OPTIMAL TESTS FOR SEPARABLE FAMILIES OF HYPOTHESES

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1. INTRODUCTION

The problem of deciding which family of distributions describes the behavior of a random sample of observations has been examined rather extensively in the goodness-of-fit and hypothesis testing literature. Given a null class of distributions, if particular alternative families of distributions are considered, then a question arises as to the largest possible power a statistical test may achieve. When a particular alternative distribution or alternative family of distributions is considered, the testing problem is essentially that of testing separable hypotheses.

For a composite null hypothesis class, tests can be constructed as a function of conditional probability integral transformations (C.P.I.T.'s), which have been considered in O'Reilly and Quesenberry [16]. Under general conditions these transformations map a random sample from an unspecified member of a continuous parametric class of distributions to a smaller set of independent random variables with uniform distributions on the unit interval. Then any statistic which measures the divergence of the transformed values from a unifrom pattern may reasonably be considered as a goodness-of-fit test statistic for the original composite null hypothesis class. Many such tests may be constructed and, indeed, many exist in the extensive goodness-of-fit literature.

In this work, it is shown that a most powerful similar test may be obtained as a function of C.P.I.T.'s for a composite goodness-of-fit null hypothesis, and sufficient conditions are given to assure that such a test is uniformly most powerful against a composite alternative class. It is further shown under general conditions that this test identifies with certain uniformly most powerful invariant tests.

2. REVIEW OF LITERATURE

The problem of testing H: $P \in \mathcal{P}_0$ *versus* K: $P \in \mathcal{P}_1$, where $\mathcal{P}_0$ and $\mathcal{P}_1$ are separable families of distributions, separable in the sense that an arbitrary member of one family can not be obtained as the limit of the members of the other family, was first considered in detail by D. R. Cox [5]. Cox was the first writer to clearly identify and point out the importance of tests of separable families and give specific tests. The term "separable" is due to him. Cox developed a general method for this testing situation based on the logarithm of the Neyman-Pearson maximum likelihood ratio (M.L.R.). The statistic he examined was

$$\ell n\ \text{M.L.R.} - E_{\mathcal{P}_0}(\ell n\ \text{M.L.R.})\big|_{\alpha=\hat{\alpha}} ,$$

where

$$\text{M.L.R.} = \sup_{\alpha} f(x_1,\ldots,x_n;\alpha)/\sup_{\beta} g(x_1,\ldots,x_n;\beta)$$

and f and g denote the probability density functions arising from $\mathcal{P}_0$ and $\mathcal{P}_1$, respectively. Cox examined the asymptotic properties of this test, and, in particular, the asymptotic variance, in order to construct a statistic whose asymptotic distribution is standard normal. In [6], he develops the test for H: Poisson *versus* K: Geometric. Jackson [13] investigated the adequacy of Cox's results for H: Lognormal *versus* K: Exponential and derives the power of the test. Jackson also compares the test with other tests and derives the test for H: Lognormal *versus* K: Gamma . Using the same principle, Atkinson [2] developed a test for a mixed model including a set of hypothesized families of distributions in order to determine which family adequately describes the data.

The M.L.R. statistics proposed by Cox lack the property of invariance, in general. Lehmann [14] gave the general theory of uniformly most powerful invariant (U.M.P.I.) tests and gave integral expressions for the location-scale parameter case. A great deal more reduction is required for particular cases. In two papers, Uthoff [20, 21] derived the U.M.P.I. test for testing two-parameter Normal *versus* Uniform, Normal *versus* Exponential, Uniform *versus* Exponential, and Normal *versus* Double Exponential. Uthoff also shows that the test for H: Normal *versus* K: Double Exponential is asymptotically equivalent to the M.L.R. test and to Geary's test [11]. In a series of papers, Dyer [8, 9, 10] investigated various test statistics, including the U.M.P.I. and M.L.R. test statistics, and compared their relative efficiencies from a discrimination point of view for several alternative families of distributions. Antle, Dumonceaux, and Haas [1] examined the M.L.R. test for several location-scale parameter families and compared its power with the power of the U.M.P.I. test. Their recommendation for using the M.L.R. test instead of the U.M.P.I. test, when the two differ, is based on the ease of computation and relatively good performance of the M.L.R. test with respect to the U.M.P.I. test. Dumonceaux and Antle [7] followed with the M.L.R. procedure for discriminating between the lognormal and Weibull distributions.

It should also be pointed out that many goodness-of-fit tests are also tests for separable hypotheses, and that even though it would intuitively be expected that tests utilizing the knowledge of the alternative class would have better power than those tests that do not, there is, in fact, no proof that this will occur. Note, for example, the remarkably strange behavior of some goodness-of-fit tests reported by Dyer [8, 9, 10].

3. GENERAL RESULTS

3.1 Introduction

Let $\mathcal{X}$ denote a Borel set of real numbers, $\mathcal{A}$ the Borel subsets of $\mathcal{X}$, and $X = (X_1, \ldots, X_n)$ denote a vector of independent and identically distributed (i.i.d.) random variables, each distributed according to an absolutely continuous distribution P on the Borel space $(\mathcal{X}, \mathcal{A})$; and, further, suppose that P is a member of a parametric class of distributions $\mathcal{P} = \{P_\theta; \theta \in \Omega\}$. The set Ω is assumed to be a K-dimensional Borel set with elements $\theta = (\theta_1, \ldots, \theta_K)$. It is also assumed that there exists a K-dimensional sufficient statistic $T = (T_1, \ldots, T_K)$ for $\mathcal{P}$ (or Ω), defined on the sample space $(\mathcal{X}^n, \mathcal{A}^n) = (\mathcal{X} \times \ldots \times \mathcal{X}, \mathcal{A} \times \ldots \times \mathcal{A})$. Also, put $\mathcal{P}^n = \{P^n; P^n = P \times \ldots \times P, P \in \mathcal{P}\}$, i.e., $\mathcal{P}^n$ is a subclass of product measures on $(\mathcal{X}^n, \mathcal{A}^n)$ corresponding to $\mathcal{P}$. The class of product measures $\mathcal{P}^n$ is also written as $\mathcal{P}^n = \{P_\theta^n; \theta \in \Omega\}$.

Let $g: \mathcal{X} \to \mathcal{X}$ be a one-to-one transformation, and let g^n be the corresponding one-to-one transformation of $\mathcal{X}^n$ onto $\mathcal{X}^n$ defined by $g^n(x_1, \ldots, x_n) = (g(x_1), \ldots, g(x_n))$. For a given g^n, suppose there exists a function $\bar{g}: \Omega \to \Omega$ (or $\mathcal{P}^n \to \mathcal{P}^n$) such that $\bar{g}\theta \in \Omega$ and $P_{\bar{g}\theta}(X \in g^n A) = P_\theta(X \in A)$ for every $A \in \mathcal{A}^n$. Let the transformation g be such that the set of transformations G and the corresponding set of transformations $\bar{G}$ are transformation groups on $\mathcal{X}$ and Ω, respectively. Let G^n denote the corresponding product transformation subgroup on $\mathcal{X}^n$.

Definition

A transformation group on a space is said to be transitive if the maximal invariant of the group is constant on the space (cf. Lehmann [14], p. 216).

Denote by $\mathcal{B}_S$ the sub σ-algebra of $\mathcal{A}$ induced by a statistic S; by $h_1 \circ h_2$ the composition of a function h_1 with a function h_2, i.e., $h_1 \circ h_2(\cdot) = h_1(h_2(\cdot))$; by I_A the indicator function of a set A. With the usual abuse of notation the same symbol, e.g., g or g^{-1} will be used to denote a point function and the corresponding set function.

Lemma 3.1

For $g\colon \mathcal{X} \to \mathcal{X}$, one-to-one, and S any statistic defined on $(\mathcal{X}^n, \mathcal{A}^n)$,

$$P_{\bar{g}\theta}(g^n A \mid g^n \mathcal{B}_S) = P_\theta(A \mid S) \text{ a.s. } P_\theta \ \forall A \in \mathcal{A}^n . \tag{3.1.1}$$

Proof

The function g^n establishes a one-to-one correspondence between $\mathcal{B}_S$ and $\mathcal{B}_{S \circ g^{-n}}$ on $\mathcal{X}^n$. In fact, $g^n \mathcal{B}_S = \mathcal{B}_{S \circ g^{-n}}$. Thus

$$\int_{g^n B} E_{\bar{g}\theta}(I_{g^n A} \mid S \circ g^{-n}) dP_{\bar{g}\theta}(y)$$

$$= \int_{g^n B} I_{g^n A} dP_{\bar{g}\theta}(y) = P_{\bar{g}\theta}(g^n A \cap g^n B) = P_{\bar{g}\theta}\{g^n(A \cap B)\}$$

$$= P_\theta(A \cap B) = \int_B I_A dP_\theta(x) , \quad \text{all } \forall B \in \mathcal{B}_S ,$$

$$= \int_B E_\theta(I_A \mid S(x)) dP_\theta(x) \qquad \forall B \in \mathcal{B}_S ,$$

$$= \int_B h_\theta(S(x))dP_\theta(x) \qquad \forall\ B \in \mathcal{B}_S \ ,$$

$$= \int_{g^n B} h_\theta(S(g^{-n}(y)))dP_{\bar{g}\theta}(y) \qquad \forall\ B \in \mathcal{B}_S \ .$$

By the Radon-Nikodym theorem, a.s. P_θ $\forall\ A \in \mathcal{A}^n$,

$$P_{\bar{g}\theta}(g^n A | g^n \mathcal{B}_S) = h_\theta(S(g^{-n}(y))) \ ,$$

$$= h_\theta(S(x)) \ ,$$

$$= P(A|S) \ .$$

Lemma 3.2

If T is a sufficient statistic for Ω (or $\mathcal{P}$), and G^n is a product transformation group on $\mathcal{X}^n$ that induces a transitive group $\bar{G}$ on Ω , then the distribution function of the conditional distribution on $\mathcal{X}^n$ for fixed T is invariant under G^n , i.e.,

$$F(x_1, \ldots, x_n | \mathcal{B}_T) = F(gx_1, \ldots, gx_n | g^n \mathcal{B}_T) \qquad (3.1.2)$$

a.s. $\mathcal{P}^n$ $\forall\ g \in G(g^n \in G^n)$.

