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SOME FURTHER CONSTRUCTIONS FOR $G_2(d)$ GRAPHS

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ABSTRACT

A $G_2(d)$ graph is a finite, undirected graph without loops or multiple edges in which each pair of vertices is adjacent to exactly d other vertices, $d \geqslant 2$. An infinite family of such graphs was given in [2]. The present paper gives some further constructions for these graphs.

1. KNOWN RESULTS

We use the notation and terminology of [2] and quote some results contained in this paper.

THEOREM A. A $G_2(d)$ graph, $d \geqslant 2$, is regular of valence n_1 such that $v - 1 = n_1(n_1 - 1)/d$ where v is the number of vertices and there exists a positive integer m such that

(i) $n_1 = d + m^2$, and

(ii) d/m is an integer with the same parity as $v - 1 - m$.

We note that a $G_2(d)$ graph with parameters (v, n, d), $d \geqslant 2$, is essentially a strongly regular graph with parameters $(v, n_1, p^1_{11}, p^2_{11})$ where $p^1_{11} = p^2_{11} = d$. By a pseudo $L_r(k)$ graph we will mean a pseudo net graph $L_r(k)$ and by an $NL_r(k)$ graph, a negative Latin Square $NL_r(k)$ graph.

THEOREM B. The existence of a pseudo $L_{r_1}(2r_1)$ and a pseudo $L_{r_2}(2r_2)$ graph implies the existence of a pseudo $L_r(2r)$ graph with $r = 2r_1r_2$.

THEOREM C. The existence of a pseudo $L_{r_1}(2r_1)$ and a $NL_{r_2}(2r_2)$ graph implies the existence of a $NL_r(2r)$ graph with $r = 2r_1r_2$.

THEOREM D. Pseudo $L_r(2r)$ and $NL_r(2r)$ graphs exist for $r = 3^m.2^{m+n-1}$, where m, n are nonnegative integers $(m, n) \neq (0, 0)$.

Noting that a pseudo $L_r(k)$ graph is a strongly regular graph with parameters $(k^2, r(k-1), k - 2 + (r - 1)(r - 2), r(r - 1))$ and a $NL_r(k)$ graph is strongly regular with parameters $(k^2, r(k + 1), -k - 2 + (r + 1)(r + 2), r(r + 1))$ we have

THEOREM E. A $G_2(d)$ graph with parameters

$$\text{(i)}\ v = 4r^2,\ n_1 = r(2r - 1),\ d = r(r - 1),$$
$$\text{(ii)}\ v = 4r^2,\ n_1 = r(2r + 1),\ d = r(r + 1),$$
$$\text{(iii)}\ v = 4r^2 - 1,\ n_1 = 2r^2,\ d = r^2,$$

exists for all $r = 3^m.2^{m+n-1}$, where m, n are non negative integers $(m, n) \neq (0, 0)$.

2. NEW CONSTRUCTIONS

We first prove some preliminary results.

Lemma 2.1. Let v, b, k, r be non negative integers, $k < v$ and $vr = bk$. Let there exist an incomplete block design D with v symbols (treatments) in b subsets (blocks) of size k such that any pair of treatments occurs together in at most one block. Then a necessary and sufficient condition that each block in D intersects precisely $k(r - 1)$ other blocks in D is that each treatment occurs exactly r times in D.

Proof. It is obvious that any two blocks in D intersect in at most one treatment. If each treatment occurs r times in D, then any block containing, say, treatments $t_1, t_2, \ldots, t_k$ intersects $(r - 1)$ other blocks containing t_i; $i = 1, 2, \ldots, k$. The sets of $(r - 1)$ blocks containing t_i and t_j are obviously disjoint, $i \neq j$. Hence this block intersects precisely $k(r - 1)$ other blocks necessarily in one treatment. This proves the sufficiency part of the theorem.

