CONJUGATE HEAT TRANSFER PROBLEMS WITH NUCLEAR HEATING

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SUMMARY

This paper deals with the exact thermal analysis of a slender canned rod with an arbitrary internal nuclear power distribution under axial forced convection turbulent flow of a constant property coolant. Three-dimensional heat conduction in the solid component is permitted, so that exact surface temperatures and heat fluxes, coolant temperature profiles and consistent internal solid temperature fields for thermoelasticity analysis can be calculated, with full allowance for end conditions. The necessity to define heat transfer coefficients a priori is avoided, and the correct spatially variable heat transfer coefficients result from the analysis.

A general formulation is developed for fully developed turbulent flow, uniform inlet temperature, constant rod conductivity and insulated end boundary conditions. Fourier series expansion in the azimuthal direction decouples the angular harmonics, and each resulting coupled two-dimensional conjugate problem for the harmonics becomes amenable to the same type of analysis. Separate general Green's function solutions for the solid and fluid regions, expressible in terms of the eigenfunctions generated by the relevant linear operators and homogeneous boundary conditions, are used to express the individual surface heat fluxes in terms of the common surface temperature. Equating heat fluxes leads to a single integral equation for the surface temperature distribution. A Fourier cosine series expansion of surface temperature and a weighted residual approximation reduce the integral equation to a set of simultaneous linear equations in the Fourier coefficients for each harmonic.

The method is illustrated by application to a simple heat generating plate problem, using published numerical data for the fluid eigenvalues and eigenfunctions.

The formulation is extended to include a temperature dependent plate conductivity, characteristic of nuclear fuel, by utilizing the fact that only the surface temperature is involved, and hence the surface conductivity admits a simple two parameter representation. An efficient iterative technique is developed for computation, and numerical results displayed.

1. Introduction

The exact thermo-mechanical analysis of a heat generating body with forced convection cooling, would involve the calculation of temperature fields in both the solid and fluid components. With turbulent flow the fluid problem is generally intractable for arbitrary geometry, but for the parallel, fully developed, turbulent flow of a constant property coolant between plates, in circular ducts and annuli, for which eddy diffusivity data is available, some progress is possible. Fortunately these geometries are not insignificant for reactor technology, both as regards reactor cores and experimental rigs.

For the static, symmetric heating and cooling of long slender convex fuel rods, the fluid and solid problems can be decoupled by the assumption of zero axial conduction in the solid, enabling the use of heat transfer coefficients either from correlations or from the solution of the fluid problem with specified heat flux or temperature. The assumption becomes less valid with rapid variation of the axial power or the surface heat transfer conditions, as with the onset of subcooled boiling, the prediction of which is itself dependent on knowledge of the fluid temperature profile near the heated wall. Internal heat conduction is significant for the unsymmetric heating and cooling of a fuel pin in a rod cluster type fuel element, particularly as regards the heat flux distribution required for subchannel analysis. As correct temperature fields are a prerequisite for correct thermal and mechanical stress analysis of canned fuel pellets, the need to solve the complete heat conduction equation further emphasises the need for critical evaluation of the can coolant heat flux and temperature relations imposed by the coolant.

Under reactor transient or accident conditions, the usual assumption of zero explicit time dependence of surface heat transfer coefficients associated with coolant flow regimes is hardly valid, even assuming specific heat flux variations by ignoring details of internal heat conduction.

For these reasons, the conjugate heat transfer problem, in which neither the surface heat flux nor the surface temperature is known a priori, but results from heat conduction in the solid and heat transport in the fluid, is significant in reactor technology. The work reported here is an investigation of comparatively simple static problems, to provide a basis of understanding and for development of mathematical - numerical techniques for more complex and transient problems.