Proof

Let $\theta \in \Omega$ be fixed and θ' be an element of Ω . Then there exists a $g^n \in G^n$ such that for the corresponding $\bar{g}$, $\theta' = \bar{g}\theta$. Let $J_x = \{(y_1, \ldots, y_n) \ ; \ y_i \le x_i \ ; \ i = 1, \ldots, n\}$ and $x = (x_1, \ldots, x_n)$. Then

$$F_\theta(x_1, \ldots, x_n | \mathcal{B}_T) = P_\theta(J_x | \mathcal{B}_T) \quad \text{a.s.} \quad P_\theta ,$$

$$= P_{\bar{g}\theta}(g^n J_x | \mathcal{B}_{T \circ g^{-n}}) \text{ a.s. } P_\theta \text{ , by Lemma 3.1,}$$

$$= P_\theta(g^n J_x | \mathcal{B}_{T \circ g^{-n}}) \text{ a.s. } P_\theta \text{ , by sufficiency of } T \text{ ,}$$

$$= F_\theta(gx_1, \ldots, gx_n | \mathcal{B}_{T \circ g^{-n}}) \text{ a.s. } P_\theta \text{ .}$$

By the sufficiency of T , the subscript θ can be omitted on F , leaving

$$F(x_1, \ldots, x_n | \mathcal{B}_T) = F(gx_1, \ldots, gx_n | \mathcal{B}_{T \circ g^{-n}}) \quad \text{a.s.} \quad \mathcal{P}^n ,$$

where the exceptional set may depend on g^n , and thus $F(x_1, \ldots, x_n | T)$ is almost invariant under G^n . However, since $\mathcal{P}^n$ is a dominated family on a Euclidean space, and both $\mathcal{X}^n$ and Ω are Euclidean sets, it follows by Lehmann [14], Theorem 4 and discussion on p. 226, that the exceptional set does not depend on g^n .

3.2 Some Properties of Conditional Probability Integral Transformations

Conditional probability integral transformations are introduced in O'Reilly and Quesenberry [16], and extended to a larger collection of classes of distributions in Quesenberry [17]. Some basic properties of the transformations given in [16] are developed here. The same results can be obtained for the transformations in [17] by the same, or very similar, arguments.

Put

$$u_1(x_1) = P(X_1 \le x_1 | T) ,$$

$$u_2(x_2) = P(X_2 \le x_2 | T, X_1) , \qquad (3.2.1)$$

$$\vdots$$

$$u_{n-K}(x_{n-K}) = P(X_{n-K} \le x_{n-K} | T, X_1, \ldots, X_{n-K-1}) ,$$

and $u(x_1, \ldots, x_{n-K}) = (u_1, \ldots, u_{n-K})$. In [16] it is shown that if the conditional distribution of $X_1, \ldots, X_{n-K}$ given T is absolutely continuous, then $(U_1, \ldots, U_{n-K}) = (u_1(X_1), \ldots, u_{n-K}(X_{n-K}))$ are independently and identically distributed $U(0,1)$ random variables. From this and a result of Basu [3], the next theorem is immediate.

Theorem 3.1

If $T = (T_1, \ldots, T_K)$ is a complete and sufficient statistic for Ω , and if the conditional distribution of $(X_1, \ldots, X_{n-K})$ given T is absolutely continuous, then $(T_1, \ldots, T_K)$ and $(U_1, \ldots, U_{n-K})$ are independent vectors.

This theorem has important applications for constructing inference procedures that may be alternatives to nonparametric or robust procedures. The sufficient statistic T contains all the information for making inferences within the family $\wp$ (or Ω), whereas the statistic $U = (U_1, \ldots, U_{n-K})$ contains information about the family $\wp$. Thus U may be used to make inferences about the class $\wp$, such as a goodness-of-fit test for the class $\wp$, and T to make a parametric test within $\wp$, and the independence exploited to assess overall error rates. Inferences based on U are considered in the following sections.

<u>Theorem 3.2</u>

If G is a group of transformations of $\mathcal{X}$ such that the induced group $\bar{G}$ on Ω is transitive, and if $\wp$ has absolute continuity rank $n - K$ and sufficient statistic $T = (T_1, \ldots, T_K)$, then u of (3.2.1) is equivalent to an invariant statistic, *i.e.*,

$$u(x_1, \ldots, x_n) = u(gx_1, \ldots, gx_n) \text{ a.s. } \wp \ \forall g \in G .$$

<u>Proof</u>

By Lemma 2.1 of [16],

$$u_j = E(E(I_{[X_j \le x_j]}|T)|X_1, \ldots, X_{j-1}),\ j = 1, \ldots, n-K \text{ a.s. } \wp .$$

By Lemma 3.2 above, $E(I_{[X_j \le x_j]}|T)$ is invariant under G . The result follows.

The following lemma is a consequence of the fact that the transforming functions of (3.2.1) are (conditional) distribution functions.

<u>Lemma 3.3</u>

In the conditional space for fixed $T = t$, there is a.s. a one-to-one correspondence between $(u_1, \ldots, u_{n-K})$ and $(x_1, \ldots, x_n)$, *i.e.*, $(U_1, \ldots, U_{n-K})$ and $(X_1, \ldots, X_n)$ are equivalent statistics in this space.

3.3 Most Powerful Similar and Most Powerful Invariant Tests for Separable Families

Using the values $(X_1, \ldots, X_n)$, consider testing the hypothesis

$$H: \; P \in \mathcal{P}_0 = \{P_\theta ; \theta \in \Omega\} , \tag{3.3.1}$$

against the composite alternative

$$K: \; P \in \mathcal{P}_1 , \; (\mathcal{P} \cap \mathcal{P}_1 = \emptyset) . \tag{3.3.2}$$

It will also sometimes be useful to consider a simple alternative

$$K': \; P = P_1 , \quad P_1 \in \mathcal{P}_1 . \tag{3.3.3}$$

Let f_θ and F_θ denote the density and distribution functions, respectively, for P_θ under H, and f_1 and F_1 denote the density and distribution functions, respectively, for P_1 of K'. If $\mathcal{P}_0$ of (3.3.1) is specified by a class of densities $\{f_\theta ; \theta \in \Omega\}$ where f_θ has a particular functional form, then H is a classical composite goodness-of-fit null hypothesis. In work in classical composite goodness-of-fit testing, it is usually (tacitly) assumed that the null hypothesis class contains all distributions for which f_θ is a density, _i.e._, that Ω in (3.3.1) is a natural parameter space.

Definition

A test φ is _similar-α_ for H of (3.3.1) if $E_{P_\theta}(\varphi) = \alpha$ $\forall \theta \in \Omega$.

Tests for composite goodness-of-fit hypotheses are traditionally required to be similar-α. This restriction on tests has obvious appeal in that, for example, if it is desired to test for normality, then all normal distributions are equally normal, so the probability of rejection should not vary on the null class.

Under H, let $U_1, \ldots, U_{n-K}$ denote the $n-K$ i.i.d. $U(0,1)$ random variables obtained by (3.2.1). Also, let f_1 denote the parent density of the sample $X_1, \ldots, X_n$ under K' of (3.3.3), and $h_1(u_1, \ldots, u_{n-K})$ the corresponding density of $U_1, \ldots, U_{n-K}$. From the remark preceding Lemma 3.1, it follows that $h_1(u_1, \ldots, u_{n-K})$ is zero a.s. except in the unit hypercube. The next lemma is a direct application of the Neyman-Pearson Lemma.

Lemma 3.4

The most powerful level-α test of H *versus* K' based on $(U_1, \ldots, U_{n-K})$ is

$$\psi(U_1, \ldots, U_{n-K}) = \begin{cases} 1, & \text{if } h_1(U_1, \ldots, U_{n-K}) > c, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3.4}$$

where c is determined by $P\{h_1(U_1, \ldots, U_{n-K}) > c\} = \alpha$, for $U_1, \ldots, U_{n-K}$ i.i.d. $U(0,1)$ random variables.

Let $\varphi = \psi \circ U$. The following theorem shows that if T is boundedly complete, then φ is a most powerful similar-α (M.P.S.-α) test for H *versus* K'.

Theorem 3.3

If T is a boundedly complete sufficient statistic for $\mathcal{P}_0$ of (3.3.1), then the test $\varphi = \psi \circ U$ above is a most powerful similar-α test for H *versus* K′ .

Proof

By Lehmann [14], Theorem 2, p. 134,

$$E_P\{\varphi(X_1,\ldots,X_n)\} = \alpha \quad \forall\ P \in \mathcal{P}_0 \quad \text{if, and only if}$$

$$E_P\{\varphi(X_1,\ldots,X_n)\,|\,T\} = \alpha \text{ a.s. } \mathcal{P}_0^T .$$

Thus to find a most powerful test in the class of similar-α tests it is sufficient to find a most powerful conditional size-α test on the conditional space of $X_1, \ldots, X_n$ given T , *i.e.*, to find the most powerful Neyman-structure test. But for $T = t$ fixed, $(X_1, \ldots, X_n)$ and $(U_1, \ldots, U_{n-K})$ are equivalent statistics by Lemma 3.3. Thus, the test φ is a most powerful similar-α test.

It will sometimes be the case that φ does not depend on P for $P \in \mathcal{P}_1$ of (3.3.2). Then, of course, φ is a uniformly most powerful (U.M.P.) similar-α test for H *versus* K . Conditions under which such tests exist are considered in the following theorem.

Theorem 3.4

If the conditions for both Theorem 3.2 and Theorem 3.3 are satisfied by $\mathcal{P}_0$, then a U.M.P. invariant level-α test exists for testing H *versus* K , provided $\bar{G}$ is also transitive on $\mathcal{P}_1$. Moreover, this test is equivalent to the U.M.P.S.-α test of Theorem 3.3.

Proof

If φ is invariant level-α, then since $\bar{G}$ is transitive it follows from Lehmann [14], Theorem 3, p. 220, that

$$E_P(\varphi) = \alpha \quad \forall\ P \in \mathcal{P}_0 ,$$

i.e., φ is a similar-α test. Thus if a test is M.P. similar-α, it will be M.P. invariant level-α, provided it is invariant. But by Theorem 3.3, a M.P. similar-α test can a.s. be written as a function of $u_1, \ldots, u_{n-K}$ only, and is $\mathcal{B}_U$-measurable and invariant a.s. by Theorem 3.2.

Thus under rather general conditions U.M.P.S.-α and U.M.P.I.-α tests identify. The two approaches of finding the U.M.P.S.-α test vary from example to example in the amount of effort required to construct the test. If $U_1, \ldots, U_{n-K}$ of (3.2.1) are rather complicated functions of $X_1, \ldots, X_n$, then the task of obtaining the marginal density of $U_1, \ldots, U_{n-K}$ required in Lemma 3.4 is a difficult one. When this occurs, the invariance approach might be superior in terms of effort required, although this is not in general true.

The next definition, and, particularly, Theorem 3.5 are motivated by a result of Dyer [8, 9, 10]. He demonstrates empirically that a number of important goodness-of-fit tests have the property that the power is less when the value of a parameter is assumed known than for the case when the same parameter is assumed unknown, under the null hypothesis. This behavior is not particularly remarkable in that none of the tests considered have any known optimal power properties. In the next definition and Theorem 3.5, natural conditions are given which assure that the power of the U.M.P.S.-α test for a smaller null hypothesis family is never less than that of the U.M.P.S.-α test for a larger family.

<u>Definition</u>

Two separable families of distributions on the same space $(\mathcal{X},\mathcal{Q})$ are said to be <u>conformable</u> with respect to a group G of transformations if the corresponding group $\bar{G}$ on the parameter space is transitive for each family.

Consider two testing problems

$$H_1: \mathcal{P}_{1H} \ \underline{\text{versus}}\ K_1: \mathcal{P}_{1K}, \tag{3.3.5}$$

and

$$H_2: \mathcal{P}_{2H} \ \underline{\text{versus}}\ K_2: \mathcal{P}_{2K}, \tag{3.3.6}$$

where $\mathcal{P}_{1H} \subset \mathcal{P}_{2H}$, $\mathcal{P}_{1K} \subset \mathcal{P}_{2K}$, and $\mathcal{P}_{iH}$ and $\mathcal{P}_{iK}$ are conformable separable families of distributions, $i = 1, 2$.

