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THE STRESS ANALYSIS OF CURVED PIPES UNDER GENERALIZED LOADING WITH A REDUCED INTEGRATION PIPE BEAM ELEMENT

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SUMMARY
This work consists on the linear-elastic stress analysis of curved pipes submitted to a generalized in-plane or out-of-plane loading, using a mixed displacement ring element. Linear polynomials were used to interpolate the global shell displacements along the longitudinal direction, while Fourier series were used along the meridional direction. The deformed shape of the curved pipe results from the superposition of the beam displacements and those of a toroidal shell.

1 - INTRODUCTION

Curved pipes are parts of processing equipment usually submitted to complex loading. The ring element described here has a displacement field that results from the superposition of a beam displacement (where the transverse section is distortion-free), with that resulting from the section distortion. We assume a selective element geometry, *i. e.*, a straight beam between two consecutive sections for the stiffness terms involving beam displacements and a toroidal shell for the transverse section distortion. In this analysis, the following assumptions are considered:

a) a semi-membrane deformation model, provided the pipe may be considered as thin-walled;

b) the shell inextensibility along the meridional direction, *i. e.*, the hoop strain $\varepsilon_\theta = 0$;

c) the factor $\rho = r/R = 0$ (r and R being the transverse section mean radius and the mean curvature radius), considering that the present pipe element combines a displacement for a straight beam ($R=\infty$) with that of a double curvature shell.

2 - THE STRAIGHT BEAM / CURVED PIPE ELEMENT DISPLACEMENTS

The beam displacements follow the Mindlin-Reissner theory. For in-plane loading, as indicated in fig. 2, the the locally based displacement vector is:

$$\{\delta^T\}_{beam\text{-}in} = \{U_i \; W_i \; \phi_i \, ; U_j \; W_j \; \phi_j\} \qquad (1\text{-a})$$

where i and j refer to each nodal section and U_i, U_j, W_i, W_j are the longitudinal and transverse in-plane nodal displacements and ϕ_i and ϕ_j are the nodal section rotations.

For out-of-plane loading the displacement vector, also in a local reference basis, is as follows:

$$\{\delta^T\}_{beam\text{-}out} = \{W_{iz} \; \phi_i \; \gamma_i \, ; W_{jz} \; \phi_j \; \gamma_j\} \qquad (1\text{-b})$$

where W_{iz} and W_{jz} are the transverse displacements normal to the curvature plane and ϕ_i, γ_i, ϕ_j, γ_j are, respectively, the section rotation and torsion, existing on the same curvature plane, as indicated in fig. 3.

The shell displacements from fig. 1 until fig. 3 are referred to a local basis {x, y, z}, having the x-axis coincident with the element centre line and the y-z plane normal to x. These displacements must be defined from a global basis $\{x_0, y_0, z_0\}$, as indicated in the same figures. The local displacements of points of the shell expressed from the two sections i and j, are:

a) for the in-plane bending:

$$U_b = U_i N_i + U_j N_j$$
$$W_b = W_i N_i + W_j N_j$$
$$\phi = \phi_i N_i + \phi_j N_j$$

b) for the out-of-plane bending:

$$W_b = W_i N_i + W_j N_j$$
$$\phi = \phi_i N_i + \phi_j N_j \qquad (2\text{-a,b})$$
$$\gamma = \gamma_i N_i + \gamma_j N_j$$

where $N_i = (1-\xi)/2$; $N_j = (1+\xi)/2$ ($\xi = -1,...1$, from section i until j) and γ, ϕ are the torsional and the bending rotations.