Now suppose that each block intersects precisely $k(r - 1)$ other blocks. Let r_i be the number of times the treatment i occurs in D. Then obviously

$$\bar{r} = \sum_{1}^{v} r_{i/v} = r.$$

Let $N = (n_{ij})$ be the usual $(0, 1)$ incidence matrix of D with v rows and b columns where $n_{ij} = 1$ or 0 according as treatment i occurs in ~~blokes~~ block j or not. Then from our hypothesis

$$N'N = kI_b + A$$

where A is an adjacency matrix of order b which is regular and of valence $k(r - 1)$. Also

$$NN' = \mathrm{diag}(r_1, \ldots, r_v) + B$$

where B is also an adjacency matrix of order v with i-th row sum $r_i(k - 1)$. Hence

$$\operatorname{tr}((NN')(NN')) = \sum_1^v r_i^2 + (k-1)\sum_1^v r_i$$
$$= \sum_1^v r_i^2 + (k-1)vr.$$

But

$$\operatorname{tr}((NN')(NN')) = \operatorname{tr}(N(N'N)N')$$
$$= \operatorname{tr} N(kI_b + A)N'$$
$$= k \operatorname{tr} NN' + \operatorname{tr} N\,A\,N'$$
$$= k \operatorname{tr} NN' + \operatorname{tr} A\,N'N$$
$$= k \operatorname{tr} NN' + \operatorname{tr}(A(kI_b + A))$$
$$= k \sum_1^v r_i + \operatorname{tr} A^2$$
$$= k\,v\,r + bk(r-1)$$
$$= v\,r(k + r - 1).$$

Hence equating the two values of the trace

$$\sum_1^v (r_i - \bar{r})^2 = \sum_1^v r_i^2 - v\,\bar{r}^2$$
$$= \sum_1^v r_i^2 - vr^2$$
$$= 0$$

which implies that each treatment occurs r times in D. This completes the proof of the lemma.

We now define the concept of an ascendent graph G^* of a strongly regular graph G with parameters $(v, n_1, p_{11}^1, p_{11}^2)$. Let (V_1, V_2) be a partition of the vertex set V of G where V_1 and V_2 respectively contain n_1^* and $v - n_1^*$ vertices. Let ∞ be a vertex not in V and let G^* be a graph with vertex set $(\infty \cup V)$. We define adjacency in G^* as follows: The vertex ∞ is adjacent only to vertices of V_1. If x, y are in V, then they are adjacent in G^* if and only if they are adjacent in G and belong both to V_1 or both to V_2, or if they are nonadjacent in G and belong one to V_1 and the other to V_2. If the graph G^* is strongly regular with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$ where v^* is necessarily $v + 1$, then G^* is said to be an ascendent of G.

We derive the conditions under which a graph G with parameters $(v, n_1, p_{11}^1, p_{11}^2)$ has an ascendent G^* with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$. We will assume that G is neither a void graph nor a complete graph i.e. $n_1 \neq 0$ and $n_2 = v - 1 - n_1 \neq 0$.

If G^* is an ascendent of G, then G is a descendent of G^* with respect to the vertex ∞ and hence from [2]

$$p_{11}^{1*} + p_{11}^{2*} = 2n_1^* - \frac{v^*}{2}$$

$$= 2n_1^* - \frac{v+1}{2} \quad . \tag{2.1}$$

$$n_1 = 2n_1^* - 2p_{11}^{2*} \quad . \tag{2.2}$$

$$p_{11}^1 = n_1 - n_1^* + p_{11}^{1*} \quad . \tag{2.3}$$

$$p_{11}^2 = n_1 - n_1^* + p_{11}^{2*} \quad . \tag{2.4}$$

From (2.2), (2.4)

$$2p_{12}^2 = 2(n_1 - p_{11}^2)$$

$$= 2(n_1^* - p_{11}^{2*})$$

$$= n_1 .$$

Hence from the usual parametric relation

$$n_1/2 = p^2_{11} = p^2_{12} \tag{2.5}$$

which implies that

$$n_2/2 = p^1_{12} = p^1_{22} \tag{2.6}$$

Also from (2.1), (2.3) and (2.4)

$$v + 1 = 4n_1 - 2p^1_{11} - 2p^2_{11}.$$

or
$$v = 6\,p^2_{11} - 2p^1_{11} - 1. \tag{2.7}$$

Thus (2.7) is a necessary condition for G to have an ascendent. It is easy to see that for a graph G with parameters $(v, n_1, p^1_{11}, p^2_{11})$ each of (2.5), (2.6), (2.7) implies the other two. It also follows from [2] that the vertex set V of G can be partitioned into (V_1, V_2) with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2, where the set V_1 is the set of vertices in G* which are adjacent to ∞. Further, each vertex in V_1 is adjacent to $p^{1*}_{11} = n_1^* - n_1 + p^1_{11}$ vertices of V_1 in G and each vertex in V_2 is adjacent to

$$\begin{aligned} p^{2*}_{12} &= n_1^* - p^{2*}_{11} \\ &= n_1 - p^2_{11} \\ &= p^2_{11} \end{aligned}$$

vertices of V_2 in G.

