

\* *Statistics Department (IAS), Australian National University. Written while visiting the Department of Statistics, University of North Carolina, and supported in part by the Office of Naval Research under Contract N00014-67-A-0321-0002.*

**SOME PROBLEMS IN THE THEORY OF POINT PROCESSES**

*by*

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0. Introduction.
  1. Generalization of a Theorem of Renyi.
  2. What is a 2-D renewal process?
  3. Concerning positive, positive-definite measures.
  4. Characterization of superpositions.
  5. The output process of a stationary GI/G/1 queue.
- References.

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## 0. INTRODUCTION

This collection of research problems has been prepared as a result of attending the ISI "Satellite Conference on Point Processes" held at IBM Watson Research Labs 2-6 August, 1971, and an invitation to speak at the 1971 Annual meeting of the Institute of Mathematical Statistics. The problems described arose from a great deal of stimulating formal and informal discussion at the Point Process Conference and it seemed desirable to give them wider dissemination. They are not in general due to me and nor can I be certain to have formulated all of them accurately as originally posed. But I hope that the spirit will be adequately conveyed.

For convenience, the notation of Daley and Vere-Jones (1972) (abbreviated D & V.J.) is used freely.

## 1. GENERALIZATION OF A THEOREM OF RENYI

It seems worthwhile starting with a problem whose solution has been (more or less) given. In the usual notation  $(\Omega, \mathcal{F}, \mathbb{P})$  will denote a point process, with sample realization  $N(A)$  defined (and finite a.s.) on bounded  $A \in \mathcal{B} \equiv \mathcal{B}(\mathbb{R})$ , the family of Borel sets of the real line  $\mathbb{R}$ . We use  $\mathcal{F}$  to denote the family of finite intervals in  $\mathbb{R}$ , and  $R(\mathcal{F})$  the ring generated by  $\mathcal{F}$ .

Renyi (1967) showed that any point process for which all the distributions  $\mathbb{P}\{N(A) = k\} \quad (A \in R(\mathcal{F}), k \in \mathbb{Z} = \{0, 1, \dots\})$  (1.1) are Poisson-distributed with mean  $\mu(A)$  where  $\mu$  is a non-atomic measure on  $\mathcal{B}$ , finite on bounded  $A \in \mathcal{B}$  is necessarily a Poisson process, i.e.,  $N(A_i)$  ( $i = 1, \dots, r$ ) are independent random variables for disjoint sets  $A_i \in R(\mathcal{F})$ .

Counter examples due to Shepp (in Goldman (1967)), Moran (1967) and Lee (1968) show that the distributions at (1.1) can be Poisson on the smaller class of sets  $A \in \mathcal{F}$  without the process itself being Poisson. Professor Matthes reported at the IBM Point Process Conference that a student of his has generalized Renyi's result in the following manner.

Let  $(\Omega, \mathcal{F}, \mathbb{P}_1)$ ,  $(\Omega, \mathcal{F}, \mathbb{P}_2)$  be two a.s. orderly (Definition 3.3 of D & V-J) point processes with

$$\mathbb{P}_1\{N(A) = k\} = \mathbb{P}_2\{N(A) = k\} \quad (\text{all } A \in \mathcal{R}(\mathcal{F}), k \in \mathbb{Z}). \quad (1.2)$$

Then  $\mathbb{P}_1 = \mathbb{P}_2$ , i.e., all the finite-dimensional distributions of the two processes coincide.

It is worth noting in this result that orderliness is assumed, and that orderliness plays an important role in Renyi's proof of his results using indicator random variables in a fashion akin to the technique in Leadbetter (1968). Without this assumption it would seem that Renyi's result, and a fortiori the one quoted by Matthes, can no longer hold, for the counter-example to Renyi's question (relaxing (1.1) to  $A \in \mathcal{F}$ ) shows that Poisson random variables  $X, Y, Z$  may have  $X+Y = Z$  without being independent.

The result quoted by Matthes also provides a partial answer to the following question which I first heard from Dr. A.J. Lawrance and given succinctly by Cox and Westcott at the Point Process Conference: Does there exist a simply stationary point process which is not strictly stationary, i.e., is it possible that the condition,

$$\mathbb{P}\{N(A_i+t) = k_i, \quad i = 1, \dots, r\} \text{ is independent of } t, \quad (1.3)$$

can be true for  $r = 1$  without being true for all integers  $r > 1$ ?

