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Summary. The present investigation is concerned with the proposal and study of a class of nonparametric paired comparison tests for the hypothesis of no differences among several objects with respect to a pair of characteristics.

1. Introduction. Let there be $t(\geq 2)$ objects which form $\binom{t}{2}$ pairs, and let the $i^{th}$ object have $n_{ij}(\geq 0)$ encounters with the $j^{th}$ object, for $i<j=1,\dots,t$ (conventionally, $n_{ij} = n_{ji}$ for $i > j$). We let

$$N = \Sigma_{i<j=1}^{t} n_{ij} = \tfrac{1}{2}\Sigma_{i\neq j=1}^{t} n_{ij}. \tag{1.1}$$

For each of the $n_{ij}$ encounters, it is judged whether the $i^{th}$ object is preferred (or not) to the $j^{th}$ object for each of the two characteristics $(\alpha,\beta)$, and the order of preference is indicated by $\succ$ (e.g., $\alpha_i \succ \alpha_j \Rightarrow$ the $i^{th}$object is preferred to the $j^{th}$ object for the characteristic $\alpha$). Thus, each encounter results in one of the following four mutually exclusive and exhaustive outcomes (the probability of ties being neglected):

$$\begin{aligned} &A_{ij}^{(1)}:\ \alpha_i \succ \alpha_j,\ \beta_i \succ \beta_j,\ A_{ij}^{(2)}:\ \alpha_i \succ \alpha_j,\ \beta_i \prec \beta_j,\\ &A_{ij}^{(3)}:\ \alpha_i \prec \alpha_j,\ \beta_i \succ \beta_j,\ A_{ij}^{(4)}:\ \alpha_i \prec \alpha_j,\ \beta_i \prec \beta_j, \end{aligned} \tag{1.2}$$

for all $i\neq j=1,\dots,t$, and we let

$$\pi_{ij\cdot k} = P\{A_{ij}^{(k)}\} \text{ for } k=1,2,3,4 \text{ and } i\neq j=1,\dots,t. \tag{1.3}$$

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Thus, the $n_{ij}$ encounters (assumed to be stochastically independent) result in the following multinomial distribution

(1.4) $$\{n_{ij}!/\prod_{k=1}^{4} n_{ij\cdot k}!\}\prod_{k=1}^{4} \pi_{ij\cdot k}^{n_{ij\cdot k}},$$

where $n_{ij\cdot k}$ is the observed frequency of the event $A_{ij}^{(k)}$, for k=1,2,3,4 and i<j=1,...,t. It may be noted that by definition

(1.5) $$\pi_{ij\cdot k} = \pi_{ji\cdot 5-k} \text{ and } n_{ij\cdot k} = n_{ji\cdot 5-k} \text{ for } k=1,\ldots,4,\ i<j=1,\ldots,t.$$

The null hypothesis to be tested relates to the equality of perference of all t objects with respect to both $(\alpha,\beta)$. This will imply that

(1.6) $$\pi_{ij\cdot k} = \pi_k \text{ for } k=1,\ldots,4 \text{ and all } i\neq j=1,\ldots,t,$$

where by virtue of (1.5), we have

(1.7) $$\pi_1 + \pi_2 = \pi_1 + \pi_3 = \pi_2 + \pi_4 = \pi_3 + \pi_4 = \tfrac{1}{2}.$$

Equivalently, (1.7) can be written as

(1.8) $$\pi_1 = \pi_4 = \tfrac{1}{4}(1+\theta),\ \pi_2 = \pi_3 = \tfrac{1}{4}(1-\theta),$$

where $\theta(0 \leq \theta \leq 1)$ is a real parameter which we shall term the association parameter. Thus, for convenience of discussion, we shall specify the null hypothesis by

(1.9) $$\begin{aligned} H_o:\ \pi_{ij\cdot k} &= \tfrac{1}{4}(1+\theta) \text{ if } k=1,4, \\ &= \tfrac{1}{4}(1-\theta), \text{ if } k=2,3, \text{ for } i<j=1,\ldots,t. \end{aligned}$$

The object of the present investigation is to consider some nonparametric tests for $H_o$ in (1.9) which may be regarded as natural extensions of some well-known tests for a single criterion only (cf. David [2, pp. 30-31]). Various properties of the proposed tests are also studied.