3 - THE SECTION DISTORTION DISPLACEMENT FIELD

In this step, the displacements $O(x,\theta)$ and $V(x,\theta)$ refer to the section ovalization, while $\Omega(x,\theta)$ defines the section warping. These may be interpolated between the two sections i and j, using linear functions N_i and N_j (eqs. (2-a,b)), combined with Fourier expansions along the coordinate θ, as shows fig. 2:

$$O(x,\theta) = O_i N_i + O_j N_j$$

For the meridional displacement, we have, following assumption b) in section 1:

$$V(x,\theta) = V_i N_i + V_j N_j \qquad \text{where: } O_i = -\partial V_i/\partial\theta \text{ and } O_j = -\partial V_j/\partial\theta$$

For the longitudinal displacement resulting from the section warping, a similar expression is used:

$$\Omega(x,\theta) = \Omega_i N_i + \Omega_j N_j$$

The terms $V_{i,\text{ or } j}$, $O_{i,\text{ or } j}$, $\Omega_{i,\text{ or } j}$ are Fourier expansions for nodal section i or j, as follows:

for in-plane bending:

$$O_i = \sum_{n=1,2,3..}^{N_\theta} a_{in} \cos(n+1)\theta = C_\theta^T \mathbf{a}_i$$
$$V_i = \sum_{n=1,2,3..}^{N_\theta} -a_{in} \frac{\sin(n+1)\theta}{n+1} = S_\theta^T \mathbf{a}_i$$
$$\Omega_i = \sum_{n=1,2,3..}^{N_\theta} b_{in} \cos(n+1)\theta = C_\theta^T \mathbf{b}_i$$

for out-of-plane bending:

$$O_i = \sum_{n=1,2,3..}^{N_\theta} a_{in} \sin(n+1)\theta = S_\theta^T \mathbf{a}_i$$
$$V_i = \sum_{n=1,2,3..}^{N_\theta} a_{in} \frac{\cos(n+1)\theta}{n+1} = C_\theta^T \mathbf{a}_i \qquad (3)$$
$$\Omega_i = \sum_{n=1,2,3..}^{N_\theta} b_{in} \sin(n+1)\theta = S_\theta^T \mathbf{b}_i$$

where N_θ is the number of terms retained in the trigonometric expansions, and:

$$S_\theta^T = \{\cos 2\theta, \cos 3\theta, ..\}; \; C_\theta^T = \{\frac{\cos 2\theta}{2}, \frac{\cos 3\theta}{3}, ..\}; \; \mathbf{a}_i^T = \{a_{i1}, a_{i2}, a_{i3}, ...\}; \; \mathbf{b}_i^T = \{b_{i1}, b_{i2}, b_{i3}, ...\}$$

Similar expressions may be written for nodal section j, changing the subscript i into j.

4 - THE DISPLACEMENT FIELD EQUATION MATRIX

The superposition of the displacements described in sections 2 and 3 leads to the local (shell) displacement vector δ, defined from the following matrix equation:

$$\delta = [N]\,\delta_e$$

where δ_e is the global displacement vector and α is the angle between the beam-pipe axis (local axis) and the x_0 axis of the global basis, as indicated in fig. 2.

In extension, the local (shell) δ and global nodal displacements δ_e are related as follows:
a) for in-plane bending:

$$\left\{\begin{matrix} u \\ v \\ w \end{matrix}\right\} = \begin{bmatrix} N_i\cos\alpha & N_i\sin\alpha & -r\cos\theta\, N_i & 0 & N_i C_\theta^T & N_j\cos\alpha & N_j\sin\alpha & -r\cos\theta\, N_j & 0 & N_j C_\theta^T \\ \sin\theta.\sin\alpha N_i & -\sin\theta.\cos\alpha N_i & 0 & N_i C_\theta^T & 0 & \sin\theta.\sin\alpha N_j & -\sin\theta.\cos\alpha N_j & 0 & N_j C_\theta^T & 0 \\ -\cos\theta.\sin\alpha N_i & \cos\theta.\cos\alpha N_i & 0 & N_i C_\theta^T & 0 & -\cos\theta.\sin\alpha N_j & \cos\theta.\cos\alpha N_j & 0 & N_j C_\theta^T & 0 \end{bmatrix} \delta_e \qquad (4)$$

