

ON THE DISTRIBUTION OF QUADRATIC FORMS

by

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## INTRODUCTION

One could easily justify the study of the distribution of quadratic forms from the standpoint that many of the tests in statistics are based on the distributions of quantities which can be thought of as special cases of either a quadratic form or functions of quadratic forms. The applications are too numerous to mention; however, we shall list a few as illustrations.

- (i) The distribution of a definite quadratic form where the components have a multivariate normal distribution.
- (ii) The problem of finding the power function of the chi-square statistic, for large samples, can be reduced to that of finding the distribution of a positive definite quadratic form in non-central normal variates.
- (iii) The distribution of a form of the serial correlation coefficient can be expressed in terms of the distribution of a ratio of two quadratic forms. See Anderson [1]<sup>1</sup>
- (iv) Of special importance is von Neumann's statistic, [19], [20], the ratio of

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1. Numbers in square brackets refer to bibliography.

the mean square successive difference to the variance, used to test whether observations are independent or whether a trend exists.

- (v) Durbin and Watson [6], use a similar statistic to test the error terms for independence in least squares regression.
- (vi) Koopmans [12], says, "Assuming a normal distribution for the random disturbance, the mathematical prerequisite for an estimation theory of stochastic processes is the study of the joint distributions of certain quadratic forms in normal variables". The problem Koopmans considers is that of estimating the serial correlation in a stationary stochastic process.
- (vii) To test hypotheses concerning variance components in the analysis of variance, we require the distribution of an indefinite quadratic form.
- (viii) Hotelling [9], shows how the distribution of the ratio of an indefinite quadratic form in non-central normal variates to a definite quadratic form could be used in the theory of selecting variates for use in prediction.

(ix) McCarthy [13], shows how the distribution of the ratio of two definite quadratic forms could be used to make an F test in the analysis of variance when the assumptions of equal variances and independence of the observations are not met.

It would make this report quite lengthy to discuss the distributions of all these functions of quadratic forms. We shall restrict ourselves to the study of a definite quadratic form in both central and non-central independent normal variates, giving an important application for each distribution. Then we shall discuss two special cases of an indefinite quadratic form. Finally, we shall discuss a few inequalities. We give below a slightly more detailed chapter-wise breakdown.

In Chapter I we shall be concerned with the distribution of a definite quadratic form in independent  $N(0,1)$  variates. Robbins [16], has treated this problem but we have carried it a bit further. Robbins and Pitman [17], have given an expression for the distribution of a linear combination of chi-square variates. We feel that we have improved on this form. We have derived an expression which depends only on the value of the determinant of the form and on the moments of the inverse quadratic form. The expression is an alternating series which converges absolutely and is such that if we stop after any even power we have an upper bound, and if we stop after any odd power, a lower bound to the

cumulative distribution function. Hotelling [10] and Gurland [8], have suggested the use of Laguerre polynomials in finding distributions of quadratic forms. A brief account of Hotelling's method will be given.

In Chapter II we have derived an expression for the distribution of a definite quadratic form in non-central independent normal variates which depends only on the value of the determinant of the form and on the moments of the inverse quadratic form in normal variates with imaginary means. This statement will be made clearer later on. This result enables us to find the power function of the chi-square statistic.

In Chapter III we have discussed the distribution of the difference between two independent chi-squares having different numbers of degrees of freedom. If the degrees of freedom are the same, the distribution becomes the same as the distribution of the sample covariance in sampling from a normal population. We have studied the properties of this distribution in some detail.

In Chapter IV we give some inequalities for the distribution of a quadratic form in  $N(0, 1)$  variates and also for the general case.

## NOTATION

All vectors are column vectors and primes indicate their transposes.

"p.d.f." stands for "probability density function";

"c.d.f." stands for "cumulative distribution function";

"r.v." stands for "random variable";

"q.f." stands for "Quadratic form" ;

" $N(\mu, \sigma)$ " stands for "a r.v. having a normal p.d.f. with mean  $\mu$  and standard deviation  $\sigma$ ".



## CHAPTER I

### THE DISTRIBUTION OF A DEFINITE QUADRATIC FORM IN INDEPENDENT CENTRAL NORMAL VARIATES

#### 1.1 The problem.

Suppose we have a q.f.  $Q_n = \frac{1}{2} Y'A Y$  in  $Y_1, \dots, Y_n$ , where the  $Y_i$  are independent  $N(0, 1)$  variates, and where  $Y' = (Y_1, \dots, Y_n)$ , where the dash denotes the transpose of the column vector  $Y$ .

Let  $F_n(t) = \Pr(Q_n \leq t)$ . Then the problem is to find  $F_n(t)$ . It

is well known that we can make an orthogonal transformation,

$Y = P X$ , say, where  $P P' = P'P = I$ , such that  $\frac{1}{2} Y'A Y = \frac{1}{2} X'P'APX =$

$\frac{1}{2} X'D_a X = \frac{1}{2} \sum_{i=1}^n a_i x_i^2$ , where  $D_a$  is a diagonal matrix having the

elements  $a_1, \dots, a_n$  in its main diagonal, and where  $a_1, \dots, a_n$  are

the latent roots of the matrix  $A$ . Under such a transformation,

$X_1, \dots, X_n$  remain independent  $N(0, 1)$ . So the problem is now to

find the  $\Pr(\frac{1}{2} \sum_{i=1}^n a_i X_i^2 \leq t)$ , where we assume that  $a_i > 0$ ,  $i=1, \dots, n$ .

#### 1.2 The solution.

Theorem 1.1. Let  $Q_n = \frac{1}{2} \sum_{i=1}^n a_i x_i^2$ , where the  $x_i$  are

independent  $N(0, 1)$ , and where  $a_i > 0$ ,  $i=1, \dots, n$ . Let  $Q_n^* = \frac{1}{2} \sum_{i=1}^n a_i^{-1} X_i^2$ .

Then,

$$(a) \quad F_n(t) = \frac{t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{n}{2}+k+1)}$$

where  $E(Q_n^*)^k$  is the  $k$ -th moment of  $Q_n^*$ .

(b) The series is absolutely convergent and therefore it is convergent.

(c) For any two positive integers  $r$  and  $s$  and every  $t > 0$ ,

$$\sum_{k=0}^{2s-2} d_k > F_n(t) > \sum_{k=0}^{2r-1} d_k, \text{ where}$$

$$d_k = \frac{t^{n/2}}{(a_1 \dots a_n)^{1/2}} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{n}{2}+k+1)} .$$

Proof.

Let  $d\mathbf{x} = dx_1 \dots dx_n$  and  $[R] = \left[ \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \leq t \right]$ , then

$$F_n(t) = (2\pi)^{-\frac{n}{2}} \int_{[R]} \dots \int \exp - \frac{1}{2} \sum_{i=1}^n a_i x_i^2 d\mathbf{x}.$$

We shall make use of the Dirichlet integral;

$$\int \dots \int \prod_{j=1}^n x_j^{\ell_j - 1} dx_j = \frac{\prod_{j=1}^n \Gamma\left(\frac{\ell_j}{p_j}\right) c_j^{\ell_j}}{\Gamma\left(\sum_{j=1}^n \frac{\ell_j}{p_j} + 1\right)},$$

where  $-\infty < x_1 < \infty$ , and  $\ell_j, c_j, p_j$  are all positive, and the

range of integration is  $\sum_{j=1}^n \frac{x_j^{p_j}}{c_j} \leq 1$ . See Edwards [7]. If we

expand the exponential in the integrand, we get

$$F_n(t) = (2\pi)^{-\frac{n}{2}} \int \dots \int_{[R]} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2} \sum_{j=1}^n x_j^2\right)^k}{k!} dx.$$

We need to evaluate integrals of the following type:

$$\int \dots \int_{[R]} \left(\sum_{j=1}^n x_j^2\right)^k dx.$$

If we expand the integrand according to the multinomial theorem, we get

$$\sum_{i_1 + \dots + i_n = k} \frac{k!}{i_1! \dots i_n!} \int \dots \int_{[R]} \prod_{j=1}^n x_j^{2i_j} dx_j.$$

We now make use of the Dirichlet integral stated earlier, where

$$c_j = \left(\frac{2t}{a_j}\right)^{1/2}, \quad p_j = 2, \quad \ell_j = (2i_j+1), \quad \text{getting}$$

$$\int \dots \int_{[R]} \left(\sum_1^n x_i^2\right) dx = \frac{(2t)^{k+\frac{n}{2}} k!}{(a_1 \dots a_n)^{1/2} \Gamma\left(\frac{n}{2}+k+1\right)} .$$

$$\sum_{i_1+\dots+i_n=k} \frac{\Gamma\left(i_1+\frac{1}{2}\right) \dots \Gamma\left(i_n+\frac{1}{2}\right)}{i_1! \dots i_n! a_1^{i_1} \dots a_n^{i_n}} .$$

The problem now is to evaluate this last expression. Recalling

that if  $X_1$  is  $N(0, 1)$ , then  $E(X_1^2)^k = \frac{2^k \Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}$ , we find that

$$\begin{aligned} E(Q_n^*)^k &= E \left[ \frac{1}{2} \sum_1^n a_i^{-1} X_i^2 \right]^k \\ &= 2^{-k} E \sum_{i_1+\dots+i_n=k} \frac{k!}{i_1! \dots i_n!} \frac{X_1^{2i_1} \dots X_n^{2i_n}}{a_1^{i_1} \dots a_n^{i_n}} \end{aligned}$$

$$\begin{aligned}
&= 2^{-k} k! \sum_{i_1 + \dots + i_n = k} \frac{E X_1^{2i_1} \dots E X_n^{2i_n}}{i_1! \dots i_n! a_1^{i_1} \dots a_n^{i_n}} \\
&= \pi^{-\frac{n}{2}} k! \sum_{i_1 + \dots + i_n = k} \frac{\Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2})}{i_1! \dots i_n! a_1^{i_1} \dots a_n^{i_n}}.
\end{aligned}$$

So that

$$\int \dots \int_{[R]} (\sum_1^n x_i^2)^k d\underline{x} = \frac{(2t)^{k + \frac{n}{2}} \pi^{\frac{n}{2}}}{(a_1 \dots a_n)^{1/2} \Gamma(\frac{n}{2} + k + 1)} E(Q_n^*)^k,$$

and

$$F_n(t) = \frac{t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{n}{2} + k + 1)}.$$

This proves part (a) of the theorem.

