A COMPARISON OF SOME APPROXIMATIONS TO THE k-SAMPLE BEHRENS-FISHER DISTRIBUTIONS

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Institute of Statistics Mimeo Series No. 704

August 1970

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Summary

Fisher's asymptotic expansion for the percentage points of the two-sample Behrens-Fisher distribution is extended to the k-sample case. Numerical values provided by the series are used to compare a simple approximation suggested by the expansion and two approximations that have been proposed previously.

1. Introduction

The construction of interval estimates for linear combinations of means is important in many applications. If the observations are normally distributed about the means with unknown, and not necessarily equal, variances it is well-known (see Lindley (1965), for example) that a Bayesian approach reduces to finding the appropriate percentage points of

$$d = \sum_1^k \xi_i \, t_{n_i} \qquad \left(\sum_1^k \xi_i^2 = 1\right), \tag{1}$$

where $\xi_1, \ldots, \xi_k$ are known constants, t_{n_i} has a Student-t distribution with n_i d.f. and $t_{n_1}, \ldots, t_{n_k}$ are mutually independent.

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* This research was supported by the National Institutes of Health Grant GM-12868

**On leave from C.S.I.R.O., Adelaide and the London School of Economics respectively.

When k=2, percentage points for some values of ξ_1, ξ_2, n_1, n_2 can be looked up in Sukhatme's tables. (See Sukhatme et al. (1951) for a detailed exposition). The tables are reproduced in Fisher and Yates (1963), which also gives tables constructed by Fisher and Healy (1956) for small odd degrees of freedom. For larger n_1, n_2 the expansion given by Fisher (1941) can be used for values of $\theta = \sin^{-1} \xi_1$ outside the range of the tables. There are no tables for $k > 2$ and even if tables did exist the number of parameters becomes so large that they would be cumbersome and difficult to use. The natural approach in these circumstances is to look for a good approximation. Approximations based on the t-distribution have been suggested by Quenouille (1958) and Patil (1965, 1969).

A comparison of different approximations requires some knowledge of the exact values, and to provide this knowledge we have extended Fisher's expansion to the k-sample case. A sketch of the development is given in the next section. Fisher's approach, based on a convolution of t variables, is difficult to extend directly, and we have adopted a different approach using the cumulant generating function. This is simpler even for k=2 and has the additional advantage of allowing a direct use of the tabulated Cornish-Fisher polynomials. A slight modification of the resulting series suggests a simple approximation closely related to that suggested by Welch in his attack on a frequentist solution to the Behrens-Fisher problem. The accuracy of this approximation and the two suggested earlier are investigated in the final section.

2. The Cornish-Fisher expansion of d_α

The cumulant generating function (c.g.f.) of t_n, $K_n(\phi) = \log \mathcal{E} \exp(i\, \phi\, t_n)$, may be expanded in an asymptotic series in n^{-1} for sufficiently large n, the terms to third order being

$$K_n(\phi) \sim -\frac{1}{2}\phi^2 + \frac{1}{4n}(\phi^4-4\phi^2) - \frac{1}{3n^2}(\phi^6-6\phi^4+6\phi^2)$$

$$+ \frac{1}{24n^3}(15\phi^8-128\phi^6+264\phi^4-96\phi^2) + .. \qquad (2)$$

We define

$$V_{ij} = \sum_{\ell=1}^{k} \xi_\ell^{2i}/n_\ell^j \, , \qquad (3)$$

following the notation of Aspin (1948) in her development of Welch's method for finding an approximate frequentist solution to the Behrens-Fisher problem (Welch (1947)). Then the c.g.f. $\Lambda(\phi)$ of $d = \sum_1^k \xi_i t_{n_i}$ $(\sum_1^k \xi_i^2=1)$ may be expressed formally as

$$\Lambda(\phi) \sim -\frac{1}{2}\phi^2 + \sum_{j=1}^{\infty} \rho_{2j}(i\phi)^{2j} \qquad (4)$$

where, to the third order,

$$\rho_2 \approx V_{11}+2V_{12}+4V_{13} \qquad = O(n_i^{-1})$$

$$\rho_4 \approx \frac{1}{4}V_{21}+2V_{22}+11V_{23} \qquad = O(n_i^{-1})$$

$$\rho_6 \approx \frac{1}{3}(V_{32}+16V_{33}) \qquad = O(n_i^{-2})$$

$$\rho_8 \approx \frac{5}{8}V_{43} \qquad = O(n_i^{-3})$$

The series (4) is of the general form considered by Fisher and Cornish (1937, 1960). Their expansion of the $100(1-\frac{\alpha}{2})\%$ point, d_α, of d in terms of the corresponding standard normal deviate u_α may be represented as

$$d_\alpha \sim u_\alpha + \sum_\pi \rho_\pi P_\pi(u_\alpha) \qquad (5)$$

(Hill and Davis (1968)), where the summation is extended over all partitions $\pi = [s_1^{t_1},\ldots,s_\ell^{t_\ell}]$ $(\sum_1^\ell s_i t_i = n)$ of all positive integers n, the $P_\pi(u_\alpha)$ are polynomials in u_α and