<u>Theorem 3.5</u>

If φ_1 is U.M.P.S.-α for (3.3.5) and φ_2 is U.M.P.S.-α for (3.3.6), then

$$E_{\mathcal{P}_{1K}}(\varphi_1) \geq E_{\mathcal{P}_{2K}}(\varphi_2) . \tag{3.3.7}$$

Proof

The class of tests that are similar-α for (3.3.6) is a subclass of the class of tests that are similar-α for (3.3.5).

3.4 Optimal Classification Rules

In practice, one may confront the problem of deciding which class among a set of classes of distributions the data $X = (X_1, \ldots, X_n)$ has arisen from. The general classification model is to base the decision as to which class the data arise from on a statistic $S = S(X_1, \ldots, X_n)$. When the set of classes of distributions consists of only two members, say $\mathcal{P}_0$ and $\mathcal{P}_1$, a classification rule is

d(n)	Outcome	Decision	
1	$S > c$	X arises from $\mathcal{P}_1$	(3.4.1)
0	$S \leq c$	X arises from $\mathcal{P}_0$.	

It is assumed that S has a continuous distribution.

It has not been mentioned, but is implied by the conditions of Theorem 3.3, that the power function of the U.M.P.S.-α test for testing H *versus* K is constant on $\mathcal{P}_1$, *i.e.*, is not a function of $P \in \mathcal{P}_1$. With this in mind, it seems reasonable to use h_1 of Lemma 3.4 as the discriminant function in (3.4.1). If equal error probabilities of misclassification are required, then the next theorem shows that the decision function based on h_1 is optimal among the class of decision functions having equal error probabilities of misclassification.

Theorem 3.6

Consider the classification rule (3.4.1). Assuming that $\wp$ satisfies the conditions of Theorem 3.3, let $d(n)$ denote the classification function based on sample size n and statistic h_1 of Lemma 3.4 with error probabilities of misclassification equal to $\alpha(n)$. Let $d'(n')$ denote a classification function based on sample size n' and a statistic S with error probabilities of misclassification equal to $\alpha'(n')$. Let $n_0 = \min_{\alpha \le \alpha_0} \{n;\ n = 1, 2, \ldots\}$. If $\alpha'(n') \le \alpha_0$, then $n' \ge n_0$.

Proof

Assume that $\alpha'(n') \le \alpha_0$ for some $n' < n_0$. By definition, $P(d(n) = 0 | \wp_1) = P(d(n) = 1 | \wp_0) \equiv \alpha(n)$ and $\alpha'(n') \equiv P(d'(n') = 1 | \wp_0) = P(d'(n') = 0 | \wp_1) \equiv \beta'(n')$. By Theorem 3.5, if $n = n'$, then $\alpha'(n') \ge \alpha(n)$, since the U.M.P.S.-$\alpha$ test is based on h_1. By the definition of n_0, $\alpha(n_0 - 1) > \alpha_0$. Therefore, $\alpha'(n_0 - 1) \ge \alpha(n_0 - 1) > \alpha_0$. Clearly $\alpha'(n') > \alpha_0 \ \forall \ n' < n_0$. By contradiction, $n' \ge n_0$.

4. APPLICATIONS

4.1 Introduction

In this section the results of Chapter 3 are applied to obtain M.P.S. and U.M.P.S.-α tests, and to show that certain tests studied previously by other writers are M.P.S. or U.M.P.S.-α tests. When the conditions of Theorem 3.4 are satisfied, the minimal sample size n_0 of Theorem 3.6 is obtained for $\alpha_0 = .10$, .05, and .01 .

In most of the testing situations considered here, $\mathcal{P}_0$ and $\mathcal{P}_1$ are location-scale parameter families, and the testing problem has already been considered from the invariance and R.M.L. approaches. When the conditions of Theorem 3.4 are satisfied, the invariance and similar approaches yield equivalent test statistics. The invariance approach is easier to use in practice in many cases. The R.M.L. is also equivalent to the best invariant test statistic in many examples, although this does not happen in general (cf. Dyer [8, 9, 10]).

Tables were generated for various test statistics by empirical methods whenever the distribution of the test statistic was mathematically intractable. For the generation of normal and lognormal variates, the algorithm developed by Chen [4] was used. Variates from other distributions considered were generated using I.B.M.'s RANDU program together with the inverse of the distribution function.

For testing H: $P \in \mathcal{P}_0$ versus K: $P \in \mathcal{P}_1$, tables of critical values and power are given for samples of size $n = 10$, 20, and 30 (with one exception) for critical values $\alpha = .10$, .05, and .01. The primary purpose of these tables is to enable the reader to compare various goodness-of-fit statistics in a variety of testing situations. These tables are not intended to provide a comprehensive study of a particular testing problem. Antle, Dumonceaux, and Hass [1] give additional tables for the Normal *versus* Cauchy, Normal *versus* Exponential, and Normal *versus* Double Exponential testing problems.

For determining n_0 of Theorem 3.6 in a particular classification problem, tables are presented giving n_0, c, and the attainable equal error probabilities for $\alpha_0 = .10$, $.05$, and $.01$.

It is interesting to note that in the examples considered the test statistic for testing two location-scale parameter families are ratios of functions of the complete and sufficient statistics for the respective families. This is particularly appealing from the computational viewpoint, in that after the respective complete and sufficient statistics have been computed, the test statistic can be computed directly from them, avoiding the additional and often cumbersome task of calculating the U-statistics.

4.2 The U.M.P.S.-α Test for the Exponential (θ,λ) Versus the Normal (μ,σ^2) Distribution

The problem of testing exponentiality *versus* normality arises in the life testing area of reliability. It is desired to determine whether failure rates follow an exponential or normal distribution before inference is undertaken. This problem may be solved by using the U.M.P.S.-α test developed below.

4.2.1 Exponential class as the null class of distributions. Let the null class of densities be

$$\lambda \exp\{-\lambda(x-\theta)\} \; I_{(\theta,\infty)}(x) \quad ; \quad -\infty < \theta < +\infty , \; \lambda > 0 .$$

Uthoff [20] derives the U.M.P. location and scale invariant test for testing this family against a Normal (μ, σ^2) alternative family. The test statistic obtained is

$$(\bar{X}-X_{(1)})/(\sum_{i=1}^{n} (X_i-\bar{X})^2/(n-1))^{1/2} . \tag{4.2.1}$$

The equivalent R.M.L. statistic is presented in Antle, Dumonceaux, and Hass [1]. An alternative method of deriving the U.M.P.S.-α test statistic is the C.P.I.T. approach based on Theorem 3.4. The C.P.I.T. transformations for a random sample, $X_1, \ldots, X_n$ from this class of densities are (from Corollary 2.1 of [16]):

$$u_{r-2} = \{1 - (z_{r-1} - z_n)/(\sum_{i=1}^{r-1} (z_i - z_n))\}^{r-2} \; ; \; r = 3, \ldots, n , \tag{4.2.2}$$

where $z_n = x_{(1)}$, the smallest sample member, and the other z's are defined as follows. Suppose $z_n = x_j$. Then $z_i = x_i$; $i = 1, \ldots, j-1$, and $z_i = x_{i+1}$; $i = j+1, \ldots, n-1$.

Let $u_1, \ldots, u_{n-2}$ be defined as in (4.2.2) and let $u_{n-1} = z_1 - z_n$, $u_n = z_n$. Then $u_i \in (0,1)$, $i = 1, \ldots, n-2$; $u_{n-1} > 0$, and $\theta < u_n < \infty$;

$$z_1 = u_n + u_{n-1} , \quad z_n = u_n , \text{ and}$$

$$z_i = u_n + u_{n-1}(1 - u_{i-1}^{1/(i-1)} \prod_{j=1}^{i-1} u_j^{1/j}) , \quad i = 2, \ldots, n-1 .$$

The Jacobian of the transformation is $J = \det((\frac{\partial z_i}{\partial u'_j}))$

$$= \det \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ -\frac{u_{n-1}}{u_1^2} & 0 & 0 & * & 1 \\ * & \frac{-u_{n-1}}{2u_1 u_2^{3/2}} & 0 & * & 1 \\ \vdots & & & & \\ * & * & \frac{-u_{n-1}}{(n-2)(\prod_{j=1}^{n-3} u_j^{1/j}) u_{n-2}^{(n-1)/(n-2)}} & * & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{u_{n-1}^{n-2}}{(n-2)![\prod_{j=1}^{n-2} u_j^{1/j}]^{n-1}} .$$

The joint distribution of $u_1, \ldots, u_{n-2}$ must be obtained under the assumption that $X_1, \ldots, X_n$ constitute a random sample from a normal distribution, (μ, σ^2) unknown. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ random variables. Then letting the z's be as described above,

$$g(z_1, \ldots, z_n) = \frac{n}{(\sigma\sqrt{2\pi})^n} \exp\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (z_i - \mu)^2\} \, I_{(-\infty, \infty)}(z_n)$$

$$\cdot \prod_{i=1}^{n-1} I_{(z_n, \infty)}(z_i) .$$

Applying the u-transformation to the z's , the joint density of $u_1, \ldots, u_n$ is

$$h(u_1,\ldots,u_n) = \frac{n}{(n-2)!(\sigma\sqrt{2\pi})^n} \frac{u_{n-1}^{n-2}}{\{\prod_{j=1}^{n-2} u_j^{1/j}\}^{n-1}}$$

$$\exp\{-[(u_n-\theta)^2 + (u_n+u_{n-1}-\theta)^2$$

$$+ \sum_{j=2}^{n-1} (u_n+a_j u_{n-1}-\theta)^2]/2\sigma^2\}$$

$$\cdot \prod_{j=1}^{n-2} \mathbb{1}_{(0,1)}(u_j) \cdot \mathbb{1}_{(0,\infty)}(u_{n-1}) \mathbb{1}_{(-\infty,\infty)}(u_n)$$

where

$$a_j = 1 - (u_{j-1}^{\frac{1}{(j-1)}} / \prod_{K=1}^{j-1} u_K^{1/K}) , \qquad j = 2, \ldots, n-1 .$$

