Abstract

SCHATZ, KYLAN. Fusion Categories: Quantum Symmetries and Categorical Obstruction Theory. (Under the Direction of Dr. Corey Jones).

A fusion category is a generalization of the representation category of a finite group. Due to their deep connection with topological quantum field theory (TQFT), fusion categories have a natural interpretation as quantum symmetries. In this work, we review the basic theory of fusion categories and some of their applications to the study of mathematical physics.

For a (2+1)D symmetry enriched topological order (SETO) described by a G-crossed extension of a modular tensor category, the preservation of bulk global onsite G-symmetry after anyon condensation requires that G-symmetry admit an equivariant algebra structure on the algebra corresponding to the condensed anyon. Phase transitions driven by anyon condensation can be represented spatially, and these condensation boundaries have their own internal topological symmetries corresponding to algebra endomorphisms. We give a description of these topological symmetries using the algebraic language of hypergroups, and investigate the coherence of bulk symmetries and boundary symmetries.

A fusion spin chain is generalization of an ordinary spin chain that arises naturally as the *boundary algebra* of a locally topologically ordered spin system. Fusion spin chains also arise as the subalgebra of operators which are symmetric under fusion categorical-symmetry, in which case they are said to be spatially realized by the input ordinary spin chain. Interestingly, there are equivalences of symmetric spin systems which cannot be detected at the level of spatial realizations; such an equivalence is called a categorical duality. We give an algebraic description of fusion spin chains and spatial realizations, and present an obstruction to the lifting of symmetric equivalences of symmetric spin systems in terms of DHR-bimodules.

Fusion Categories: Quantum Symmetries and Categorical Obstruction Theory.

by
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Biography

Kylan Schatz was born in Raleigh, North Carolina, to their parents, Bobbie and Rudi Schatz. Their early years were marked by challenges in forming connections with both peers and adult role models. However, high school became a turning point, where they developed meaningful relationships and discovered a deeper appreciation for mathematics. Inspired by the enthusiasm of their high school math teacher – a fellow graduate of the NC State mathematics program – Kylan's passion for the subject flourished, and they became increasingly invested in math pedagogy. Outside of the classroom, they were actively involved in student leadership and service, serving as an executive member of the student council and helping to kickstart a local tutoring program for elementary school students.

In 2016, Kylan began their undergraduate studies at NC State University, majoring in mathematics, later adding computer science as a second major, and Spanish and statistics as minors. They continued their commitment to service and community engagement as an executive member of both the College of Sciences Council and the Society for Undergraduate Mathematics, organizing outreach and social events across the Triangle area. In July 2019, Kylan's father, Rudi, died by suicide after years of struggle. In the wake of this loss, Kylan faced the compounding challenges of a global pandemic, unaddressed mental health struggles, grief, dysphoria, and job loss. Despite these hardships, they pursued graduate studies in mathematics at NC State, completing their first year of the Master's program entirely online. Seeking connection, they became actively involved in the Graduate Student Association and the American Mathematical Society chapter, finding support through academic and service communities.

Kylan later transitioned into the PhD program, quickly forming close bonds with their peers. They became actively involved in departmental service, joining the Diversity, Equity and Inclusion committee and helping to organize the Mathematics Teaching and Learning seminar. Beyond academics, they prioritized cultivating both personal well-being and community, developing a love for cooking, gardening, and rock climbing. Now, as they reach the culmination of their PhD, Kylan reflects with gratitude on the resilience, growth, and connections that have shaped their journey. They are the happiest they have ever been – having found both community and a deepened understanding of themself.

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Introduction

Mathematical Physics

Liquid water and ice are two distinct macroscopic manifestations of the same microscopic physical system; a free sea of water molecules can change suddenly into a crystalline lattice by freezing. The transition between these states can be *explained by observing microscopic physical symmetries*. An observer of a fluid could move in any direction and see the same configuration of molecules, while an observer of a crystal would need to take discrete steps along the lattice to preserve their viewpoint. As each translational symmetry of the solid is also a symmetry of the liquid, we say the system becomes less symmetric when it freezes. Such a change in a system to become more ordered and less symmetric is a physical phenomenon known as *symmetry breaking*.

