ANALYSIS OF NONLINEAR THERMOVISCOELASTIC AND THERMOPLASTIC BEHAVIOR OF SOLIDS OF REVOLUTION BY THE FINITE ELEMENT METHOD

J.T. ODEN, T.J. CHUNG, J.E. KEY,

Division of Graduate Programs and Research,
The University of Alabama, Huntsville, Alabama, U.S.A.

ABSTRACT

General finite element models for the analysis of a broad class of nonlinear problems in finite thermoelasticity, thermoviscoelasticity, and thermoplasticity are derived. Numerical results obtained from a number of example problems are given.

1. INTRODUCTION

Constitutive equations describing specific materials are necessarily valid for only specific ranges of deformation, temperature, rates-of-loading, and stress. Thus, when one describes a material as elastic, he must have in mind a generally limited set of conditions to which the material is to be subjected. Indeed, virtually all of the traditional structural engineering materials which are treated as elastic, can respond quite inelastically under conditions not uncommon in their practical lifetime.

One can construct a more general theory, however, which characterizes certain materials for a much broader range of conditions than classical elasticity (or, for that matter, finite elasticity, thermoelasticity, or viscoelasticity) and which, under the appropriate special conditions, still can depict purely elastic response if it exists. One such theory is that proposed by Coleman [1] for thermomechanically simple materials; i.e. materials characterized by constitutive equations for the free energy, stress, entropy and heat flux for which the response at a given instant of time is determined by the history of the deformation and temperature. Such materials have the property that for rest histories they do indeed respond as a hyperelastic solid; i.e., for appropriate temperature and strain histories, every simple material will appear to be elastic.

One must ultimately pay the price, of course, for the greater flexibility of more general theories of material behavior. Constitutive equations for simple materials which are sufficiently general to include the nonlinear thermal, and viscous effects mentioned earlier are so highly complex that practically no quantitative information exists on their application to specific problems. It is toward resolving this situation that the present paper is directed. Specifically, this paper is concerned with the numerical analysis of a class of nonlinear problems in thermoelasticity and thermoviscoelasticity which can be identified within the framework of the theory of thermomechanically simple materials. We step outside of this framework in certain cases, however, since we also wish to incorporate in the analysis some measure of thermoplastic behavior. The finite element method is used to obtain discrete models of such materials, and a number of techniques are used to solve the resulting systems of nonlinear differential and integrodifferential equations.

So as to keep the scope of the paper within manageable bounds, we consider applications of the general methods developed to only three subclasses of problems: finite thermoelasticity, transient nonlinear response and heat generation in thermorheologically simple materials, and dynamic, thermoplastic response of thin shells. We also limit numerical examples to bodies of revolution.

2. THERMOMECHANICAL PRELIMINARIES

We shall now outline the essential features of the theory of thermomechanically simple materials that are the basis of the considerations in this paper. More detailed discussions can be found (see Oden [27]).

We wish to trace the motion of a material body B from a natural reference configuration C_o to a current configuration C. As usual, we establish a fixed spatial frame $\underset{\sim}{z}$ in C_o, represented by the cartesian system z_i, and an intrinsic material frame $\underset{\sim}{x}$ so that x^i denote particle labels and x^i and z_i coincide at time $t = t_o$ in C_o. The functions $z_i(\underset{\sim}{x},t)$ define the motion of time t, $z_i(\underset{\sim}{x},t) - x_i \equiv u_i(\underset{\sim}{x},t)$ are the components of displacement, T_o is a uniform reference temperature in C_o, $T(\underset{\sim}{x},t) + T_o$ is the absolute temperature in C, $2\gamma_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}$ ($u_{i,j} \equiv \partial u_i/\partial x_j$) is the Green-Saint Venant strain tensor, and $G_{ij} = \delta_{ij} + 2\gamma_{ij}$ is the Green-Saint Venant deformation tensor (see Oden [2]). The response of the body at time t is determined by the total history of the temperature $T + T_o$ at t and by the motion at t as represented by the total history of displacement field $\underset{\sim}{u}$ and its gradients or, equivalently, the histories of the tensors $\underset{\sim}{\gamma}$ and $\underset{\sim}{G}$.

We assume that the body is composed of a thermomechanically simple material; that is, a material whose response (e.g., free energy, stress, entropy, heat flux) at time t is determined by the total history $T^t(s) \equiv T(\underset{\sim}{x},t-s)$ of the temperature change and the history of, say, $\underset{\sim}{\gamma}(\underset{\sim}{x},t)$. To characterize a thermomechanically simple material, only two constitutive equations are needed; one giving the free energy φ as a functional of the histories $T^t(s)$ and $\underset{\sim}{\gamma}^t(s)$ and another giving the heat flux vector $\underset{\sim}{q}$ as a functional of $T^t(s)$ and $\underset{\sim}{\gamma}^t(s)$ and a function of the current temperature gradient g(t):

$$\begin{aligned} \varphi &= \overset{\infty}{\underset{s=0}{\bar{\varphi}}}(\underset{\sim}{\gamma}^t(s), T^t(s)) \\ \underset{\sim}{q} &= \overset{\infty}{\underset{s=0}{\bar{Q}}}(\underset{\sim}{\gamma}^t(s), T^t(s); \underset{\sim}{g}(t)) \end{aligned} \tag{2.1}$$

Here s is a real parameter belonging to the half-open interval $[0,\infty)$ and the absence of $\underset{\sim}{g}(t)$ from the functional $\bar{\varphi}(\ )$ arises from the postulate that φ must satisfy the Clausius-Duhem inequality.

Now $f^t(s) = f(t-s)$ defines the total history of a quantity f. It is convenient, however, to decompose the total history into the pair $(f_r^t(s); f(t))$ where $f_r^t(s)$ is the restriction of $f(t-s)$ to $(0,\infty)$ and $f(t) = f^t(0)$ is the current value of $f(t-s)$. The restriction $f_r^t(s)$ is called the past history. Then, instead of (2.1), we have

$$\begin{aligned} \varphi &= \overset{\infty}{\underset{s=0}{\bar{\varphi}}}(\underset{\sim}{\gamma}_r^t(s), T_r^t(s); \underset{\sim}{\gamma}(t), T(t)) \\ \underset{\sim}{q} &= \overset{\infty}{\underset{s=0}{\bar{Q}}}(\underset{\sim}{\gamma}_r^t(s), T_r^t(s); \underset{\sim}{\gamma}(t), T(t), \underset{\sim}{g}(t)) \end{aligned} \tag{2.2}$$

The functionals $\Phi(\ )$ and $\underset{\sim}{Q}(\ )$ are assumed to be frame-indifferent; i.e., to be invariant under observer transformations.