2. The likelihood ratio test and the spurious degrees of freedom. By (1.4), the unrestricted maximum of the likelihood function comes out as

$$(2.1) \qquad L(\hat{\Omega}) = \prod_{i<j=1}^{t} \left[ \left\{ n_{ij}! / \prod_{k=1}^{4} n_{ij\cdot k}! \right\} \prod_{k=1}^{4} (n_{ij\cdot k}/n_{ij})^{n_{ij\cdot k}} \right],$$

(where we adopt the convention that $x^0 = 1$ for $x = 0$). Under $H_o$ in (1.9), the likelihood function is

$$(2.2) \qquad \left[ \prod_{i<j=1}^{t} \left\{ n_{ij}! / \prod_{k=1}^{4} n_{ij\cdot k}! \right\} \right] 4^{-N} (1+\theta)^{N_1} \; (1-\theta)^{N-N_1},$$

where $N_1 = \Sigma_{i<j=1}^{t} (n_{ij\cdot 1} + n_{ij\cdot 4})$. Thus, the maximum likelihood estimator of $\theta$ is

$$(2.3) \qquad \hat{\theta}_N = (2N_1 - N)/N.$$

Hence, the maximum of the likelihood function under $H_o$ is

$$(2.4) \qquad L(\hat{\omega}) = \left[ \prod_{i<j=1}^{t} \left\{ n_{ij}! / \prod_{k=1}^{4} n_{ij\cdot k}! \right\} \right] (N_1/N)^{N_1} \; [(N-N_1)/N]^{N-N_1} .$$

Thus, the likelihood ratio (L.R.) criterion is

$$(2.5) \qquad \lambda_N = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{N_1^{N_1} (N-N_1)^{N-N_1}}{N^N} \cdot \prod_{i<j=1}^{t} \left\{ \frac{n_{ij}^{\,n_{ij}}}{\prod_{k=1}^{4} n_{ij\cdot k}^{\,n_{ij\cdot k}}} \right\} .$$

Now, under $H_o$ in (1.9), the distribution of $N_1$ is given by

(2.6) $$\binom{N}{N_1} 2^{-N} (1+\theta)^{N_1} (1-\theta)^{N-N_1}: \quad 0 \leq N_1 \leq N,$$

and hence, under $H_o$, the likelihood function conditional on $N_1$ or $\hat{\theta}_N$ is

(2.7) $$\binom{N}{N_1}^{-1} 2^{-N} \prod_{i<j=1}^{t} \left\{ n_{ij}! / \prod_{k=1}^{4} n_{ij \cdot k}! \right\} .$$

Since (2.7) is a completely known probability function, it is always possible to find a value of $\lambda_N$, say $\lambda_\epsilon(\hat{\theta}_N)$, such that

(2.8) $$P\{\lambda_N \leq \lambda_\epsilon(\hat{\theta}_N) | H_o, \hat{\theta}_N\} = \epsilon: \; 0 < \epsilon < 1,$$

$\epsilon$ being the preassigned level of significance of the test. (2.8) implies that

(2.9) $$P\{\lambda_N \leq \lambda_\epsilon(\hat{\theta}_N) | H_o\} = E_{\hat{\theta}_N} \{P[\lambda_N \leq \lambda_\epsilon(\hat{\theta}_N) | H_o, \hat{\theta}_N]\} = \epsilon.$$

Hence, the test based on the critical region: $\lambda_N \leq \lambda_\epsilon(\hat{\theta}_N)$ is a similar size $\epsilon$ test for $H_o$ in (1.9). [More precisely, we require in (2.8) and (2.9) a randomized test procedure to make the probabilities strictly equal to $\epsilon$; otherwise, we may take the probabilities as $\leq \epsilon$]. For large $n_{ij}$'s, using the results of Wald [5] we conclude that under $H_o$ in (1.9), $-2 \log \lambda_N$ has asymptotically a $\chi^2$ distribution with $3\binom{t}{2}-1$ degrees of freedom. Thus, asymptotically

(2.10) $$-2 \log \lambda_\epsilon(\hat{\theta}_N) \cong \chi^2_{3\binom{t}{2}-1, \epsilon},$$

where $\chi^2_{r,\epsilon}$ is the upper 100 $\epsilon$ % point of a $\chi^2$ distribution with r.d.f.

The L. R. test considered above is really a test for the identity of $\binom{t}{2}$ multinomial laws in (1.4) under the further specifications in (1.9), and hence carries $3\binom{t}{2}-1$ d.f. In actual practice, we are often not interested in such a broad class of alternatives but rather in a comprehensive test for an analysis of variance problem posed below. Let us define

$$\pi_{ij}(\alpha) = P\{\alpha_i \succ \alpha_j\} = \pi_{ij\cdot 1} + \pi_{ij\cdot 2}, \tag{2.11}$$

$$\pi_{ij}(\beta) = P\{\beta_i \succ \beta_j\} = \pi_{ij\cdot 1} + \pi_{ij\cdot 3}, \quad i \neq j=1,\ldots,t. \tag{2.12}$$