2. General Theory

The linear problem of a canned, cylindrical fuel element, with temperature independent conductivity and insulated ends, cooled by the fully developed symmetric annular turbulent flow of a constant property fluid is governed by the following equations and can-coolant interface conditions:-

$$
\begin{aligned}
&L_f T_f(r,\theta,z) + Q(r,\theta,z) = 0 \;; \qquad 0 < r < r_i, \qquad 0 < z < L \\
&L_c T_c(r,\theta,z) = \rho C_p \, v(r) \frac{\partial}{\partial z} T_c(r,\theta,z) \;; \qquad r_i < r < r_o, \qquad 0 < z < L \\
&[T_s(\theta,z)]_f = T_f(r_i,\theta,z) = T_c(r_i,\theta,z) = [T_s(\theta,z)]_c \\
&[q_s(\theta,z)]_f = q_f(r_i,\theta,z) = q_c(r_i,\theta,z) = [q_s(\theta,z)]_c
\end{aligned}
\qquad (1)
$$

Additional boundary conditions include $T_c(r,\theta,0) = 0$, $\frac{d}{dr} T_c(r_o,\theta,z) = 0$, and $v(r_o)$ or $\frac{dv(r_o)}{dr} = 0$. With T_f, θ, T_c etc. $\equiv \sum_{-\infty}^{+\infty} (T_f^n(r,z), Q^n(r,z), T_c^n(r,z) \text{ etc.})\, e^{in\theta}$, the angular mode equations are uncoupled, and eq.(1) is replaced, for each n by:-

$$
\begin{aligned}
&L_f^n T_f^n(r,z) + k(r) \frac{\partial^2}{\partial z^2} T_f^n(r,z) + Q^n(r,z) = 0 \\
&L_c^n T_c^n(r,z) = \rho C_p\, v(r) \frac{\partial}{\partial z} T_c^n(r,z) \\
&[T_s^n(z)]_f = [T_s^n(z)]_c \\
&[q_s^n(z)]_f = [q_s^n(z)]_c
\end{aligned}
\tag{2}
$$

where

$$
L_f^n \equiv \frac{1}{r}\frac{d}{dr}\left(r\, k(r) \frac{d}{dr}\right) - \frac{n^2 k(r)}{r^2} \tag{3}
$$

$$
L_c^n = \rho c \left[\frac{1}{r}\frac{d}{dr}\left(r(\kappa + \varepsilon_r(r)) \frac{d}{dr}\right) - \frac{n^2}{r^2}(\kappa + \varepsilon_\theta(r))\right] \tag{4}
$$

Without loss of generality eq.(3) includes a fuel can interface temperature discontinuity of the form $(q_r)_I = C_i(\delta_r T)_I$. Since the fuel thermal problem is linear, a formal solution may be written in terms of the Green's function defined by the operator $[L_f^n + k(r) \frac{\partial^2}{\partial z^2}]$ with zero on r_i and zero gradient at z=0 and L, leading to a surface heat flux

$$
[q_s^n(z)]_f = \int_0^{r_i}\int_0^L Q^n(r',z')\, G_Q^n(z,r',z')\, r'dr'dz' + \int_0^L T_s^n(z')\, G_T^n(z\ z')\, dz' ,
$$

i.e.,

$$
[q_s^n(z)]_f = q_Q^n(z) + \int_0^L T_s^n(z')\, G_T^n(z,z')\, dz' \tag{5}
$$

Similarly the fluid problem leads to

$$
[q_s^n(z)]_c = \int_0^z T_s^n(z')\, F^n(z,z')\, dz' \tag{6}
$$

Equality of heat fluxes gives the form of the integral equation for the unknown common surface temperature for each of the angular nodes, i.e.,

$$
\int_0^z T_s^n(z')\, F^n(z\ z')dz' = q_Q^n(z) + \int_0^L T_s^n(z')\, G_T^n(z\ z')dz' \tag{7}
$$

3. The Form of the Fuel and Coolant Kernels

The functions $q_Q^n(z)$ and $G_T^n(z,z')$ may be developed using the expressions

$$
T_f^n(r,z),\ Q_f^n(r,z) = \sum_N (T_f^{n,N}(r),\ Q_f^{n,N}(r)) \cos\left(\frac{N\pi z}{L}\right)
$$

so

$$\left(L_f^n - k(r)\,\frac{N^2\pi^2}{L^2}\right) T_f^{n,N}(r) + Q^{n,N}(r) = 0 \tag{8}$$