## 2. WHAT IS A 2-D RENEWAL PROCESS?

There exist in the literature various multidimensional renewal theorems (e.g. Bickel and Yahav (1965 a,b), Farrell (1964-1966) and Mode (1967)) that are more one-dimensional in flavour as far as we are concerned here, for what is done is to investigate e.g. first passage problems for sums of independent identically distributed (i.i.d.) vector-valued random variables. When the mean is non-zero (which is the case with a univariate renewal process), these sums are (roughly speaking) contained within a cone whose axis has the direction of the mean vector, so that in any region far away from the origin, there will not any point of the process unless the region is in the neighbourhood of the cone. By the question which heads this section, we mean that we seek a point process in  $R^2$  constructed via one or more sets of i.i.d. random variables such that all parts of  $R^2$  are likely to contain points of the process. (In the language of the rain-drop problem, if each point is taken as the centre of a sufficiently large circle (= rain-drop), then there is to be positive probability of being able to pass through circles without jumping from any such circle to infinity in any direction: see e.g. Roberts (1967).)

In constructing realizations of a stationary point process in  $R$ , it is often simpler to condition the process on having a point at the origin and then to use non-negative random variables as the distances between consecutive points. The probability measure of a point process conditioned in this way is the so-called Palm measure: the stationary point process corresponding to it can be obtained by relocating the origin at random on a large interval  $(-T,0)$  and taking the limit  $T \rightarrow \infty$ . For a stationary orderly process in  $R^2$  with a finite mean number of points in any bounded region the same remarks apply (c.f. Papangelou (1972)), except of course that in order to be stationary, we need

$\mathbb{P}\{N(A_i+t) = k_i, i = 1, \dots, r\}$  is independent of  $t \in \mathbb{R}^2$  (2.1)

to be true for all bounded  $A_i \in \mathcal{B}(\mathbb{R}^2)$  and (let it be emphasized)  $t \in \mathbb{R}^2$ . Neglecting the degenerate case that  $\mathbb{P}$  is a zero measure (and hence not a probability measure), we cannot obtain (2.1) from the 2-D renewal processes indicated above.

A construction that is commonly used for homogeneous Poisson processes in  $\mathbb{R}^2$  is to locate the points at the origin and at points with polar coordinates  $(r_n, \theta_n)$ , where  $\{\theta_n\}$  ( $n = 1, 2, \dots$ ) are i.i.d. random variables uniformly distributed on  $(0, 2\pi)$ , and  $r_n^2 = X_1 + \dots + X_n$  where  $\{X_n\}$  are i.i.d. negative exponential random variables. In other words, points are located at random on the perimeters of concentric circles, the annular areas between neighbouring circles being i.i.d. exponential random variables. A possible generalization of this construction is to allow the  $\{X_n\}$  to be any family of i.i.d. non-negative random variables with finite mean, but unfortunately appealing to Grigelionis' (1970) theorem on superpositions of point processes indicates that the distribution of  $N(A_i+t)$  ( $i = 1, \dots, r$ ) as  $t \rightarrow \infty$  converges to Poisson (a proof of this assertion is, I suspect, messy).

Another possible construction which does lead to the required stationarity properties in  $\mathbb{R}^2$  without being Poisson is to locate points at  $(S_m, T_n)$  ( $m, n = 1, 2, \dots$ ) where  $S_m = X_1 + \dots + X_m$  and  $T_n = Y_1 + \dots + Y_n$  where  $\{X_n\}$  and  $\{Y_n\}$  are independent sets of i.i.d. random variables, but of course this process is degenerate in that points are located at the vertices of the rectangles defined in the plane by the lines  $x = S_m$  and  $y = T_n$  ( $m, n = 1, 2, \dots$ ). It could be called simply the direct product of two renewal processes.

A third possibility is to use independent sets of i.i.d. non-negative random variables and let the points be  $(S_{mn}, T_{mn})$  ( $m, n = 1, 2, \dots$ ) where  $S_{mn} = X_{mn1} + \dots + X_{mnm}$ ,  $T_{mn} = T_{mn1} + \dots + T_{mnn}$  and  $\{X_{mni}\}$ ,  $\{Y_{mnj}\}$

( $i = 1, \dots, m; j = 1, \dots, n; m, n = 1, 2, \dots$ ) are the i.i.d. variables. However, appeal to Grigelionis' (1970) results again appears to show that, provided  $X$  and  $Y$  have non-arithmetic distributions, the distribution of  $N(A+t)$  for large  $t$  is again Poisson.