Under $H_o$ in (1.9), $\pi_{ij}(\alpha) = \pi_{ij}(\beta) = \frac{1}{2}$ for all $i \neq j=1,\ldots,t$. Thus, if we want to test for the homogeneity of $\pi_{ij}(\alpha)$'s and of $\pi_{ij}(\beta)$'s, the number of d.f. cannot exceed $2[\binom{t}{2}-1]$. Further, preference behavior is frequently stochastically transitive, i.e. if $\pi_{ij}(\alpha) \geq \frac{1}{2}$ and $\pi_{jk}(\alpha) \geq \frac{1}{2}$, then $\pi_{ik}(\alpha) \geq \frac{1}{2}$. Indeed, stronger restrictions on the preference probabilities are often reasonable and the $\binom{t}{2}$ quantities $\pi_{ij}(\alpha)$ may be expressible in terms of only t parameters, say $\pi_i^*(\alpha)$, $i=1,\ldots,t$. In a similar manner, we may proceed with $\pi_{ij}(\beta)$'s. Thus, for the homogeneity of the parameters in (2.11) and in (2.12), we can have alternative tests carrying only $2(t-1)$ d.f. To sum up, the L. R. test will have a much broader scope, but, for the specific purpose of testing homogeneity of the parameters in (2.11) and (2.12), it carries many spurious d.f. [It is well-known (cf. [3]) that in $\chi^2$ tests the effect of increasing the degrees of freedom is to reduce the power of the test unless the noncentrality parameter increases at a sufficiently fast rate to compensate; in the present situation, the spurious d.f. may not contribute much to the noncentrality parameter.]

The alternative test to be proposed is a natural generalization of the paired comparison test for a single characteristic (cf. David [2, pp. 30-31]) and will be appropriate if the characteristics $(\alpha, \beta)$ can be related to an underlying bivariate trait or random variable $(X,Y)$ in whose locations we are interested.

Fortunately, this is possible in a wide range of problems involving paired comparisons. In such a setting, the feasibility and optimality of an appropriate L.R. test requires knowledge of the underlying bivariate distribution. This in turn limits the scope of the inferences and usually renders quite complicated the solution of the likelihood equations necessary for the evaluation of the L. R. criterion. On the other hand, our proposed methods appear to be quite simple and reasonably efficient.

3. A permutationally distribution free test. We rewrite (2.7) as

$$\left[\binom{N}{N_1}^{-1} \prod_{i<j=1}^{t} \binom{n_{ij}}{n_{ij}^{(1)}}\right] \prod_{i<j=1}^{t} \left\{\left[\binom{n_{ij}^{(1)}}{n_{ij\cdot 1}} (\tfrac{1}{2})^{n_{ij}^{(1)}}\right]\left[\binom{n_{ij}^{(2)}}{n_{ij\cdot 2}} (\tfrac{1}{2})^{n_{ij}^{(2)}}\right]\right\}, \tag{3.1}$$

where $n_{ij}^{(1)} = n_{ij\cdot 1} + n_{ij\cdot 4}$, $n_{ij}^{(2)} = n_{ij\cdot 2} + n_{ij\cdot 3}$, for i j=1,...,t. Thus, the first factor of (3.1) is a (generalized) hypergeometric distribution, while the second factor is the product of t(t-1) independent binomial distributions; all these distributions are simple and well-tabulated. Let us now define

$$\begin{aligned} u_{ij} &= [n_{ij\cdot 1} + n_{ij\cdot 2} - n_{ij\cdot 3} - n_{ij\cdot 4}]/n_{ij}^{\frac{1}{2}}, \text{ if } n_{ij} > 0, \\ &= 0, \text{ otherwise;} \end{aligned} \tag{3.2}$$

$$\begin{aligned} v_{ij} &= [n_{ij\cdot 1} - n_{ij\cdot 2} + n_{ij\cdot 3} - n_{ij\cdot 4}]/n_{ij}^{\frac{1}{2}} \text{ if } n_{ij} > 0, \\ &= 0, \text{ otherwise, for } i \neq j = 1, \dots, t. \end{aligned} \tag{3.3}$$

Also let

$$T_{N,i}^{(1)} = \sum_{\substack{j=1 \\ \neq i}}^{t} u_{ij}, \quad T_{N,i}^{(2)} = \sum_{\substack{j=1 \\ \neq i}}^{t} v_{ij}, \text{ for } i=1,\dots,t. \tag{3.4}$$

It may be noted that by definition

(3.5) $$\Sigma_{i=1}^{t} T_{N,i}^{(k)} = 0 \text{ for } k=1,2.$$

[If $n_{ij} = n$ for all $i<j=1,\ldots,t$, we may simplify (3.4) a little further. On characteristic $\alpha(\beta)$, denote the score of the $i^{th}$ object in its $n_{ij}$ (=n) encounters with the $j^{th}$ object by $a_{ij}(b_{ij})$, so that $a_{ij} = n_{ij\cdot 1} + n_{ij\cdot 2}$ ($b_{ij} = n_{ij\cdot 1} + n_{ij\cdot 3}$), for $i \neq j=1,\ldots,t$. Let