Direct solution of eq.(8), or solution via the eigenfunctions ϕ_i^n defined by

$$(L_f^n + K_i^n)\,\phi_i^n = 0\ ; \qquad \phi_i^n(r_i) = 0$$

leads to

$$T_f^{n,N}(r) = T_s^{n,N}\, f^{n,N}(r) + T_Q^{n,N}(r)$$

with

$$f^{n,N}(r_i) = 1\,, \qquad T_Q^{n,N}(r_i) = 0\,.$$

Hence

$$[q_s^n(z)]_f = \sum_N \left\{ \frac{-2k(r_i)}{L(1+\delta_{N,o})} \left[\frac{df^{n,N}(r)}{dr}\right]_{r_i} \cos\left(\frac{N\pi z}{L}\right) \int_0^L T_s^n(z')\cos\left(\frac{N\pi z'}{L}\right) dz' \right.$$

$$\left. - k(r_i)\left[\frac{d\,T_Q^{n,N}(r)}{dr}\right]_{r_i} \cos\left(\frac{N\pi z}{L}\right) \right\}$$

Then the required functions are of the form

$$q_Q^n(z) = \sum_N q_Q^{n,N} \cos\left(\frac{N\pi z}{L}\right)$$

$$G_T^n(z,z') = \sum_N G_T^{n,N} \cos\left(\frac{N\pi z}{L}\right) \cos\left(\frac{N\pi z'}{L'}\right) \tag{9}$$

For the coolant, eigenfunctions ψ_i^n defined by

$$L_c^n\,\psi_i^n(r) + \lambda_i^n\,\rho C_p\, v(r)\,\psi_i^n(r) = 0$$

$$\psi_i^n(r_i) = \left[\frac{d\psi_i^n(r)}{dr}\right]_{r_o} = 0$$

with the property $\int_{r_i}^{r_o} \rho C_p\, v(r)\,\psi_i^n\,\psi_j^n\, r\,dr = \delta_{ij}$ enables a solution to be developed as

$$T_c^n(r,z) = \sum T_c^{n,i}(z)\,\psi_i^n(r)\ ; \qquad T_c^{n,i}(z) = \int_{r_i}^{r_o} T_c^n(r,z)\,\rho C_p\, v\,\psi_i^n\, r\,dr$$

and gives

$$T_c^{n,i}(z) = -\int_0^z k_c\, T_s^n(z') \left[\frac{d\psi_i^n(r)}{dr}\right]_{r_i} e^{-\lambda_i^n(z-z')}\, dz'\ .$$

Hence the required coolant kernel is

$$F^n(z,z') = \sum_i \{ k_c \left[\frac{d\psi_i^n}{dr} \right]_{r_i} \}^2 e^{-\lambda_i^n(z-z')} \qquad (10)$$

From the formulation, knowledge of the surface temperature distribution is sufficient to define completely the temperature fields in fuel and coolant.

4. Application to Plate Problem

The two dimensional problem of a plate type element is governed by equations formally the same as eq.(2) with n=0, r replaced by x and $L_f = \frac{\partial}{\partial x} (k(x) \frac{\partial}{\partial x})$ etc. The simpler plate problem with $Q = Q(z)$ may be used to develop methods of solution, particularly to deal with temperature dependent conductivity, an essential property of most nuclear fuels.