To show absolute convergence we note that if

$a_1 \geq a_2 \geq \dots \geq a_n > 0$ , then

$$Q_n^* = \frac{1}{2} \left( \frac{X_1^2}{a_1} + \dots + \frac{X_n^2}{a_n} \right) \leq \frac{1}{2a_n} \sum_1^n X_i^2, \text{ and}$$

$$E(Q_n^*)^k \leq \frac{1}{a_n^k} E\left(\frac{1}{2} \sum_{i=1}^n X_i^2\right)^k = \frac{\Gamma(\frac{n}{2}+k)}{a_n^k \Gamma(\frac{n}{2})}, \text{ so that}$$

$$F_n(t) \leq \frac{t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+k+1) a_n^k \Gamma(\frac{n}{2})}$$

$$\leq \frac{t^{n/2}}{(a_1 \dots a_n)^{1/2}} \frac{e^{t/a_n}}{\Gamma(\frac{n}{2}+1)} < \infty, \text{ for finite } t.$$

This proves part (b) of the theorem.

The bounds we obtain are based on the fact that if  $r$  and  $s$  are

any two positive integers  $\geq 1$ , then for  $z > 0$ ,  $\sum_{k=0}^{2s-2} \frac{(-z)^k}{k!} > e^{-z} >$

$$\sum_{k=0}^{2r-1} \frac{(-z)^k}{k!}.$$

This proves part (c) of the theorem.

In the case where some of the latent roots are zero, i.e. when the form is positive semi-definite of rank  $r$ , say, we need only replace  $n$  by  $r$  in the theorem and in the proof.

Remarks.

(1) The moments of  $Q_n^*$ ,  $E(Q_n^*)^k$ , are easy to obtain from the

cumulants of  $Q_n^*$ . The  $r$ -th cumulant of  $Q_n^*$  is  $k_r(Q_n^*) = \frac{(r-1)!}{2} \sum_{i=1}^n a_i^{-r}$ .

From Kendall [11], we have expressions for the first ten moments in terms of the cumulants.

(ii) Let  $S_r$  be the sum of the first  $r+1$  terms of the series for  $F_n(t)$ . Then  $S_0, S_2, S_4, \dots$  is a sequence of upper bounds and  $S_1, S_3, S_5, \dots$  is a sequence of lower bounds to  $F_n(t)$ . If  $E_{2k+2}$  is the error committed by stopping with  $S_{2k+1}$ , then

$$E_{2k+2} \leq \text{l.u.b. } S_{2r} - \text{g.l.b. } S_{2r+1}, \quad r=0,1,\dots,k; k=0,1,\dots,$$

$$E_{2k+1} \leq \text{l.u.b. } S_{2r} - \text{g.l.b. } S_{2r-1}, \quad r=1,2,\dots,k; k=1,2,\dots$$

The values of l.u.b.  $S_{2r}$  and g.l.b.  $S_{2r+1}$  depend on the values of the latent roots. We note that  $E_{2k+2} \leq S_{2k} - S_{2k+1}$  = the last term included, and  $E_{2k+1} \leq S_{2k} - S_{2k-1}$  = the last term included. Hence, the error is less than the last term included and it is positive if we take an odd number of terms and negative if we take an even number of terms.

(iii) The above theorem seems to be in several ways an improvement over the method given in Robbins [16].

### 1.3 An application: The distribution of a sum of squares in dependent variates.

Suppose that  $X_1, \dots, X_n$  have a joint multivariate normal distribution with zero means and covariance matrix equal to  $A^{-1}$ .

Then the problem is, what is the distribution of  $\frac{1}{2} \sum_1^n X_1^2$ ? Now

$$\Pr\left[\frac{1}{2} X'X \leq t\right] = \frac{|A|^{1/2}}{(2\pi)^{n/2}} \int_{\frac{1}{2} X'X \leq t} \dots \int \text{Exp} - \frac{1}{2} X'A X \, d\underline{x}.$$

Make an orthogonal transformation,  $X = L Y$ , say where  $L L' = L'L = I$ .

Then  $X'X = Y'Y$  and  $X'A X = Y'L'A L Y = \sum_1^n a_1 Y_1^2$ , where  $a_1, \dots, a_n$  are

the latent roots of the matrix  $A$ . Now make the transformation

$z_1^2 = a_1 Y_1^2$ , getting

$$\Pr\left[\frac{1}{2} X'X \leq t\right] = (2\pi)^{-\frac{n}{2}} \int_{\frac{1}{2} \sum_1^n a_1^{-1} z_1^2 \leq t} \dots \int \text{Exp} - \frac{1}{2} z'z \, d\underline{z}$$

$$= \Pr\left[Q_n^* \leq t\right], \text{ and we can make use of}$$

theorem 1.1.

Remark.

Combining the results of theorem 1.1 and the above application it is easy to show that we could find the distribution of a definite q.f.,  $X'A X$ , in  $X_1, \dots, X_n$  where  $X_1, \dots, X_n$  have a multivariate normal distribution with covariance matrix  $B^{-1}$ , and this distribution involves as parameters the characteristic roots of  $AB^{-1}$ .



We shall now state, without proof, an obvious corollary to theorem 1.1, obtained by letting the first  $m_1$  latent roots be  $a_1$ , the next  $m_2$  latent roots be  $a_2$ , etc.

Corollary 1.1

Let  $S_r = \frac{1}{2}(a_1 X_{m_1}^2 + \dots + a_r X_{m_r}^2)$ , where the  $X_{m_i}^2$  are independent r.v.'s having a central chi-square distribution with  $m_i$  degrees of freedom. Let  $S_r^* = \frac{1}{2}(a_1^{-1} X_{m_1}^2 + \dots + a_r^{-1} X_{m_r}^2)$ ,  $M = \sum_{i=1}^r m_i$ ,  $a_i > 0$ ,  $G_r(t) = \Pr(S_r \leq t)$ , then

$$G_r(t) = \frac{t^{M/2}}{(a_1 \dots a_r)^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{M}{2} + k + 1)},$$

where  $E(S_r^*)^k$  is the  $k$ -th moment of  $S_r^*$ . The series is absolutely convergent, and for any two positive integers  $s$  and  $j$  and every  $t > 0$ ,

$$\sum_{k=0}^{2s-2} d_k > G_r(t) > \sum_{k=0}^{2j-1} d_k, \text{ where}$$

$$d_k = \frac{t^{M/2}}{(a_1^{m_1} \dots a_r^{m_r})^{1/2}} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{M}{2} + k + 1)} .$$

Remarks.

(i) The moments of  $S_r^*$  are easy to obtain from the cumulants. The  $j$ -th cumulant of  $S_r^*$  is  $k_j(S_r^*) = \frac{(j-1)!}{2} \sum_{i=1}^r m_i a_i^{-j}$ .

We can find the first ten moments in terms of the cumulants from Kendall [11].

(ii) The above corollary gives a method which seems to be an improvement over the one given in Robbins and Pitman [17].

1.4 Hotelling's method of Laguerre polynomials.

In this section we shall give a very brief account of a method suggested by Hotelling [10]. Let  $Q_n = \frac{1}{2}(a_1 X_1^2 + \dots + a_n X_n^2)$ , where the  $X_i$  are independent  $N(0, 1)$  variates and where  $a_i > 0$ ,  $i=1, \dots, n$ . Let  $g(q)$  be the p.d.f. of  $Q_n$  and let

$$f(x) = \frac{e^{-x} x^{m-1}}{\Gamma(m)} , \quad x > 0 , \quad \text{where } m = \frac{n}{2} .$$

Then the suggested expansion for  $g(q)$  is the Laguerre series

$$g(q) = f(q) \sum_{r=0}^{\infty} b_r L_r^{(m-1)}(q),$$

where  $L_r^{(m-1)}(q)$ ,  $r=0, 1, \dots$ , is the sequence of Laguerre polynomials satisfying the relation

$$\int_0^{\infty} f(x) L_i^{(m-1)}(x) L_j^{(m-1)}(x) dx = \binom{i+m-1}{i} \delta_{ij}$$

$$i, j = 0, 1, \dots, \text{ for } m > 0.$$

The Laguerre polynomial  $L_s^{(m-1)}(q)$  has the following explicit representation:

$$L_s^{(m-1)}(q) = \sum_{v=0}^s \binom{s+m-1}{s-v} \frac{(-q)^v}{v!}, \quad s=0, 1, \dots$$

See Sgego [18]. It follows from the orthogonality condition that

$$b_s = \frac{s! \Gamma(m)}{\Gamma(s+m)} \int_0^{\infty} g(q) L_s^{(m-1)}(q) dq,$$

and so  $b_s$  is a linear function of the moments of  $Q_n$ . The series

$$g(q) = f(q) (1 + b_1 L_1(q) + b_2 L_2(q) + \dots)$$

converges uniformly over the whole real axis and so we can integrate term by term.