(3.6) $$a_i = \sum_{\substack{j=1 \\ \neq i}}^{t} a_{ij} \text{ and } b_i = \sum_{\substack{j=1 \\ \neq i}}^{t} b_{ij}, \quad i=1,\ldots,t$$

be the total scores. Then

(3.7) $$T_{N,i}^{(1)} = n^{-\frac{1}{2}}[2a_i - n(t-1)], \quad T_{N,i}^{(2)} = n^{-\frac{1}{2}}[2b_i - n(t-1)], \text{ for } i=1,\ldots,t.$$

Thus, the $T_{N,i}$'s are related to standardized deviations of the total scores from their expectations].

To formulate the test in a convenient way, we further define

(3.8) $$Z_{N,i}^{(1)} = \tfrac{1}{2}(T_{N,i}^{(1)} - T_{N,i}^{(2)}) = \sum_{\substack{j=1 \\ \neq i}}^{t} n_{ij}^{-\frac{1}{2}}(n_{ij\cdot 2} - n_{ij\cdot 3})$$

(3.9) $$Z_{N,i}^{(2)} = \tfrac{1}{2}(T_{N,i}^{(1)} + T_{N,i}^{(2)}) = \sum_{\substack{j=1 \\ \neq i}}^{t} n_{ij}^{-\frac{1}{2}}(n_{ij\cdot 1} - n_{ij\cdot 4}),$$

for $i=1,\ldots,t$. From (3.1), we get after some simple manipulations that

(3.10) $$E\{Z_{N,i}^{(k)} | H_o, \hat{\theta}_N\} = 0 \text{ for } k=1,2;\ i=1,\dots,t;$$

(3.11) $$E\{Z_{N,i}^{(k)} \cdot Z_{N,j}^{(q)} | H_o, \hat{\theta}_N\} = \delta_{kq}(\delta_{ij}t-1)N_k/N,$$

for $k,q=1,2$; $i,j=1,\dots,t$, where $\delta_{uv}$ is the Kronecker delta and $N_2 = N-N_1$. Thus considering the linearly independent set of random variables $\{Z_{N,i}^{(k)},\ k=1,2;\ i=1, \dots, t-1\}$, taking the reciprocal of the covariance matrix as a suitable discriminant of their quadratic form, and finally symmetrizing, we obtain an appropriate test statistic as

(3.12) $$D_N = \frac{1}{t} \sum_{k=1}^{2} \sum_{i=1}^{t} (N/N_k)[Z_{N,i}^{(k)}]^2$$
$$= \frac{1}{t(1-\hat{\theta}_N^2)} \left\{ \sum_{i=1}^{t} [(T_{N,i}^{(1)})^2 - 2\hat{\theta}_N T_{N,i}^{(1)} T_{N,i}^{(2)} + (T_{N,i}^{(2)})^2] \right\},$$

where $\hat{\theta}_N = (N_1-N_2)/N$. Since $D_N$ is a positive semidefinite quadratic form, it will increase stochastically with increasing heterogeneity among the $T_{N,i}^{(1)}$ and/or $T_{N,i}^{(2)}$ $(i=1,\dots,t)$. Hence, it seems natural to consider the following test procedure:

(3.13) $$\text{if } D_N \geq D_{N,\epsilon}, \text{ reject } H_o \text{ in (1.9)},$$
$$< D_{N,\epsilon}, \text{ accept } H_o,$$

where $P\{D_N \geq D_{N,\epsilon} | H_o, \hat{\theta}_N\} = \epsilon$. As in (2.9), the unconditional level of significance will also be equal to $\epsilon$. For small values of $n_{ij}$'s, we may use (3.1) directly to compute the value of $D_{N,\epsilon}$, while for large $n_{ij}$'s, we have the following.

THEOREM 3.1. Under $H_o$ in (1.9), the conditional distribution of $D_N$ given $\hat{\theta}_N$ is asymptotically a $\chi^2$ distribution with $2(t-1)$ d.f.