It is perhaps appropriate to comment on the non-negativity of the random variables. I have tried (Daley (1970)) to look at the distribution of the distances between points in a point process with sample realizations conditioned by having a point at the origin and defined by the points  $\{S_n\} \cup \{T_n\}$  ( $n = 1, 2, \dots$ ) where  $S_n = X_1 + \dots + X_n$  and  $T_n = -(X_{-1} + \dots + X_{-n})$  and  $\{X_n\}, \{X_{-n}\}$  are two independent sets of given i.i.d. random variables. (Given a plant at the origin, a daughter plant is located at  $S_1$ , grand-daughter plant at  $S_2, \dots$ , while the mother plant is at  $T_{-1}$ , the grand-mother plant at  $T_{-2}, \dots$ .) Only in the case of a two-sided exponential distribution for  $X_n$  (i.e.,  $\text{pr}\{X_n < -x\} = pe^{-\alpha x}$ ,  $\text{pr}\{X_n > x\} = (1-p)e^{-\beta x}$  for  $x > 0$  with  $\alpha\beta E(X_n) = (1-p)\alpha - p\beta \neq 0$ ) were explicit formulae obtained.

One of the nice features of the 1-D renewal process we should like to reproduce in  $R^2$  is that a renewal process (in  $R$ ) has

$$\frac{\text{var}(N(I))}{E(N(I))} \approx \frac{\text{var}(X)}{(E(X))^2} \quad (\text{large intervals } I)$$

where the random variable  $X$  has the distribution of times between events. A construction which at least allows  $\text{var}(N(A))/E(N(A))$  ( $A \in \mathcal{B}(R^2)$ ) to take any prescribed value for large circles  $A$  (or more generally, convex sets in  $R^2$  with large minor diameter  $\equiv \sup\{d: \exists \text{ a circle of diameter } d \subseteq A\}$ ) is to start from a lattice array of points (e.g., all points  $(m, n)$  for all positive or negative integers  $m, n$ ) and, with these as centres, assign a cluster independently about each centre (i.e., a randomly dispersed collection of a finite random number of points). If  $\tilde{n}(A)$  denotes a generic sample realization

of a cluster (and the clusters will be assumed to be i.i.d.), then

$$\frac{\text{var}(N(A))}{E(N(A))} \approx \frac{\text{var}(\tilde{n}(R^2))}{E(\tilde{n}(R))}.$$

However, the underlying regularity of the lattice location of the parent process would still be present (it may not be apparent to the eye, but it would be revealed by spectral analysis: see Example 7 and equation (53) of Daley (1971) for 1-D analogues).

### 3. CONCERNING POSITIVE, POSITIVE-DEFINITE MEASURES

Let  $N(\cdot)$  be a weakly stationary point process, i.e. it has finite second moment  $M_2(A) = E(N^2(A))$  (bounded  $A \in \mathcal{B}$ ) which, along with the first moment measure  $M(A) = E(N(A))$  (bounded  $A \in \mathcal{B}$ ) is invariant under translation of the set  $A$  ( $M(A+t) - M(A) = M_2(A+t) - M_2(A) = 0$  (bounded  $A \in \mathcal{B}$ )). Then it is known (see e.g. §3.1 of D & V-J) that  $M(A) = m|A|$  ( $|A|$  = Lebesgue measure of  $A$ ) and  $M_2(\cdot)$  is determined by the reduced covariance measure  $C(\cdot)$  which,

in the simplest case where  $A$  is an interval, is

$$V(y) \equiv \text{var}(N(0,y]) = \int_{\mathbb{R}} (y-|u|)^+ C(du) = \alpha y + 2 \int_{0+}^y (y-u) C(du) \quad (3.1)$$

where  $\alpha = C(\{0\}) = V'(0+) = \lim_{h \rightarrow 0} h^{-1} M_2(0,h]$ . Further, the measure

$C(du) + m^2 du \equiv C_1(du)$  is easily shown to be positive (for  $C_1(du) dt =$

$E(N(dt)N(u+du))$ ) and positive-definite in the sense that for every bounded

Borel function  $\phi$  of compact support in  $\mathbb{R}$ ,  $\int_{\mathbb{R}} C_1(dx) \int_{\mathbb{R}} \overline{\phi(x+u)} \phi(u) du \geq 0$ .

Let  $P_+$  denote the class of all positive, positive-definite measures. Then

the properties of  $P_+$  known to us are the following (and we use  $S$  to denote

the class of all test functions  $\phi$  infinitely differentiable with

$x^n \phi^{(m)}(x) \rightarrow 0$  ( $|x| \rightarrow \infty$ ) for all  $m, n \in \mathbb{Z}$ ).