Integrating $h(u_1, \ldots, u_n)$ with respect to u_{n-1} and u_n , the joint density of $u_1, \ldots, u_{n-2}$ is, with

$$\alpha = 1 + \sum_{j=2}^{n-1} a_j^2 , \qquad \beta = 1 + \sum_{j=2}^{n-1} a_j ,$$

$$\frac{n}{(n-2)!(\sigma\sqrt{2n})^n (\prod_{j=1}^{n-2} u_j^{1/j})^{n-1}} \int_0^\infty y^{n-2} \exp\{-y^2(\alpha-\beta^2/n)/2\sigma^2$$

$$\int_0^\infty \exp\{-n(x-(\beta u_{n-1}-n\theta)/n)^2/2\sigma^2\}dxdy$$

$$\cdot \prod_{i=1}^{n-2} \mathbb{1}_{(0,1)}(u_i)$$

$$= \frac{}{(n-2)!(\sigma\sqrt{2n})^{n-1} (\prod_{j=1}^{n-2} u_j^{1/j})^{n-1}} \int_0^\infty y^{n-2} \exp\{-y^2(\alpha-\beta^2/n)2\sigma^2\}dy$$

$$\cdot \prod_{i=1}^{n-2} \mathbb{1}_{(0,1)}(u_i)$$

$$= \frac{\sqrt{n}\,\Gamma((n-1)/2)}{2\pi^{(n-1)/2}\Gamma(n-1)} \cdot \{[\prod_{j=1}^{n-2} u_j^{1/j}](\alpha-\beta^2/n)^{1/2}\}^{-(n-1)} \prod_{i=1}^{n-2} \mathbb{I}_{(0,1)}(u_i) \; . \tag{4.2.3}$$

Expressing (4.2.3) in terms of $z_1, \ldots, z_n$ (ignoring the constant coefficient), (4.2.3) reduces to

$$\{\sum_{i=1}^{n} (z_i - z_n)^2 / [\sum_{i=1}^{n} (z_i - z_n)]^2 - 1/n\}^{-(n-1)/2} \; , \tag{4.2.4}$$

and, finally, in terms of the original X variables,

$$\{\sum_{i=1}^{n} (x_i - x_{(1)}) / (\sum_{i=1}^{n} (x_i - \bar{x})^2)^{1/2}\}^{n-1} \; . \tag{4.2.5}$$

By Theorem 3.4, the hypothesis of exponentiality is rejected if expression (4.2.5) exceeds a critical value c, or, equivalently, if

$$T_{e,n} \equiv \frac{\bar{X} - X_{(1)}}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 / n}} > c \; , \tag{4.2.6}$$

where $P(T_{e,n} > c \mid \mathcal{P}_0) = \alpha$.

As mentioned at the end of Section 4.1, $T_{e,n}$ is a ratio of functions of the respective complete and sufficient statistics, and, in particular, is a ratio of estimators of dispersion for the respective families. Also, the statistic $T_{e,n}$ is independent of the complete and sufficient statistics of the respective families.

4.2.2 Normal class as the null class of distributions. Let the null class of densities be

$$(\sqrt{2\pi}\,\sigma)^{-1} \exp\{-(x-\mu)^2/2\sigma^2\} \, \mathbb{I}_{(-\infty,\infty)}(x) \; ; \; \sigma > 0 \; , \; -\infty < \mu < \infty \; .$$

The C.P.I.T. transformations for a random sample $X_1, \ldots, X_n$ from this class of densities are (from [16]):

$$U_i = G_{i-2}\{(i-2)^{1/2}(X_i-\bar{X}_i)/[(i-1)S_i^2 + (X_i-\bar{X}_i)^2]^{1/2}\} \;, \; i = 3, \ldots, n \;,$$

where G_{i-2} is the Student-t distribution function with $n - 2$ degrees of freedom,

$$\bar{X}_i = \sum_{j=1}^{i} X_j/i \;, \quad \text{and} \quad S_i^2 = \sum_{j=1}^{i} (X_j - \bar{X}_i)^2/i \;.$$

Since the U's are invariant with respect to linear transformations of the X's , the U.M.P.S.-α and U.M.P.I.-α tests are equivalent by Theorem 3.4. The test statistic is simply the reciprocal of (4.2.6), with the resulting test being to reject normality if

$$\frac{\sqrt{\sum_{i=1}^{n} (X_i-\bar{X})^2/n}}{\bar{X} - X_{(1)}} \geq c \;, \tag{4.2.7}$$

or, equivalently, if $T_{e,n} < c$, when $P(T_{e,n} < c|\mathcal{P}_0) = \alpha$. This test is equivalent to the R.M.L. test presented in Antle, Dumonceaux, and Hass [1]. Also, the remarks in the paragraph following (4.2.6) apply here.

Tables of critical values and power for the test statistic $T_{e,n}$ for samples of size $n = 10(5)30$ are presented in [1]. The table values were obtained by simulation using 5000 samples of size n for each n .

<u>4.2.3 Discrimination between the Exponential (θ,λ) and Normal (μ,σ^2) distributions</u>. If it is desired to use the statistic $T_{e,n}$ to discriminate between exponentiality and normality, then it may be reasonable to require α = P(Type I Error) and β = P(Type II Error) to be equal. Table 4.2.1 gives the minimal sample size n_0 required to obtain $\alpha = \beta \leq \alpha_0$ for $\alpha_0 = .10$, .05, and .01 . Table values were obtained by simulation using 5000 samples. Normality is accepted if $T_{e,n}$ exceeds the critical value c ; otherwise exponentiality is accepted.

Entries that appear in this and other tables which are derived by simulation are presented with two or three digits to the right of the decimal. By investigating samples of 5,000, 10,000 and 30,000, it was generally observed that entries based on 5,000 and 10,000 samples differed only in the second digit, and that entries based on 30,000 samples differed only in the third digit with those based on 10,000 samples.

Table 4.2.1

α_0	α	n_0	c
.10	.098	14	1.356
.05	.046	20	1.386
.01	.009	33	1.419

4.3 The U.M.P.S.-α Test For the Exponential $(0,\lambda)$ Versus the Uniform $(0,\theta)$ Distribution

4.3.1 Exponential class as the null class of distributions.

Let the null class of densities be

$$\lambda^{-1}\exp(-\lambda x) \prod_{(0,\infty)}(x) \ ; \quad \lambda > 0 \ .$$

A maximal invariant under the group G of transformations $g(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)$, $c > 0$, is $(X_1/X_n \ , \ldots, X_{n-1}/X_n)$, since both classes have the positive real line as a region of support. The U.M.P.I. test rejects the hypothesis of exponentiality whenever , [14],

$$\int_0^\infty v^{n-1} f_1(vx_1,\ldots,vx_n)dv \Big/ \int_0^\infty v^{n-1} f_0(vx_1,\ldots,vx_n)dv > c \ , \qquad (4.3.1)$$

where f_1 is the Uniform $(0,\theta)$ density and f_0 is the Exponential $(0,\lambda)$ density. Evaluating the numerator of (4.3.1),

$$\theta^{-n} \int_0^\infty v^{n-1} \prod_{]0,\theta/X(n)[}(v)\ dv = X_{(n)}^{-n}/n \ . \qquad (4.3.2)$$

Evaluating the denominator of (4.3.1),

$$\lambda^{-n} \int_0^\infty v^{n-1}\exp(-v \sum_{i=1}^{n} X_i/\lambda)dv = \sqrt{n}\, (\sum_{i=1}^{n} X_i)^{-n} \ . \qquad (4.3.3)$$

The test then reduces to rejecting exponentiality if $T_{e,u}^{(0)} \equiv \bar{X}/X_{(n)} > c$, and accepting otherwise, where $P(T_{e,u}^{(0)} > c|\rho_0) = \alpha$. This test is readily shown to be equivalent to the R.M.L. test.

As mentioned at the end of Section 4.1, $T_{e,u}^{(0)}$ is a ratio of functions of the respective complete and sufficient statistics, and, in particular, is a ratio of estimators of dispersion for the respective families. Also, the statistic $T_{e,u}^{(0)}$ is independent of the complete and sufficient statistics of the respective families.

4.3.2 Uniform class as the null class of distributions. Let the null class of densities be

$$\theta^{-1} \prod_{(0,\theta)}(x) \;; \qquad \theta > 0 \,.$$

As in Section 4.2.2, the U.M.P.S.-α test is to reject uniformity if $T_{e,u}^{(0)} < c$ and accept otherwise, where $P(T_{e,u}^{(0)} < c|\rho_0) = \alpha$. This test is equivalent to the R.M.L. test. Also, the remarks in the paragraph preceding this section apply here.

4.3.3 Distribution of $T_{e,u}^{(0)}$ when $X_1, \ldots, X_n$ are i.i.d. Uniform $(0,\theta)$ random variables. Let $z_n = x_{(n)}$, the largest sample member, and the other z's be defined as follows. Suppose $z_n = x_j$. Then $z_i = x_i$; $i = 1, \ldots, j-1$, and $z_i = x_{i+1}$; $i = j+1, \ldots, n-1$. Then

$$\bar{X}/X_{(n)} = (1 + \sum_{i=1}^{n-1} z_i/z_n)/n \,. \tag{4.3.4}$$

The joint density of $z_1, \ldots, z_n$ is

$$g(z_1,\ldots,z_n) = n\theta^{-n} \prod_{(0,\theta)}(z_n) \prod_{i=1}^{n-1} \prod_{(0,z_n)}(z_i) \,. \tag{4.3.5}$$

Let $t_n = z_n$ and $t_i = z_i/z_n$; $i = 1, \ldots, n-1$. Then

$$\bar{X}/X_{(n)} = (1 + \sum_{i=1}^{n-1} t_i)/n \,. \tag{4.3.6}$$

The joint density of $t_1, \ldots, t_n$ is

$$h(t_1,\ldots,t_n) = n\theta^{-n} t_n^{n-1} \underset{(0,\theta)}{I}(t_n) \prod_{i=1}^{n-1} \underset{(0,1)}{I}(t_i) . \qquad (4.3.7)$$

Observing that $t_1, \ldots, t_n$ are independent and that $t_1, \ldots, t_{n-1}$ are i.i.d. Uniform (0,1) random variables, $\bar{X}/X_{(n)}$ has the same distribution as $\{1 + \text{sum of } (n-1) \text{ i.i.d. Uniform } (0,1) \text{ random variables}\}/n$. Using the Central Limit Theorem, $T_{e,u}^{(0)}$ is approximately normally distributed with mean $(n+1)/2n$ and variance $(n-1)/12n^2$. Therefore, for large enough n, the critical value c in 4.3.2 is approximately

$$c \doteq \{n + 1 - z_\alpha \sqrt{(n-1)/3}\}/2n , \qquad (4.3.8)$$

where $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(.)$ is the standard normal distribution function.

The distributional properties of a sum of i.i.d. Uniform (0,1) random variables have received considerable attention in the literature of statistics. It is well known that the convergence to normality of such a sum is very rapid, the approximation being good for many purposes for samples as small as $n = 5$. For the purpose of examining the tails of the distribution, however, larger sample sizes may be required. For $n = 10$ and $\alpha = .01$, the critical value obtained by simulation is 0.351. Using (4.3.8), $c \doteq 0.349$ with empirical power 0.0093. The comparisons improve for $\alpha = .05$ and .10.