PROOF. By virtue of (3.10) and (3.11), it suffices to show that

(3.14) $$Z_N = \Sigma_{k=1}^{2} \Sigma_{i=1}^{t} c_{ik} Z_{N,i}^{(k)} \quad (\text{where } \Sigma_{i=1}^{t} c_{ik} = 0 \text{ for } k=1,2)$$

has (under $H_o$ and conditioned on $\hat{\theta}_N$) asymptotically a normal distribution with zero mean and a finite variance

(3.15) $$t\{\sum_{k=1}^{2} (N_k/N) \sum_{i=1}^{t} c_{ik}^2\} > 0.$$

(3.15) follows from (3.11) and (3.14). To prove the asymptotic normality of $Z_N$, we rewrite it as

(3.16) $$2 \sum_{k=1}^{2} \sum_{i<j=1}^{t} (c_{ik}-c_{jk})(n_{ij}^{(k)}/n_{ij})^{\frac{1}{2}}(n_{ij\cdot k}-\tfrac{1}{2}n_{ij}^{(k)})/(n_{ij}^{(k)})^{\frac{1}{2}}.$$

For the asymptotic theory we also assume that

(3.17) $$\lim_{N=\infty} n_{ij}/N = \rho_{ij} \cdot 0 < \rho_{ij} < 1, \text{ for all } i < j=1,\dots,t.$$

Now, from (3.1), it follows that conditioned on $\{n_{ij}^{(k)}, 1 \leq i < j \leq t, k=1,2\}$ $2\{n_{ij\cdot k} - (\frac{1}{2})n_{ij}^{(k)}\}/\{n_{ij}^{(k)}\}^{\frac{1}{2}}$ $(1 \leq i < j \leq t, k=1,2)$ are all stochastically independent standardized binomial variables, and hence, asymptotically (under (3.17)), they have jointly a t(t-1) variable normal distribution with a null mean vector and a unit dispersion matrix. Thus, conditioned on $\{n_{ij}^{(k)}, k=1,2, 1 \leq i < j \leq t\}$, $Z_N$ in (3.14)( or (3.16)) has asymptotically a normal distribution with zero mean and variance

(3.18) $$\sum_{k=1}^{2} \sum_{i<j=1}^{t} (c_{ik}-c_{jk})^2 \, n_{ij}^{(k)}/n_{ij}.$$

Now, the joint distribution of $n_{ij}^{(k)}$ $i < j=1,\ldots,t$, $k=1,2$ is given by the first factor of (3.1), and hence by Tchebysheff's inequality it can be shown that

$$n_{ij}^{(k)}/n_{ij} = N_k/N + O_p(n_{ij}^{-\frac{1}{2}}) \text{ for all } i < j=1,\ldots,t;\ k=1,2. \tag{3.19}$$

Thus, the difference between (3.15) and (3.18) is $O_p(N^{-\frac{1}{2}})$. The rest of the proof will follow readily by the method of characteristic functions, and hence is omitted.

REMARK. The condition (3.17) can be replaced by the following. We denote the incidence matrix by $((n_{ij}))_{i,j=1,\ldots,p}$, where $n_{ii} = 0$ for $i=1,\ldots,p$, and $n_{ij} = n_{ji}$ for $i < j=1,\ldots,p$. A sufficient condition for theorem 3.1 to hold is that each row (or column) of this matrix contains at least one non-zero $n_{ij}$ and only the set of non-zero $n_{ij}$'s satisfies (3.17). The proof of this follows on the same lines. This extension covers also some incomplete block designs where not all possible pairs are considered.

THEOREM 3.2. *Under $H_o$ in (1.9), $D_N$ has also (unconditionally) asymptotically a $\chi^2$ distribution with $2(t-1)$ d.f.*

PROOF. As in the proof of theorem 3.1, we express $Z_N$ in (3.14) as

$$\sum_{i<j=1}^{t}\left\{\sum_{k=1}^{2}(c_{ij}-c_{jk})(n_{ij\cdot k}-n_{ij\cdot 5-k})/n_{ij}^{\frac{1}{2}}\right\}, \tag{3.20}$$

which is a linear function of $\binom{t}{2}$ independent sets of multinomial variates, and hence has asymptotically a normal distribution with zero mean and variance (under $H_o$)

$$(t/2)\{(1+\theta)\sum_{i=1}^{t} c_{i1}^2 + (1-\theta)\sum_{i=1}^{t} c_{i2}^2\}. \tag{3.21}$$

Consequently, by routine analysis we find that under $H_o$,

$$(3.22) \qquad D_N^* = \frac{1}{t(1-\theta^2)} \left\{ \sum_{i=1}^{t} \left[ (T_{N,i}^{(1)})^2 - 2\theta T_{N,i}^{(1)} T_{N,i}^{(2)} + (T_{N,i}^{(2)})^2 \right] \right\}$$

has asymptotically a $\chi^2$ distribution with 2(t-1) d.f. Now, under $H_o$, $\hat{\theta}_N$ in (2.3) is the minimum variance unbiased (MVU) estimator of $\theta$, and hence, from (3.12) and (3.22), it is easily seen that under $H_o$,

$$(3.23) \qquad D_N \overset{P}{\sim} D_N^* .$$

Hence, the theorem.