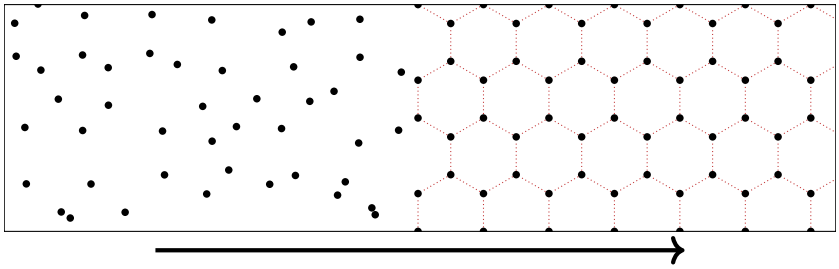


Figure 1: A system exhibiting symmetry breaking.

The approach for explaining phase transitions via the breaking of symmetry encompasses the *Landau paradigm*. Exotic substances which exhibit phase transitions not explained by symmetry breaking have only recently been discovered. One class of examples are the *topological phases of matter*, named in reference to their invariance under small local perturbation. The unique behaviors of these materials have brought advances to the world of superconductors and quantum computation [BBCW19; Del19] making an extension of the Landau

paradigm to explain topological phase transitions desirable.

Figure 2: The interactions of quasiparticles could be used to encode the behavior of a quantum computer which is inherently fault tolerant [Kit03].

The appropriate analogue of a classical symmetry to explain topological phase transitions is a *fusion category* [EGNO16], a type of *categorical symmetry* which generalizes the representation theory of a finite group. These symmetries are interesting algebraic objects with rich internal structure, the additional complexity of which has hindered progress towards a total classification. By studying the structure of categorical symmetries, it is possible to deduce physically necessary conditions on a system. Such a mathematical necessity imposed onto a physical system is known as a *categorical obstruction*.

Topological Orders and Their Symmetries

A *topological order* is a schema for assigning an appropriate categorical symmetry to a gapped topological phase of matter. There is an expected rough correspondence between $(n+1)$D gapped topological phases and *fusion n-categories* representing bulk excitations [Joh22]. Associated to a (2+1)D gapped topological phase is a nondegenerate fusion 2-category which can be identified with a *modular tensor category*, the objects of which represent quasiparticles known as *anyons*.

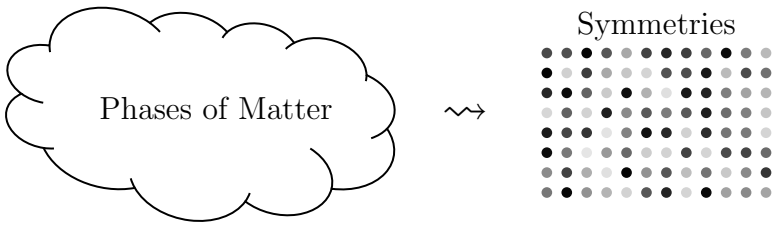


Figure 3: An order assigns a symmetry to a phase of matter in a way which is coarse.

Certain anyons can initiate a topological phase transition through the process of *anyon*

condensation [Bur18]. Such a phase transition can be spatially represented as a boundary. Condensable anyons are given by connected, separable, *commutative algebra objects* internal to the bulk modular tensor category [Kon14]. In the unitary case, *commutative Q-system objects* [CPJP22] are instead considered. Excitations on the condensation boundary are described by *right modules* over the algebra associated to that anyon. The categorical obstruction associated to this phenomenon has been studied: an action must admit *equivariant structure* on the associated algebra which is coherent with the structure of that algebra [BJLP19].

The boundary has its own internal topological symmetries, namely *string operator symmetries*. These symmetries have the structure of a *hypergroup* [SW03] and holographically represent the dual string-net model; associated to each string operator symmetry is an *algebra endomorphism* on the algebra representing the boundary. The endomorphism ring can be equipped with a *convolution product* [BD20], a generalization of the convolution of finite group characters. Such an operation gives $\mathrm{End}(A)$ the structure of a hypergroup which can be naturally identified with the hypergroup of boundary operators.

