

A NOTE ON THE P-ARY REPRESENTATION OF INTEGERS

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This article gives a new form of the p-ary representation of any non-negative integer for prime p, which is a generalization of Ikeda's formula for p = 2 [1].

For any non-negative integer N, the usual formula of the binary representation is given by

$$N = \sum_{k=0}^s C_k \cdot 2^k$$

where $C_k = \left[\frac{N}{2^k} \right] - 2 \left[\frac{N}{2^{k+1}} \right]$, $k=0, 1, \dots, s$; $s = \max \{h; 2^h \leq N\}$.

On the other hand, Ikeda [1] showed that N is expanded in the form

$$N = \sum_{k=0}^s \binom{N}{2^k} \cdot 2^k$$

where, for any positive integer R, $R_2 = 0$ if R is even, and = 1 if R is odd.

This implies, by the uniqueness of the binary representation, that

$$(1) \quad \frac{N}{2^k} = \binom{N}{2^k} \pmod{2}.$$

The purpose of this article is to prove the following identity.

$$(2) \quad \left[\frac{N}{p^k} \right] - p \left[\frac{N}{p^{k+1}} \right] = \binom{N}{p^k}_p,$$

or, equivalently,

$$(3) \quad \left[\frac{N}{p^k} \right] = \binom{N}{p^k}_p \pmod{p},$$

where N and k are non-negative integers, p is any given prime, $[]$ is the ordinary Gauss symbol, and $\binom{N}{n}_p$ denotes the remainder when $\binom{N}{n}$ is divided by p . Here, we use the convention that the number of n -combinations out of N , $\binom{N}{n}$, is equal to zero if $n > N$.

PROOF OF (2)

(I) In the case $0 \leq N < p^k$, it is evident that

$$\left[\frac{N}{p^k} \right] - p \left[\frac{N}{p^{k+1}} \right] = 0 = \binom{N}{p^k}_p$$

for any non-negative integer k .

(II) In the case $p^k \leq N < p^{k+1}$, N is expressed uniquely in the following form

$$N = jp^k + i,$$

for some integers j and i ; $1 \leq j \leq p-1$ and $0 \leq i \leq p^k-1$.

Then, for the left hand side of (2) it is easy to see that

$$\left[\frac{jp^k + i}{p^k} \right] - p \left[\frac{jp^k + i}{p^{k+1}} \right] = j$$

for any non-negative k . Thus, putting

$$I_{j,i}^k = \binom{N}{p^k}_p = \binom{jp^k + i}{p^k}_p$$

we have to show that the relation

$$(4) \quad I_{j,i}^k = j$$

holds true for any non-negative integer k .

In the first place, we shall prove the relation (4) when $i=0$, that is,

$$I_{j,0}^k = j, \text{ or equivalently, } \binom{jp^k}{p^k} = j \pmod{p}.$$

Since

$$\binom{jp^k}{p^k} = j \binom{jp^k - 1}{p^k - 1},$$

it suffices to prove that the relation

$$(5) \quad \binom{jp^k - 1}{p^k - 1} \equiv 1 \pmod{p}$$

holds true for any non-negative k . When $k = 0$, this relation is trivial.

Let us consider the expansion

$$(6) \quad \binom{jp^k - 1}{p^k - 1} = \prod_{h=1}^{p^k-1} \frac{jp^k - h}{h}$$

for any positive k .

In the above expansion, h is expressed in general as

$$h = p^{m(h)} \cdot \delta(h)$$

where $\delta(h)$ is prime to p and $m(h)$ is an integer such that $0 \leq m(h) \leq k-1$.

If h is prime to p , then $m(h) = 0$ and hence $h = \delta(h)$.

Thus, it follows from (6) that

$$(7) \quad \binom{jp^k - 1}{p^k - 1} = \frac{\prod_{h=1}^{p^k-1} (jp^{k-m(h)} - \delta(h))}{\prod_{h=1}^{p^k-1} \delta(h)}$$

Here, we have

$$\prod_{h=1}^{p^k-1} (jp^{k-m(h)} - \delta(h)) = p \cdot Q(p) + (-1)^{p^k-1} \prod_{h=1}^{p^k-1} \delta(h),$$

where $Q(p)$ is a polynomial in p .

Thus, we obtain

$$\binom{jp^k - 1}{p^k - 1} = \frac{p \cdot Q(p)}{\prod_{h=1}^{p^k-1} \delta(h)} + (-1)^{p^k-1}.$$

Recalling that $p^k - 1$ is even when $p \geq 3$, we thus have,

$$\binom{jp^k - 1}{p^k - 1} = \frac{p \cdot Q(p)}{\prod_{h=1}^{p^k-1} \delta(h)} + 1.$$

Since $\prod_{h=1}^{p^k-1} \delta(h)$ is prime to p , it divides $Q(p)$, and hence we have (5) for $p \geq 3$.

When $p = 2$, it holds that

$$\binom{j \cdot 2^k - 1}{2^k - 1} = 2 \cdot \frac{Q(2)}{\prod_{h=1}^{2^k-1} \delta(h)} - 1 \equiv 1 \pmod{2}.$$

Thus, the relation (5) is true for any prime p .

In the second place, we shall prove the relations

$$(8) \quad I_{j,1}^k = I_{j,i-1}^k$$

for $i=1,2,\dots,p^k-1$ and $j=1,2,\dots,p-1$, ($k \geq 1$).

Putting, as before, $h = p^{m(h)} \cdot \delta(h)$ for $h=1,2,\dots,p^k-1$, we have

$$\begin{aligned} \binom{jp^k + i}{p^k} - \binom{jp^k + i - 1}{p^k} &= \prod_{h=1}^{p^k-1} \frac{jp^k + i - h}{h} \\ &= p^{k-m(i)} \cdot \frac{j \prod_{h=1, h \neq i}^{p^k-1} (jp^{k-m(h)} + p^{m(i)-m(h)} \cdot \delta(i) - \delta(h))}{\prod_{h=1}^{p^k-1} \delta(h)} \end{aligned}$$

Since $\prod_{h=1}^{p^k-1} \delta(h)$ is prime to p and $0 \leq m(i) \leq k-1$, the second factor of the last expression of the above equalities should be a positive integer.

Hence

$$\binom{jp^k + i}{p^k} - \binom{jp^k + i - 1}{p^k} = 0 \pmod{p},$$

which proves (8).

This completes the proof of (4) in the case (II).

(III) In the case $p^{k+1} \leq N$, we may put for some q, j and i ,

$$N = qp^{k+1} + jp^k + i,$$

where q is a positive integer, $j=0,1,\dots,p-1$ and $i=0,1,\dots,p^k-1$.

Let us put

$$J_{j,i}^k = \binom{qp^{k+1} + jp^k + i}{p^k}.$$

Then, by a similar argument in the case (II), we have

$$J_{j,0}^k = J_{j,1}^k = \dots = J_{j,p^k-1}^k, \quad j=0,1,\dots,p-1$$

for any k positive and

$$J_{j,0}^k = j, \quad j=0,1,\dots,p-1$$

for any non-negative k . This implies that (2) is true in the case (III).

Hence, in all the cases, we proved the relations (2) for any prime number p and non-negative integer k .

REFERENCE

- [1] S. Ikeka, "A note on the Hamming weight of binary vectors", to appear in the Journal of the College of Industrial Technology, Nihon University, Tokyo, Vol.2, No.1 (1968).