Conversely, suppose (2.7) is satisfied for G and further there exists a partition (V_1, V_2) of the vertex set V of G with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2, such that each vertex in V_1 is adjacent to

$n_1^* - n_1 + p_{11}^1$ vertices in V_1 and each vertex in V_2 is adjacent to p_{11}^2 vertices in V_2. Then by using arguments similar to those in [2], it can be shown that G* is a strongly regular graph with parameters (2.2), (2.3), (2.4) provided p_{11}^{1*}, p_{11}^{2*} are non negative integers. From the relation

$$n_1^* \; p_{12}^{1*} = n_2^* \; p_{11}^{1*}$$

it follows that n_1^* satisfies the equation

$$f(x) \equiv x^2 + x(p_{11}^1 - 5p_{11}^2) + v\, p_{11}^2 = 0.$$

The above equation is easily seen to have real positive roots. Since n_1^* is necessarily an integer a further necessary condition for G to have an ascendent G* is that the above equation has an integral solution.

We easily verify that

$$f'(n_1 - p_{11}^1) = -(p_{11}^1 + p_{11}^2) < 0 .$$

$$f'(n_1 - p_{11}^2) = -(1 + p_{12}^1 + p_{11}^2) < 0 .$$

$$f(n_1 - p_{11}^2) = p_{11}^2 \; p_{12}^1 > 0 .$$

$$f(n_1 - p_{11}^1) = p_{11}^2 \; (p_{11}^1 - 1) .$$

Further, it is easily seen that if $p_{11}^1 = 0$, then $f(x) = 0$ has no integral solution. Hence, if the equation has an integral solution then we can assume that $p_{11}^1 \geq 1$ and then the above relations imply that

$$p_{11}^{1*} = n_1^* - n_1 + p_{11}^1 \geq 0.$$

$$p_{11}^{2*} = n_1^* - n_1 + p_{11}^2 > 0 .$$

Thus the condition that $f(x) = 0$ has an integral solution n_1^* is necessary and sufficient for nonnegativeness of p_{11}^{1*}, p_{11}^{2*}. We can, therefore, state the following theorem.

THEOREM 2.1. Let G be a strongly regular graph with parameters $(v, n_1, p_{11}^1, p_{11}^2)$. Then G has an ascendent G* with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$ if and only if the following parametric and structural conditions (P) and (S) are satisfied in G.

(P) $v = 6\, p_{11}^2 - 2\, p_{11}^1 - 1.$

(S) The equation

$$x^2 + x\,(p_{11}^1 - 5\, p_{11}^2) + v\, p_{11}^2 = 0.$$

has an integral solution n_1^* and there exists a partition (V_1, V_2) of the vertex set V of G with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2 such that every vertex in V_1 has $n_1^* - n_1 + p_{11}^1$ adjacent vertices in V_1 and every vertex in V_2 has p_{11}^2 adjacent vertices in V_2.

The parameters of G* are then given by

$$v^* = v + 1, \quad n_1^*, \quad p_{11}^{1*} = n_1^* - n_1 + p_{11}^1, \quad p_{11}^{2*} = n_1^* - n_1 + p_{11}^2.$$