Remarks.

(i) The main drawback in using the above expansion is that no convenient bound is known for the error committed by stopping after a certain number of terms.

(ii) Hotelling suggests that the above series be used to find the p.d.f. of the ratio of a definite quadratic form to a sum of squares and the p.d.f. of an indefinite form by convolution, since an indefinite form is the difference of two definite forms.

## CHAPTER II

### THE DISTRIBUTION OF A DEFINITE QUADRATIC FORM IN INDEPENDENT NON-CENTRAL NORMAL VARIATES

#### 2.1 The problem.

Suppose we have a q. f.  $Q_n = \frac{1}{2} Y'AY$  in  $Y_1, \dots, Y_n$  where the  $Y_i$  are independent  $N(\xi_i, 1)$  variates, and where  $Y' = (Y_1, \dots, Y_n)$ .

Let

$$G_n(t; \xi_1, \dots, \xi_n) = G_n(t; \underline{\xi}) = \Pr(Q_n \leq t)$$

then the problem is to find  $G_n(t; \underline{\xi})$ . Let us make an orthogonal transformation  $Y = \Gamma X$ , say, where  $\Gamma'\Gamma = \Gamma\Gamma' = I$ , such that

$$Y'AY = X'\Gamma'AX = X'D_a X = \sum_1^n a_i X_i^2, \quad ,$$

where  $D_a$  is a diagonal matrix having the elements  $a_1, \dots, a_n$  in the main diagonal, where  $a_1, \dots, a_n$  are the latent roots of the matrix  $A$ .

If  $EY = \xi$ , then  $EX = \Gamma'\xi = \mu$  (say), where  $\xi' = (\xi_1, \dots, \xi_n)$  and  $\mu' = (\mu_1, \dots, \mu_n)$ . Hence, under such a transformation  $X_1, \dots, X_n$  remain independent with the same variances as  $Y_1, \dots, Y_n$ , but the

mean of  $X_i$  is now  $\mu_i = (\gamma_{1i}\xi_1 + \dots + \gamma_{ni}\xi_n)$ , where  $(\gamma_{1i}, \dots, \gamma_{ni})$  are the elements of the  $i$ -th row of  $\Gamma'$ .

So the problem is now to find the  $\Pr \left[ \frac{1}{2} \sum_{i=1}^n a_i X_i^2 \leq t \right]$ , where the  $X_i$  are independent  $N(\mu_i, 1)$ .

## 2.2 The solution.

Theorem 2.1 Let  $Q_n = \frac{1}{2} \sum_{i=1}^n a_i X_i^2$ , where the  $X_i$  are independent  $N(\mu_i, 1)$

and where  $a_1 \geq a_2 \geq \dots \geq a_n > 0$ . Let  $Q_n^{**} = \frac{1}{2} \sum_{k=1}^n a_n^{-1} Y_k^2$ , where the

$Y_k$  are independent  $N(\mu_k, 1)$ ,  $i = \sqrt{-1}$ . Then

$$(a) \quad G_n(t; \underline{\mu}) = \frac{e^{-\lambda} t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{s=0}^{\infty} \frac{(-t)^s}{s!} \frac{c_s}{\Gamma(\frac{n}{2} + s + 1)},$$

where  $c_s = E(Q_n^{**})^s$ , and  $\lambda = \frac{1}{2} \sum_{i=1}^n \mu_i^2$ .

(b) The series is absolutely convergent and therefore it is convergent.

(c) For any two positive integers  $r$  and  $k$  and every  $t > 0$ ,

$$\sum_{s=0}^{2r} d_s > G_n(t; \underline{\mu}) > \sum_{s=0}^{2k-1} d_s, \text{ where}$$

$$d_s = \frac{e^{-\lambda} t^{n/2}}{(a_1 \dots a_n)^{1/2}} \frac{(-t)^s}{s!} \frac{c_s}{\Gamma(\frac{n}{2} + s + 1)}$$

Proof. We know that if the  $X_i$  are independent  $N(\mu_i, 1)$  and if

$$\lambda = \frac{1}{2} \sum_1^n \mu_i^2, \text{ and } Y = \frac{1}{2} \sum_1^n X_i^2, \text{ then the p.d.f. of } Y \text{ is}$$

$$e^{-\lambda} e^{-y} y^{\frac{n}{2}-1} \sum_{m=0}^{\infty} \frac{(\lambda y)^m}{m! \Gamma(m + \frac{n}{2})}$$

Let  $[R] = [ \sum_1^n a_i y_i \leq t ]$ ,  $dy = dy_1 \dots dy_n$ , and  $\lambda_j = \frac{1}{2} \mu_j^2$ , then

$$G_n(t; \underline{\mu}) = \int_{[R]} \dots \int \prod_{j=1}^n e^{-\lambda_j} e^{-y_j} y_j^{-\frac{1}{2}} \sum_{i_j=0}^{\infty} \frac{(\lambda_j y_j)^{i_j}}{i_j! \Gamma(i_j + \frac{1}{2})} dy_j$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_n=k} \frac{\lambda_1^{i_1} \dots \lambda_n^{i_n}}{i_1! \dots i_n! \Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2})} \cdot I(t; \underline{a}; n),$$

where

$$I(t; \underline{a}; n) = \int \dots \int_{[R]} \text{Exp} \left[ - \sum_{i=1}^n y_i \prod_{j=1}^n y_j^{i_j} \right]^{-\frac{1}{2}} dy_j.$$

Expanding the exponential and making use of the Dirichlet integral stated in (1.2), we obtain:

$$G_n(t; \underline{a}; n) = e^{-\lambda} \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_n=k} \frac{\lambda_1^{i_1} \dots \lambda_n^{i_n}}{i_1! \dots i_n! \Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2})}.$$

$$\sum_{r=0}^{\infty} \sum_{j_1+\dots+j_n=r} \frac{(-1)^{r+k} t^{r+k} \Gamma(i_1+j_1+\frac{1}{2}) \dots \Gamma(i_n+j_n+\frac{1}{2})}{j_1! \dots j_n! a_1^{i_1+j_1+\frac{1}{2}} \dots a_n^{i_n+j_n+\frac{1}{2}} \Gamma(k+r+\frac{n}{2}+1)}.$$



This can be rewritten as

$$G_n(t; \underline{\mu}) = \frac{e^{-\lambda t n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{s=0}^{\infty} c_s \frac{(-t)^s}{s! \Gamma(\frac{n}{2} + s + 1)}, \text{ where}$$

$$c_s = \sum_{k=0}^s \sum_{i_1 + \dots + i_n = k} \sum_{j_1 + \dots + j_n = s} d,$$

$$\text{where } d = \frac{s! \lambda_1^{i_1} \dots \lambda_n^{i_n} \Gamma(j_1 + \frac{1}{2}) \dots \Gamma(j_n + \frac{1}{2}) (-1)^k}{a_1^{j_1} \dots a_n^{j_n} i_1! \dots i_n! (j_1 - i_1)! \dots (j_n - i_n)! \Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2})}$$

The problem now is to evaluate  $c_s$ .

Let  $Q_n^{**} = \frac{1}{2} \sum_{k=1}^n a_k^{-1} Y_k^2$ , where the  $Y_k$  are independent  $N(i\mu_k, 1)$ ,  $i = \sqrt{-1}$ ,

then

$$EY^{2r} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} Y^{2r} \text{Exp} - \frac{1}{2}(Y-i\mu)^2 dY$$

$$= \pi^{-1/2} \sum_{j=0}^r \binom{2r}{2j} (-\mu^2)^j 2^{r-j} \Gamma(r-j + \frac{1}{2})$$

$$= \sum_{j=0}^r \binom{r}{j} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(j + \frac{1}{2})} (-\mu^2)^j 2^{r-j}, \quad \text{and}$$

$$E(Q_n^{**})^s = 2^{-s} \sum_{j_1 + \dots + j_n = s} \frac{s!}{j_1! \dots j_n!} \frac{E Y_1^{2j_1} \dots Y_n^{2j_n}}{a_1^{j_1} \dots a_n^{j_n}}$$

$$= 2^{-s} \sum_{j_1 + \dots + j_n = s} \frac{s! 2^{s-k}}{j_1! \dots j_n! a_1^{j_1} \dots a_n^{j_n}} \sum_{i_1=0}^{j_1} \dots \sum_{i_n=0}^{j_n} h, \quad h,$$

$$\text{where } h = \binom{j_1}{i_1} \dots \binom{j_n}{i_n} \frac{\Gamma(j_1 + \frac{1}{2}) \dots \Gamma(j_n + \frac{1}{2}) (-\mu_1^2)^{i_1} \dots (-\mu_n^2)^{i_n}}{\Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2})}$$

$$= \sum_{k=0}^s \sum_{i_1 + \dots + i_n = k} \sum_{j_1 + \dots + j_n = s} d$$

$$\text{where } d = \frac{s! \lambda_1^{i_1} \dots \lambda_n^{i_n} (+1)^k \Gamma(j_1 + \frac{1}{2}) \dots \Gamma(j_n + \frac{1}{2})}{i_1! \dots i_n! (j_1 - i_1)! \dots (j_n - i_n)! \Gamma(i_1 + \frac{1}{2}) \dots \Gamma(i_n + \frac{1}{2}) a_1^{j_1} \dots a_n^{j_n}} .$$

But this is just  $c_s$ . Hence,

$$G_n(t; \underline{\mu}) = \frac{e^{-\lambda} t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{s=0}^{\infty} \frac{(-t)^s}{s!} \frac{E(Q_n^{**})^s}{\Gamma(\frac{n}{2} + s + 1)}$$

This proves (a) of the theorem.