4.3.4 Distribution of $T_{e,u}^{(0)}$ when $X_1, \ldots, X_n$ are i.i.d. Exponential $(0,\lambda)$ random variables. Let $z_1, \ldots, z_n$ be defined as in Section 4.3.3 and consider again $\bar{X}/X_{(n)}$ in (4.3.1). The joint density of $z_1, \ldots, z_n$ is

$$g(z_1,\ldots,z_n) = n\lambda^{-n}\exp(-\sum_{i=1}^{n} z_i/\lambda) \prod_{(0,\infty)}(z_n) \prod_{i=1}^{n-1} \prod_{(0,z_n)}(z_i) \, . \tag{4.3.9}$$

Let $t_n = z_n$ and $t_i = z_i/z_n$; $i = 1, \ldots, n-1$. Then

$$\bar{X}/X_{(n)} = (1 + \sum_{i=1}^{n-1} t_i)/n \, . \tag{4.3.10}$$

The joint density of $t_1, \ldots, t_n$ is

$$h(t_1,\ldots,t_n) = n\lambda^{-n} t_n^{n-1} \exp\{-t_n(1+\sum_{i=1}^{n-1} t_i)/\lambda\}$$

$$\cdot \prod_{(0,\infty)}(t_n) \prod_{i=1}^{n-1} \prod_{(0,1)}(t_i) \, . \tag{4.3.11}$$

The marginal distribution of $t_1, \ldots, t_{n-1}$ is

$$h^*(t_1,\ldots,t_{n-1}) = n!(1+\sum_{i=1}^{n-1} t_i)^{-n} \prod_{i=1}^{n-1} \prod_{(0,1)}(t_i) \, . \tag{4.3.12}$$

The distribution of $Y = \sum_{i=1}^{n-1} t_i$ is

$$P(y) = n!(1+y)^{-n}\eta_{n-1}(y) \tag{4.3.13}$$

where $\eta_n(\cdot)$ is the density function of the sum of n i.i.d. Uniform (0,1) random variables. Finally, the density of $R = T_{e,u}^{(0)}$ is

$$k(r) = n!n(nr)^{-n}\eta_{n-1}(nr-1) \, . \tag{4.3.14}$$

The asymptotic properties of this distribution are not known. This density is quite complicated for even small n , and is, therefore, only recommended for use for very small n .

4.3.5 Critical values of $T_{e,u}^{(0)}$ and power for various sample sizes. Table 4.3.1 gives critical values of $T_{e,u}^{(0)}$ for the test H: Exponential $(0,\lambda)$ versus K: Uniform $(0,\theta)$ and for the test H: Uniform $(0,\theta)$ versus K: Exponential $(0,\lambda)$. For $n = 20$ and 30 the normal approximation was used to obtain critical values and power under the assumption of uniformity. All other entries were obtained by simulation, using 30,000 samples.

Table 4.3.1

Critical Values And Power of The U.M.P. Similar-α Test For Discriminating Between Exponential and Uniform Distributions

H: Exponential $(0,\lambda)$ K: Uniform $(0,\theta)$

Reject H if $T_{e,u}^{(0)} > c$

	α=.01		α=.05		α=.10	
n	c	Power	c	Power	c	Power
10	.594	.30	.524	.61	.487	.76
20	.471	.80	.419	.95	.389	.98
30	.415	.98	.371	1.00	.343	1.00

H: Uniform $(0,\theta)$ K: Exponential $(0,\lambda)$

Reject H if $T_{e,u}^{(0)} < c$

	α=.01		α=.05		α=.10	
n	c	Power	c	Power	c	Power
10	.351	.47	.407	.69	.438	.79
20	.379	.88	.422	.96	.444	.98
30	.396	.98	.431	.99	.450	1.00

4.3.6 Discrimination between the Exponential $(0,\lambda)$ and Uniform $(0,\theta)$ distributions. If it is desired to use the statistic $T_{e,u}^{(0)}$ to discriminate between exponentiality and uniformity, then it may be reasonable to require the error probabilities of misclassification to be equal. Table 4.3.2 gives the minimal sample size n_0 required to obtain $\alpha = \beta \leq \alpha_0$ for α_0 = .10, .05, and .01 . Table values were obtained by simulation using 5000 samples, and by using the normal approximation developed in 4.3.3. Uniformity is accepted if $T^{(0)}_{e,u}$ exceeds the critical value c ; otherwise exponentiality is accepted.

Table 4.3.2

α_0	α	n_0	c
.10	.100	14	.440
.05	.048	20	.420
.01	.010	34	.401

4.4 Tests For the Exponential $(0,\lambda)$ Versus the Lognormal $(0,\sigma^2)$ Distribution

The problem of deciding whether data is exponentially or log-normally distributed arises in the study of survival times of micro-organisms which have been exposed to a disinfectant or poison (*cf*.. Irwin [12]). Cox [5,6] developed tests for the testing problem and presented the asymptotic distribution of the test statistic. For testing H: Lognormal $(0,\sigma^2)$ *versus* K: Exponential $(0,\lambda)$, the test statistic given by Cox is

$$T_\ell = \ell n(\hat{\beta}/\beta_{\hat{\alpha}})/[(\exp(\hat{\alpha}_2) - 1 - \hat{\alpha}_2 - \tfrac{1}{2}\hat{\alpha}^2)/n]^{1/2} , \qquad (4.4.1)$$

where

$$\hat{\alpha}_1 = \sum_{i=1}^{n} \ell n\, X_i/n \ ,$$

$$\hat{\alpha}_2 = \sum_{i=1}^{n} (\ell n\, X_i - \hat{\alpha}_1)^2/n \ ,$$

$$\hat{\beta} = \bar{X} \ ,$$

$$\beta_{\hat{\alpha}} = \exp\{\hat{\alpha}_1 + \frac{1}{2}\hat{\alpha}_2\} \ .$$

Lognormality is rejected if $T_\ell > z_\alpha$, where $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(.)$ is the standard normal distribution function.

For testing H: Exponential $(0,\theta)$ _versus_ K: Lognormal $(0,\sigma^2)$, the test statistic given by Cox is

$$T_e = -\{(\hat{\alpha}_1 - \alpha_{1,\hat{\beta}}) + \frac{1}{2}\ \ell n(\hat{\alpha}_2/\psi'(1))\}/0.925 \ , \qquad (4.4.2)$$

where

$$\alpha_{1,\beta} = \ell n\, \beta + \psi(1) \ ,$$

$$\psi(t) = d\ell n \,\Gamma(t)/dt \ ,$$

$$\psi(1) = \text{Euler's constant} \ ,$$

$$\psi'(1) = \pi^2/6 \ ,$$

and $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\beta}$ are as described above. Exponentiality is rejected if $T_e > z_\alpha$, where $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(.)$ is the standard normal distribution function.

It is clear that interchanging H and K will not, in general, have the effect of inverting the test statistic if Cox's method is used, nor will the test statistics developed by this method be invariant, in general.

Srinivasan [19] has also considered the testing problem H: Exponential $(0,\lambda)$ <u>versus</u> K: Lognormal $(0,\sigma^2)$. The statistic proposed by Srinivasan is

$$D_n = \sup_t |S_n(t) - \tilde{F}(t;\lambda)| \ , \qquad (4.4.3)$$

where

$$\tilde{F}(t;\lambda) = \{1 - (1 - t/n\bar{X})^{n-1}\} \prod_{(0,n\bar{X})}(t) + \prod_{[n\bar{X},\infty)}(t) \ ,$$

the M.V.U.E. of the distribution function under H , and $S_n(t)$ is the empirical distribution function. Exponentiality is rejected if D_n exceeds a specified critical value.

For corrections to and comments about Srinivasan's results, see Schafer, Finkelstein, and Collins [18] and Moore [15].

<u>4.4.1 σ known</u>. Let the null class of densities be

$$\lambda^{-1} \exp(-x/\lambda) \prod_{(0,\infty)}(x) \ ; \qquad \lambda > 0 \ ,$$

and the alternative class of densities be

$$(x\sigma\sqrt{2\pi})^{-1} \exp\{-(\ell n\ x)^2/2\sigma^2\} \prod_{(0,\infty)}(x) \ ; \qquad \sigma > 0 \ .$$

The U.M.P.I.-α and U.M.P.S.-α test is given by (4.3.1), where f_1 is the lognormal density function and f_0 is the exponential density function. Evaluating the numerator of (4.3.1),

$$\left(\prod_{i=1}^{n} x_i\right)^{-1}\int_0^{\infty} v^{-1}\exp\{-\sum_{i=1}^{n}(\ell n\, x_i + \ell n\, v)^2/2\sigma^2\}dv$$

$$= \left(\prod_{i=1}^{n} x_i\right)^{-1}\int_{-\infty}^{\infty} \exp\{-\sum_{i=1}^{n}(t+\ell n\, x_i)^2/2\sigma^2\}dt \text{ , letting } t = \ell n\, v \text{ ,}$$

$$= (2\pi\sigma/n)^n\left(\prod_{i=1}^{n} x_i\right)^{-1}\exp\{-[\sum_{i=1}^{n}\ell n^2 x_i - (\sum_{i=1}^{n}\ell n\, x_i)^2/n]/2\sigma^2\} \text{ .} \quad (4.4.4)$$

The denominator of (4.3.1) is given in (4.3.3). The test reduces to rejecting exponentiality if

$$\left(\sum_{i=1}^{n} X_i\right)^n\left(\prod_{i=1}^{n} X_i\right)^{-1} \exp\{-[\sum_{i=1}^{n}\ell n^2 X_i - (\sum_{i=1}^{n}\ell n X_i)^2/n]2\sigma^2\} > c(\sigma) \text{ ,}$$

or, equivalently, if

$$T_{e,\ell}(\sigma) \equiv \ell n(\sum_{i=1}^{n} X_i) - (\sum_{i=1}^{n}\ell n\, X_i)/n$$

$$- [\sum_{i=1}^{n}\ell n^2 X_i - (\sum_{i=1}^{n}\ell n\, X_i)^2/n]/2n\sigma^2 > c(\sigma) \text{ ,} \quad (4.4.5)$$

where $P(T_{e,\ell}(\sigma) > c(\sigma)|\mathcal{P}_0) = \alpha$. It is readily shown that this test is equivalent to the R.M.L. test.

4.4.2 Critical values of $T_{e,\ell}(\sigma)$ and power for $n = 10, 20$. Table 4.4.1 gives critical values of $T_{e,\ell}(\sigma)$ and power for various values of σ for testing H: Exponential $(0,\lambda)$ *versus* K: Lognormal $(0,\sigma^2)$, σ known. All entries were obtained by simulation using 5000 samples.