The complete characterization of the material requires specification of the stress $\underset{\sim}{\sigma}$ and the entropy density η as functionals of $\underset{\sim}{\gamma}^t(s), T^t(s)$ (and possibly $\underset{\sim}{g}(t)$). However, if these functionals and those of (2.1) are to satisfy the Clausius-Duhem inequality, then the stress and entropy functionals are determined by the free energy in the sense that

$$\begin{aligned} \underset{\sim}{\sigma} &= \partial_{\underset{\sim}{\gamma}} \overset{\infty}{\underset{s=0}{\Phi}}(-) \equiv \sum_{s=0}^{\infty} (\underset{\sim}{\gamma}_s^t(s), T_s^t(s); \underset{\sim}{\gamma}(t), T(t)) \\ \eta &= -\partial_T \overset{\infty}{\underset{s=0}{\Phi}}(-) \equiv \sum_{s=0}^{\infty} \mathfrak{N}(\underset{\sim}{\gamma}_s^t(s), T_s^t(s); \underset{\sim}{\gamma}(t), T(t)) \end{aligned} \tag{2.3}$$

Here $\partial_{\underset{\sim}{\gamma}}$ and ∂_T denote partial differentiation with respect to $\underset{\sim}{\gamma}$ and T and $\underset{\sim}{\sigma}$ is the second Piola-Kirchhoff stress tensor. Likewise, if $d(\underset{\sim}{x},t)$ denotes the internal dissipation as defined by $\operatorname{tr} \underset{\sim}{\sigma}\dot{\underset{\sim}{\gamma}} - \rho(\dot{\varphi} + \eta\dot{T})$, ρ being the mass density, then d is also determined by the free energy by way of the relation

$$d = \delta_{\underset{\sim}{\gamma}} \overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t(s), T_s^t(s); \underset{\sim}{\gamma}(t), T(t) \,|\, \dot{\underset{\sim}{\gamma}}_s^t(s)) + \delta_T \overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t(s), T_s^t(s); \underset{\sim}{\gamma}(t), T(t) \,|\, \dot{T}_s^t(s)) \tag{2.4}$$

Here $\delta_{\underset{\sim}{\gamma}}$ and δ_T denote Frechet derivatives of the functional $\Phi(-)$ with respect to the past histories $\underset{\sim}{\gamma}_s^t$ and T_s^t, respectively; i.e.

$$\begin{aligned} \delta_{\underset{\sim}{\gamma}} \overset{\infty}{\underset{s=0}{\Phi}}(- \,|\, \underset{\sim}{\Gamma}_s^t) &\equiv \lim_{\alpha \to 0} \frac{1}{\alpha}\Big[\overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t + \alpha\underset{\sim}{\Gamma}_s^t, T_s^t, \underset{\sim}{\gamma}, T) - \overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t, T_s^t, \underset{\sim}{\gamma}, T)\Big] \\ \delta_T \overset{\infty}{\underset{s=0}{\Phi}}(- \,|\, H_s^t) &= \lim_{\alpha \to 0} \frac{1}{\alpha}\Big[\overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t, T_s^t + \alpha H_s^t; \underset{\sim}{\gamma}, T) - \overset{\infty}{\underset{s=0}{\Phi}}(\underset{\sim}{\gamma}_s^t, T_s^t; \underset{\sim}{\gamma}, T)\Big] \end{aligned} \tag{2.5}$$

Here $\underset{\sim}{\Gamma}_s^t$ and H_s^t are arbitrary past strain and temperature histories and each functional is linear in the quantity following the vertical stroke. By taking $\alpha\underset{\sim}{\Gamma}_s^t = \underset{\sim}{\gamma}_s^t(s+\alpha s)$, $\alpha H_s^t = H_s^t(s+\alpha s)$, we find that $\dot{\underset{\sim}{\gamma}}_s^t(s) = -d\underset{\sim}{\gamma}(t-s)/ds$ and $\dot{T}_s^t(s) = -dT(t-s)/ds$ in (2.4).

The theory of thermomechanically simple materials just outlined is sufficiently broad to emcompass a variety of interesting special cases. For example, various theories of heat conduction and thermoviscoelasticity are generated by retaining appropriate terms in expansions of $\Phi(\ )$ in the histories $\underset{\sim}{\gamma}_s^t$ and T_s^t (see Oden [2]). Among applications that we shall consider are those for which $\Phi(\ )$ is a quadratic functional in $\underset{\sim}{\gamma}_s^t$ and T_s^t which corresponds to an isotropic, thermorheologically simple material (Oden, Oden and Armstrong, Cost [2], [3], [4]).

$$\begin{aligned} \Phi[-] = \Phi_o &+ \int_0^{\zeta} D^{ij}(\zeta - \hat{s}) \frac{\partial \hat{\gamma}_{ij}}{\partial \hat{s}}\, d\hat{s} - \int_0^{\zeta} f(\zeta - s) \frac{\partial \hat{T}}{\partial \hat{s}}\, d\hat{s} \\ &+ \frac{1}{6}\delta^{ij}\delta^{mn} \int_0^{\zeta}\int_0^{\zeta} [3K(\zeta - \hat{s}_1, \zeta - \hat{s}_2) - 2G(\zeta - \hat{s}_1, \zeta - \hat{s}_2)] \times \end{aligned}$$

$$\times \frac{\partial \hat{\gamma}_{ij}}{\partial \hat{s}_1} \frac{\partial \hat{\gamma}_{mn}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2 + \frac{1}{4} (\delta^{ij}\delta^{mn} + \delta^{im}\delta^{jn}) \int_0^{\zeta}\int_0^{\zeta} 2$$

$$\times G(\zeta-\hat{s}_1, \zeta-\hat{s}_2) \frac{\partial \hat{\gamma}_{ij}}{\partial \hat{s}_1} \frac{\partial \hat{\gamma}_{mn}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2 - \delta^{ij} \int_0^{\zeta}\int_0^{\zeta} 3\alpha K(\zeta-\hat{s}_1, \zeta-\hat{s}_2)$$

$$\times \frac{\partial \hat{\gamma}_{ij}}{\partial \hat{s}_1} \frac{\partial \hat{T}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2 - \frac{1}{2} \int_0^{\zeta}\int_0^{\zeta} m(\zeta-\hat{s}_1, \zeta-\hat{s}_2) \frac{\partial \hat{T}}{\partial \hat{s}_1} \frac{\partial \hat{T}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2 \qquad (2.5a)$$