$$\rho_\pi = \prod_{i=1}^{\ell} \rho_{s_i}^{t_i}/t_i\ ! \ . \tag{6}$$

Fisher and Cornish tabulated sufficient P_π to give the expansion (5) to $O(n_i^{-3})$. The additional polynomials required for higher orders may be constructed using the methods given in Hill and Davis (1968, sections 4 and 5). A valid asymptotic series is obtained when terms of like order are grouped. We have finally

$$d_\alpha \sim u_\alpha + \sum_{r=1}^{\infty} Q_r(u_\alpha) \tag{7}$$

where $Q_r(u_\alpha) = O(n_i^{-r})$. Expressions for $Q_r(u)$ have been given in Davis and Scott (1970) for r=1,...,5. A computer program giving d_α to fifth-order terms is available. Inparticular

$$\begin{aligned} Q_1(u) &= u[V_{11} + \tfrac{1}{4} V_{21}(u^2-3)], \\ Q_2(u) &= u[2V_{12} - \tfrac{1}{2} V_{11}^2 + (2V_{22} - \tfrac{3}{4} V_{11}V_{21})(u^2-3) \\ &\quad + \tfrac{1}{3} V_{32}(u^4-10u^2+15) - \tfrac{3}{32} V_{21}^2(3u^4-24u^2+29)]. \end{aligned} \tag{8}$$

The result up to the fourth order reduces to that of Fisher (1941, Table 2) in the two-sample case with $\xi_1 = \sin\theta$, $\xi_2 = \cos\theta$. The fifth order term was found necessary to get reasonable accuracy for values of n_i as small as 6. Sukhatme et al (1951) also found this in their work, and extended Fisher's expansion to $O(n_i^{-5})$ for k=2. It is difficult to check directly, since they do not give the expression for the extra term, but our numerical results agree essentially with theirs. We note that the term of highest order in cos θ and sin θ in Q_r corresponds to

$$V_{21}^r = \left(\frac{\sin^4\theta}{n_1} + \frac{\cos^4\theta}{n_2}\right)^r .$$

Fisher's homogeneous trigonometric polynomials are obtained by multiplying the remaining terms by positive powers of $(\sin^2\theta + \cos^2\theta)$. The unexplained check observed by Fisher (1941, p. 155) when we formally take $\cos^2\theta = -\sin^2\theta$ is explained by the fact that $Q_r(u)$ then reduces to a multiple of $(\frac{1}{n_1} + \frac{1}{n_2})^r P_{4r}(u)$.

3. An approximation to the posterior distribution of d

In his approach to finding a frequentist solution to the Behrens-Fisher problem, Welch (1947) suggests comparing the observed value of d with

$$d^*_\alpha \doteq u_\alpha[1+ \frac{1}{4} V_{21}(u^2_\alpha+1)- \frac{1}{4} V_{22}(u^2_\alpha+1) + \frac{1}{3} V_{32}(u^4_\alpha+5u^2_\alpha+3) - \frac{1}{32} V^2_{21}(9u^4_\alpha+32u^2_\alpha+15) +\ldots] \quad (9)$$

A comparison of the special case of the t-distribution ($k=1$, $\xi_1=1$) for which $V_{ij}=n_1^{-j}$ suggests approximating d^*_α by the corresponding percentage point of t_ν with $\nu = V_{21}^{-1} = (\sum_1^k \xi_j^4/n_j)^{-1}$. This differs from expression (9) only in terms of $O(n_i^{-2})$ and Welch has shown that the approximation works very well in practice for reasonable n_i.

The form of expansion (7) as it stands does not lead to a similar approximation for d_α. However if we express the expansion in terms of the standardized deviate d_α/s where s^2 is the variance of d,

$$s^2 = \sum_1^k \xi_j^2\, n_j/(n_j-2) \sim 1+2V_{11}+4V_{12}+8V_{13} +\ldots, \quad (10)$$

it is found that all products formed from V_{11}, V_{12}, V_{13} etc. are removed to give

$$\frac{d_\alpha}{s} \sim u_\alpha[1+\frac{1}{4} V_{21}(u^2_\alpha-3)+(2V_{22}-V_{11}V_{21})(u^2_\alpha-3) + \frac{1}{3} V_{32}(u^4_\alpha-10u^2_\alpha+15) - \frac{3}{32} V^2_{21}(3u^4_\alpha-24u^2_\alpha+29)+\ldots]$$

Comparison with the special case of t_n now leads directly to the simple approximation to the posterior distribution of d:

$$d \approx t_\nu s\sqrt{\frac{\nu-2}{\nu}} \tag{11}$$

where

$$\nu = V_{21}^{-1} = \left(\sum_1^k \xi_j^4/n_j\right)^{-1}. \tag{12}$$

The degrees of freedom are the same as in Welch's approximation, but the approximation is for the standardized d in terms of the standardized t.