(1°) Given  $\gamma \in \mathcal{P}_+$  there exists one and only one measure  $\mu$  such that for all  $\phi \in \mathcal{S}$ ,

$$\int_{\mathbb{R}} \phi(x) \gamma(dx) = \int_{\mathbb{R}} \tilde{\phi}(\theta) \mu(d\theta) \quad (3.2)$$

where

$$\tilde{\phi}(\theta) = \int_{\mathbb{R}} e^{ix\theta} \phi(x) dx.$$

(2°) This  $\mu \in \mathcal{P}_+$ .

(3°)  $\gamma$  need not be totally finite but  $\gamma(x, x+1]$  is uniformly bounded on  $x \in \mathbb{R}$ .

(4°)  $\lim_{x \rightarrow \infty} x^{-1} \gamma(-x, x]$  exists and equals  $2\mu(\{0\})$ .

Returning to the point process context, such properties as we know for  $V(\cdot)$  or  $C(\cdot)$  are consequences of  $C_1(\cdot)$  being in  $\mathcal{P}_+$  with the exception of the inequality

$$(5^\circ) \quad \alpha \geq m$$

(equality holds if  $N(\cdot)$  is orderly). If we allow  $N(\cdot)$  to be a (non-negative) random measure then all of the preceding remains true with the possible exception of (5°). We therefore arrive at the questions implicit in Daley (1971) and Vere-Jones (1972): to state them we denote by  $\mathcal{P}_+^{\text{RM}}$  and  $\mathcal{P}_+^{\text{PP}}$  the subsets of  $\mathcal{P}_+$  arising from weakly stationary random measures and point processes respectively. By taking  $N_1$  and  $N_2$  to be orderly weakly stationary point processes with  $E(N_1(A)) = E(N_2(A))$  and defining a random measure by  $\Lambda = pN_1 + (1-p)N_2$  ( $0 < p < 1$ ) we find that  $\lim_{h \rightarrow 0} h^{-1} \text{var}(\Lambda(0, h]) = p^2 m + (1-p)^2 m < m = pm + (1-p)m = E(\Lambda(0, 1])$ , so  $\Lambda$  does not satisfy (5°). Thus,

$$\mathcal{P}_+^{\text{PP}} \subset \mathcal{P}_+^{\text{RM}} \subseteq \mathcal{P}_+.$$

The questions are as follows.

(1) Given any  $\gamma \in \mathcal{P}_+^{\text{PP}}$  or  $\mathcal{P}_+^{\text{RM}}$ , how can we construct a point process or random measure respectively with  $C_1 = \gamma$ ?

- (ii) Is  $P_+^{RM} = P_+$ ? If not, how is  $P_+^{RM}$  characterized?
- (iii) What criteria (if any) other than (5°) can be used to distinguish between  $P_+^{RM}$  and  $P_+^{PP}$ ?

There are some examples of the effect of operations on point processes on  $C(\cdot)$  and its related spectral measure (this is essentially  $\nu(A) \equiv \mu(A) - m^2 \delta_{\{0\}}(A)$ ) in Daley (1971). Perhaps the nicest characterization of a subset of  $P_+^{PP}$  is due to Milne and Westcott (1972) who show via the class of Gauss-Poisson point processes that to any totally finite symmetric measure  $\mu$  on  $\mathbb{R}$  with an atom at 0 equal to  $\mu_0$  (say), there is a stationary orderly point process with  $m = \mu_0$  and  $C = \mu$ . In particular then, any non-negative integrable function can be the covariance density function  $C(\cdot)$  of a point process (when  $C - m\delta_{\{0\}}$  is absolutely continuous with respect to Lebesgue measure, its density function  $c(\cdot)$  is the covariance density function of the process).

#### 4. CHARACTERIZATION OF SUPERPOSITIONS

Suppose that  $N_1$  and  $N_2$  are independent Poisson processes: then it is well known that their sum or superposition

$$N = N_1 + N_2$$

is also a Poisson process. A natural question to ask is whether there are any other families of point processes whose type is preserved under superposition (assuming independence of  $N_1$  and  $N_2$  of course). As far as characterization of  $N_1$ ,  $N_2$  or  $N$  by means of independence (or Markovian dependence) of the intervals between points is concerned, it seems that the Poisson is the only possibility. Çinlar (1972) has reviewed the literature in the area.

The problem which remains is that of proving (or disproving) the result making minimal assumptions on  $N_1$  and  $N_2$ .

*Conjecture:* Let  $N_1$  and  $N_2$  be independent stationary orderly point processes with their stationary superposition process having its Palm measure characterized by its intervals being Markov-dependent of finite order. Then  $N_1$ ,  $N_2$  and  $N$  are Poisson processes.