Consider initially a single region constant conductivity plate element, insulated at both ends and cooled symmetrically by a coolant flowing in parallel channels. Due to symmetry, only half of the plate needs to be considered, with $\frac{\partial}{\partial x} T_f(0,z) = 0$. Let x_i be the half thickness of the plate, $x_o - x_i$ the coolant channel width, and suppose $\frac{\partial}{\partial x} T_c(r_o,z) = 0$. A direct solution of eq.(8) leads to

$$f^N(x) = \cosh\left(\frac{N\pi x}{L}\right) / \cosh\left(\frac{N\pi x_i}{L}\right)$$

and

$$T_Q^N = - Q^N \left[\cosh\left(\frac{N\pi x}{L}\right) / \cosh\left(\frac{N\pi x_i}{L}\right) - 1\right] / (kN^2\pi^2 / L^2)$$

with

$$Q(z) = \sum_N Q^N \cos\left(\frac{N\pi z}{L}\right) .$$

The fluid equations for flow in a parallel plate channel with surface temperature specified on one wall and perfect insulation on the other wall leads to a wall heat flux equation formally the same as eq.(6) and the kernel the same as eq.(10). The data associated with the eigenfunctions $\phi_i(x)$, and needed for the calculation of $[q_s(z)]_f$, have been calculated and tabulated by Hatton and Quarmby [1] for a few given Reynolds numbers and Prandtl numbers.

The integral equation expressing heat flux continuity at the fuel coolant interface is

$$\sum_N \left[\frac{-2k}{L(1+\delta_{n,o})} \left(\frac{N\pi}{L}\right) \int_0^L T_s(z') \cos \frac{N\pi z'}{L} dz' + Q^N / \left(\frac{N\pi}{L}\right)\right] \tanh\left(\frac{N\pi x_i}{L}\right) \cos\left(\frac{N\pi z}{L}\right)$$

$$= \sum_i \{ k_c \left[\frac{d\psi_i}{dx}\right]_{x_i} \}^2 \int_0^z T_z(z') e^{-\lambda_i(z-z')} dz' \qquad (11)$$

Writing $T_s = f(z)I_+(z)$, to satisfy the zero entrance temperature condition, and expanding $f(z) = \sum T_s^N \cos \frac{N\pi z}{L}$, eq.(11) becomes

$$\sum_{N=0} \left[- k\left(\frac{N\pi}{L}\right) T_s^N + Q^N / \left(\frac{N\pi}{L}\right)\right] \tanh\left(\frac{N\pi x_i}{L}\right) \cos\left(\frac{N\pi z}{L}\right) = \sum_{N=0} \sum_i \{ k_c \left[\frac{d\psi_i}{dx}\right]_{x_i} \}^2 \left(\frac{1}{\lambda_i} e^{-\lambda_i z}\right) T_s^N$$

$$+ \sum_i \{ k_c \left[\frac{d\psi_i}{dx}\right]_{x_i} \}^2 \frac{1}{\lambda_i} \int_0^z \sum_{N=1} \left(\frac{N\pi}{L}\right) T_s^N \sin\left(\frac{N\pi z'}{L}\right) e^{-\lambda_i(z-z')} dz' \qquad (12)$$

Multiplying eq.(12) by $\int_0^L \cos\frac{M\pi z}{L}\,dz$, $M = 0,1 \ldots K$ and carrying out the integrations gives

$$\frac{(1+\delta_{M,o})L}{2}\left[-k\alpha_M T_s^M + Q^M/\alpha_M\right]\tanh(\alpha_M x_i) = \sum_{N=0}^{K}\sum_i d_{i,M}\,\lambda_i\left[1-(-1)^M e^{-\lambda_i L}\right]T_s^N$$

$$-\sum_{N=1}^{K}\sum_i \alpha_N^2 d_{i,N}\left[-\frac{L}{2}\delta_{N,M} + \lambda_i\{1-(-1)^{N+M}\}/(\alpha_N^2-\alpha_M^2)\right.$$

$$\left.+\{1-(-1)^M e^{-\lambda_i L}\}\lambda_i/(\lambda_i^2+\alpha_M^2)\right]T_s^N \qquad (13)$$

where $\alpha_N = \frac{N\pi}{L}$; $d_{i,N} = \{k_c\left[\frac{d\psi_i}{dx}\right]_{x_i}\}^2 / \left[(\lambda_i^2+\alpha_N^2)\,\lambda_i\right]$.