4. A comparison with other approximations

Quenouille (1958) has suggested approximating the distribution of d by a t-distribution with the same variance. Patil (1965, 1969) has proposed using a multiple of a t-variate with the multiplying constant and degrees of freedom chosen by equating the first four moments. All three approximations can be expressed in the form (11), $d \approx t_\nu s\sqrt{\frac{\nu-2}{\nu}}$, for different values of ν. The values of ν are:

$$\nu_{DS} = \left[\sum_1^k \xi_j^4/n_j\right]^{-1} \tag{13}$$

for the approximation of section 3 (D-S);

$$\nu_Q = \left[\sum_1^k \xi_j^2\, n_j/(n_j-2)\right] / \left[\sum_1^k \xi_j^2/(n_j-2)\right] \tag{14}$$

for Quenouille's approximation (Q);

$$\nu_P = 4 + \left[\sum_1^k \xi_j^2\, n_j/(n_j-2)\right]^2 / \left[\sum_1^k \xi_j^4\, n_j^2/(n_j-2)^2(n_j-4)\right] \tag{15}$$

for Patils' approximation (P).

The approximations all become more accurate for fixed $\xi_1,\ldots,\xi_k$ as $\min_{1\le j\le k}\{n_j\}$

increases. As k increases with $\min_{1 \le j \le k} \{n_j\} \ge 3$, d/s converges in distribution to a standard normal random variable provided $\max_{1 \le j \le k} \xi_j \to 0$ (see, for example, Hájek and Šidák (1969, p. 153)). Since $\nu_{DS} \ge 3/\max\{\xi_i\}$ the D-S approximation also approaches the appropriate normal percentage point under these conditions and the error converges to zero as k increases. The same is true of the P approximation if $\min\{n_j\} \ge 5$. The value of ν_Q, however, remains finite so that the error in the Q approximation does not converge to zero as k increases.

The results of a numerical investigation into the accuracy of the three approximations are summarized in Tables 1 and 2. Table 1 gives the maximum absolute error of the approximations over the range of values of $\Theta = \sin^{-1} \xi_1$ used in Sukhatme's Tabulation (1938, 1951) for k=2 and n_1, n_2 = 6, 8, 12, 24, ∞. Table 2 covers the special case $n_1=n_2=\ldots=n_k$ for k=3,4,5,10 and gives the maximum absolute error of the approximations over the grid of all $\{\xi_1, \xi_2,\ldots,\xi_k\}$ with $10\xi_i^2$ a positive integer. All the approximations perform exceptionally well at the 5% level, not so well at the 1% level. As expected, the accuracy improves as the degrees of freedom increase for all the approximations. The accuracy of the D-S and P approximations improves ultimately as k increases, but the Q approximation seems to get worse. The performances of the D-S and P approximations are very similar and it is difficult to give a clear-cut recommendation for one or the other. The D-S approximation is simpler and more accurate at the 5% level, while the P approximation becomes better at the 1% level. It was found that the true value at the 1% level always lay between the values given by the P and D-S approximations for the range of parameters considered. The Q approximation is not as accurate as the other two. However it is simpler to use since it is based directly on the t-distribution, and it gives reasonable values at the 5% level.

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Table 1

Maximum absolute error of the approximations for k=2

α		.05			.01		
n_1	n_2	D-S	P	Q	D-S	P	Q
6	6	.008	.011	.012	.10	.05	.19
8		.009	.015	.015	.08	.06	.17
12		.012	.019	.020	.07	.07	.15
24		.013	.022	.024	.09	.08	.11
∞		.013	.028	.027	.10	.09	.08
8	8	.004	.006	.014	.05	.02	.15
12		.004	.009	.014	.04	.03	.12
24		.005	.011	.016	.04	.03	.10
∞		.005	.014	.017	.05	.03	.06
12	12	.001	.002	.012	.02	.01	.10
24		.001	.004	.011	.02	.01	.07
∞		.001	.005	.010	.02	.01	.04
24	24	.000	.001	.007	.00	.00	.05
∞		.000	.001	.004	.01	.00	.02

Table 2

Maximum absolute error of the approximations for $k \geq 3$, $n_i = n$

α		.05			.01		
k	n_2	D-S	P	Q	D-S	P	Q
3	6	.012	.017	.018	.10	.07	.28
	8	.004	.009	.020	.05	.03	.21
	12	.003	.004	.017	.02	.01	.14
	24	.001	.001	.010	.00	.00	.07
4	6	.010	.019	.022	.10	.08	.33
	8	.004	.010	.024	.05	.03	.24
	12	.002	.004	.020	.02	.01	.16
	24	.001	.001	.012	.00	.00	.08
5	6	.007	.020	.024	.09	.08	.36
	8	.004	.010	.027	.04	.03	.27
	12	.002	.004	.023	.02	.01	.17
	24	.001	.001	.013	.00	.00	.08
10	6	.006	.005	.032	.06	.02	.45
	8	.003	.002	.034	.03	.00	.32
	12	.001	.000	.027	.01	.00	.20
	24	.000	.000	.015	.00	.00	.10