where: $\delta_e^T = \left\{ \overline{U}_i \ \overline{W}_i \ \phi_i \ \mathbf{a}_i^T \ \mathbf{b}_i^T \ ; \ \overline{U}_j \ \overline{W}_j \ \phi_j \ \mathbf{a}_j^T \ \mathbf{b}_j^T \right\}$

b) for out-of-plane bending:

$$\left\{\begin{matrix} u \\ v \\ w \end{matrix}\right\} = \begin{bmatrix} 0 & -r\sin\theta.\sin\alpha N_i & r\sin\theta.\cos\alpha N_i & 0 & N_i S_\theta^T & 0 & -r\sin\theta.\sin\alpha N_j & r\sin\theta.\cos\alpha N_j & 0 & N_j S_\theta^T \\ N_i\cos\theta & r N_i\cos\alpha & r N_i\sin\alpha & N_i C_\theta^T & 0 & N_j\cos\theta & r N_j\cos\alpha & r N_j\sin\alpha & N_j C_\theta^T & 0 \\ N_i\sin\theta & 0 & 0 & N_i S_\theta^T & 0 & N_j\sin\theta & 0 & 0 & N_j S_\theta^T & 0 \end{bmatrix} \delta_e \qquad (5)$$

where: $\delta_e^T = \left\{ \overline{W}_i \ \overline{\phi}_{x_{0i}} \overline{\phi}_{y_{0i}} \ a_i^T \ b_i^T \ ; \ \overline{W}_j \ \overline{\phi}_{x_{0j}} \overline{\phi}_{y_{0j}} \ a_j^T \ b_j^T \right\}$

In the two previous nodal displacement vectors we find beam terms (figs. 2 and 3), and subvectors $\mathbf{a}_i^T$, $\mathbf{a}_j^T$ and $\mathbf{b}_i^T$, $\mathbf{b}_j^T$ concerning the ovalization and warping of the transverse section.

5 - THE STIFFNESS MATRIX CALCULATION.

The semi-membrane model and the assumption b) in section 1 lead to the vector:

$$\varepsilon^T = \{ \varepsilon_{xx}, \gamma_{x\theta}, k_{\theta\theta} \} \qquad (6)$$

where ε_{xx} is the longitudinal membrane strain, $\gamma_{x\theta}$ is the shear membrane strain and $k_{\theta\theta}$ is the meridional curvature variation, related to the section ovalization. In a matrix form, the deformation vector ε in (6) is calculated from the equation:

$$\varepsilon = [B]\, \delta_e \qquad (7)$$

The matrix [B] results from:

$$[B] = [L]\,[N] \qquad (8)$$

where [L] is a differential operator and [N] is the shape function matrix in (4) or (5).
The calculation of the deformation matrix [B] is developed in two steps:

a) first we use in eq. (8) the differential operator $[L]_{beam}$, for a straight pipe (having an infinite curvature radius, this is, R -> ∞). This generates terms in ε (eq. (7)) involving a straight pipe-beam deformation. $[L]_{beam}$ multiplies in [N] ((4) or (5)) the columns number 1, 2, 3 and 6, 7, 8.

b) In eq. (8) we use now $[L]_{curved}$, modified for a curved pipe. This multiplies in [N], (eqs. (4) or (5)), the columns number 4, 5 and 9, 10. The two [L] operators are as follows:

$$[L]_{beam} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [L]_{curved} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{-\sin\theta}{R} & \frac{\cos\theta}{R} \\ \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial x} & 0 \\ 0 & -\frac{1}{r^2}\frac{\partial}{\partial \theta} & \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \end{bmatrix} \qquad (9)$$

The stiffness matrix [K] of the element is calculated from the matrix equation:

$$[K] = \int_0^L \int_0^{2\pi} [B]^T [D] [B] \, dx \, rd\theta \qquad (10)$$

The integration in (9) is extended to the element surface S, having used a modified elasticity matrix [D] for thin semi-membrane shells as followed by Wilczek (1984). A one-point reduced gaussian integration was used for the variable x, while an exact one was used for θ. An accurate solution is obtained with a simple two nodes pipe element, as reported by Hughes (1987).