If  $k_r(Q_n)$  is the  $r$ -th cumulant of  $Q_n$ , it is not difficult to show that

$$k_r(Q_n) = \frac{(r-1)!}{2} \sum_{j=1}^n a_j^r (1 + r\mu_j^2) .$$

Hence, to find the  $r$ -th cumulant of  $Q_n^{**}$  we must replace  $\mu_j$  by  $i\mu_j$  and

$$a_j \text{ by } a_j^{-1}, \text{ getting, } k_r(Q_n^{**}) = \frac{(r-1)!}{2} \sum_{j=1}^n a_j^{-r} (1 - r\mu_j^2) . \text{ There-}$$

fore,  $c_s$  is the  $s$ -th moment of a r.v. whose  $s$ -th cumulant is

$$k_s(Q_n^{**}) .$$

If  $Q_n^* = \frac{1}{2} \sum_{i=1}^n a_i^{-1} X_i^2$ , where the  $X_i$  are independent  $N(0, 1)$ ,

then  $k_r(Q_n^*) = \frac{(r-1)!}{2} \sum_{j=1}^r a_j^{-r}$ . Therefore,  $k_r(Q_n^{**}) \leq k_r(Q_n^*)$ ,

and it follows that  $E(Q_n^{**})^s \leq E(Q_n^*)^s \leq \frac{\Gamma(\frac{n}{2} + s)}{a_n^s \Gamma(\frac{n}{2})}$ . Consequently,

$$G_n(t; \mu) \leq \frac{e^{-\lambda} t^{n/2}}{(a_1 \dots a_n)^{1/2}} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{\Gamma(\frac{n}{2} + s)}{a_n^s \Gamma(\frac{n}{2} + s + 1)}$$

$$\leq \frac{e^{-\lambda} t^{n/2} e^{t/a_n}}{(a_1 \dots a_n)^{1/2} \Gamma(\frac{n}{2} + 1)} < \infty \text{ for finite } t.$$

This proves (b) of the theorem. The bounds stated in theorem 2.1 are based on the fact that if  $r$  and  $s$  are any two positive integers  $\geq 1$ , then for  $z > 0$ ,

$$\sum_{k=0}^{2s} \frac{(-z)^k}{k!} > e^{-z} > \sum_{k=0}^{2r-1} \frac{(-z)^k}{k!}.$$

This proves (c) of the theorem.

Theorem 2.2. Under the conditions of theorem 2.1,

$$G_n(t; \underline{\mu}) \geq e^{-\lambda} G_n(t; \underline{0}) \quad ,$$

where  $G_n(t; \underline{0})$  equals the  $F_n(t)$  of theorem 1.1.

Proof.

$$G_n(t; \underline{\mu}) = \int_{\underline{R}} \dots \int_{\underline{R}} \prod_{i=1}^n \frac{e^{-\lambda_i} e^{-y_i} y_i^{-1/2}}{\Gamma(\frac{1}{2})} \sum_{j_i=0}^{\infty} \frac{(\lambda_i y_i)^{j_i}}{j_i! \Gamma(j_i + \frac{1}{2})} dy_1.$$

Now,

$$\int_{\underline{R}} \dots \int_{\underline{R}} \prod_{i=1}^n \frac{e^{-y_i} y_i^{\frac{2j_i+1}{2} - 1}}{\Gamma(\frac{2j_i+1}{2})} dy_1 = \Pr \left[ \frac{1}{2} (a_1 x_{2j_1+1}^2 + \dots + a_n x_{2j_n+1}^2) \leq t \right],$$

where  $x_{2j_m+1}^2$  is a r.v. having a central chi-square distribution

with  $2j_m+1$  degrees of freedom. Hence,

$$G_n(t; \underline{\mu}) = e^{-\lambda} \sum_{k=0}^{\infty} \sum_{j_1+\dots+j_n=k} \frac{\lambda_1^{j_1} \dots \lambda_n^{j_n}}{j_1! \dots j_n!} .$$

$$\Pr \left[ \frac{1}{2} (a_1 x_{2j_1+1}^2 + \dots + a_n x_{2j_n+1}^2) \leq t \right],$$

and so,

$$G_n(t; \underline{\mu}) \geq e^{-\lambda} G_n(t; \underline{0}) \quad .$$

We note that equality is attained for any given  $t$ , by putting  $\lambda = 0$  (in which case  $\underline{\mu} = \underline{0}$ ).

Theorem 2.3. Under the conditions of theorem 2.1,

$$G_n(t; \underline{\mu}) \leq G_n(t; \underline{0}) \quad .$$

Proof.

Using a special case of a more general unpublished result due to Sigeity Moriguti, we find that

$$\int_{\lfloor ax^2 \leq t \rfloor} \exp - \frac{1}{2}(x-\mu)^2 dx \leq \int_{\lfloor ax^2 \leq t \rfloor} \exp - \frac{1}{2} x^2 dx \quad .$$

We can generalize the above inequality to the case  $n = 2$  as follows:

$$\begin{aligned} & \int_{\lfloor a_1 x_1^2 + a_2 x_2^2 \leq t \rfloor} \exp - \frac{1}{2} \lfloor (x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 \rfloor dx_1 dx_2 \\ &= \int_{\lfloor a_2 x_2^2 \leq t \rfloor} \int_{\lfloor a_1 x_1^2 \leq t - a_2 x_2^2 \rfloor} \exp - \frac{1}{2} (x_1 - \mu_1)^2 dx_1 \exp - \frac{1}{2} (x_2 - \mu_2)^2 dx_2 \end{aligned}$$

$$\begin{aligned}
&\leq \int_{[a_2 x_2^2 \leq t]} \int_{[a_1 x_1^2 \leq t - a_2 x_2^2]} \exp - \frac{1}{2} x_1^2 dx_1 \exp - \frac{1}{2} (x_2 - \mu)^2 dx_2 \\
&\leq \int_{[a_2 x_2^2 \leq t]} \int_{[a_1 x_1^2 \leq t - a_2 x_2^2]} \exp - \frac{1}{2} x_1^2 dx_1 \exp - \frac{1}{2} x_2^2 dx_2 \\
&= \int_{[a_1 x_1^2 + a_2 x_2^2 \leq t]} \exp - \frac{1}{2} (x_1^2 + x_2^2) dx_1 dx_2 .
\end{aligned}$$

Generalizing the method used for the case  $n=2$ , it is not difficult to show that the above inequality could be established for  $n=3, 4, \dots$ . This completes the proof. Combining the results of theorems 2.2 and 2.3 we have the very useful inequality:

$$G_n(t; \underline{0}) \geq G_n(t; \underline{\mu}) \geq e^{-\lambda} G_n(t; \underline{0}).$$

The proof of theorem 2.3 was suggested to the writer by Professor S. N. Roy.

Remarks.

(1) The introduction of an imaginary mean in theorem 2.1 is merely a mathematical convenience; we could have omitted this formulation altogether and merely stated that  $c_s$  is the  $s$ -th

moment of a r.v. whose  $s$ -th cumulant is  $k_s(Q_n^{**})$ .

(ii) The moments  $c_s$  are easy to compute when we know the cumulants. Kendall [11], gives the first ten moments in terms of the cumulants.

(iii) It can be shown that  $G_n(t; \mu)$  is invariant under an orthogonal transformation  $Y = \Gamma X$ , say, such that  $\Gamma' A \Gamma = D_a$  and  $\Gamma' \Gamma = \Gamma \Gamma' = I$ .

### 2.3 An application: The power function of the chi-square statistic.

Suppose that the observations from a random experiment can fall into any one of  $k$  cells and that the expected number of observations in the  $i$ -th cell under  $H_0$  and  $H_1$  is  $m_i^0$  and  $m_i$ , respectively.

That is:  $H_0: m_1^0, \dots, m_k^0$ , and  $H_1: m_1, \dots, m_k$ . Let

$$\chi_0^2 = \sum_{i=1}^k \frac{(n_i - m_i^0)^2}{m_i^0}, \quad \text{and}$$

$$\chi^2 = \sum_{i=1}^k \frac{(n_i - m_i)^2}{m_i}, \quad \text{where}$$

$n_i$  is the observed number of observations in the  $i$ -th cell. If

$1 - \beta$  is the power of the test, then



$$\beta = \Pr \left[ \chi_0^2 \leq t \mid m_1, \dots, m_k \right]$$

$$= (2\pi)^{-k/2} \int \cdots \int_{\chi_0^2 \leq t} \exp \left[ -\frac{1}{2} \chi_0^2 \right] \prod_{i=1}^k d \left( \frac{n_1 - m_1}{\sqrt{m_1}} \right)$$

approximately, for sufficiently large  $n_1$ . Putting  $x_i = \frac{n_1 - m_1}{\sqrt{m_1}}$ ,

$a_i/2 = m_1/m_1^0$ ,  $\mu_i = \frac{m_1 - m_1^0}{\sqrt{m_1}}$ , and letting  $x_i + \mu_i = y_i$ , we get

$$\Pr \left[ \chi_0^2 \leq t \mid m_1, \dots, m_k \right] = (2\pi)^{-(k/2)} \int \cdots \int_{\sum_{i=1}^k a_i y_i^2 \leq t} \exp \left[ -\frac{1}{2} \sum_{i=1}^k (y_i - \mu_i)^2 \right] \prod_{i=1}^k dy_i$$

$$= G_k(t; \underline{\mu}) .$$

Hence, the application of theorem 2.1 will give us the power function of the chi-square statistic.