$$t^{ij} = D^{ij}(0) + \delta^{im}\delta^{jn} \int_0^t 2G(f(t)-\hat{s}) \frac{\partial \gamma_{mn}}{\partial \hat{s}} \, d\hat{s} + \frac{1}{3} \delta^{ij} \int_0^t M(f(t)-\hat{s})$$

$$\times \frac{\partial \gamma_{kk}}{d\hat{s}} \, d\hat{s} - \delta^{ij} \int_0^t \alpha 3K(f(t)-\hat{s}) \frac{\partial T}{\partial \hat{s}} \, d\hat{s} \qquad (2.5b)$$

$$\xi = f(0) + 3\alpha K(0)\gamma_{ii} + M(0)T(t) - \int_0^t \frac{\partial}{\partial \hat{s}} [3\alpha K(\hat{s})]\gamma_{ii}(t-\hat{s})ds - \int_0^t \frac{\partial}{\partial \hat{s}} [m(\hat{s})]T(t-\hat{s})d\hat{s} \qquad (2.5c)$$

$$d = -\frac{1}{6} \int_0^t\int_0^t \frac{\partial}{\partial t} [M(t(t) - \hat{s}_1, f(t) - \hat{s}_2)] \frac{\partial \gamma_{ii}}{\partial \hat{s}_1} \frac{\partial \gamma_{jj}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2$$

$$- \frac{1}{2} \int_0^t\int_0^t \frac{\partial}{\partial t} [2G(f(t) - \hat{s}_1, f(t) - \hat{s}_2)] \frac{\partial \gamma_{ij}}{\partial \hat{s}_1} \frac{\partial \gamma_{ij}}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2$$

$$+ \int_0^t\int_0^t \frac{\partial}{\partial t} [3\alpha K(f(t) - \hat{s}_1, f(t) - \hat{s}_2)] \frac{\partial \gamma_{ij}}{\partial \hat{s}_1} \frac{\partial T}{\partial \hat{s}_2} \, d\hat{s}_1 d\hat{s}_2$$

$$+ \frac{1}{2} \int_0^t\int_0^t \frac{\partial}{\partial t} [m(f(t) - \hat{s}_1, f(t) - \hat{s}_2)] \frac{\partial T}{\partial \hat{s}_1} \frac{\partial T}{\partial \hat{s}_2} \qquad (2.5d)$$

Here ζ is the reduced time; i.e. in certain materials the variation of various relaxation moduli with the logarithm of time experiences a shift in time for various uniform temperature, which suggests a change of variables $\zeta = f[T(\underline{x},t)]$. $K(\,)$, $G,(\,)$, etc. are relaxation moduli. For further details, see References [2,3,4].

Alternately, (2.2), (2.3), and (2.4) define a general theory of thermoelasticity if we set $d \equiv 0$ and assume that φ is a differentiable function of only the current strain and temperature; i.e.

$$\varphi = \Phi(\underset{\sim}{\gamma}, T) \tag{2.6}$$

Then (2.3) gives

$$\underset{\sim}{\sigma} = \frac{\partial \Phi(\underset{\sim}{\gamma}, t)}{\partial \underset{\sim}{\gamma}} \qquad \eta = - \frac{\partial \Phi(\underset{\sim}{\gamma}, T)}{\partial T} \tag{2.7a,b}$$

In this case it is still possible that $\underset{\sim}{q}$ is a functional of the histories of $\underset{\sim}{\gamma}$ and T, but it is more plausible in a theory of thermoelasticity to regard $\underset{\sim}{q}$ as only a function of the current values of $\underset{\sim}{\gamma}$, T, and $\underset{\sim}{g}$. Obviously, if $T = 0$ for all t, (2.6) and (2.7) describe a hyperelastic material.

We remark that in the case of an isotropic, incompressible, thermoelastic material, $\Phi(\underset{\sim}{\gamma}, T)$ determines the stress only to within an arbitrary hydrostatic pressure $h = h(x,t)$. $\Phi(\underset{\sim}{\gamma}, T)$ is then generally expressible in terms of the first two principal invariants I_1, I_2 of the deformation tensor $\underset{\sim}{G}$, the third invariant being unity. Instead of (2.7a) we then have

$$\underset{\sim}{\sigma} = \frac{\partial \Phi(I_1, I_2, T)}{\partial \underset{\sim}{\gamma}} + h \frac{\partial I_3}{\partial \underset{\sim}{\gamma}} \tag{2.8}$$

while $I_3 = 1$ throughout B.

3. FINITE ELEMENT APPROXIMATIONS

The method by which one makes a transition from a general continuum theory to a consistent, finite-element model of the continuum is well documented (see Oden, Oden and Aguirre-Ramirez, Oden and Poe, Oden [2], [5], [6], [7]) and only a brief discussion is warranted here. As usual, we decompose the continuum into a finite number of elements connected continuously together at prescribed nodes and interelement boundaries. We then isolate a typical element e, containing N_e nodes, and approximate the local displacement and temperature fields according to

$$\underset{\sim}{u}_e = \psi_N(\underset{\sim}{x})\underset{\sim}{u}^N \qquad T_e = \varphi_N(\underset{\sim}{x})T^N \tag{3.1a,b}$$

Here N is summed from 1 to N_e, $\psi_N(\underset{\sim}{x})$ and $\varphi_N(\underset{\sim}{x})$ are local interpolation functions, $\underset{\sim}{u}^N$ and T^N are the displacements and temperatures at node $\underset{\sim}{x}^N$, and $\psi_N(\underset{\sim}{x}^M) = \varphi_N(\underset{\sim}{x}^M) = \delta_N^M$. We do, of course, use higher-order representations wherein derivatives of $\underset{\sim}{u}^N$ and T^N are specified at the nodes, but for simplicity we discuss only first order representations in this section. Also, the character of $\psi_N(\underset{\sim}{x})$ and $\varphi_N(\underset{\sim}{x})$ may be quite different; e.g., an approximation of T of lower order than $\underset{\sim}{u}$ may be used. Once these local fields are defined, the approximate fields over the entire discrete model are obtained by connecting elements together in the usual fashion.

From (3.1) we compute the local strain and absolute temperature,

$$2\gamma_{ij}^{(e)} = \psi_{N,i} u_j^N + \psi_{N,j} u_i^N + \psi_{N,i}\psi_{M,j} u_k^N u_k^M \qquad \theta^{(e)} = T_e + \varphi_N T^N \tag{3.2a,b}$$

From (3.2a), all other local kinematical quantities follow (i.e., G^o_{ij}, I^o_1, I^o_2, I^o_3, $\underset{\sim}{\gamma}^t_r(s)$, etc.).