It is obvious that any two blocks of a BIBD with $\lambda = 1$ have at most one treatment in common. Consider a BIBD with $r = 2k + 1$, $\lambda = 1$. Then the values of v and b are given by $v = 2k^2 - k$ and $b = 4k^2 - 1$. Consider the blocks as vertices of a graph G and define two blocks as adjacent or nonadjacent according as they have a treatment in common or not. Then [2] G is strongly regular with parameters $(4k^2 - 1, 2k^2, k^2, k^2)$ and satisfies the condition (P) of the above theorem. Also the equation $f(x) = 0$ has integral solutions $k(2k - 1)$ and $k(2k + 1)$. Take $n_1^* = k(2k - 1)$. If the $4k^2 - 1$ blocks can be partitioned into sets V_1 and V_2 of $k(2k - 1)$ and $(k + 1)(2k - 1)$ blocks respectively such that each block in V_1 is adjacent to $k^2 - k$ blocks in V_1 and each block in V_2 is adjacent to k^2 blocks in V_2, then the condition (S) is also satisfied. From Lemma 2.1 this means that the set V_1 (respectively V_2) contains each of the $2k^2 - k$ treatments exactly k (respectively $k + 1$) times.

We note that a BIBD with $r = 2k + 1$, $\lambda = 1$ is a partial geometry $(r, k, t) = (2k + 1, k, k)$. The graph G is then [1] the graph of the dual configuration and is also a partial geometry $(k, 2k + 1, k)$. We can, therefore, state the following theorem.

THEOREM 2.2. Let G be the graph of the dual of a BIBD with $r = 2k + 1$, $\lambda = 1$. Then G has an ascendent G* which is a pseudo $L_k(2k)$ graph if and only if the $4k^2 - 1$ blocks of the BIBD can be partitioned into sets V_1 and V_2 of $k(2k - 1)$ and $(k + 1)(2k - 1)$ blocks respectively such that each of the $2k^2 - k$ treatments of the BIBD occur k times in V_1 and $k + 1$ times in V_2.

BIBD's having the structure of the above theorem exist for $k = 5$ and 7. See for example Appendix I in [4]. Hence we have the following result.

COROLLARY. Pseudo $L_5(10)$ and pseudo $L_7(14)$ graphs exist.

Goethals and Seidel [3] have constructed a pseudo $L_5(10)$ graph in precisely the same manner.

Using the Corollary and Theorems B, C and D we have the following theorem

THEOREM 2.3. (I) pseudo $L_r(2r)$ graphs exist for all $r = 3^m\ 5^a\ 7^c\ 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers $(m, n, a, c) \neq (0, 0, 0, 0)$.

(II) $NL_r(2r)$ graphs exist for $r = 5^a\ 7^c\ 2^{a+c}$ where, a, c are non negative integers and for $r = 3^m\ 5^a\ 7^c\ 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers and $(m, n) \neq (0, 0)$.

The proof is similar to that of Theorem 9.3 and 9.5 in [2] and is omitted.

Finally, noting that pseudo $L_r(2r)$ and $NL_r(2r)$ graphs are $G_2(d)$ graphs we have

THEOREM 2.4. $G_2(d)$ graphs with the following parameters exist

$$\text{(i)}\ v = 4r^2, \quad n_1 = r(2r - 1), \quad d = r(r - 1);$$

$$\text{(ii)}\ v = 4r^2 - 1, \quad n_1 = 2r^2, \quad d = r^2;$$

for all $r = 3^m\ 5^a\ 7^c\ 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers $(m, n, a, c) \neq (0, 0, 0, 0)$.

$$\text{(iii)}\ v = 4r^2, \quad n_1 = r(2r + 1), \quad d = r(r + 1);$$

with $r = 5^a\ 7^c\ 2^{a+c}$, where a, c are non negative integers and with $r = 3^m\ 5^a\ 7^c\ 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers and $(m, n) \neq (0, 0)$.

We remark that since our construction is essentially by a composition method, any new $G_2(d)$ graph can be utilised in conjunction with the above theorem to enlarge such a family considerably.

REFERENCES

1. BOSE, R.C., "Strongly Regular Graphs, partial geometries and partially balanced designs," Pacific J. Math. 13(1963), pp. 389 - 418.

2. BOSE, R.C. AND SHRIKHANDE, S.S., "Graphs in which each pair of vertices is adjacent to the same number d of other vertices," To appear in Studia Scientiarum Mathematicarum Hungarica.

3. GOETHALS, J.M. AND SEIDEL, J.J., "Strongly regular graphs derived from combinatorial designs," To appear in Canad. J. Math.

4. Hall Marshall, "Combinatorial Theory," Blaisdell(1967).