## CHAPTER III

### THE DISTRIBUTION OF AN INDEFINITE QUADRATIC FORM IN INDEPENDENT CENTRAL NORMAL VARIATES

#### 3.1 Special case 1.

As a preliminary step to the study of the distribution of an indefinite q.f., this section is concerned with the distribution of the difference between two independent chi-squares having different numbers of degrees of freedom. If the degrees of freedom are the same, the distribution becomes the same as that of the sample covariance in sampling from a normal population. We shall study some of the properties of this distribution in order to anticipate the behavior of the distribution of a more general indefinite q.f. Others [15], [4], [2] have considered this problem from a slightly different standpoint.

The main results of this section are:

- (i) Recurrence relations 3.1.4, 3.1.5, 3.1.6, 3.1.7, 3.1.20
- (ii) Inequalities 3.1.9, 3.1.21
- (iii) Further properties 3.1.12, 3.1.13, 3.1.15, 3.1.16
- (iv) An application of a result due to Berry [3], 3.1.19 .

If  $T_{n,m} = X_n - Y_m$ , where  $X_n$  and  $Y_m$  are independently distributed with p.d.f.  $h_n(x)$  and  $h_m(x)$  respectively, where

$$h_n(x) = \frac{1}{\Gamma(\frac{n}{2})} e^{-x} x^{\frac{n}{2} - 1}, \quad x > 0,$$

then if the p.d.f. of  $T_{n,m}$  is  $g_{n,m}(t)$ ,

$$g_{n,m}(t) = \int_0^{\infty} h_n(x+t) h_m(x) dx, \quad t > 0,$$

$$= \int_0^{\infty} h_n(x) h_m(x-t) dx, \quad t < 0.$$

We see that given the p.d.f. for  $t > 0$ , to get it for  $t < 0$ , replace  $t$  by  $-t$  and interchange  $n$  and  $m$ . Therefore, we shall consider only the case  $t > 0$ , for definiteness.

Hence,

$$3.1.1 \quad g_{n,m}(t) = \frac{e^{-t}}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \int_0^{\infty} e^{-2x(x+t)} x^{\frac{n}{2}-1} x^{\frac{m}{2}-1} dx, \quad t > 0.$$

The moment generating function of  $T_{n,m}$  is  $M(\theta) = Ee^{t\theta} =$   
 $(1-\theta)^{-\frac{n}{2}} (1+\theta)^{-\frac{m}{2}}$ . From  $M(\theta)$  we see that

(i) If  $n, m \rightarrow \infty$  so that  $\frac{n}{m} \rightarrow 1$ , then  $T_{n,m}$  is

asymptotically normal  $(\frac{n-m}{2}, \sqrt{\frac{n+m}{2}})$ .

(ii) If  $n \rightarrow \infty$ , but  $m$  remains finite, then  $T_{n,m}$  is

asymptotically normal  $(\frac{n-m}{2}, \sqrt{n})$ .

(iii) If  $m \rightarrow \infty$ , but  $n$  remains finite, then  $T_{n,m}$  is

asymptotically normal  $(\frac{n-m}{2}, \sqrt{m})$ .

In 3.1.1 let  $x/t = y$ , getting

$$3.1.2 \quad g_{n,m}(t) = \frac{e^{-t} t^{\frac{n+m}{2} - 1}}{\Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \int_0^{\infty} e^{-2yt} (1+y)^{\frac{n}{2}-1} y^{\frac{m}{2}-1} dy.$$

Let  $n/2 = p$ ,  $m/2 = q$ , and let

$$3.1.3 \quad I(p,q) = \int_0^{\infty} e^{-2yt} (1+y)^p y^q dy.$$

Integrate 3.1.3 by parts three separate times as follows:

- (i)  $u = (1+y)^p y^q$ ,  $dv = e^{-2yt} dy$ ,  
(ii)  $u = (1+y)^p e^{-2yt}$ ,  $dv = y^q dy$ ,  
(iii)  $u = (1+y)^p y^{q-1}$ ,  $dv = e^{-2yt} y dy$ , getting

$$(i)' \quad I(p,q) = \left(\frac{1}{2t}\right) [ -q I(p,q-1) + p I(p-1,q) ],$$

$$m > 0, q > 0,$$

$$(ii)' \quad I(p,q) = -\left(\frac{p}{q+1}\right) I(p-1,q+1) + \left(\frac{2t}{q+1}\right) I(p,q+1),$$

$$m > -2, \quad q+1 > 0,$$

$$(iii)' \quad I(p,q) = \left(\frac{1}{4t^2}\right) \int (q-1) I(p,q-2) + p I(p-1,q-1)$$

$$+ 2t(q-1) I(p,q-1) + 2tp I(p-1,q) \int,$$

$$m > 2, \quad q-1 > 0, \text{ respectively.}$$

The restrictions placed on  $m$  and  $q$  are to prevent the integrated part from becoming infinite. Equate (i)' to (iii)' getting

$$(iv) \quad I(p+1,q) = \left(\frac{q}{2t}\right) I(p+1,q-1) + \left(\frac{p+1}{2t}\right) I(p,q).$$

In (i)' replace  $p$  by  $p+1$  and  $q$  by  $q+1$  and then substitute from (ii)' and (iv) into (i)' getting

$$(v) \quad 4t^2 I(p+1,q+1) = 2(p+1)(q+1) I(p,q) + q(q+1) I(p+1,q-1)$$

$$+ p(p+1) I(p-1,q+1).$$

Now  $q(q+1) I(p+1,q-1) + p(p+1) I(p-1,q+1) = q(q+1) I(p-1,q-1) + \int p(p+1)+q(q+1) \int I(p-1,q+1) + 2q(q+1) I(p-1,q)$ , and  $I(p-1,q+1) + I(p-1,q) = I(p,q)$ . Substitute into (v) getting

$$4t^2 I(p+1, q+1) = \int^{2(p+1)(q+1)+p(p+1)+q(q+1)} I(p, q) \\ + q(q+1) I(p-1, q-1) + \int^{q(q+1)-p(p+1)} I(p-1, q),$$

so that finally,

$$3.1.4 \quad \varepsilon_{n+4, m+4}(t) = \int^{\frac{n(n+2)+m(m+2)+2(n+2)(m+2)}{4(n+2)(m+2)}} \varepsilon_{n+2, m+2}(t) \\ + \frac{t^2}{n(n+2)} \varepsilon_{n, m}(t) + \int^{\frac{m(m+2)-n(n+2)}{2n(n+2)(m+2)}} t \varepsilon_{n, m+2}(t); \quad t > 0.$$

The same relation holds for  $t < 0$  if we replace  $t$  by  $-t$  and interchange  $n$  and  $m$ . Note that if  $n=m$ , the last term vanishes and the relation reduces to

$$\varepsilon_{n+4, n+4}(t) = \left(\frac{n+1}{n+2}\right) \varepsilon_{n+2, n+2}(t) + \frac{t^2}{n(n+2)} \varepsilon_{n, n}(t).$$

From the first integration by parts we get the simple relation

$$3.1.5 \quad \varepsilon_{n+2, m+2}(t) = \frac{1}{2} \int \varepsilon_{n+2, m}(t) + \varepsilon_{n, m+2}(t), \text{ for all } t.$$

We now make use of the p.d.f.'s to obtain the c.d.f.'s.

For  $x > 0$ ,

$$\Pr \int T_{n, m} > x = \int_x^{\infty} \varepsilon_{n, m}(t) dt$$

$$= \frac{1}{\Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \int_x^{\infty} e^{-t} \int_0^{\infty} e^{-2y(y+t)} \frac{n-1}{2} \frac{m-1}{2} y^{\frac{n}{2}-1} dy dt.$$

Integrate by parts where  $dv = e^{-t} dt$ , and  $u = \int_0^{\infty} e^{-2y(y+t)} \frac{n-1}{2} \frac{m-1}{2} y^{\frac{n}{2}-1} dy$ ,

getting

$$\Pr[T_{n,m} > x] = g_{n,m}(x) + \Pr[T_{n-2,m} > x].$$

Hence, if  $n$  is even,

$$3.1.6 \quad \Pr[T_{n,m} > x] = \sum_{j=0}^{(n-2)/2} g_{n-2j,m}(x), \text{ where}$$

$$g_{2,m}(x) = \Pr[T_{2,m} > x] = \frac{e^{-x}}{2^{m/2}}.$$

If  $n$  is odd

$$3.1.7 \quad \Pr[T_{n,m} > x] = \sum_{j=0}^{(n-3)/2} g_{n-2j,m}(x) + \Pr[T_{1,m} > x].$$

From 3.1.6 and 3.1.7 it is seen that if we have a table of  $g_{n,m}(t)$

and if we know  $\Pr[T_{1,m} > x]$  for all  $m$ , we can find  $\Pr[T_{n,m} > x]$

for all  $n$  and  $m$ .

Our first objective, then, is to be able to find  $g_{n,m}(t)$  for any  $n$  and  $m$ . If we consider a rectangular table of  $g_{n,m}(t)$  having  $n$  columns and  $m$  rows, we will find that given  $g_{n,m}(t)$  for  $n,m=1,2,3,4,5$  and  $g_{2,m}(t)$  for  $m=1,2,\dots$ , we can complete the table using 3.1.4 and 3.1.5.