Associated to a gapped boundary to the vacuum phase is a *Lagrangian algebra*, and its endomorphism hypergroup can be identified with the fusion ring of the boundary theory in a canonical way [BJ22]. As a consequence, there is an induced action of each full fusion subcategory of the boundary theory on the associated Lagrangian algebra object.

String Nets and QCA

Much of the algebraic study of (2+1)D topologically ordered phases of matter can be described using *string-nets*. Arising from the operators on the boundary of a string-net model is a *fusion categorical spin chain* [JNPW23]. Fusion spin chains arise from the study of symmetric classical spin chains; given a fusion categorical symmetry on a spin chain, its fixed points define a fusion spin chain which is *spatially represented* by the input classical spin chain. On a spin chain one can define a local discrete time dynamics called *quantum cellular automata (QCA)*, and states of the model are equivalent in some meaningful way if they are related by QCA.

Outline of Thesis

In Chapter 1, we present some preliminaries on the theory of fusion categories and categorical algebra.

In Chapter 2, we investigate the boundary symmetries of (2+1)D topological orders. We answer the naturally arising question of when the induced actions on the boundary are coherent with the preservation of symmetry under transition driven by anyon condensation. We give a concrete formulation of these symmetries and the associated obstruction, giving a classification in the case of condensation to the trivial phase. The content of this Chapter is adapted from [Sch24].

In Chapter 3, we investigate fusion categorical symmetries on spin chains. We give a formulation for *categorical duality* of spin chains, motivated by the classical example of Kramers-Wannier duality. Lastly, we present an obstruction to the lifting of fusion symmetric QCA, and give a description of realizable group symmetric QCA. The content of this Chapter is adapted from [JSW24] with permission from the coauthors.

Bibliographic Notes

Some of the first appearances of monoidal categories in the mathematical literature are credited to Bénabou, Mac Lane, Eilenberg, and Kelly [EK65] in the 1960s [Bén63; Lan63; EK65]. This work was essentially contemporaneous with the initial study of symmetric monoidal categories. Braided monoidal categories were first studied in generality by Joyal and Street in the 1980s [JS85; JS93].

Linear, rigid monoidal categories (tensor categories) gained favor in the 1980s due to their utility as a tool for mathematical physics. Some early names in the study of tensor categories are Saavedra-Rivano, Kelly, Deligne, and Milne [Saa72; Kel64; DM82]. Fusion categories became increasingly popular in the early 2000s, some early names in the field being Ocneanu (unpublished), Etingof, Nikshych, Ostrik, Calaque, Müger, Drinfeld, and Gelaki [ENO05; CE08; Müg10].

Topological quantum field theory (TQFT) has its first origins in string theory, with

Edward Witten being a founding member for the "cohomological" study of TQFT [Wit88]. Functorial TQFT was first axiomatized by Michael Atiyah in the 1980s [Ati88]. Modular tensor categories were first introduced by Turaev [Tur92], and their relation with TQFT is reviewed in the early work by Bojko and Kirrilov [BK00].

Some other works which review the connection between mathematical physics and tensor categories include those by Witten, Baez, Lauda, Coeke, Kassel, Moore, Seiberg, and Delaney [Wit09; Bae06; BL11; Coe10; Kas95; MS89; Del19].