Table 4.4.1

Critical Values and Power of $T_{e,\ell}(\sigma)$ Test For Discriminating Between Exponential and Lognormal Distributions

H: Exponential $(0,\lambda)$ K: Lognormal $(0,\sigma^2)$, σ known

Reject H if $T_{e,\ell}(\sigma) > c(\sigma)$

		α=.01		α=.05		α=.10	
n	σ	c(σ)	Power	c(σ)	Power	c(σ)	Power
	0.4	1.63	.93	1.16	1.00	.84	1.00
	0.6	2.07	.38	1.91	.80	1.80	.92
	0.8	2.24	.08	2.18	.36	2.14	.53
10	1.0	2.38	.08	2.32	.21	2.30	.35
	1.4	2.75	.24	2.63	.40	2.58	.53
	2.0	3.07	.64	2.91	.78	2.84	.85
	2.4	3.20	.81	3.02	.90	2.93	.94
	0.4	1.63	1.00	1.16	1.00	.83	1.00
	0.6	2.52	.93	2.34	1.00	2.23	1.00
	0.8	2.84	.43	2.78	.76	2.73	.89
20	1.0	3.02	.23	2.99	.45	2.96	.63
	1.4	3.36	.48	3.28	.68	3.25	.77
	2.0	3.65	.91	3.54	.96	3.49	.98
	2.4	3.75	.98	3.63	.99	3.58	1.00

4.4.3 σ unknown. Since σ is not a location or scale parameter, σ occurs in (4.4.4). Consequently, the left-hand side of expression (4.4.4) is not a statistic unless σ is known. When σ is unknown, test statistics can be formed by replacing σ in (4.4.4) with $\hat{\sigma}$, a sample estimator of σ . In order to obtain an invariant test, the expression

$$\hat{\sigma}^n \exp\{-[\sum_{i=1}^{n} \ell n^2 X_i - (\sum_{i=1}^{n} \ell n\, X_i)^2/n]/2\hat{\sigma}^2\} \qquad (4.4.6)$$

must be invariant with respect to the group G of transformations $g(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)$, $c > 0$. The estimator

$$\hat{\sigma}^2 = [\sum_{i=1}^{n} \ell n^2 X_i - (\sum_{i=1}^{n} \ell n\, X_i)^2/n]/(n-1) \qquad (4.4.7)$$

satisfies the invariance requirement and, furthermore, is an unbiased estimator of σ^2 if the underlying distribution is lognormal. Replacing σ^2 by $\hat{\sigma}^2$ in (4.4.4), a test for H: Exponential $(0,\lambda)$ versus K: Lognormal $(0,\sigma^2)$ is given by rejecting exponentiality if

$$(\sum_{i=1}^{n} X_i)^n (\prod_{i=1}^{n} X_i)^{-1} [\sum_{i=1}^{n} \ell n^2\, X_i - (\sum_{i=1}^{n} \ell n\, X_i)^2/n]^{-(n-1)/2} > c ,$$

or, equivalently, if

$$T^*_{e,\ell} \equiv \ell n(\sum_{i=1}^{n} X_i) - (\sum_{i=1}^{n} \ell n\, X_i)/n$$

$$- (n-1)\ell n[\sum_{i=1}^{n} \ell n^2 X_i - (\sum_{i=1}^{n} \ell n\, X_i)^2/n]/2n > c , \qquad (4.4.8)$$

where $P(T^*_{e,\ell} > c|\mathcal{P}_0) = \alpha$. The R.M.L. test rejects exponentiality if

$$(\sum_{i=1}^{n} X_i)^n \{(\sum_{i=1}^{n} \ell n^2\, X_i)^{n/2} \prod_{i=1}^{n} X_i\}^{-1} \geq c , \qquad (4.4.9)$$

and does not have the property of invariance.

4.4.4 Critical values of $T^*_{e,\ell}$ and power for n = 10, 20. Table 4.4.2 gives critical values of $T^*_{e,\ell}$ for the test H: Exponential versus K: Lognormal. All entries were obtained by simulation using 5000 samples.

Table 4.4.2

Critical Values And Power of $T^*_{e,\ell}$ For Discriminating Between Exponential And Lognormal Distributions

H: Exponential K: Lognormal

Reject H if $T^*_{e,\ell} > c$

n	c	σ	α=.01 Power	α=.05 Power	α=.10 Power
10	α = .01	0.4	.92	.99	1.00
	c = 2.02	0.6	.34	.69	.83
		0.8	.08	.26	.43
	α = ..05	1.0	.04	.13	.26
	c = 1.90	1.4	.17	.29	.39
	α = .10	2.0	.55	.67	.75
	c = 1.84	2.4	.75	.83	.87
20	α = ..01	0.4	1.00	1.00	1.00
	c = 2.17	0.6	.85	.98	.99
		0.8	.27	.63	.78
	α = .05	1.0	.12	.35	.52
	c = 2.10	1.4	.40	.55	.66
	α = .10	2.0	.86	.91	.93
	c = 2.07	2.4	.97	.98	.98

4.5 The U.M.P.S.-α Test for the Uniform (θ_1,θ_2) Versus the Normal (μ,σ) Distribution

4.5.1 Uniform class as the null class of distributions. Let the null class of densities be

$$(\theta_2 - \theta_1)^{-1} \, I_{(\theta_1,\theta_2)}(x) \; ; \qquad -\infty < \theta_1 < \theta_2 < +\infty \, .$$

Uthoff [20] derives the U.M.P. location and scale invariant test for testing this family against a Normal (μ,σ^2) alternative family. The test statistic obtained is

$$(X_{(n)} - X_{(1)})/(\sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1))^{1/2} \, . \qquad (4.5.1)$$

An alternative method of deriving the U.M.P.S.-α test statistic is the C.P.I.T. approach based on Theorem 3.4. The C.P.I.T. transformations for a random sample $X_1, \ldots, X_n$ from this class of densities are (from Corollory 2.1 of [16]):

$$u_i = (z_i - z_1)/(z_n - z_1) \ ; \qquad i = 2, \ldots, n-1 \ , \tag{4.5.2}$$

where $z_1 = x_{(1)}$, $z_n = x_{(n)}$, and the other z's are defined as follows. Suppose $z_1 = x_j$ and $z_n = x_k$, $j \neq k$. Then $z_i = x_{i-1}$, $i = 2, \ldots, j$; $z_i = x_i$, $i = j+1, \ldots, k-1$; and $z_i = x_{i+1}$, $i = k, \ldots, n-1$. To find the U.M.P. similar-α test the joint distribution of $u_2, \ldots, u_{n-1}$ must be obtained under the assumption that $X_1, \ldots, X_n$ constitute a random sample from a Normal (μ, σ^2) distribution. Without loss of generality, let $(\mu, \sigma^2) = (0,1)$.

Let $z_1, \ldots, z_n$ be as described in the preceding paragraph. The joint density of $z_1, \ldots, z_n$ is (ignoring constant coefficients)

$$f(z_1, \ldots, z_n) \propto \exp\Big(-\sum_{i=1}^{n} z_i^2/2\Big) \, I_{(-\infty,\infty)}(z_1) \, I_{(z_1,\infty)}(z_n) \cdot \prod_{i=2}^{n-1} I_{(z_1,z_n)}(z_i) \ . \tag{4.5.3}$$

Let $u_1 = z_1$, $u_n = z_n - z_1$, and $u_2, \ldots, u_{n-1}$ be as described in (4.5.2). The joint density of $u_1, \ldots, u_n$ is (ignoring constant coefficients)

$$g(u_1, \ldots, u_n) \propto u_n^{n-2} \exp\Big\{-\Big[n u_1^2 + 2u_1 u_n \Big(1 + \sum_{i=2}^{n-1} u_i\Big) + u_n^2 \Big(1 + \sum_{i=2}^{n-1} u_i^2\Big)\Big]/2\Big\}$$

$$\cdot \, I_{(-\infty,\infty)}(u_1) \, I_{(0,\infty)}(u_n) \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \ .$$

Integrating $g(u_1, \ldots, u_n)$ with respect to u_1 and u_n, the joint density of $u_2, \ldots, u_{n-1}$ is (ignoring constant coefficients)

$$
\begin{aligned}
h(u_2,\ldots,u_{n-1}) \propto \int_0^\infty t^{n-2} \exp\{-t^2(1+\sum_{i=2}^{n-1} u_i^2)/2\} \ . \\
\cdot \int_{-\infty}^{\infty} \exp\{-(nv^2+2vt(1+\sum_{i=2}^{n-1} u_i))/2\} dv dt \\
\cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \\
\propto \int_0^\infty t^{n-2} \exp\{-t^2((1+\sum_{i=2}^{n-1} u_i^2) \\
-(1+\sum_{i=2}^{n-1} u_i)^2/n)/2\} dt \cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \\
\propto \int_0^\infty r^{(n-1)/2-1} \exp\{-r((1+\sum_{i=2}^{n-1} u_i^2) \\
-(1+\sum_{i=2}^{n-1} u_i)^2/n\} dr \cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \\
\propto \{(1+\sum_{i=2}^{n-1} u_i^2) - (1+\sum_{i=2}^{n-1} u_i)^2/n\}^{-(n-1)/2} \\
\cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \ . \qquad (4.5.4)
\end{aligned}
$$

Expressing (4.5.4) in terms of $z_1, \ldots, z_n$, (4.5.4) reduces to

$$(z_n - z_1)^2/(\sum_{i=1}^{n} (z_i - \bar{z})^2)^{(n-1)/2} \ , \qquad (4.5.5)$$

and, finally, in terms of the original X variables,

$$\{(x_{(n)} - x_{(1)})/(\sum_{i=1}^{n} (x_i - \bar{x})^2)^{1/2}\}^{n-1} \ . \qquad (4.5.6)$$

By Theorem 3.4 the hypothesis of uniformity is rejected if

$$\{(X_{(n)}-X_{(1)})/(\sum_{i=1}^{n}(X_i-\bar{X})^2)^{1/2}\}^{n-1} > c \ , \text{ or equivalently,}$$

if

$$T_{u,n} \equiv \frac{X_{(n)} - X_{(1)}}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2}} > c \ , \qquad (4.5.7)$$

where $P(T_{u,n} > c|\mathcal{P}_0) = \alpha$.

As mentioned at the end of Section 4.1, $T_{u,n}$ is a ratio of functions of the respective complete and sufficient statistics, and in particular is a ratio of estimators of dispersion for the respective families. Also, the statistic $T_{u,n}$ is independent of the complete and sufficient statistics of the respective families.

4.5.2 Normal class as the null class of distributions. Let the null class of densities be

$$(\sqrt{2\pi}\,\sigma)^{-1}\exp\{-(x-\mu)^2/2\sigma^2\} \prod_{(-\infty,\infty)}(x) \ ; \ \sigma > 0 \ , \ -\infty < \mu < +\infty \ .$$

As in Section 4.2.2, the U.M.P.S.-α test is to reject uniformity if $T_{u,n} < c$ and to accept otherwise, where $P(T_{u,n} < c|\mathcal{P}_0) = \alpha$. The remarks in the paragraph following (4.5.7) also apply here.

4.5.3 Critical values of $T_{u,n}$ and power for various sample sizes. Table 4.5.1 gives critical values of $T_{u,n}$ for the test H: Uniform *versus* K: Exponential and for the test H: Normal *versus* K: Uniform . All entries were obtained by simulation using 30,000 samples.

Table 4.5.1

Critical Values and Power of $T_{u,n}$ Test for Discriminating Between Uniform and Normal Distributions

H: Uniform (θ_1, θ_2) K: Normal (μ, σ^2)

Reject H if $T_{u,n} > c$

n	α=.01 c	α=.01 Power	α=.05 c	α=.05 Power	α=.10 c	α=.10 Power
10	1.215	.07	1.138	.22	1.096	.35
20	.903	.33	.840	.59	.810	.72
30	.728	.65	.687	.83	.667	.90

H: Normal (μ, σ^2) K: Uniform (θ_1, θ_2)

Reject H if $T_{u,n} < c$

n	α=.01 c	α=.01 Power	α=.05 c	α=.05 Power	α=.10 c	α=.10 Power
10	.837	.07	.890	.22	.921	.34
20	.687	.27	.728	.54	.755	.69
30	.607	.53	.644	.80	.666	.90

4.5.4 Discrimination between the Uniform (θ_1, θ_2) and Normal (μ, σ^2) distributions. If it is desired to use the statistic $T_{u,n}$ to discriminate between uniformity and normality, then it may be reasonable to require the error probabilities of misclassification to be equal. Table 4.5.2 gives the minimal sample size n_0 required to obtain $\alpha = \beta \leq \alpha_0$ for $\alpha_0 = .10, .05$, and $.01$. Table values were obtained by simulation using 5000 samples. Normality is accepted if $T_{u,n}$ exceeds the critical value c ; otherwise uniformity is accepted.