First of all we can fill in the second column using the fact that

$$g_{2,m}(t) = \frac{e^{-t}}{2^{m/2}}, \quad m=1,2,\dots, \quad \text{and}$$

$$g_{4,m}(t) = \frac{e^{-t}}{2^{m/2}} \left(t + \frac{m}{4}\right), \quad m=1,2,3,4,5.$$

If we let  $n=m$ , we obtain, as we shall see later, that

$$g_{n,n}(t) = \frac{\left(\frac{t}{2}\right)^{(n-1)/2}}{\pi^{1/2} \Gamma\left(\frac{n}{2}\right)} K_{(n-1)/2}(t).$$

Letting  $n=1,3,5$  we find that  $g_{1,1}(t) = \frac{1}{\pi} K_0(t)$ ,  $g_{2,3}(t) = \frac{t}{\pi} K_1(t)$ , and  $g_{5,5}(t) = \frac{t^2}{3\pi} K_2(t)$ , where  $K_n(t)$  is the modified Bessel function of the second kind of order  $n$ . See McLachlan [14], and Watson [21].



We shall now find the p.d.f. for cases where  $n=m+2$ . If we complete the square in 3.1.1 and let  $n=m+2$ , we get

$$\mathfrak{E}_{m+2,m}(t) = \frac{\left(\frac{t}{2}\right)^m}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)} \int_1^{\infty} e^{-ty} (y^2-1)^{\frac{m}{2}-1} dy + \int_1^{\infty} e^{-ty} (y^2-1)^{\frac{m}{2}-1} y dy$$

Using the fact that

$$K_n(t) = \frac{\pi^{1/2} \left(\frac{t}{2}\right)^n}{\Gamma\left(n+\frac{1}{2}\right)} \int_1^{\infty} e^{-ty} (y^2-1)^{n-\frac{1}{2}} dy, \text{ and}$$

$\frac{d}{dt} \int t^{-n} K_n(t) dt = -t^{-n} K_{n+1}(t)$ , we get that

$$\mathfrak{E}_{m+2,m}(t) = \frac{\left(\frac{t}{2}\right)^{\frac{m+1}{2}}}{\pi^{1/2} \Gamma\left(\frac{m}{2}+1\right)} \int \left[ K_{(m-1)/2}(t) + K_{(m+1)/2}(t) \right]$$

If we now put  $m=1,3$  in the above expression we find that

$$\mathfrak{E}_{3,1}(t) = \frac{t}{\pi} \int \left[ K_0(t) + K_1(t) \right], \text{ and}$$

$$g_{5,3}(t) = \frac{t^2}{3\pi} [K_1(t) + K_2(t)].$$

We can now use 3.1.5 to find  $g_{5,1}(t)$ ,  $g_{3,1}(t)$ ,  $g_{1,5}(t)$ , and  $g_{3,5}(t)$ . There are still six values remaining to be found, namely,  $g_{1,2}(t)$ ,  $g_{1,4}(t)$ ,  $g_{3,2}(t)$ ,  $g_{3,4}(t)$ ,  $g_{5,2}(t)$  and  $g_{5,4}(t)$  which can be expressed in terms of the incomplete gamma function. In fact we find that

$$g_{n,2}(t) = \frac{e^{-t}}{2^{n/2} \Gamma(\frac{n}{2})} \int_{2t}^{\infty} e^{-y} y^{\frac{n}{2}-1} dy, \quad \text{and}$$

$$g_{n,4}(t) = \frac{e^{-t}}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}+1}} \left[ \int_{2t}^{\infty} e^{-y} y^{\frac{n}{2}} dy - 2t \int_{2t}^{\infty} e^{-y} y^{\frac{n}{2}-1} dy \right].$$

Letting  $n=3,5$  we get  $g_{3,2}(t)$ ,  $g_{5,2}(t)$ ,  $g_{3,4}(t)$ , and  $g_{5,4}(t)$ .

Using 3.1.5 we obtain  $g_{1,2}(t)$  and  $g_{1,4}(t)$ . Now we have all

$g_{n,m}(t)$  for  $n,m=1,2,3,4,5$ , and together with  $g_{2,m}(t)$ , 3.1.4 and

3.1.5 we can get all the remaining  $g_{n,m}(t)$ .

Our second objective, then, is to find  $\Pr[T_{1,m} > x]$  for all  $m$ . We shall first give a method of evaluating this when  $m$

is even, and then a method for any  $m$ .

Evaluation of  $\Pr\{T_{1,m} > x\}$ , when  $m$  is even

$$\Pr\{T_{1,m} > x\} = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})} \int_x^{\infty} e^{-t} H(t) dt, \text{ where}$$

$$H(t) = \int_0^{\infty} e^{-2y} y^{\frac{m}{2}-1} (y+t)^{-\frac{1}{2}} dy.$$

Expand  $H(t)$  in a Maclaurin series about  $t=0$ . Now

$$H^{(k)}(t) = (-1)^k \pi^{-\frac{1}{2}} \Gamma(k+\frac{1}{2}) \int_0^{\infty} e^{-2y} y^{\frac{m}{2}-1} (y+t)^{-(k+\frac{1}{2})} dy,$$

and  $H^{(k)}(0) = (-1)^k \pi^{-\frac{1}{2}} \Gamma(k+\frac{1}{2}) \Gamma(\frac{m}{2}-k-\frac{1}{2}) 2^{k+\frac{1}{2}-\frac{m}{2}}.$

We may write

$$H(t) = \sum_{k=0}^r \frac{t^k}{k!} H^{(k)}(0) + R_r(t), \text{ where}$$

$$R_r(t) = \frac{1}{r!} \int_0^t (t-y)^r H^{(r+1)}(y) dy.$$

Note that  $|H^{(r+1)}(y)| \leq |H^{(r+1)}(0)|$ , so that

$$|R_r(t)| \leq \frac{t^{r+1}}{(r+1)!} |H^{(r+1)}(0)|.$$

Therefore,

$$\left| \Pr\{T_{1,m} > x\} - \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})} \sum_{k=0}^r \frac{H^{(k)}(0)}{k!} \int_x^{\infty} e^{-t} dt \right| \leq$$

$$\frac{|H^{(r+1)}(0)|}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})(r+1)!} \int_x^{\infty} e^{-t} t^{r+1} dt.$$

Hence, the error committed by stopping after any term is less in magnitude than the first term neglected.

Evaluation of  $\Pr\{T_{1,m} > x\}$  for any  $m$ .

$$\Pr\{T_{1,m} > x\} = \frac{1}{\pi^{1/2} \Gamma(\frac{m}{2})} \int_x^{\infty} e^{-t} t^{\frac{m-1}{2}} \int_0^{\infty} e^{-2yt} y^{\frac{m}{2}-1} (1+y)^{-\frac{1}{2}} dy dt.$$

Let  $H(t) = (1+t)^{-\frac{1}{2}}$ , then

$$H(t) = \pi^{-\frac{1}{2}} \sum_{j=0}^r \frac{(-t)^j}{j!} \Gamma(j + \frac{1}{2}) + R_r(t), \text{ where}$$

$$R_r(t) = \frac{1}{r!} \int_0^t (t-y)^r H^{(r+1)}(y) dy, \text{ and where}$$

$$H^{(r+1)}(y) = (-1)^{r+1} \pi^{-\frac{1}{2}} \Gamma(r+\frac{3}{2}) (1+t)^{-(r+\frac{3}{2})}. \text{ Then}$$

Then  $|H^{(r+1)}(y)| \leq |H^{(r+1)}(0)|$ , so that

$$|R_r(t)| \leq \frac{t^{r+1}}{(r+1)!} |H^{(r+1)}(0)|.$$

Therefore

$$\left| \Pr\{T_{1,m} > x\} - \frac{1}{\pi \Gamma(\frac{m}{2})} \sum_{j=0}^r \frac{(-1)^j}{j!} \frac{\Gamma(j+\frac{1}{2}) \Gamma(\frac{m}{2}+j)}{2^{\frac{m}{2}+j}} I_{j+\frac{1}{2}}(x) \right| \leq$$

$$\frac{\Gamma(r+\frac{3}{2}) \Gamma(\frac{m}{2}+r+1)}{\pi \Gamma(\frac{m}{2}) (r+1)! 2^{\frac{m}{2}+r+1}} I_{r+\frac{3}{2}}(x), \text{ where}$$

$$I_{j+\frac{1}{2}}(x) = \int_x^{\infty} e^{-t} t^{-(j+\frac{1}{2})} dt, \quad j=0,1,\dots$$

Hence, the error committed is less in magnitude than the first term neglected. We find that

$$I_{j+\frac{1}{2}}(x) = \frac{e^{-x} x^{-(j+\frac{1}{2})}}{j-\frac{1}{2}} - \frac{1}{j-\frac{1}{2}} I_{j-\frac{1}{2}}(x) \quad j=0,1,\dots$$

By repeating the above recurrence relation several times, we

find that all  $I_{j+\frac{1}{2}}(x)$  can be made to fall back on  $\int_x^{\infty} e^{-t} t^{1/2} dt$ ,

which we can get from a table of the incomplete gamma function.

Briefly summarizing, then, we have shown how to evaluate any  $g_{n,m}(t)$  and  $\Pr\{T_{1,m} > x\}$  so that we can use 3.1.6 and 3.1.7 to find any c.d.f.