6 - PIPE BENDS WITH TANGENT TERMINATIONS

In this analysis, we consider a long straight pipe, where the ovalization and warping result from edge effects. These displacements are defined form functions of x and θ, as for the curved pipe. For the determination of these functions a variational method is used. Proceeding this way, we get the most accurate functions for the ovalization $O(x,\theta)$ and warping $\Omega(x,\theta)$ of the transverse section, minimizing the elastic potential energy involved in the straight pipe deformation. Such functions will define the ovalization decay, their shape being assumed as follows:

$$O(x,\theta) = \sum_{i=1}^{N_\theta} a_i(x)\cos(i+1)\theta \;;\quad V(x,\theta) = \sum_{i=1}^{N_\theta} -\frac{a_i(x)}{(i+1)}\sin(i+1)\theta \;;\quad \Omega(x,\theta) = \sum_{i=1}^{N_\theta} b_i(x)\cos(i+1)\theta$$

where $O(x,\theta)$ and $V(x,\theta)$ are, respectively, the transverse and the meridional displacement, both resulting from the pipe ovalization and $\Omega(x,\theta)$ is the longitudinal displacement as a consequence of the transverse section warping. The transverse section of the straight pipe is assumed inextensible; for that reason, $O(x,\theta)$ and $V(x,\theta)$ are related as in section 3. All these functions do not depend on the bending of the pipe; this problem may be treated following the usual beam theory, having a distortion-free transverse section.

Considering a long tangent pipe element, a matrix notation for the displacement field resulting from the ovalization and warping is:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 0 & \{E_a C_\theta\}^T \\ \{E_a S_\theta\}^T & 0 \\ \{E_b C_\theta\}^T & 0 \end{bmatrix} * \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \end{Bmatrix} \qquad (11)$$

where, in the previous equation, $\mathbf{a}_0^T = \{a_{01}, a_{02}, a_{03}, \ldots (N_\theta \text{ terms})\}$ are ovalization displacements and $\mathbf{b}_0^T = \{b_{01}, b_{02}, b_{03}, \ldots (N_\theta \text{ terms})\}$ are warping displacements. The subvectors in the shape function matrix in the second member include x-depending functions and are as follows:

$\{E_a C_\theta\}^T = \{a_1(x)\cos 2\theta, a_2(x)\cos 3\theta, \ldots (N_\theta \text{ terms})\}$

} (ovalization shape functions)

$\{E_a S_\theta\}^T = \{-a_1(x)\sin 2\theta / 2, -a_2(x)\sin 3\theta / 3, \ldots (N_\theta \text{ terms})\}$

$\{E_b C_\theta\}^T = \{b_1(x)\cos 2\theta, b_2(x)\cos 3\theta, \ldots (N_\theta \text{ terms})\}$ (warping shape functions)

The deformations of a straight pipe are the same as for the straight pipe-beam, eq. (6):

$$\varepsilon^T = \{ \varepsilon_{xx}, \gamma_{x\theta}, k_{\theta\theta} \} \qquad (6)$$

The meaning of the elements in the previous vector are as in eq. (6), section 5; the differential operator matrix [L] is the same as $[L]_{beam}$ in (9), but now with terms involving the meridional curvature variation as a result of the section ovalization.
The constitutive matrix [D] is the same as for the curved pipe, following the semi-membrane model and the zero hoop strain.