Eq.(13) is a set of K+1 algebraic equations in the unknowns T_s^K . That is,

$$A\,T_s = b \qquad (14)$$

This can be solved by a direct matrix inversion technique, or by an iterative routine based on $A = D + B$, where D is diagonal, i.e.,

$$(T_s)_{n+1} = D^{-1}[b - B(T_s)_n] \text{ with } (T_s)_o = 0 .$$

Convergence is guaranteed because A is diagonally dominant.

5. Temperature Dependent Conductivity

The thermal problem of a temperature dependent conductivity fuel is non-linear and can be solved by iterative techniques. Suppose $k = k(T)$, the transformation

$$Z(x,z) = \int_{T_o}^{T(x,z)} k(T')\,dT' \qquad (15)$$

gives

$$L_f T = \frac{\partial^2}{\partial x^2} Z .$$

The operator on the new temperature variable Z is again linear and treatments similar to outlined before lead to

$$A_1 Z_s + A_2 T_s = b \qquad (16)$$

corresponding to eq.(14). Note that A_1 is diagonal and elements of the vector Z_s are the coefficient of the cosine series expansion of $Z(x_i,z)$. Then, providing eq.(15) can be translated into a matrix form,

$$Z_s = GT_s = (K + H)T_s \qquad (17)$$

where K is diagonal, the following efficient iterative scheme can be derived by eliminating Z_s from eqs.(16) and (17) :-

$$(T_s)_{n+1} = [D_2 + A_1K_n]^{-1}[b - (A_1H_n + B_2)(T_s)_n] , \qquad (18)$$

again $A_2 = D_2 + B_2$. Note that the inversion only involves the diagonal matrix.

If $k(T)$ is simple, analytical methods can be used to obtain eq.(17) and K_n and H_n are readily obtainable from $(T_s)_n$. For more complex cases, numerical integration can be used. However, the proposed techniques involved only $k(T_s)$. Since the variation in T_s is small, a two parameter representation of $k(T_s)$ is permissible. For example,

$$k(T_s) = \alpha + \beta T_s .$$

Integration of eq.(15) gives

$$Z_s = \alpha(T_s - T_0) + \beta(T_s^2 - T_0^2) / 2 .$$

Then

$$Z_s^0 = \{\alpha + \beta \left[(T_s^0)_n + 2T_0\right] / 2\} T_s^0 + \sum_{m=1} \left[\frac{\beta}{4} (T_s^m)_n\right] T_s^m$$

$$Z_s^m = \{\alpha + \beta \left[T_0 + (T_s^0)_n\right] \} T_s^m + \frac{\beta}{4} \sum_{l=1} \sum_{k=1} (\delta_{l+k,m} + \delta_{|l-k|,m})(T_s^l)_n T_s^k .$$

5. Illustrative Examples

Two single region plate problems were used. Fig.1 shows the axial variations of T_s, q_s and Nusselt number, obtained for a problem of constant conductivity. The heat source, $Q(z) = a_1 + \exp(-a_2 z) \sin(a_3 z + a_4)$, was chosen to simulate a distorted chopped sine power distribution with $a_1 = 0.4$, $a_2 = 0.05$, $a_3 = \pi/50$, and $a_4 = \pi/12$. The system dimensions expressed in terms of the half plate thickness were $x_i = 1.0$, $x_o - x_i = 1.0$ and $L = 50$. The system constants were $k = 30$ Btu/hr/ft/^{0}F, $k_c = 0.0183$ Btu/hr/ft/^{0}F, Re = 73612, and Pr = 1.0. These constants approximate a turbulent flow air-steel system.

The variables in Fig.1 are dimensionless. $Q(z)$ and q_s are normalized by their average values $\bar{Q}$ and $\bar{q}_s$. The half plate thickness is selected as the characteristic length and $T_s k/(\bar{Q}\, x_i^2)$ is the dimensionless temperature. Note that based on these dimensionless variables q_s should be the same as Q if axial conduction is neglected. The difference between these two curves shows the significance of axial conduction.