Table 4.5.2

α_0	α	n_0	c
.10	.100	31	.657
.05	.048	41	.587
.01	.009	68	.467

4.6 The U.M.P.S.-α Test for the Uniform (θ_1,θ_2) Versus the Exponential (θ,λ) Distribution

4.6.1 Uniform class as the null class of distributions. Let the null class of densities be

$$(\theta_2 - \theta_1)^{-1} \prod_{(\theta_1,\theta_2)}(x) \;;\quad -\infty < \theta_1 < \theta_2 < +\infty \,.$$

Uthoff [20] derives the U.M.P. location and scale invariant test for testing this family against an Exponential (θ,λ) alternative family. The test statistic obtained is

$$(\bar{X} - X_{(1)})/(X_{(n)} - X_{(1)}) \,, \tag{4.6.1}$$

which is easily seen to be equivalent to the R.M.L. test statistic. An alternative method of deriving the U.M.P.S.-α test statistic is the C.P.I.T. approach based on Theorem 3.4. The C.P.I.T. transformations for a random sample $X_1, \ldots, X_n$ from this class of densities are given in (4.5.2). To find the U.M.P.S.-α test the joint distribution of $u_2, \ldots, u_{n-1}$ must be obtained under the assumption that $X_1, \ldots, X_n$ constitute a random sample from an Exponential (θ,λ) distribution. Without loss of generality, let $(\theta,\lambda) = (0,1)$.

Let $z_1, \ldots, z_n$ be defined as in (4.5.2). The joint density of $z_1, \ldots, z_n$ is (ignoring constant coefficients)

$$f(z_1,\ldots,z_n) \propto \exp(-\sum_{i=1}^{n} z_i)\, I_{(0,\infty)}(z_1)\, I_{(z_1,\infty)}(z_n) \prod_{i=2}^{n-1} I_{(z_1,z_n)}(z_i) \;. \tag{4.6.2}$$

Let $u_1 = z_1$, $u_n = z_n - z_1$, and $u_i = (z_i - z_1)/(z_n - z_1)$; $i = 2, \ldots, n-1$. The joint density of $u_1, \ldots, u_n$ is (ignoring constant coefficients)

$$g(u_1,\ldots,u_n) \propto u_n^{n-2} \exp\{-nu_1 - u_n(1 + \sum_{i=2}^{n-2} u_i)\}$$

$$\cdot\, I_{(0,\infty)}(u_1)\, I_{(0,\infty)}(u_n) \prod_{i=2}^{n-2} I_{(0,1)}(u_i) \;.$$

Integrating $g(u_1, \ldots, u_n)$ with respect to u_1 and u_n , the joint density of $u_2, \ldots, u_{n-1}$ is (ignoring constant coefficients)

$$h(u_2,\ldots,u_{n-1}) \propto \int_0^\infty t^{n-2} \exp\{-t(1 + \sum_{i=2}^{n-1} u_i)\}$$

$$\cdot \int_0^\infty e^{-nv} dv dt \cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i)$$

$$\propto \int_0^\infty t^{n-2} \exp\{-t(1 + \sum_{i=2}^{n-1} u_i)\} dt \cdot \prod_{i=2}^{n-1} I_{(0,1)}(u_i)$$

$$\propto (1 + \sum_{i=2}^{n-1} u_i)^{-(n-1)} \prod_{i=2}^{n-1} I_{(0,1)}(u_i) \;. \tag{4.6.3}$$

Expressing (4.6.3) in terms of $z_1, \ldots, z_n$, (4.6.3) reduces to

$$(\sum_{i=1}^{n} (z_i - z_1)/(z_n - z_1))^{-(n-1)}$$

$$= \{(z_n - z_1)/\sum_{i=1}^{n} (z_i - z_1)\}^{n-1} \;, \tag{4.6.4}$$

and, finally, in terms of the original X variables,

$$\{(x_{(n)}-x_{(1)})/\sum_{i=1}^{n}(x_i - x_{(1)})\}^{n-1} . \tag{4.6.5}$$

By Theorem 3.4, the hypothesis of uniformity is rejected if

$$\{(X_{(n)}-X_{(1)})/\sum_{i=1}^{n}(X_i-X_{(1)})\}^{n-1} > c \text{ , or equivalently,}$$

if

$$T_{e,u} \equiv \frac{\bar{X} - X_{(1)}}{X_{(n)}-X_{(1)}} < c , \tag{4.6.6}$$

where

$$P(T_{e,u} < c|\mathcal{P}_0) = \alpha .$$

As mentioned at the end of Section 4.1, $T_{e,u}$ is a ratio of functions of the respective complete and sufficient statistics, and in particular is a ratio of estimators of dispersion for the respective families. Also, the statistic $T_{e,u}$ is independent of the complete and sufficient statistics for the respective families.

4.6.2 Exponential class as the null class of distributions. Let the null class of densities be

$$\lambda^{-1}\exp\{-\lambda(x-\theta)\} \prod_{(\theta,\infty)}(x) \; ; \quad -\infty < \theta < +\infty , \qquad \lambda > 0 .$$

As in 4.2.2, the U.M.P.S.-α test is to reject exponentiality if $T_{e,u} > c$ and to accept otherwise, where $P(T_{e,u} > c|\mathcal{P}_0) = \alpha$. The remarks in the paragraph following (4.6.6) also apply here.

4.6.3 Distribution of $T_{e,u}$ when $X_1, \ldots, X_n$ are i.i.d. uniform (θ_1,θ_2) random variables. Let $U_2, \ldots, U_{n-1}$ be as described in (4.5.2). As mentioned at the beginning of Section 3.2, $U_2, \ldots, U_{n-1}$ are i.i.d. Uniform (0,1) random variables. Observing that

$$\sum_{i=2}^{n-1} U_i = n\{(\bar{X}-X_{(1)})/(X_{(n)}-X_{(1)}) - 1\} , \tag{4.6.7}$$

and applying the Central Limit Theorem, for large n the distribution of $\sum_{i=2}^{n-1} U_i$ is approximately normal with mean $(n-2)/2$ and variance $(n-2)/12$. Therefore, for large n, the distribution of $T_{e,u} \equiv (\bar{X} - X_{(1)})/(X_{(n)} - X_{(1)})$ is approximately normal with mean $1/2$ and variance $(n-2)/12n^2$, and the critical value c in (4.6.6) is approximately

$$c = \frac{1}{2} - z_\alpha \cdot \sqrt{(n-2)/3}\ /2n , \tag{4.6.8}$$

where $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(.)$ is the standard normal distribution function.

The distributional properties of a sum of i.i.d. Uniform (0,1) random variables have received considerable attention in the literature of statistics. It is well known that the convergence to normality of such a sum is very rapid, the approximation being good for many purposes for samples as small as $n = 5$. For the purpose of examining the tails of the distribution, however, larger samples may be required. For $n = 10$ and $\alpha = .01$, the critical value obtained by simulation is 0.315. Using (4.6.8), $c = 0.310$ with empirical power 0.0091. The comparisons improve for $\alpha = .05$ and .10 .

4.6.4 Distribution of $T_{e,u}$ when $X_1,\ldots,X_n$ are i.i.d. Exponential (θ,λ) random variables. Let $U_2, \ldots, U_{n-1}$ be as described in (4.5.2). The density of $Y = \sum_{i=2}^{n-1} U_i$ is, using (4.6.3),

$$f(y) = 2^{-1}(n-1)!(1+y)^{-(n-1)}\eta_{n-2}(y) , \tag{4.6.9}$$

where $\eta_n(\cdot)$ is the density function of the sum of n i.i.d. Uniform (0,1) random variables. Using relation (4.6.7), the density of $R = T_{e,u}$ is

$$g(r) = (n/2)(n-1)!(nr)^{-(n-1)}\eta_{n-2}(nr+1) . \tag{4.6.10}$$

The asymptotic properties of this distribution are not known. This density is quite complicated for even small n, and is, therefore, only recommended for use for very small n.

4.6.5 Critical values of $T_{e,u}$ and power for various sample sizes. Table 4.6.1 gives critical values of $T_{e,u}$ for the test H: Uniform *versus* K: Exponential and for the test H: Exponential *versus* K: Uniform. For $n \geq 20$, the normal approximation was used for critical values and power under the assumption of uniformity. Other entries were obtained by simulation, using 30,000 samples of size n for each n.

4.6.6 Discrimination between the Uniform (θ_1,θ_2) and Exponential (θ,λ) distributions. If it is desired to use the statistic $T_{e,u}$ to discriminate between uniformity and exponentiality, then it may be reasonable to require the error probabilities of misclassification to be equal. Table 4.6.2 gives the minimal sample size n_0 required to

Table 4.6.1

Critical values and power of $T_{e,u}$ test for discriminating between uniform and exponential distributions

H: Uniform (θ_1,θ_2) K: Exponential (θ,λ)

Reject H if $T_{e,u} < c$

n	α=.01 c	Power	α=.05 c	Power	α=.10 c	Power
10	.315	.42	.366	.64	.395	.75
20	.357	.85	.398	.94	.420	.97
30	.382	.98	.416	.99	.435	1.00

H: Exponential (θ,λ) K: Uniform (θ_1,θ_2)

Reject H if $T_{e,u} > c$

n	α=.01 c	Power	α=.-5 c	Power	α=.10 c	Power
10	.558	.24	.489	.55	.454	.71
20	.455	.77	.404	.94	.376	.98
30	.406	.97	.361	1.00	.338	1.00

Table 4.6.2

α_0	α	n_0	c
.10	.095	15	.409
.05	.049	21	.401
.01	.010	36	.391

obtain $\alpha = \beta \le \alpha_0$ for $\alpha_0 = .10$, .05 , and .01 . Table values were obtained by simulation, using 5000 samples, and by using the normal approximation developed in Section 4.6.3. Exponentiality is accepted if $T_{e,u}$ exceeds the critical value c ; otherwise uniformity is accepted.

It is interesting to note that the sample sizes in Table 4.6.2 are larger than the corresponding sample sizes in Table 4.3.2, and the power values in Table 4.6.1 are smaller than the corresponding values in Table 4.3.1. This is a direct consequence of the result stated in Theorem 3.5.