Let now $\tilde{\theta}_N$ be any other consistent estimator of $\theta$, and by substituting $\tilde{\theta}_N$ for $\theta$ in $D_N^*$ (in (3.22)), define a statistic $\tilde{D}_N$. It follows from theorem 3.2 that under $H_o$ in (1.9), $\tilde{D}_N \overset{P}{\sim} D_N^*$, and hence has asymptotically a $\chi^2$ distribution with 2(t-1) d.f. Thus, we may consider a large sample test:

$$(3.24) \qquad \text{if } \tilde{D}_N \begin{array}{l} \geq \chi^2_{2(t-1),\epsilon}, \text{ reject } H_o \text{ in (1.9)} \\ < \chi^2_{2(t-1),\epsilon}, \text{ accept } H_o, \end{array}$$

which will have level of significance $\epsilon(0 < \epsilon < 1)$. From theorem 3.1 and (3.23) it follows that $D_{N,\epsilon}$, defined in (3.13), also asymptotically reduces to $\chi^2_{2(t-1),\epsilon}$. Hence under $H_o$, the two tests in (3.13) and (3.24) will be equivalent. We shall see later on that this asymptotic equivalence also holds for some non-null cases. However, we may add a few remarks here. First, $\hat{\theta}_N$ is the MVU estimator of $\theta$ (under $H_o$), and hence, among the class of all possible $\tilde{D}_N$, $D_N$ in (3.12) is optimal in a certain sense. Second, the exact null distribution of $D_N$ given $\hat{\theta}_N$ (and under $H_o$) can be obtained from (3.1). Hence, for small samples, some exact (conditional) test can be constructed, a task which may be considerably more difficult for other $\tilde{D}_N$. Finally, $\hat{\theta}_N$ is really a very simple estimator of $\theta$. All these considerations argue for the use of the test given in (3.12) and (3.13).

4. Performance characteristics of the tests. We shall now prove the consistency of the tests against appropriate alternatives and also study their power properties. When the $\binom{t}{2}$ multinomial laws of the type (1.4) are not all identical, let us define

$$\bar{\pi}_{k,N} = \frac{1}{2N} \sum_{i \neq j=1}^{t} n_{ij} \pi_{ij \cdot k} \text{ for } k=1,2,3,4, \tag{4.1}$$

where $\pi_{ij\cdot k}$'s are defined in (1.3) and (1.5). (1.5) and (4.1) imply that

$$\bar{\pi}_{1,N} = \bar{\pi}_{4,N},\ \bar{\pi}_{2,N} = \bar{\pi}_{3,N}, \text{ and } \bar{\pi}_{1,N} + \bar{\pi}_{2,N} = \bar{\pi}_{1,N} + \bar{\pi}_{3,N} = \tfrac{1}{2}, \tag{4.2}$$

whatever be the $\binom{t}{2}$ multinomial laws. Let us therefore write

$$\bar{\pi}_{1,N} = \bar{\pi}_{4,N} = \tfrac{1}{4}(1+\bar{\theta}_N),\ \bar{\pi}_{2,N} = \bar{\pi}_{3,N} = \tfrac{1}{4}(1-\bar{\theta}_N). \tag{4.3}$$

Like $\theta$ in (1.8), $\bar{\theta}_N$ also lies in the interval (0,1).

THEOREM 4.1. Whatever be the $\pi_{ij\cdot k}$'s ($i < j=1,\ldots,t$, $k=1,2,3,4$), $\hat{\theta}_N$ in (2.3) is the MVU estimator of $\bar{\theta}_N$ in (4.3).

PROOF. Writing $\hat{\theta}_N$ in (2.3) equivalently as

$$\frac{1}{2N} \sum_{i \neq j=1}^{t} (n_{ij\cdot 1} - n_{ij\cdot 2} - n_{ij\cdot 3} + n_{ij\cdot 4}), \tag{4.4}$$

the unbiasedness of $\hat{\theta}_N$ as an estimator of $\bar{\theta}_N$, follows readily. Also, for multinomial distributions of the type (1.4), $(n_{ij\cdot 1}-n_{ij\cdot 2}-n_{ij\cdot 3}+n_{ij\cdot 4})/n_{ij}$ is the

MVU estimator of $(\pi_{ij\cdot 1} - \pi_{ij\cdot 2} - \pi_{ij\cdot 3} + \pi_{ij\cdot 4})$, for all $i < j=1,\ldots,t$, and these estimators are all independent. The rest of the proof is simple and is omitted. Hence, the theorem.