To arrive at equations of motion and heat conduction governing the element, we follow an established procedure (Oden, Oden and Aguirre-Ramirez, Oden [2], [5], [7]) and apply the principle of conservation of energy to the element subjected to the fields (3.1). By arguing that the result must hold for arbitrary nodal velocities and temperatures, we arrive at the equations of motion of the element,

$$m_{NM}\ddot{u}^M_i + \int_{U_{o(e)}} \sigma^{\alpha j}\psi_{N,j}(\delta_{\alpha i} + \psi_{M,i}u^M_\alpha)dU_o = P_{Ni} \tag{3.3}$$

and the equations of heat conduction for the element,

$$\int_{U_{o(e)}} \{(T_o + \varphi_M T^M)\dot{\eta}\psi_N - q^i\varphi_{N,i}\}dU_o = q_N + d_N \tag{3.4}$$

Here $U_{o(e)}$ is the initial volume of the element, $\sigma^{\alpha j}$ and q^i are the contravariant components of $\underset{\sim}{\sigma}$ and $\underset{\sim}{q}$, m_{NM} is the mass matrix, and P_{Ni}, q_N, and d_N are the generalized forces, normal heat flux, and dissipation at node N:

$$m_{NM} = \int_{U_{o(e)}} \rho_o\psi_N\psi_M dU_o \qquad q_N = \int_{U_{o(e)}} \rho_o r\varphi_N dU_o \int_{A_{o(e)}} q^i n_i \varphi_N dA_o$$

$$P_{Ni} = \int_{U_{o(e)}} \rho_o F_i\psi_N dU_o + \int_{A_{o(e)}} S^i\psi_N dA_o$$

$$d_N = \int_{U_o} d\varphi_N dU_o \tag{3.5}$$

In these equations, ρ_o is the initial mass density, r is the heat supply density from internal sources, n_i are components of a unit normal to the initial surface A_o, F_i are the components of body force, and S^i the surface tractions.

To apply (3.3) and (3.4) to specific problems, we need only introduce the appropriate constitutive equations for stress, entropy, and heat flux. In the case of thermomechanically simple materials, we have

$$m_{NM}\ddot{u}^M_i + \int_{U_{o(e)}} \frac{\partial}{\partial\gamma_{\alpha j}} \overset{\infty}{\underset{s=0}{\Phi}} (\underset{\sim}{\gamma}^t_r(s), T^t_r(s); \underset{\sim}{\gamma}(t), T(t))\psi_{N,j}(\delta_{\alpha i} + \psi_{N,i}u^M_\alpha)dU_o = P_{Ni} \tag{3.6}$$

and the equations of heat conduction are

$$- \{\varphi_N(T_o + \varphi_M T^M) \frac{d}{dt} \frac{\partial}{\partial T} \overset{\infty}{\underset{s=0}{\Psi}}(\underline{\gamma}_r^t(s), T_r^t(s); \underline{\gamma}(t), T(t))$$

$$\upsilon_o$$

$$- \overset{\infty}{\underset{s=0}{Q}}{}^t(\underline{\gamma}_r^t(s), T_r^t(s); \underline{\gamma}(t), T(t), \underline{g}(t)\varphi_{N,i})\}d\upsilon_o = q_N + d_N \qquad (3.7)$$

We examine specific applications of these equations (and certain alternate forms of them appropriate for shells and bodies of revolution) in the next section.

4. APPLICATIONS

We now describe applications of the theory presented previously to a rather broad class of transient and quasi-static nonlinear problems in finite thermoelasticity, thermoviscoelasticity, and, after certain additional considerations, transient thermoelastoplasticity. Applications are restricted to bodies and thin shells of revolution and fall within the scope of rather general computer programs in various stages of development at the UAH Research Institute.

Finite Thermoelasticity. We are not aware of any solutions that exist to problems in finite thermoelasticity. We consider here the general problem of finite deformations of incompressible, isotropic, thermoelastic bodies of revolution. Let $(x^1,x^2,x^3) = x^i$ denote the cylindrical coordinates (r,z,θ). Then the strains are

$$2\gamma_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\lambda,\alpha}u_{\lambda,\beta}$$

$$\gamma_{\alpha 3} = 0 \qquad \gamma_{33} = \frac{1}{2}(\lambda^2-1) \qquad (4.1)$$

where $\alpha,\beta,\lambda = 1,2$, only axisymmetric deformations are considered, and $\lambda = 1 + u_1/r$ is the circumferential extension ratio.

We shall use a finite element model composed of triangular elements of revolution over each of which the hydrostatic pressure and local temperature are assumed to be uniform whereas the local displacement field is linear in r and z. Then

$$\varphi_N(r,z) = a_N + b_{N\alpha}x^\alpha \qquad (4.2)$$

$N=1,2,3$ where a_N and $b_{N\alpha}$ are known in terms of local coordinates.

Now we need to know the free energy Ψ as a function of the invariants $I_1=3 + 2\gamma_{\alpha\alpha} + 2\gamma_{33}$, $I_2=\lambda^2 I_1 - \lambda^4 + I_3$, $I_3=\lambda^2\varphi = 1(\lambda=1 + u_1/r, \varphi = 1 + 2\gamma_{\alpha\alpha} + 2\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\gamma_{\alpha\lambda}\gamma_{\lambda\mu})$, and the absolute temperature $\theta=T_o + T(r,z)$. To arrive at a possible form for Ψ, we expand Ψ about a state of zero strain and uniform temperature T_o in powers of I_1-3, I_2-3, and $\theta-T_o$ and retain only linear terms in I_1,I_2, and quadratic terms in γ_{ij} and T. This leads us to a Mooney-type constitutive law,

$$\Psi(I_1,I_2,T) = C_1(I_1-3) + C_2(I_2-3) + \beta T I_2 + \mu T \qquad (4.3)$$

where C_1,C_2,β, and μ are material constants.