### Inequalities.

In what follows, under this heading, we shall discuss certain inequalities related to the distribution of the difference between two independent chi-squares considered in the preceding sub-section, 3.1, of section 3.

If  $n = 1$ ,

$$g_{1,m}(t) = \frac{e^{-t} t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})} \int_0^{\infty} e^{-2y} \left(1 + \frac{y}{t}\right)^{\frac{1}{2}} y^{\frac{m}{2}-1} dy,$$

and since  $e^{-\frac{y}{2t}} \leq \left(1 + \frac{y}{t}\right)^{-\frac{1}{2}} \leq 1$ , it follows that

$$3.1.8 \quad 2^{-\frac{m}{2}} h_1(t) \left(1 + \frac{1}{4t}\right)^{-\frac{m}{2}} \leq g_{1,m}(t) \leq 2^{-\frac{m}{2}} h_1(t), \quad t > 0,$$

and

$$3.1.9 \quad 2^{-\frac{m}{2}} [1 - H_1(x)] \geq \Pr[T_{1,m} > x] \geq 2^{-\frac{m}{2}} \left(1 + \frac{1}{4x}\right)^{-\frac{m}{2}} [1 - H_1(x)],$$

where  $H_n(x) = \int_0^x h_n(t) dt, \quad x > 0.$

If  $n, m > 2$ , then since

$$(y+t)^{\frac{n+m}{2}-2} \geq (y+t)^{\frac{n}{2}-1} y^{\frac{m}{2}-1} \geq y^{\frac{n+m}{2}-2}, \quad \text{it follows that}$$

3.1.10

$$\frac{e^{-t}}{2^{\frac{n+m}{2}-1} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} \int_0^{\infty} e^{-y} y^{\frac{n+m}{2}-2} dy \geq g_{n,m}(t) \geq \frac{e^{-t} \Gamma(\frac{n+m}{2} - 1)}{2^{\frac{n+m}{2}-1} \Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})} .$$

Again, since  $1 \leq (1 + \frac{y}{t})^{\frac{n}{2}-1} \leq e^{\frac{(n-2)y}{2t}}$ , it follows from 3.1.1 that,

for  $t \geq (\frac{n-2}{4})$ ,

$$3.1.11 \quad 2^{\frac{m}{2}} h_n(t) \leq g_{n,m}(t) \leq 2^{\frac{m}{2}} h_n(t) \left[ 1 - \frac{(n-2)}{4t} \right]^{\frac{m}{2}}$$

Now letting  $t \rightarrow \infty$  in 3.1.8 and 3.1.11 we have

$$3.1.12 \quad g_{n,m}(t) \sim 2^{\frac{m}{2}} h_n(t), \text{ so that}$$

$g_{n,m}(t)$  has the same order of contact at  $+\infty$  as the p.d.f. of  $\chi^2$  with  $n$  degrees of freedom. Similarly,  $g_{n,m}(t)$  has the same order of contact at  $-\infty$  as the p.d.f. of  $\chi^2$  with  $m$  degrees of freedom.

In 3.1.10 letting  $t \rightarrow 0$ , we have



$$3.1.13 \quad g_{n,m}(0) = \frac{\Gamma(\frac{n+m}{2} - 1)}{2^{\frac{n+m}{2}-1} \Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}, \text{ and so}$$

the frequency curve does not meet the origin.

The case  $n=m$ .

Under this heading we shall consider certain properties of the p.d.f. and c.d.f. of  $T_{n,n}$ . In this case the p.d.f. of  $T_{n,n}$  is the same as that of the sum of products of pairs of independent  $N(0, 1)$  variates.

On the p.d.f. of  $T_{n,n}$ .

The p.d.f. is symmetric and so we shall consider  $g_{n,n}(t)$  for  $t \geq 0$ . If we put  $n=m$  in 3.1.13, 23 get

$$3.1.14 \quad g_{n,n}(0) = \frac{\Gamma(n-1)}{2^{n-1}\Gamma(\frac{n}{2})} = \frac{\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}.$$

We can show that  $g_{n,n}(0)$  is a decreasing function of  $n$ .

Differentiating 3.1.14 we have

$$\frac{2\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} g'_{n,n}(0) = \frac{\Gamma'(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2})} - \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})}.$$

From Cramer ([5], p. 131), we have that  $\Gamma'(n)/\Gamma(n)$  is an increasing function and so  $g_{n,n}(0)$  is a decreasing function of  $n$ . That is,

$$3.1.15 \quad \frac{d}{dn} g_{n,n}(0) < 0.$$

Finally, it is easy to show that

$$3.1.16 \quad \frac{d}{dt} g_{n,n}(t) = 0 \quad \text{at } t=0, \text{ and}$$

consequently that  $g_{n,n}(t)$  has its maximum at the origin.

If in 3.1.1 we put  $n=m$ , and let  $2x = t(y-1)$ , we get

$$3.1.17 \quad g_{n,n}(t) = \frac{\left(\frac{t}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} K_{\frac{n-1}{2}}(t) .$$

If  $n=1$ ,  $g_{1,1}(t) = \frac{1}{\pi} K_0(t)$ . It is known that  $K_0(t)$  is asymptotic to both axes and has a logarithmic singularity at the origin. Using a well-known expansion for  $K_0(t)$ , [14], we find that

$$3.1.18 \quad g_{1,1}(t) \sim \frac{1}{\pi} \log \frac{2}{t}, \quad \text{as } t \rightarrow 0.$$

The moment generating function of  $T_{n,n}$  becomes

$$M(\theta) = (1-\theta^2)^{-\frac{n}{2}}, \text{ and so}$$

$$E T_{n,n}^{2s} = \frac{(2s)!}{s!} \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2})}$$

It is worth noting that if we expand  $T_{n,n}^{2s}$  and use the known moments of  $\chi^2$ , we get as a by-product that

$$\sum_{j=0}^{2s} \binom{2s}{j} \Gamma(\frac{n}{2}+2s-j) \Gamma(\frac{n}{2}+j) (-1)^j = \frac{(2s)!}{s!} \Gamma(\frac{n}{2}+s) \Gamma(\frac{n}{2}),$$

where  $s=0,1,\dots$  and  $n=1,2,\dots$

On the c.d.f. of  $T_{n,n}$ .

For large values of  $n$  we may wish to use the normal approximation, since  $T_{n,n}$  is asymptotically normal  $(0, \sqrt{n})$ . It is

here appropriate to use a result due to Berry [3]. Let  $X = \sum_{i=1}^n x_i$ ,

where the  $x_i$  are independent r.v.'s. Let  $EX = \alpha$ ,  $\text{Var } X = \sigma^2$ , the

c.d.f. of  $X$  be  $F(x)$ ,  $\lambda(X_1) = E |X_1|^3 / \text{Var } X_1$ ,  $\Lambda = \max_1 \lambda(X_1)$ , and

$$G(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt, \text{ then}$$

$$\sup_{-\infty < x < \infty} \left| F(x) - G\left(\frac{x-\alpha}{\sigma}\right) \right| \leq \frac{C \wedge}{\sigma}, \text{ where}$$

$$(2\pi)^{-\frac{1}{2}} \leq C \leq 1.88, \text{ according to Berry.}$$

If we let  $U_1 = \frac{1}{2}(X_1^2 - Y_1^2)$ , then we may write  $T_{n,n} = \sum_1^n U_1$ ,

where  $X_1$  and  $Y_1$  are independent  $N(0, 1)$  variates. Putting  $n=m=1$ ,

and  $2x = t(\sec \theta - 1)$  in 3.1.1, we find that  $E |U_1|^3 = 8/\pi$ .

If  $F_n(x)$  is the c.d.f. of  $T_{n,n}$ , then

$$3.1.19 \quad \sup_{-\infty < x < \infty} \left| F_n(x\sqrt{n}) - G(x) \right| \leq \frac{8C}{\pi\sqrt{n}}$$

Consider next  $\int_x^{\infty} t^n K_n(t) dt$ . Integrate by parts where

$U = t K_n(t)$ ,  $dV = t^{n-1} dt$ , and use the relation  $t K_n'(t) = nK_n(t) - t K_{n+1}(t)$ , getting

$$\int_x^{\infty} t^{n+1} K_{n+1}(t) dt = x^{n+1} K_n(x) + (2n+1) \int_x^{\infty} t^n K_n(t) dt.$$

Replacing  $n$  by  $(n-1)/2$ , we have

$$\begin{aligned} 3.1.20 \quad F_{n+2}(x) &= F_n(x) - \frac{2^{(n+1)/2}}{n\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})} K_{(n-1)/2}(x) \\ &= F_n(x) - \frac{x}{n} \mathfrak{E}_{n,n}(x) \end{aligned}$$

Hence, knowing that  $\mathfrak{E}_{2,2}(x) = \frac{e^{-x}}{2}$ ,  $\mathfrak{E}_{4,4}(x) = \frac{e^{-x}}{4}(1+x)$ ,

$F_2(x) = 1 - \frac{e^{-x}}{2}$ , we can obtain all  $F_{2n+2}(x)$ . Similarly, having a table of the Bessel functions  $K_0(x)$ ,  $K_1(x)$ , ... we can get all  $F_{2n+1}(x)$ , if we know  $F_1(x)$ .

Evaluation of  $F_1(x)$ .