Since the multinomial law in (1.4) carries 3 d.f., we may write

(4.5) $$\pi_{ij\cdot 1} = \tfrac{1}{4}[(1 + \Delta_{ij})(1 + \epsilon_{ij}) + (\bar{\theta}_N + \eta_{ij})],$$

(4.6) $$\pi_{ij\cdot 1} = \tfrac{1}{4}[(1 + \Delta_{ij})(1 - \epsilon_{ij}) - (\bar{\theta}_N + \eta_{ij})],$$

(4.7) $$\pi_{ij\cdot 3} = \tfrac{1}{4}[(1 - \Delta_{ij})(1 + \epsilon_{ij}) - (\bar{\theta}_N + \eta_{ij})],$$

(4.8) $$\pi_{ij\cdot 4} = \tfrac{1}{4}[(1 - \Delta_{ij})(1 - \epsilon_{ij}) + (\bar{\theta}_N + \eta_{ij})], \text{ for } i \neq j=1,\ldots,t,$$

where $(\Delta_{ij}, \epsilon_{ij}, \eta_{ij})$ are all real parameters which can assume values in the unit interval (0,1). The $\Delta$'s and $\epsilon$'s account for heterogeneity of locations and the $\eta$'s for heterogeneity of association. Thus, from (1.5) and (4.3), we have $\Delta_{ji} = -\Delta_{ij}$, $\epsilon_{ji} = -\epsilon_{ij}$, $\eta_{ij} = \eta_{ji}$, so that

(4.9) $$\frac{1}{2N}\sum_{i \neq j=1}^{t} n_{ij}\Delta_{ij} = \frac{1}{2N}\sum_{i \neq j=1}^{t} n_{ij}\epsilon_{ij} = \frac{1}{2N}\sum_{i \neq j=1}^{t} n_{ij}\eta_{ij} = 0.$$

Using (3.17), let us then define

(4.10) $$\Delta_{i\cdot} = \sum_{\substack{j=1 \\ \neq i}}^{t} \rho_{ij}^{\frac{1}{2}}\Delta_{ij}, \quad \epsilon_{i\cdot} = \sum_{\substack{j=1 \\ \neq i}}^{t} \rho_{ij}^{\frac{1}{2}}\epsilon_{ij}, \text{ for } i=1,\ldots,t,$$

and let

(4.11) $$\Delta^2 = \Sigma_{i=1}^{t} \Delta_{i\cdot}^2, \quad \epsilon^2 = \Sigma_{i=1}^{t} \epsilon_{i\cdot}^2 .$$

THEOREM 4.2. The test based on $D_N$ is consistent against the set of alternatives $\Delta^2 + \epsilon^2 > 0$.

PROOF. We rewrite $D_N$ in (3.12) in the alternative form

$$D_N = \frac{1}{t}\left\{\sum_{i=1}^{t}\left[T_{N,i}^{(k)}\right]^2 + \frac{1}{1-\hat{\theta}_N^2}\sum_{i=1}^{t}\left[T_{N,i}^{(k)} - \hat{\theta}_N T_{N,i}^{(q)}\right]^2\right\}, \tag{4.12}$$

where (k,q) is any permutation of (1,2). Now, using (3.2), (3.3), (3.17), (4.5) through (4.11), it can be shown that

$$\frac{1}{N}\sum_{i=1}^{t}[T_{N,i}^{(1)}]^2 \xrightarrow{P} \Delta^2, \quad \frac{1}{N}\sum_{i=1}^{t}[T_{N,i}^{(2)}]^2 \xrightarrow{P} \epsilon^2. \tag{4.13}$$

Thus, if $\Delta^2 + \epsilon^2 > 0$ it follows from (4.12) and (4.13) that $D_N$ can be made stochastically indefinitely large as $N\to\infty$. Hence, the theorem follows from (3.13) and theorem 3.1 (whereby $D_{N,\epsilon}$ in (3.13) tends to $\chi^2_{2(t-1),\epsilon}$).

In view of the consistency of the test, we shall consider alternatives infinitely close to the null hypothesis to study its asymptotic power. By an adaptation to the categorical situation of Pitman's [4] types of alternatives, we let in (4.5) through (4.8), $\bar{\theta}_N = \theta$ and

$$H_N:\ \Delta_{ij} = N^{-\frac{1}{2}}\mu_{ij},\ \epsilon_{ij} = N^{-\frac{1}{2}}\nu_{ij} \text{ and } \eta_{ij} = N^{-\frac{1}{2}}\xi_{ij}, \tag{4.14}$$

where $\mu_{ij}$, $\nu_{ij}$ and $\xi_{ij}$ ($i\neq j=1,\dots,t$) are all real and finite. Further, as in (4.10), we write

$$\mu_{i\cdot} = \sum_{\substack{j=1\\ \neq i}}^{t}\rho_{ij}^{\frac{1}{2}}\mu_{ij},\quad \nu_{i\cdot} = \sum_{\substack{j=1\\ \neq i}}^{t}\rho_{ij}^{\frac{1}{2}}\nu_{ij}, \text{ for } i=1,\dots,t. \tag{4.15}$$

Then we have the following.