The stiffness matrix is as indicated in eq. (10); however, the expressions for the shape functions in x, eq. (11), are unknown. For their determination, the procedure by Millard and Roche (1984), consisting of a variational method was adopted. These authors determined the expressions for the unknown functions in the displacement field (11) minimizing a functional U, the elastic deformation energy, in order to each set of *ith* order of the unknown functions $a_i(x)$ and $b_i(x)$ in (11). For the minimization U we adopted a "weak" variational formulation, as described by Millard and Roche (1984), leading to the following general solutions for the unknown functions:

$$a_i(x) = e^{-\lambda_i x}(C_{1i}\cos\mu_i x + C_{2i}\sin\mu_i x)$$
$$b_i(x) = e^{-\lambda_i x}(C_{3i}\cos\mu_i x + C_{4i}\sin\mu_i x) \qquad \text{(for any } i = 1, 2, \ldots N_\theta \text{ terms)}$$

where, in these expressions, λ_i and μ_i are the real and the imaginary coefficients of the complex *ith* order solution for the fourth degree characteristic equation resulting from the minimization of energy U, and C_{1i}, C_{2i}, C_{3i}, C_{4i} are constants of integration. The calculation of the constants $C_{1i}...C_{4i}$ results from the compatibility conditions at the edge (between the curved and the tangent pipes) and from the first degree x-functions in (2-a, b), these having zero second x-derivates:

$$C_{1i} = a_{0i} \quad C_{3i} = b_{0i} \quad C_{2i} = a_{0i}(\lambda_i^2 - \mu_i^2)/(\lambda_i\mu_i) \quad C_{4i} = b_{0i}(\lambda_i^2 - \mu_i^2)/(\lambda_i\mu_i) \qquad (12)$$

The tangent pipe is assumed long enough, so that the distortion on edge i could not propagate until the opposed end j; therefore, the stiffness matrix for a long straight pipe, involving only ovalization and warping displacements, will have zero off-diagonal elements given the uncoupled edge effects:

$$[K] = \begin{bmatrix} [K_{ii}] & [0] \\ [0] & [K_{jj}] \end{bmatrix} \qquad (13)$$

7 - EXAMPLES

The examples analysed here consider situations of in-plane and out-of-plane loads and were studied with detail by Wilczek (1984). Figs. 4 and 5 refer to pure bending, resulting from continuous sine load acting on the end section. Fig. 6 presents a less typical load case, involving concentrated forces. A good agreement for the results of the stresses is observed, particulary for the last case, considering that numerical difficulties arise from concentrated loads in sections close to the disturbed one.

8 - CONCLUSIONS

An inexpensive and accurate pipe element, having a simple and effective definition, was formulated. This element solves the generalized in-plane and out-of-plane loading problems on curved pipes with considerable economy of the global number of degrees of freedom.

9 - REFERENCES

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Wilczek, E., (1984), "Statische Berechnung Eines Rohrkrümmers mit Realen Randbedingungen" Ph. D. Thesis, Technischen Hochschule Aachen, Aachen.

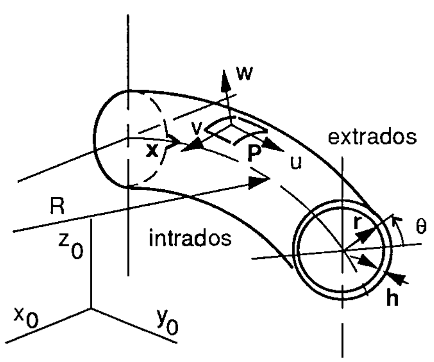

Fig. 1 - Basis and coordinate notations in a toroidal shell.

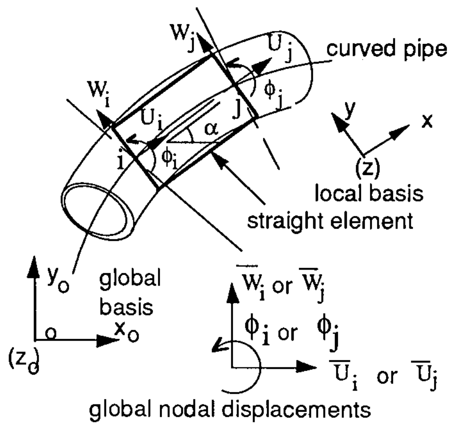

Fig. 2 - Beam displacements for in-plane bending.