4.7 The U.M.P.S.-α Test for the Uniform $(0,\theta)$ Versus the Right Triangular $(0,\theta)$ Distribution

4.7.1 Right triangular class as the null class of distributions.

Let the null class of densities be

$$2\theta^{-2}x \, I_{(0,\theta)}(x) \; ; \qquad \theta > 0 \; .$$

Since both classes are scale parameter classes, the U.M.P.I.-α and U.M.P.S.-α test is given by (4.3.1), where f_0 is the right triangular density function and f_0 is the uniform density function. Evaluating the denominator of (4.3.1),

$$(2\theta^{-2})^n \prod_{i=1}^{n} x_i \int_0^{\infty} v^{2n-1} \, I_{(0,\theta/X_{(n)})}(v)\, dv = 2^{n-1} \prod_{i=1}^{n} x_i / n x_{(n)}^{2n} \; . \qquad (4.7.1)$$

The numerator of (4.3.1) is given in (4.3.2). The test is then to reject right triangularity if $\prod_{i=1}^{n} (X_{(n)}/X_i) > c$, or equivalently, if

$$T_{u,r} \equiv \ln X_{(n)} - \sum_{i=1}^{n} \ln X_i/n > c \,, \tag{4.7.2}$$

where $P(T_{u,r} > c|\mathcal{P}_0) = \alpha$. This test is equivalent to the R.M.L. test.

As mentioned at the end of Section 4.1, the U.M.P. similar-α test is based on the ratio of functions of the respective complete and sufficient statistics. Also, $T_{u,r}$, the logarithm of this ratio, is independent of the complete and sufficient statistics of the respective families.

<u>4.7.2 Uniform class as the null class of distributions.</u> Let the null class of densities be

$$\theta^{-1} \, I_{(0,\theta)}(x) \;; \qquad \theta > 0 \,.$$

As in Section 4.2.2, the U.M.P.S.-α test is to reject uniformity if $T_{u,r} < c$ and accept otherwise, where $P(T_{u,r} < c|\mathcal{P}_0) = \alpha$. The remarks in the paragraph following (4.7.2) also apply here.

<u>4.7.3 Distribution of $T_{u,r}$ when $X_1, \ldots, X_n$ are i.i.d. right triangular random variables.</u> Let $z_n = x_{(n)}$, the largest sample member, and the other z's be defined as follows. Suppose $z_n = x_j$. Then $z_i = x_i$; $i = 1, \ldots, j-1$, and $z_i = x_{i+1}$; $i = j+1, \ldots, n-1$. The joint density of $z_1, \ldots, z_n$ is

$$f(z_1,\ldots,z_n) = n(2\theta^{-2})^n \prod_{i=1}^{n} z_i \, I_{(0,\theta)}(z_n) \prod_{i=1}^{n-1} I_{(0,z_n)}(z_i) \,. \tag{4.7.3}$$

Let $y_i = z_i/z_n$; $i = 1, \ldots, n-1$, and $y_n = z_n$. The joint density of $y_1, \ldots, y_n$ is

$$g(y_1,\ldots,y_n) = n(2\theta^{-2})^n y_n^{2n-1} \prod_{i=1}^{n-1} y_i \, I_{(0,\theta)}(y_n) \prod_{i=1}^{n-1} I_{(0,1)}(y_i) \; . \tag{4.7.4}$$

The marginal density of $y_1, \ldots, y_{n-1}$ is

$$h(y_i,\ldots,y_{n-1}) = n(2\theta^{-2})^n \prod_{i=1}^{n-1} y_i \, I_{(0,1)}(y_i) \int_0^\theta v^{2n-1} \, dv$$

$$= 2^{n-1} \prod_{i=1}^{n-1} y_i \, I_{(0,1)}(y_i) \; . \tag{4.7.5}$$

Let $t_i = -\sum_{j=1}^{i} \ell n \, y_j$; $i = 1, \ldots, n-1$. Note that $t_{n-1} = nT_{u,r}$. The joint density of $t_1, \ldots, t_{n-1}$ is

$$k(t_1,\ldots,t_{n-1}) = 2^{n-1}\exp(-2t_{n-1}) \, I_{(0,\infty)}(t_{n-1}) \prod_{i=1}^{n-2} I_{(0,t_{i+1})}(t_i) \; . \tag{4.7.6}$$

The marginal density of t_{n-1} is

$$p(t) = (\Gamma n)^{-1} 2^{n-1} t^{n-1} \exp(-2t) \, I_{(0,\infty)}(t) \; , \tag{4.7.7}$$

so $4nT_{u,r}$ is a chi-square random variable with $2n$ degrees of freedom. The distribution function of $T_{u,r}$ is

$$F(t) = P(T_{u,r} \le t) = P(\chi^2_{2n} \le 4nt) \; . \tag{4.7.8}$$

4.7.4 Distribution of $T_{u,r}$ when $X_1, \ldots, X_n$ are i.i.d. Uniform $(0,\theta)$ random variables. Let $z_1, \ldots, z_n$ be defined as in Section 4.7.4. The joint density of $z_1, \ldots, z_n$ is

$$f(z_1,\ldots,z_n) = n\theta^{-n} \, I_{(0,\theta)}(z_n) \prod_{i=1}^{n-1} I_{(0,z_n)}(z_i) \; . \tag{4.7.9}$$

Let $y_i = z_i/z_n$; $i = 1, \ldots, n-1$, and $y_n = z_n$. The joint density of $y_1, \ldots, y_n$ is

$$g(y_1,\ldots,y_n) = n\theta^{-n} y_n^{n-1} I_{(0,\theta)}(y_n) \prod_{i=1}^{n-1} I_{(0,1)}(y_i) , \qquad (4.7.10)$$

so $Y_1,\ldots,Y_{n-1}$ are i.i.d. Uniform (0,1) random variables.

Let $t_i = -\sum_{j=1}^{i} \ln Y_j$; $i = 1, \ldots, n-1$. Note that $t_{n-1} = nT_{u,r}$. The joint density of $t_1, \ldots, t_{n-1}$ is

$$h(t_1,\ldots,t_{n-1}) = \exp(-t_{n-1}) I_{(0,\infty)}(t_{n-1}) \prod_{i=1}^{n-2} I_{(0,\, t_{i+1})}(t_i) . \qquad (4.7.11)$$

The marginal density of t_{n-1} is

$$p(t) = (\Gamma n)^{-1} t^{n-1} \exp(-t) \, I_{(0,\infty)}(t) , \qquad (4.7.12)$$

so $2nT_{u,r}$ is a chi-square random variable with $2n$ degrees of freedom. The distribution function of $T_{u,r}$ is

$$G(t) = P(T_{u,r} \le t) = P(\chi^2_{2n} \le 2nt) . \qquad (4.7.13)$$

4.7.5 <u>Critical values of $T_{u,r}$ and power for various sample sizes.</u>

Table 4.7.1 gives critical values of $T_{u,r}$ for the test H: Uniform *versus* K: Right Triangular and for the test H: Right Triangular *versus* K: Uniform. Formulae (4.7.8) and (4.7.13) were used to obtain the entries in the table.

4.7.6 <u>Discrimination between the Right Triangular and Uniform distributions</u>. If it is desired to use the statistic $T_{u,r}$ to discriminate between right triangularity and uniformity, then it may be reasonable to require the error probabilities of misclassification to be equal. Table 4.7.2 gives the minimal sample size n_0 required to obtain $\alpha = \beta \le \alpha_0$ for $\alpha_0 = .10, .05,$ and $.01$. Table values

Table 4.7.1

Critical Values and Power of $T_{u,r}$ Test for Discriminating Between Uniform and Right Triangular Distributions

H: Uniform K: Right Triangular

Reject H if $T_{u,r} < c$

n	$\alpha = .01$ c	Power	$\alpha = .05$ c	Power	$\alpha = .10$ c	Power
10	.4130	.316	.5425	.643	.6221	.794
20	.5541	.706	.6627	.918	.7263	.968
30	.6248	.908	.7198	.986	.7743	.996
40	.6693	.977	.7549	.998	.8035	1.000
50	.7007	.995	.7793	1.000	.8236	1.000
∞	1	1	1	1	1	1

H: Right Triangular K: Uniform

Reject H if $T_{u,r} > c$

n	$\alpha = .01$ c	Power	$\alpha = .05$ c	Power	$\alpha = .10$ c	Power
10	.9392	.536	.7853	.735	.7103	.820
20	.7961	.818	.6970	.926	.6476	.959
30	.7365	.937	.6590	.981	.6200	.991
40	.7021	.980	.6367	.995	.6036	.998
50	.6790	.994	.6217	.999	.5925	1.000
∞	.5	1	.5	1	.5	1

. .Table 4.7.2

α_0	α	n_0	c
.10	.09250	15	.67786
.05	.04981	23	.68316
.01	.00975	46	.68815

were obtained by using formulae (4.7.8) and (4.7.13). Uniformity is accepted if $T_{u,r}$ exceeds the critical value c ; otherwise right triangularity is accepted.

4.8 The U.M.P.S.-α Test for the Pareto Versus the Lognormal Distribution

Let $X_1, \ldots, X_n$ be i.i.d. according to one of the two following classes of densities:

$$f_1(x) = \theta_1 \theta_2^{\theta_1} x^{-(\theta_1+1)} \prod_{(\theta_2, \infty)}(x) \; ; \quad \theta_1, \theta_2 > 0 \text{ (Pareto density)},$$

$$f_2(x) = (\sqrt{2\pi}\,\sigma x)^{-1} \exp\{-(\ln X - \mu)^2/2\sigma^2\} \prod_{(0, \infty)}(x) \quad \text{(Lognormal density)} .$$

Let $Y_i = \ln X_i$, $i = 1, \ldots, n$. Then $Y_1, \ldots, Y_n$ are i.i.d. according to either the Exponential $(\ln \theta_2, \theta_1)$ or the Normal (μ, σ^2) distribution. The U.M.P.S.-α test for this testing problem is given in 4.1. It is readily shown that this test is equivalent to the R.M.L. test.

SUMMARY

Under rather general conditions it has been shown that a most powerful similar test may be obtained as a function of C.P.I.T.'s for a composite goodness-of-fit null hypothesis. Sufficient conditions are given to assure that such a test is uniformly most powerful against a composite alternative class. It is further shown under general conditions that this test identifies with certain uniformly most powerful invariant tests. Also, it is shown that if certain conditions are satisfied then the addition of information about the parameters of either the null or alternative classes results in a revised test with power at least as great as the test based on the original information.

The usefulness of such optimal tests is limited in practice by the following considerations. The separable hypotheses testing problem consists essentially of two parametric classes of distributions, viz., the null hypothesis class and the alternative class. When either of these classes is changed, a different test statistic with different null distribution will obtain, in general. Thus, each new testing problem requires a new set of significance points. Since for many cases the only practical way to obtain significance points is by Monte Carlo simulation, the computational effort and resultant tables would be large if tests are to be available for many cases of interest.

In practice, one often uses the same test statistic for a particular null hypothesis class against all alternative classes.

Also, the C.P.I.T. approach mentioned earlier can be used to establish a specified size test for many different testing problems and only one set of significance points is required. Either of these types of tests will, in general, be suboptimal for a particular problem. When such a suboptimal test is used, the power of the foregoing optimal test will provide a least upper bound for the power of such tests.

Finally, it was shown that under general conditions the test statistic used to construct the most powerful similar test for a particular testing problem can be used to construct an optimal classification rule.

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