We have  $\pi[1 - F_1(x)] = \int_x^{\infty} K_0(t) dt$ . Using tables of

$K_0(t)$  we could evaluate  $F_1(x)$  by numerical quadrature; however, we give an alternative method here. Integrate by parts where  $U = 1/t$ ,  $dV = tK_0(t) dt$ , and use the relation  $\int t^n K_{n-1}(t) dt = -t^n K_n(t)$ ,  $n=1,2,\dots$ , getting

$$\int_x^{\infty} K_0(t) dt = K_1(x) - \int_x^{\infty} K_1(t) \frac{dt}{t}.$$

If we carry out this integration by parts repeatedly, we find by induction that we get an alternating series, in which the error committed by stopping with a given term is less in magnitude than the first term neglected. If  $d_r$  is the  $r$ -th term of the series, then

$$d_r = \frac{(-1)^{r+1} 2^r \Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2}) x^{r-1}} K_r(x), \quad r=1,2,\dots$$

Furthermore, if  $s$  and  $k$  are any two positive integers,

$$3.1.21 \quad \sum_1^{2s-1} d_r \geq \pi [1 - F_1(x)] \geq \sum_1^{2k} d_r.$$

### 3.2 Special case 2.

In this section we shall give an expression for the p.d.f. of an indefinite q.f. when the latent roots of the matrix of the q.f. are equal in pairs.

#### Theorem 3.2.

Let

$$Q_n = \frac{1}{2}(a_1 Y_1 + \dots + a_k Y_k - a_{k+1} Y_{k+1} - \dots - a_n Y_n),$$

where the  $a_i > 0$ , and the  $Y_i$  are independent r.v.'s each having a chi-square distribution with two degrees of freedom. If  $f(q)$  is the p.d.f. of  $Q_n$ , then

$$f(q) = \sum_{j=1}^k e^{-\frac{q}{a_j}} a_j^{n-2} \prod_{\substack{i=1 \\ i \neq j}}^k (a_j - a_i)^{-1} \prod_{s=k+1}^n (a_j + a_s)^{-1}, \quad q > 0$$

$$= \sum_{j=k+1}^n e^{\frac{q}{a_j}} a_j^{n-2} \prod_{\substack{i=1 \\ i \neq j}}^k (a_j + a_i)^{-1} \prod_{s=k+1}^n (a_j - a_s)^{-1}, \quad q < 0.$$

#### Proof.

The moment generating function of  $Q_n$  is

$M(t) = \prod_{j=1}^n (1 - a_j t) (1 + a_{k+1} t) \dots (1 + a_n t)^{-1}$ . Then

$$f(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itq} M(it) dt.$$

We can evaluate  $f(q)$  using contour integration. Let us take as our contour in the complex plane, the real line from  $-R$  to  $+R$  and then the semi-circle of radius  $R$ , in the lower half-plane if  $q > 0$ , and in the upper half-plane when  $q < 0$ . In both cases the value of the integral around the semi-circle tends to zero as  $R \rightarrow \infty$ , if  $n \geq 2$ . Hence

$$f(q) = 2\pi i \int \text{Sum of residues of the integrand at } t = -\frac{1}{a_j}, j=1, \dots, k,$$

$q > 0$ , and

$$= 2\pi i \int \text{Sum of residues of the integrand at } t = \frac{1}{a_j}, j=k+1, \dots, n,$$

$q < 0$ .

Evaluating the residues, we get the form stated in the theorem.



## CHAPTER IV

### FURTHER BOUNDS ON THE C.D.F. OF A DEFINITE QUADRATIC FORM IN INDEPENDENT IDENTICALLY DISTRIBUTED VARIATES - EACH CENTRAL NORMAL OR MORE GENERAL

#### 4.1 The central normal case.

If we make use of the p.d.f. of  $Q_{2m}$  when the latent roots of the matrix of  $Q_{2m}$  are equal in pairs, we can obtain some convenient bounds on the p.d.f. of  $Q_{2n}$  when the latent roots are not necessarily equal in pairs. Let

$Q_{2m} = \frac{1}{2}(a_1 X_1^2 + \dots + a_m X_m^2 + a_{m+1} X_{m+1}^2 + \dots + a_{2m} X_{2m}^2)$ , where  $a_i > 0$ ,  $i=1, \dots, 2m$ , and where  $a_1 = a_{m+1}$ ,  $a_2 = a_{m+2}$ ,  $\dots$ ,  $a_m = a_{2m}$ , then the moment generating function of  $Q_{2m}$  is

$M(t) = \prod_{i=1}^m (1 - a_i t)^{-1}$ , and hence the p.d.f.

of  $Q_{2m}$  is  $h(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itq} M(it) dt$ .

Using the calculus of residues we readily find that

$$4.1.1 \quad h(q) = \sum_{j=1}^m e^{-\frac{q}{a_j}} a_j^{m-2} \prod_{\substack{k=1 \\ k \neq j}}^m (a_j - a_k)^{-1}$$

Suppose now that we want the p.d.f. of

$$Q_{2n} = \frac{1}{2}(a_1 X_1^2 + \dots + a_{2n} X_{2n}^2), \text{ where } a_1 \geq a_2 \geq \dots \geq a_{2n} > 0.$$

We form the two expressions

$$Q_U = \frac{1}{2} \int [a_1(x_1^2 + x_2^2) + a_3(x_3^2 + x_4^2) + \dots + a_{2n-1}(x_{2n-1}^2 + x_{2n}^2)]_-, \text{ and}$$

$$Q_L = \frac{1}{2} \int [a_2(x_1^2 + x_2^2) + a_4(x_3^2 + x_4^2) + \dots + a_{2n}(x_{2n-1}^2 + x_{2n}^2)]_-, \text{ so that}$$

$$Q_U \geq Q_{2n} \geq Q_L .$$

We can find the p.d.f. for  $Q_U$  and  $Q_L$  by using 4.1 so that we have bounds on the c.d.f. of  $Q_{2n}$ . Let  $f_U(q)$ ,  $f(q)$ ,  $f_L(q)$  be the p.d.f.'s of  $Q_U$ ,  $Q_{2n}$ ,  $Q_L$  respectively. Then

$$\int_0^t f_U(q) dq \leq \Pr [Q_{2n} \leq t] \leq \int_0^t f_L(q) dq, \text{ where}$$

$$f_U(q) = \sum_{j=1}^n e^{-\frac{q}{a_{2j-1}}} a_{2j-1}^{n-1} \prod_{\substack{k=1 \\ k \neq j}}^n (a_{2j-1} - a_{2k-1})^{-1}, \text{ and}$$

$$f_L(q) = \sum_{j=1}^n e^{-\frac{q}{a_{2j}}} a_{2j}^{n-1} \prod_{\substack{k=1 \\ k \neq j}}^n (a_{2j} - a_{2k})^{-1}.$$

Remarks.

(i) The above inequality was suggested by Professor Hotelling.

(ii) The above method could be extended to cover the case of an indefinite q.f.

4.2 The general case.

In this section we shall discuss briefly a system of inequalities for the distribution of a definite q.f. Let

$$Q_n = \frac{1}{2}(a_1 Y_1^2 + \dots + a_n Y_n^2), \text{ where}$$

$$\Pr [Y_i^2 \geq c] = p, \quad i=1, \dots, n. \text{ Then}$$

$$\Pr(Y_1^2 < c, \dots, Y_n^2 < c) = (1-p)^n. \text{ It follows that}$$

$$\Pr \left[ Q_n < \frac{c}{2}(a_1 + \dots + a_n) \right] \geq (1-p)^n .$$

Let  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ , getting the following system of inequalities:

$$\Pr \left[ Q_n < \frac{c}{2} a_n \right] \leq (1-p)^n ,$$

$$\Pr \left[ Q_n < \frac{c}{2}(a_{n-1} + a_n) \right] \leq (1-p)^n + \binom{n}{1}(1-p)^{n-1}p ,$$

$$\Pr \left[ Q_n < \frac{c}{2}(a_{n-2} + a_{n-1} + a_n) \right] \leq (1-p)^n + \binom{n}{1}(1-p)^{n-1}p + \binom{n}{2}(1-p)^{n-2}p^2 ,$$

. . . . .

$$\Pr \left[ Q_n < \frac{c}{2}(a_1 + \dots + a_n) \right] \leq 1-p^n$$

The above system was suggested to the writer by Professor Harold Hotelling. The following improvements are due to Professor S. N. Roy.

Suppose that the distribution of  $Y_i^2$  is known and that

$$\Pr(Y_i^2 \geq \frac{2c}{a_n}) = P_1 ,$$

$$\Pr(Y_i^2 \geq \frac{2c}{a_{n-1} + a_n}) = P_2 ,$$

$$\Pr(Y_1^2 \geq \frac{2c}{a_{n-2} + a_{n-1} + a_n}) = P_3,$$

. . . . .

$$\Pr(Y_1^2 \geq \frac{2c}{a_1 + \dots + a_n}) = P_n.$$

Then it follows that:

$$(1-p_n)^n \leq \Pr(Q_n < c) \leq (1-p_1)^n,$$

$$(1-p_n)^n \leq \Pr(Q_n < c) \leq (1-p_2)^n + \binom{n}{1} (1-p_2)^{n-1} p_2,$$

$$(1-p_n)^n \leq \Pr(Q_n < c) \leq (1-p_3)^n + \binom{n}{1} (1-p_3)^{n-1} p_3 + \binom{n}{2} (1-p_3)^{n-2} p_3^2,$$

. . . . .

$$(1-p_n)^n \leq \Pr(Q_n < c) \leq 1-p_n^n.$$

Hence we have one lower bound and a whole system of upper bounds. Obviously, we would want the least upper bound, in practice. Incidentally,  $p_1 \leq p_2 \leq \dots \leq p_n$ .

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