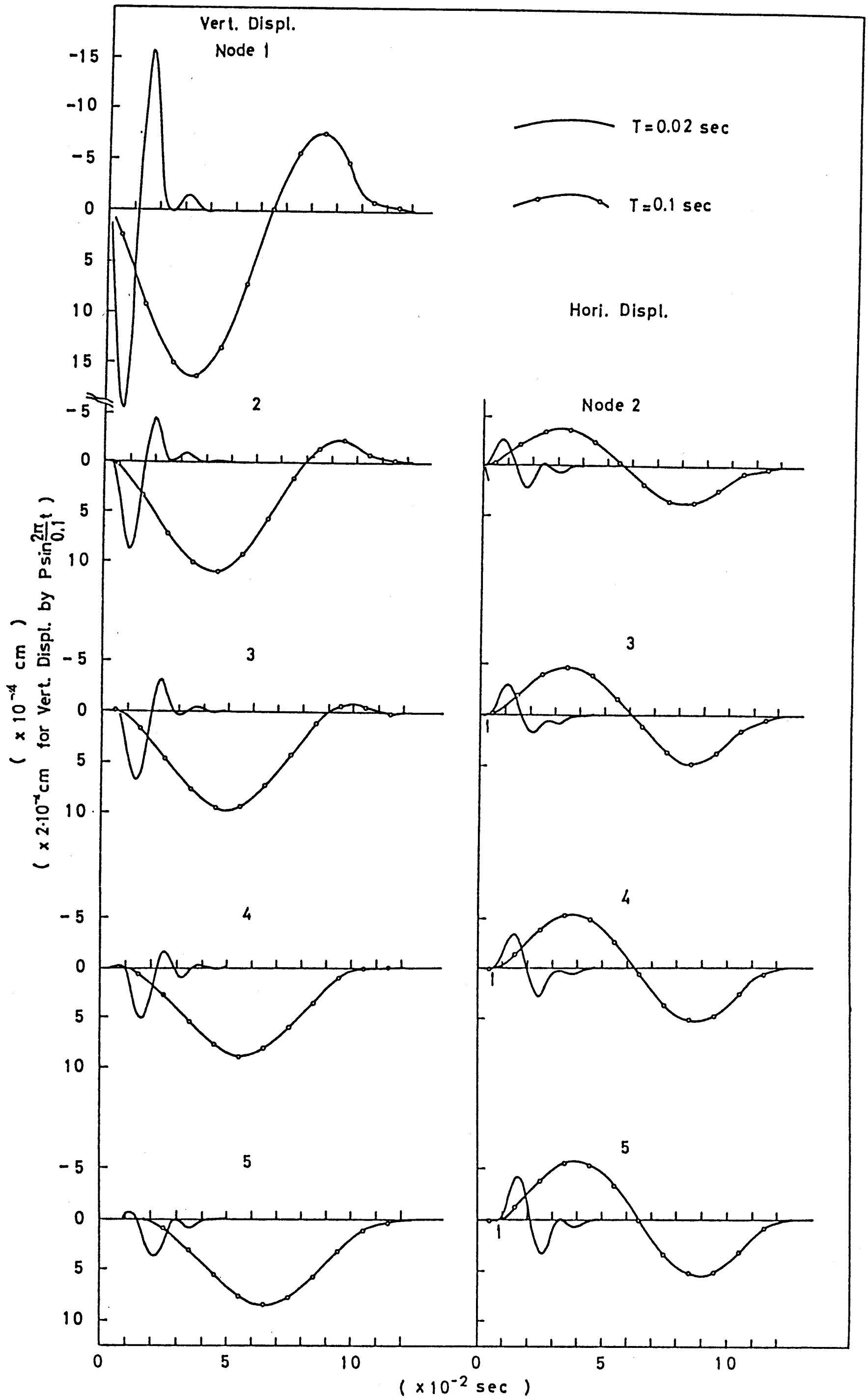

Fig.4 Displacement Response to Line Load $P\sin\frac{2\pi}{T}t$

($\lambda=\mu=80000$ kgf/cm² P =100 kgf)