Substituting (4.1) - (4.3) into (2.8) and (3.3) and ignoring inertia terms, we obtain

$$2\upsilon_o\{(C_1+\lambda^2C_2)b_{N\beta}(\delta^\beta_\alpha+b_{M\beta}u^M_\alpha)+[C_1+2C_2(1+\gamma^\beta_\beta)+h\varphi]\frac{\lambda}{\bar r}(\delta^{N1}+\delta^{N2}+\delta^{N3})\delta^{1\alpha}$$

$$+(C_2+\lambda^2h)b_{N\lambda}(\delta^\beta_\alpha+b_{M\beta}u^M_\alpha)[\delta^\beta_\lambda+\gamma_{\rho\mu}(\epsilon^{\beta\rho}\epsilon^{\lambda\mu}+\epsilon^{\rho\lambda}\epsilon^{\mu\beta})]-q_{Nk}=p_{Nk} \tag{4.4}$$

Here we have used the approximation $\lambda = 1+\bar u_1/\bar r$, where $\bar u_1 = (u^1_1+u^2_1+u^3_1)/3$ and $\bar r = (r^1+r^2+r^3)/3$; $h = h_o$ is a uniform element hydrostatic pressure and q_{Nk} is a generalized thermal load, the precise form of which we shall give subsequently. To these equations we must add the incompressibility condition, $\upsilon_{(e)}-\upsilon_{o(e)}=0$, where $\upsilon_{(e)}$ and $\upsilon_{o(e)}$ are the volumes of the deformed and undeformed elements; i.e.

$$\upsilon_{(e)} = 2\pi\bar R\bar A \qquad \upsilon_{o(e)} = 2\pi FA_o \tag{4.5a,b}$$

where $\bar R = \bar r+\bar u$, A_o is the underformed area of the element in the r, z-plane, and

$$A = \left|\frac{1}{2}\sum_{p=1}^{3}\epsilon_{MNP}\epsilon^{\alpha\beta3}(x^M_\alpha+u^M_\alpha)(x^N_\beta+u^N_\beta)\right| \tag{4.7}$$

We account for the fact that surface tractions may depend upon the displacements and their gradients in the manner described in earlier papers (e.g., Oden and Key, and Oden [8], [9]). However, in uncoupled quasi-static thermoelasticity, it is convenient to define a "thermal load" as follows. Observe that for quasi-static behavior of an element of the material (4.3), (3.3) becomes

$$P_{Nk} = \int_{\upsilon_{o(e)}}\left(\frac{\partial\Phi}{\partial I_\alpha}\cdot\frac{\partial I_\alpha}{\partial\gamma_{jk}}+h\frac{\partial I_3}{\partial\gamma_{ij}}+\beta T\delta_{ij}\right)\psi_{N,j}(\delta_{ik}+\psi_{M,k}u^M_i)d\upsilon_o \tag{4.8}$$

Let the first three terms in parentheses be denoted $q^{ij}_{(m)}$ (m for mechanical) and let the mechanical loads be zero. Then (4.8) can be written

$$q_{Nk} = \int_{\upsilon_{o(e)}}q^{ij}_{(m)}\psi_{N,j}(\delta_{ik}+\psi_{M,k}u^M_i)d\upsilon_o \tag{4.9}$$

where q_{Nk} is the generalized thermal load

$$q_{Nk} \equiv -\int_{\upsilon_{o(e)}}\beta T\psi_{N,i}(\delta_{ik}+\psi_{M,k}u^M_i)d\upsilon_o \tag{4.10}$$

The systems of nonlinear equations generated by application of (4.4) [or (4.9)] can thus be solved using the incremental loading technique, as in the case of purely mechanical loading (Oden and Key [8], [10]), by incrementing a prescribed temperature distribution.

As a representative example, we consider the nonlinear behavior of the irregular thermoelastic container shown in Fig. 1. The material is specified as the modified Mooney type (see eq. (4.3)) with material constants C_1 = 80 psi, C_2 = 20 psi and $\beta = -300 \times 10^{-5}(C_1 + C_2)$. The body was supported in the z-direction along edge DE. A temperature of 1040° along edge CD. Dimensions of the undeformed body are indicated in the figure.

The finite-element representation involved 134 finite elements that were connected at 85 nodal points. This corresponds to 299 unknowns, 134 element hydrostatic pressures and 165 components of nodal displacements. The method of incremental loading was used to solve the system of nonlinear equations, and twenty thermal load increments were applied. Approximate gradients ($\partial f_i / \partial x$), computed by finite differences, with Δx_j = .00001, were used in the recurrance formulas. Solution of the problem required 36 minutes on the Univac 1108 computer. Computed stress profiles for $T_{interior}$ = 1040°F are given in Fig. 2a, and the deformed shape for various temperatures is shown in Fig. 2b.

Thermoviscoelastic Cylinder. As a second example, we now consider the transient response of a thick-walled cylinder composed of a thermorheologically simple material. Infinitesimal strains are assumed and we consider the material to obey a form of the constitutive law (2.5) for which the material kernals G, K, $\mathfrak{m}$, and $\varkappa$ are given by decaying Prony series. Complete details of this analysis are given in the paper by Oden and Armstrong [11]. The analysis involved the solution of very large systems of nonlinear integrodifferential equations. Figures 3 and 4 contain the computed radial stress waves for the isothermal and nonisothermal cases and the temperature generated for the case of a purely mechanical pressure loading of 10h(t) psi, h(t) being the unit step function. Convective heat transfer conditions were imposed on the outer and inner boundaries.

Nonlinear Shells. Although the basic theories discussed in Section 2 apply equally well to all types of shell structures we shall limit our presentation to the dynamic elastoplastic response of geometric nonlinear shells of revolution. To deal with a nonlinear response we consider large displacements but small strains. Under these assumptions, the total strain tensor for a shell is $e_{\alpha\beta} + z\chi_{\alpha\beta}$, where $e_{\alpha\beta}$ is the middle surface membrane strain tensor,

$$e_{\alpha\beta} = \frac{1}{2}\{u_{\alpha|\beta} + u_{\beta|\alpha} - 2u^3 b_{\alpha\beta} + u^3_{,\alpha} u^3_{,\beta} + u^\lambda b_{\alpha\lambda} u^\gamma b_{\beta\gamma}\}$$

and $\chi_{\alpha\beta}$ is the bending strain tensor,

$$\chi_{\alpha\beta} = -u^3_{|\alpha\beta} - u_{\lambda|\beta} b^\lambda_\alpha - u_\lambda b^\lambda_{\alpha|\beta} - b^\mu_\beta u_{\mu|\alpha} + b^\mu_\beta b_{\mu\alpha} u^3$$

All Greek indices range from 1 to 2; z is the distance to the middle surface; commas and strokes represent, respectively, the ordinary and covariant differentiations; $b_{\alpha\beta}$ is the second fundamental tensor; u^α represents the displacement corresponding to the surface coordinate x^α; and u^3 is the normal displacement.