THEOREM 4.3. Under $\{H_N\}$ in (4.14), $D_N$ has asymptotically a noncentral $\chi^2$ distribution with 2(t-1) d.f. and the noncentrality parameter

$$\frac{1}{t(1-\theta^2)}\left\{\sum_{i=1}^{t}\left[\mu_{i.}^2 - 2\theta\mu_{i.}\nu_{i.} + \nu_{i.}^2\right]\right\} . \tag{4.16}$$

PROOF. Under $\{H_N\}$ in (4.14), the joint distribution of $(u_{ij}, v_{ij})$, defined by (3.2) and (3.3), can be shown to be asymptotically bivariate normal with means $(\rho_{ij}^{\frac{1}{2}}\mu_{ij}, \rho_{ij}^{\frac{1}{2}}\nu_{ij})$, variances unity, and correlation coefficient equal to $\theta$, for all $i < j=1,\dots,t$. Thus $(T_{N,i}^{(1)}, T_{N,i}^{(2)})$ with $i=1,\dots,t-1$ (defined by (3.4)), will have asymptotically a 2(t-1) variate normal distribution with means $(\mu_{i.}, \nu_{i.})$, defined by (4.15), variances equal to (t-1), covariances between $T_{N,i}^{(1)}, T_{N,j}^{(2)}$ equal to $(\delta_{ij}t-1)\theta$, for $i,j=1,\dots,t-1$, and finally, covariances between $T_{N,i}^{(k)}, T_{N,j}^{(k)}$ equal -1 for all $i\neq j=1,\dots,t-1$, $k=1,2$, where $\delta_{ij}$ is the usual Kronecker delta. Consequently, by routine methods we see that $D_N^*$, defined in (3.22), has asymptotically a noncentral $\chi^2$ distribution with 2(t-1) d.f. and the noncentrality parameter in (4.16). Finally, by theorem 4.1, $\hat{\theta}_N$ converges to $\theta$ (under $\{H_N\}$), and hence $D_N \overset{P}{\sim} D_N^*$. Hence the theorem follows.

It may be noted that under (4.14), the test based on $\tilde{D}_N$ in (3.24) will be asymptotically power equivalent to the one based on $D_N$ in (3.13). Finally, as compared to the parametrically optimum paired comparison test (for bivariate normal distribution) based on Hotelling's $T^2$-test, the efficiency of the proposed paired comparison test will be the same as that of the bivariate median test (cf. Chatterjee and Sen [1]) for the c sample problem. For brevity, these results are not reproduced again.

5. A further remark. We have so far considered the case of paired characteristics only. The general case of p-tuple characteristics $(\alpha^{(1)},\dots,\alpha^{(p)})$ for some $p \geq 1$,

can be tackled in a similar way. In this case, there will be $2^p$ possible outcomes (as compared to 4 in (1.2)), and there will be $\binom{p}{2}$ two-way marginal tables. For $(\alpha^{(k)}, \alpha^{(q)})$ the same structure holds as in (1.8), (1.9); the corresponding $\theta$ is denoted by $\theta_{kq}$, $k \neq q = 1, \ldots, p$, and the estimators by $\hat{\theta}_{N,kq}$ (defined as in (2.3)), for $k \neq q = 1, \ldots, p$. We write

$$\text{(5.1)} \qquad \underset{\sim}{\Theta} = ((\theta_{kq}))_{k,q=1,\ldots,p}, \quad \hat{\underset{\sim}{\Theta}}_N = ((\hat{\theta}_{N,kq}))_{k,q=1,\ldots,p},$$

where conventionally $\theta_{kk} = \hat{\theta}_{N,kk} = 1$ for all $k=1,\ldots,p$. Let $\hat{\underset{\sim}{\Theta}}_N^{-1}$ be the reciprocal matrix of $\hat{\underset{\sim}{\Theta}}_N$ and define $T_{N,i}^{(k)}$ as in (3.4), for all $k=1,\ldots,p$, $i=1,\ldots,t$. Then the test statistic will be

$$\text{(5.2)} \qquad D_{N(p)} = \frac{1}{t} \sum_{k=1}^{p} \sum_{q=1}^{p} \hat{\theta}_N^{kq} \sum_{i=1}^{t} T_{N,i}^{(k)} T_{N,i}^{(q)} .$$

It can be shown that under $H_o$ of homogeneity of the t objects, $D_{N(p)}$ will have a known permutation distribution which asymptotically reduces to $\chi^2$ distribution with p(t-1) d.f. Hence, a test procedure essentially similar to (3.13) and (3.24) can be proposed. Because of the similarity of approach, the details are omitted.

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