The best result to date appears to be that of Mecke (1967) who shows that if  $N_1$  and  $N$  are stationary renewal processes and the distributions of the intervals between points have continuous density functions which are right-continuous at the origin, then the conjectured statement is true. Earlier, McFadden established the result assuming  $N_1$ ,  $N_2$  and  $N$  to be stationary renewal processes with finite variance for the intervals between points, and Ambartzumian weakened these assumptions by allowing  $N$  to have Markov-dependent intervals.

Notice that since a Poisson random variable  $Z$  which is the sum of independent random variables  $X$  and  $Y$  necessarily implies that  $X$  and  $Y$  are Poisson, assuming  $N$  to be Poisson implies, by Renyi's (1967) result, that  $N_1$  and  $N_2$  are also Poisson processes (and for this, there is no need to assume stationarity).

A problem related to the above is to assume that  $N$  is a completely random process (or, process without after-effects): Does this imply (asks Professor Matthes) that  $N_1$  and  $N_2$  are also completely random processes?

## 5. THE OUTPUT PROCESS OF A STATIONARY GI/G/1 QUEUE

The final area to be discussed here is at first sight hardly point process theory, yet its interest lies deeper than its title. A queueing system can be regarded as an operator for point processes, transforming the input process to the output process. From this point of view, an infinite server system is the same as random translation of all the points. The case of a single server system is more complex, and in Daley (1968), we attempted to review as much as was then known. Here we shall extend that slightly and indicate what seems to be the main unsolved problem.

We take a conventional stable GI/G/1 system (renewal input process, i.i.d. service times independent of the input process, single server, service in order of arrival (though this is not necessary),  $E(\text{service time}) < E(\text{inter-arrival time}) < \infty$ ) and assume that it is specified to be in its stationary state. In order to discuss inter-arrival and inter-departure intervals, we condition the system to have an input -- customer  $C_0$  say -- at time  $t = 0$ , and it is then convenient to denote by  $T_n$  the time between the inputs of  $C_n$  and  $C_{n+1}$ , and by  $D_n$  the time between the departures of  $C_{n-1}$  and  $C_n$ .  $S_n$  denotes the service time of  $C_n$ .

It has long been known that when  $\{T_n\}$  are i.i.d. negative exponential random variables (i.e., the input is Poisson), the same is true of  $\{D_n\}$  if and only if  $\{S_n\}$  are also i.i.d. negative exponential random variables (i.e. the output is Poisson). More generally, we conjecture that  $\{D_n\}$  are independent random variables - hence the output process is a renewal process and if the system is part of a series of queues with the output feeding into another system as an input, some closed form algebraic results for the latter system may possibly be obtainable - if and only if either  $\{T_n\}$  and  $\{S_n\}$  are independent sequences of i.i.d. negative exponential random variables, or

else there is a constant  $c$  such that  $\text{pr}\{S_n=c\} = 1$  (all  $n$ ) and  $\text{pr}\{T_n \geq c\} = 1$ . In the latter event, if  $N_1(\cdot)$  and  $N_2(\cdot)$  denote the input and output point processes respectively, then  $\mathbb{P}\{N_1(A)=k\} = \mathbb{P}\{N_2(A+c)=k\}$  (all  $A \in \mathcal{B}$ ,  $k \in \mathbb{Z}$ ). So far as attempted proofs of this conjecture are concerned, we have tried as a first step to show by using Wiener-Hopf decompositions (see Kingman (1966)) that  $\{D_n\}$  and  $\{T_n\}$  must have the same distribution.

A simpler result shown in Daley (1968) that lends credence to the conjecture is that when  $\{S_n\}$  are negative exponential variates,  $\{D_n\}$  are i.i.d. if and only if the input is Poisson.

As for second-order properties, with  $\{S_n\}$  negative exponential variates,  $\{D_n\}$  may be uncorrelated without necessarily being independent, while with a Poisson input process and no assumption on  $\{S_n\}$ ,  $\{D_n\}$  are uncorrelated only if they have a negative exponential distribution, when necessarily so do  $\{S_n\}$  and the output is Poisson. Since then, we have shown in unpublished work that if the input process is a stationary renewal process (as distinct from an interval stationary input process as above - see §2.2 of D & V-J) then if  $\{S_n\}$  follow the negative exponential distribution, the output process  $N_2(\cdot)$  has  $\text{cov}(N_2(A), N_2(B)) = 0$  for disjoint  $A, B \in \mathcal{B}$  if and only if the input process is Poisson (when also  $N_2(\cdot)$  is Poisson). This is in strong contrast with the possibilities for  $T_n$  when  $\{D_n\}$  are uncorrelated. On the other hand, we have not yet been able to show that a similar assertion holds true with a Poisson input and arbitrary distribution for  $\{S_n\}$ .

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