In order to account for plastic behavior, we divide the thickness of the shell into layers and derive the equations of motion in incremental form, applying the yield conditions for each layer of the shell thickness. The Prandtle-Reuss flow rule, applicable only to this small layer but not to the entire thickness, may be given by

$$\dot{\gamma}^{p}_{\alpha\beta} = \frac{\partial f}{\partial \sigma^{\alpha\beta}} \dot{\lambda} = Z_{\alpha\beta} d\bar{\epsilon}_{(p)} \tag{4.11}$$

Here $f = f(\sigma^{\alpha\beta})$ is the Huber-Mises yield function and λ the normality parameter, and $d\bar{\epsilon}_{(p)}$ is the incremental equivalent yield strain. It can be shown that the incremental stress tensor $\dot{\sigma}^{\alpha\beta}$ and incremental strain tensor $\dot{\gamma}_{\alpha\beta}$ are related by

$$\dot{\sigma}^{\alpha\beta} = H^{\alpha\beta\lambda\mu}\dot{\gamma}_{\lambda\mu} \tag{4.12}$$

where

$$H^{\alpha\beta\lambda\mu} = E^{\alpha\beta\lambda\mu} - \frac{Z_{\gamma\theta}Z_{\delta\eta}E^{\alpha\beta\gamma\theta}E^{\delta\eta\lambda\mu}}{E_p + Z_{\omega\kappa}Z_{\xi\zeta}E^{\omega\kappa\xi\zeta}} \tag{4.13}$$

and $E_p = d\bar{\sigma}/d\bar{\epsilon}_p$ is the instantaneous plastic tangent modulus in which $d\bar{\sigma}$ represents the equivalent yield stress, and $E^{\alpha\beta\gamma\theta}$ is the contravariant tensor of elastic moduli in surfaces parallel to the middle surface.

Following the usual procedure, we approximate the local displacement fields over the element

$$w_i = \psi_N(x) w_i^{\theta} \tag{4.14}$$

In the present study we use one dimensional meridional line element with 4 degrees of freedom at each node for the axisymmetric shell. Components of the generalized displacements consist of meridional, tangential, and transverse translations, and meridional rotation. The interpolation functions are based on linear variations of meridional and tangential displacements and a cubic variation of transverse displacement.

The stress tensor is given by

$$\sigma^{\alpha\beta} = q^{\alpha\beta}_{(m)} + q^{\alpha\beta}_{(b)} \tag{4.15}$$

where $q^{\alpha\beta}_{(m)} = E^{\alpha\beta\lambda\mu}e_{\lambda\mu}$ and $q^{\alpha\beta}_{(b)} = zE^{\alpha\beta\lambda\mu}\chi_{\lambda\mu}$. We obtain

$$\frac{\partial \gamma_{\alpha\beta}}{\partial w_K^N} = \frac{\partial e_{\alpha\beta}}{\partial w_K^N} + z\frac{\partial \chi_{\alpha\beta}}{\partial w_K^N} = A^K_{N\alpha\beta} + C_{NM\alpha\beta}w_K^M + zB^K_{N\alpha\beta} \tag{4.16}$$

where $A^K_{N\alpha\beta}$ and $C_{NM\alpha\beta}w_K^M$ represent, respectively, derivatives of the linear and nonlinear membrane strain interpolation functions; and $B^K_{N\alpha\beta}$ denotes the derivative of the bending strain interpolation function.

At this point we induce a perturbation or an increment, $\delta w_K^N = \Delta_K^N$, so that

$$\delta\left(\frac{\partial \gamma_{\alpha\beta}}{\partial w_K^N}\right) = C_{NM\alpha\beta}\Delta_K^M$$

With these results, the incremental form of the finite element equation of motion becomes

$$\delta(m_{NM}\ddot{w}_K^M) + \delta\left(\int_U \sigma^{\alpha\beta}\frac{\partial \gamma_{\alpha\beta}}{\partial w_K^N}\,dU\right) = \delta P_{NK} \tag{4.17}$$

or

$$m_{NM}\ddot{\Delta}_K^M + \int_U \sigma^{\alpha\beta}C_{NM\alpha\beta}dU\,\Delta_K^M + \int_U \delta\sigma^{\alpha\beta}(A^K_{N\alpha\beta} + C_{NM\alpha\beta}w_K^M + zB^K_{N\alpha\beta})dU = \delta P_{NK} \tag{4.18}$$

where m_{NM} = mass matrix, and $\delta\sigma^{\alpha\beta} = H^{\alpha\beta\lambda\mu}\delta\gamma_{\lambda\mu}$ from (4.16). We rewrite (4.18) in a more convenient form,

$$m_{NM}\ddot{\Delta}_K^M + K_{NM}^{(I)}\Delta_K^M + K_{NM}^{(II)}\Delta_K^M + K_{NM}^{(III)}\Delta_K^M = \delta P_{NK} \tag{4.19}$$

Here $K_{NM}^{(I)}$ is the linear elastic stiffness matrix, $K_{NM}^{(II)}$ is the incremental geometric stiffness matrix, $K_{NM}^{(III)}$ is the incremental plastic tangent stiffness matrix, and δP_{NM} is the applied dynamic load within an incremental time integration step. Specific equations for $K_{NM}^{(II)}$,... etc. can be obtained from (4.16) and (4.18), but, because of space limitations, we do not present them here.

A complete dynamic response analysis under various trial dynamic loads (impulsive, step, periodic, or any other form of dynamic load) may be performed with various types of numerical time integration for equations of motion. Numerical instability is noted in problems dealing with highly nonlinear equations and extremely large dynamic loads. In the present study the constant acceleration method is used.

It is convenient to rearrange the incremental equation of motion (4.19) in the matrix form for an arbitrary time step i,

$$\underset{\sim}{M}\ddot{\underset{\sim}{\Delta}}_i + \underset{\sim}{K}^{(I)}\underset{\sim}{\Delta}_i = \delta\underset{\sim}{P}_i + \delta\bar{\underset{\sim}{P}}_i + \delta\bar{\bar{\underset{\sim}{P}}}_i$$

in which

$$\delta\bar{\underset{\sim}{P}}_i = \underset{\sim}{K}^{(II)}\underset{\sim}{\Delta}_{i-1},\quad \delta\bar{\bar{\underset{\sim}{P}}}_i = \underset{\sim}{K}^{(III)}\underset{\sim}{\Delta}_{i-1}$$

so that the geometric stiffness matrix and the plastic tangent stiffness matrix serve as load vectors. Initially $\Delta\bar{\underset{\sim}{P}}_i = \Delta\bar{\bar{\underset{\sim}{P}}}_i = 0$, which reduces the problem to a simple linear elastic response analysis. Either the geometric nonlinearity or the material nonlinearity alone may be handled, or both effects can be incorporated. For only the geometric nonlinearity we calculate $\Delta\bar{\underset{\sim}{P}}_i$ based on displacements and stresses determined in the previous time step and direct iterations are continued until convergence. The analysis of elastoplastic response consists of calculating the equivalent yield stresses at each time increment to determine incremental plastic loads, $\Delta\bar{\bar{\underset{\sim}{P}}}_i$ satisfying the yield criteria and associated flow rule. Since the stresses vary through the shell thickness it is necessary to divide the thickness into a number of layers and check the status of yielding for each layer and for each element of the shell. All elements in which unloading takes place are determined at each time step and are excluded from the contribution to plastic loads. For a given applied load and given time increment, an updating of the plastic load vector continues with each process dependent on the previous histories until satisfactory convergence is reached. Once convergence is obtained, the current histories are carried over to the next time increment, and updating of

plastic loads is once again repeated until convergence.