For comparison, the surface temperature based on two conventional methods and the conjugate approach are shown in Fig.2. One method uses the assumption of no axial conduction and the asymptotic heat transfer coefficient, and the other uses no axial conduction but the theoretically correct heat transfer coefficient. These curves show clearly the reduction of the maximum surface temperature when axial conduction is considered.

A comparison of surface heat fluxes calculated from the plate and fluid solutions illustrates the accuracy of the results. Both fluxes can be evaluated once T_s is known. The greatest differences are expected near the entrance where rapid changes occur. Using K = 40 in the expansion of T_s, the difference is 0.5% at a point 0.25 plate thickness away from the entrance, and decreases rapidly to 0.08% at a distance of 5 plate thickness.

The results obtained for a case of temperature dependent conductivity are shown in Fig.3. The conductivity k(Btu/hr/ft/^{0}F) = 4.14 - 0.00181 T(^{0}F) is a linear approximation of the conductivity of UO_2 at the temperature range of 480 - 1000^{0}F. The other constants used were x_i = 1/10", $x_o - x_i$ = 2/10", L = 10", Re - 73612, Pr = 1.0, k_c = 0.0183 Btu/hr/ft/^{0}F T_0 = 480^{0}F, and $Q(z) = 6 \sin(6\pi z/10 - \pi/4) \times 10^5$ Btu/hr/ft^3.

The fact that q_s in Fig.3 differs only slightly from a sine curve is expected, as axial conduction in a poor conductor is negligible where the spatial power variation is small. In a dynamic situation when sharp changes in the power distribution are involved, a technique such as the proposed one, capable to handle more than one dimensional conduction is needed. However the results demonstrate the ability of the proposed technique to handle temperature dependent conductivity.

6. Conclusion

The method developed, i.e., the integral equation formulation and the numerical solution technique, is capable of solving the conjugate problem posed quite accurately, but its utility in its present form is limited by the requirements for fluid eigenvalues and eigenfunctions. Alternative ways of solving the fluid problem must be developed before extension to the more interesting transient situation is feasible. The illustrative example indicates that in the static problem, significant differences can exist between correct surface temperatures and heat fluxes and those calculated by assuming zero axial conduction.

7. References

[1] A.P. Hatton and A. Quarmby. "The effect of axially varying and unsymmetrical boundary conditions on heat transfer with turbulent flow between parallel plates". Int.J.Heat & Mass Transfer, 6 (1963), 903-913.

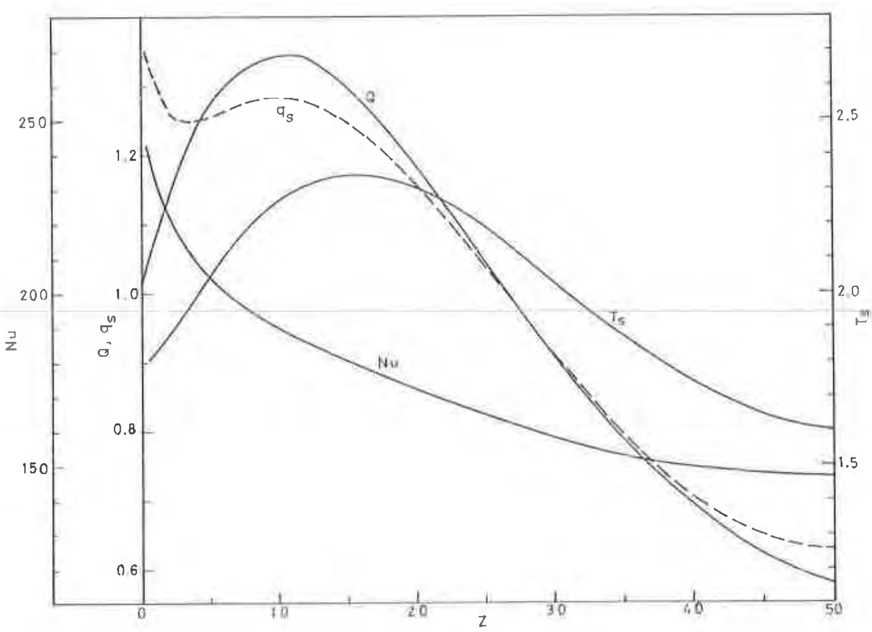

Fig.1. Constant conductivity plates.