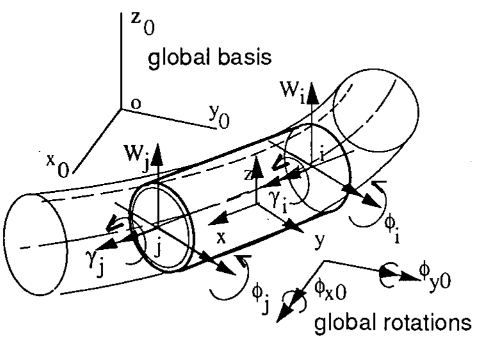

Fig. 3 - Beam displacements for out-of plane bending.

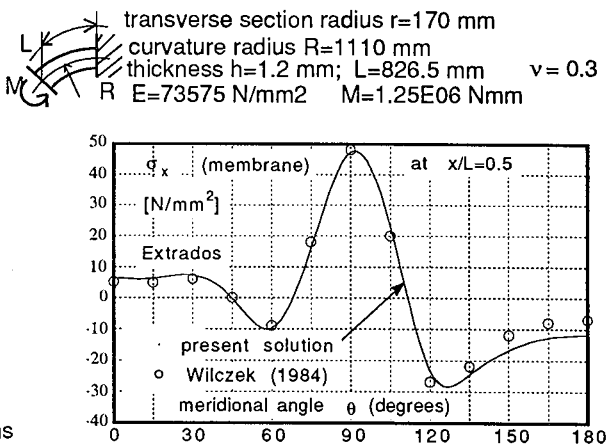

Fig. 4 - In-plane bending of a curved pipe having thin flanges.

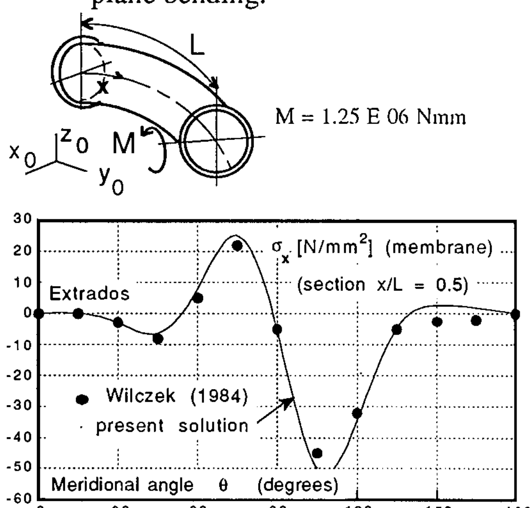

Fig. 5 - Out-of-plane bending of the same pipe as in fig. 4.

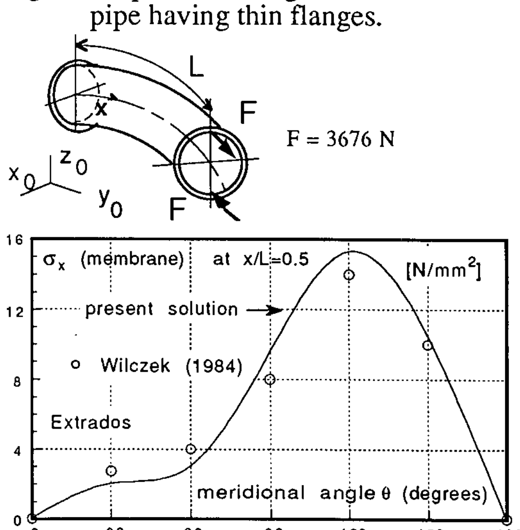

Fig. 6 - Out-of-plane bending from a pair of forces (pipe as in fig. 4).