In order to verify the theory and procedure described above, a spherical cap is subjected to an impulsive load of 1,000 psi of infinite duration, transverse to the middle surface. A yield stress of 24,000 psi and a plastic modulus of 21,000 psi are used. A total of five ring elements was used in this analysis. A time integration increment of 10^{-6} second produced a stable solution. The transverse response of the apex is shown in Fig. 5. It is seen that the elastoplastic responses are considerably higher than the elastic responses. Irregularities adjacent to peaks are regarded as a consequence of local yielding. The vibration nodes of the one-half side of the spherical cap at selected transient states are shown in Fig. 6. The yielded regions corresponding to these nodes are represented by the shaded area in Fig. 7. It is demonstrated that the present scheme is efficient in the elastoplastic dynamic snalysis.

Acknowledgement. Support of the work reported in this paper by the U.S. Air Force of Scientific Research under Contract No. F44620-69-C-0124 is gratefully acknowledged.

REFERENCES

[1] COLEMAN, B. D., "Thermodynamics of Materials with Memory," Archives for Rational Mechanics and Analysis, Vol. 17, pp. 1-46, (1964).

[2] ODEN, J. T., Finite Elements of Nonlinear Continua, McGraw-Hill Book Co., New York (1971).

[3] ODEN, J. T., ARMSTRONG, W. H., "Analysis of Nonlinear Dynamic Coupled Thermoviscoelasticity Problems by the Finite Element Method," Journal of Computers and Structures (to appear).

[4] COST, T. L., "Thermomechanical Coupling Phenomena in Non-Isothermal Viscoelastic Solids, Ph.D. Thesis, University of Alabama, University, (1969).

[5] ODEN, J. T., AGUIRRE-RAMIREZ, G., "Formulation of General Discrete Models of Thermomechanical Behavior of Materials with Memory," International Journal of Solids and Structures, Vol. 5, No. 10, pp. 1077-1093 (1969).

[6] ODEN, J. T., POE, J., "On the Numerical Solution of a Class of Problems in Dynamic Coupled Thermoelasticity," Proceedings, 5th SECTAM, Developments in Theoretical and Applied Mechanics, Ed. by D. Frederick, Pergamon Press, Oxford (1970).

[7] ODEN, J. T., "Finite Element Formulation of Problems of Finite Deformation and Irreversible Thermodynamics of Nonlinear Continua - A Survey and Extension of Recent Developments," Recent Advances in Matrix Methods of Structural Analysis and Design, Ed. by R. H. Gallagher, Y. Yamada, and J. T. Oden, University of Alabama Press, University, Alabama, pp. 693-724 (1971).

[8] ODEN, J. T., KEY, J. E., "Numerical Analysis of Finite Axisymmetric Deformations of Incompressible Elastic Solids of Revolution," International Journal of Solids and Structures, Vol. 6, pp. 497-518 (1970).

[9] ODEN, J. T., "Note on an Approximate Method for Computing Nonconservative Forces on Finitely Deformed Finite Elements," AIAA Journal, Vol. 8, No. 11, pp. 2088-2090 (1970).

[10] ODEN, J. T., KEY, J. E., "Analysis of Finite Deformations of Elastic Solids by the Finite Element Method," Proceedings, IUTAM Symposium High Speed Computing of Elastic Structures, Liege, Belgium (1970).

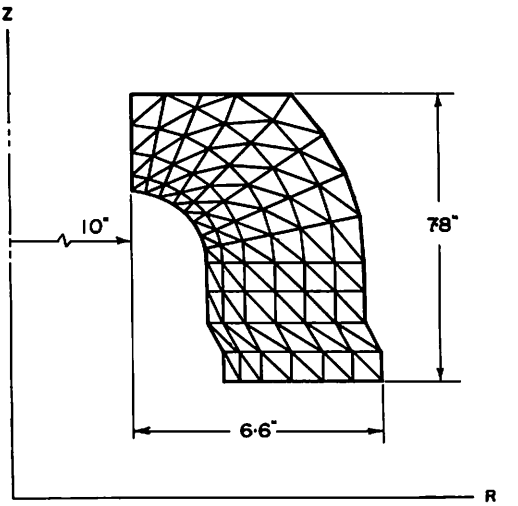


Figure 1. Finite element Model of Quadrant of Thermoelastic Solid of Revolution.

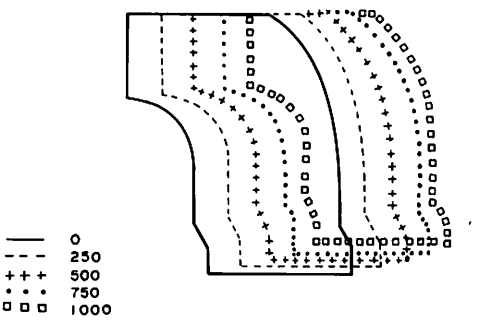
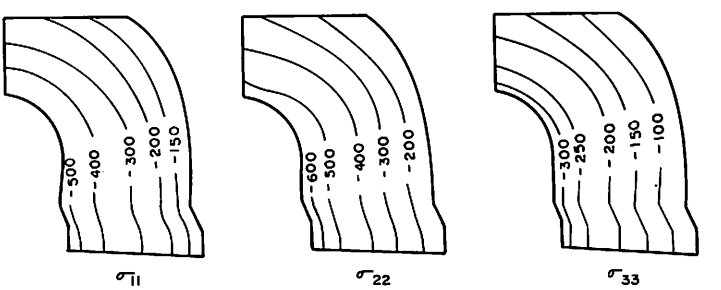


Figure 2(a). Displacement Profiles for Various Values of Internal Temperatures;



(b) Stress Contours for $T_{interior}$=1,000°F (stress in psi).