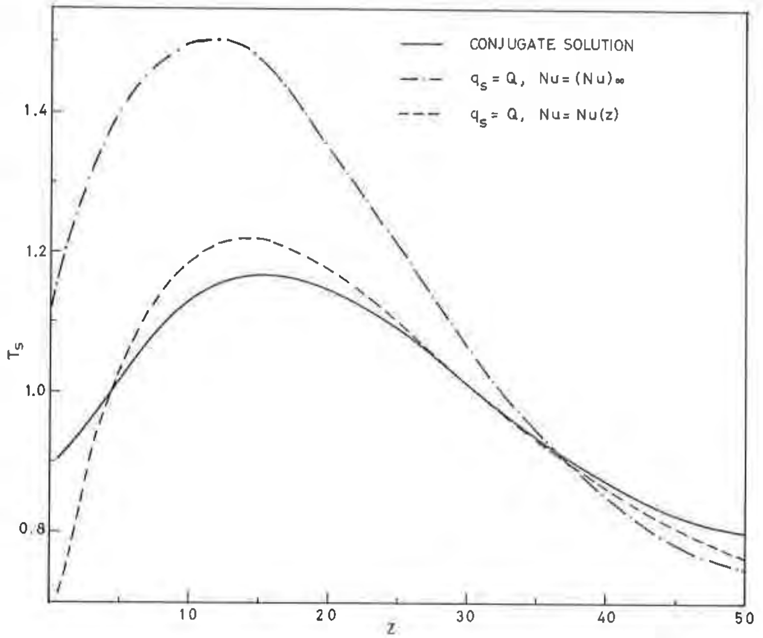

Fig.2. Comparison of surface temperatures.

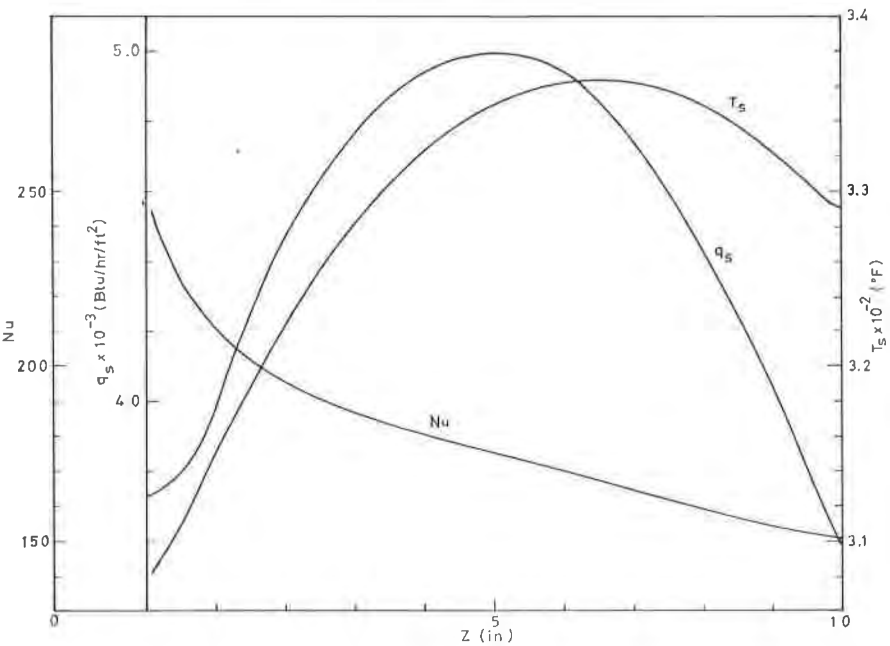

Fig.3. Variable conductivity plates