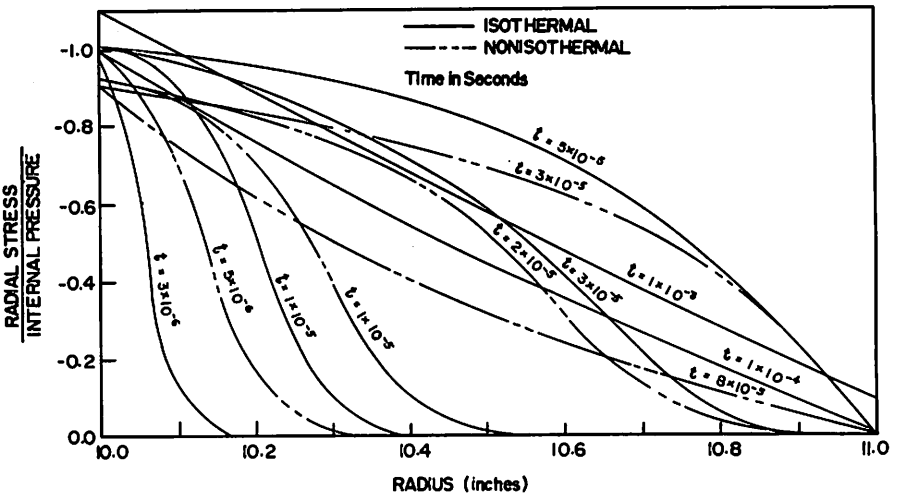


Figure 3. Radial Stress Waves in Thermorheologically Simple Cylinder under Internal Pressure

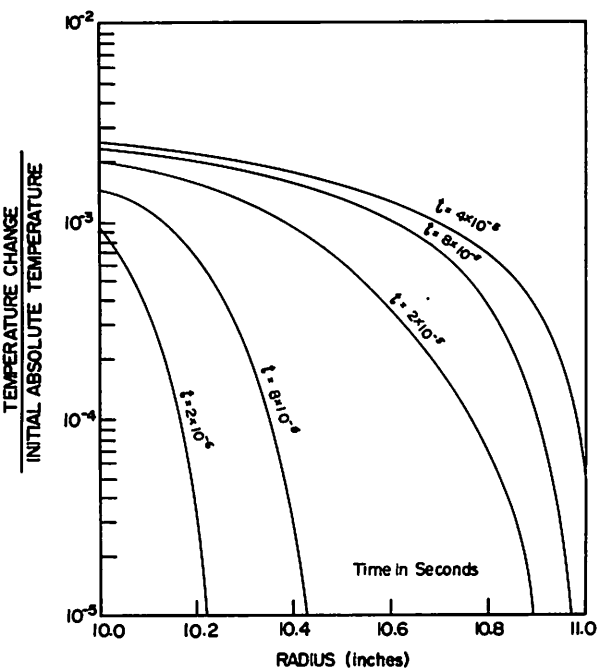


Figure 4. Nondimensional Temperature Distributions at Various Real Times.

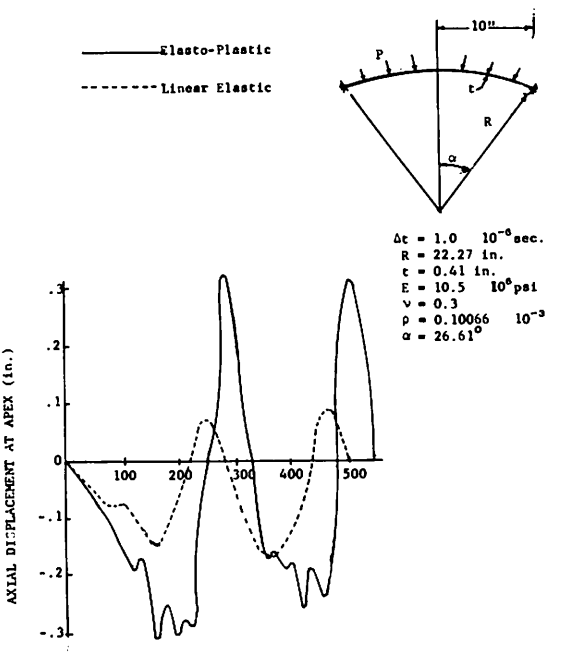


Figure 5. Comparison of Elasto-plastic and Linear Elastic Transverse Response at the Apex.

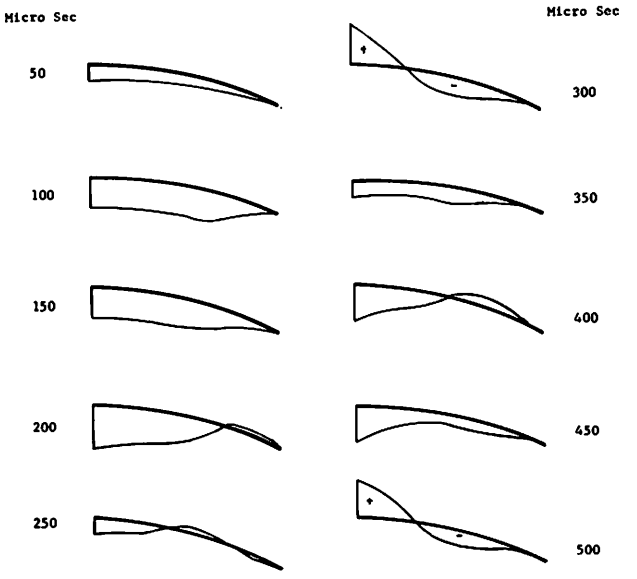


Figure 6. Various Vibration Modes of Half Structure for Problem in Fig. 5.

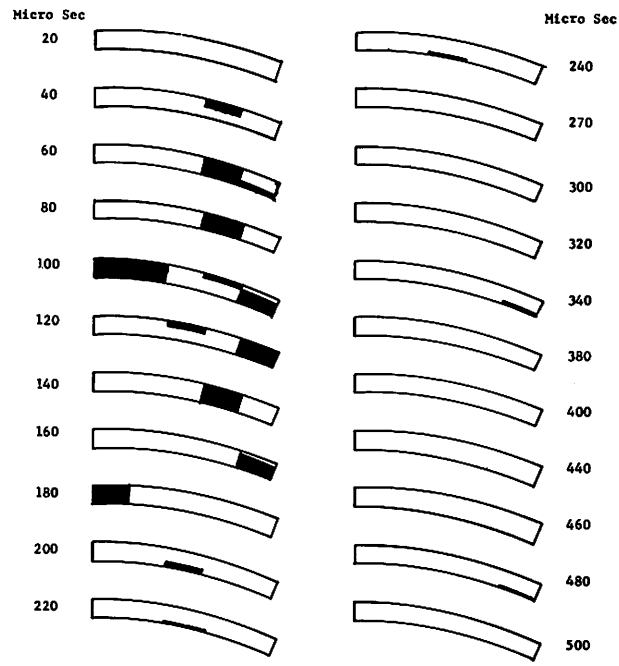


Figure 7. Yielded Regions of Spherical Cap in Fig. 5 (6 layers through thickness).