Chapter 1

Preliminaries

1.1 Categories

A *category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ ([Rie16, Definition 1.1.1], [Mit65, Section 1.1]) is a class of *objects* $\mathrm{Ob}(\mathcal{C})$ together with a class of *morphisms* $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{Mor}_{\mathcal{C}}(X, Y) : X, Y \in \mathrm{Ob}(\mathcal{C})\}$ such that:

- Composition of morphisms is well defined, i.e.: for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and for any $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Y, Z)$, there exists $(g \circ f) \in \mathrm{Mor}(X, Z)$,
- Identity morphisms exist, i.e.: for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_Y \in \mathrm{Mor}(Y, Y)$ such that for any $f \in \mathrm{Mor}(Y, Z)$ and $g \in \mathrm{Mor}(X, Y)$, $f \circ \mathrm{id}_Y = f$ and $\mathrm{id}_Y \circ g = g$.
- Composition of morphisms is associative, i.e.: for all $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and for any $f \in \mathrm{Mor}(X, Y)$, $g \in \mathrm{Mor}(Y, Z)$, and $h \in \mathrm{Mor}(Z, W)$, $(h \circ g) \circ f = h \circ (g \circ f)$.

As opposed to classical algebra, in a category it makes more sense to study *isomorphisms of objects* and *equality of morphisms*. Equivalence of morphisms can be studied algebraically or graphically using *commutative diagrams*, i.e. diagrams of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \curvearrowright & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[k]{} & W \end{array} \qquad \Longleftrightarrow \qquad h \circ f = k \circ g; \tag{1.1}$$

the left side is read 'the diagram commutes.' Constructing large commutative diagrams gives a "topological" method for equating morphisms in a category. In vague terms: we may

interpret a commutative diagram as a sort of cell-complex in which, by the above axiomatics, homotopy classes of paths give equivalence classes of maps.

An *isomorphism* ([Rie16, Definition 1.1.9] [Mit65, Section 1.5]) is a morphism with both left and right inverses. That is, it is $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ such that there exist $f', f'' \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ such that:

$$f' \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f'' = \mathrm{id}_Y \,. \tag{1.2}$$

A *monomorphism (monic morphism)* ([Rie16, Definition 1.2.7], [Mit65, Section 1.5]) is a categorical analogue of injection: it is a morphism $m \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ such that for any $g_1, g_2 \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$:

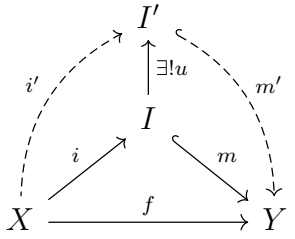
$$m \circ g_1 = m \circ g_2 \implies g_1 = g_2. \tag{1.3}$$

Similarly, an *epimorphism (epic morphism)* is a categorical analogue of surjection: it is a morphism $e \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ such that for any $g_1, g_2 \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$:

$$g_1 \circ e = g_2 \circ e \implies g_1 = g_2. \tag{1.4}$$

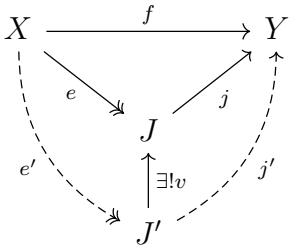
Remark 1.1. It is clear that every isomorphism is both monic and epic. However, the converse is not generically true: a morphism being both monic and epic *does not* imply that it is an isomorphism.

We may extend the classical algebraic notion of "sub-objects" to category theory by describing objects in context *together with special morphisms*. For example, given $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, the *image of* f [Mit65, Section 1.10] is a pair $\mathrm{Im}(f) := (I, m)$ of an object and a monomorphism $m : I \to Y$ such that there exists unique morphism $i : X \to I$ with $f = m \circ i$, which is universal in the sense: for any other object I' and monomorphism $m' : I' \to Y$ for which $f = m' \circ i'$, there is unique $u : I \to I'$ for which $m = m' \circ u$:



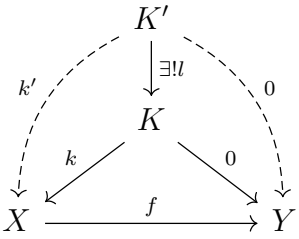
(1.5)

The *coimage of* f is a pair $\mathrm{Coim}(f) := (J, e)$ of an object and an epimorphism $e : X \to J$ such that there exists unique morphism $j : J \to Y$ with $f = j \circ e$, and is universal in the sense: for any other object J' and epimorphism $e' : X' \to J$ for which $f = j' \circ e'$, there is unique $v : J' \to J$ for which $e = v \circ e'$:



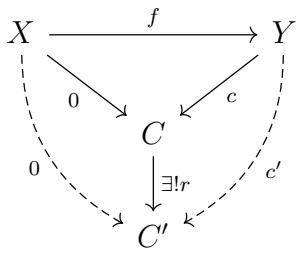
(1.6)

The *kernel of* f ([EGNO16, Section 1.3], [Mit65, Section 1.13]) is a pair $\mathrm{Ker}(f) := (K, k)$ of an object and a morphism $k : K \to X$ such that $f \circ k = 0$ that is universal in the sense: if $k' : K' \to X$ such that $f \circ k' = 0$, then there is unique $l : K' \to K$ such that $kl = k'$:



(1.7)

Similarly, the *cokernel of* f is a pair $\mathrm{Coker}(f) := (C, c)$ of an object and a morphism $c : Y \to C$ such that $c \circ f = 0$ that is universal in the sense: if $c' : Y \to C'$ such that $c' \circ f = 0$, then there is unique $r : C \to C'$ such that $rc = c'$:



. (1.8)

Every kernel is necessarily a monomorphism and every cokernel is necessarily an epimorphism [Rie16]. An *Abelian category* ([EGNO16, Definition 1.3.1], [Rie16, Definition E.5.1], [Mit65, Section 1.20]) is a category with direct sums and a zero object, with all kernels and cokernels such that every morphism has a canonical decomposition, that is: for every $\varphi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ there exists an exact sequence:

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C \ , \tag{1.9}$$

such that:

$$ji = \varphi, \qquad (K, k) = \mathrm{Ker}(\varphi), \quad (C, c) = \mathrm{Coker}(\varphi), \tag{1.10}$$

$$\text{and} \qquad (I, i) = \mathrm{Coker}(k), \quad (I, j) = \mathrm{Ker}(c). \tag{1.11}$$

A (nonzero) object X in an Abelian category $\mathcal{C}$ is called *simple* [EGNO16, Definition 1.5.1] if 0 and X are its only subobjects. An object X is called *semisimple* if it is a direct sum of simple objects. The category $\mathcal{C}$ is called *semisimple* if every object in $\mathcal{C}$ is semisimple. The category $\mathcal{C}$ is instead called *finitely semisimple* if each object can be written as a finite sum of simple objects.

A *linear category over field* $\Bbbk$ [EGNO16, Definition 1.2.2] is a category $\mathcal{C}$ with direct sums, a zero object, and whose morphisms have the structure of vector spaces denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and composition is $\Bbbk$-linear. When there is no risk of ambiguity, we write "linear" instead of "$\Bbbk$-linear."

A linear Abelian category is said to be *finite* [EGNO16, Definition 1.8.6] if its morphism spaces are finite dimensional, there are finitely many isomorphism classes of simple objects, objects have finite length, and there are "enough projectives."

A *functor* $F : \mathcal{C} \to \mathcal{D}$ ([Rie16, Definition 1.3.1], [Mit65, Section 2.3]) is a tuple of maps which takes $\mathrm{Ob}(\mathcal{C}) \ni X \mapsto F(X) \in \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}_{\mathcal{C}}(X, Y) \ni f \mapsto F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ in a way that is *functorial*, i.e.: $F(g \circ f) = F(g) \circ F(f)$, and which sends identities to identities.

A *linear functor* [EGNO16, Definition 1.2.3] is a functor $F : \mathcal{C} \to \mathcal{D}$ between linear categories for which the maps of morphism spaces:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \tag{1.12}$$

are linear transformations.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\eta : F \to G$ ([Rie16, Definition 1.4.1], [Mit65, Section 2.9]) is a class of morphisms:

$$\eta = \{\eta_X : F(X) \to G(X)\}$$

such that for any two objects X, Y and morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & \curvearrowright & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array} . \tag{1.13}$$

A *natural isomorphism* is a natural transformation whose component morphisms are all isomorphism.

Let $F : \mathcal{C} \to \mathcal{D}$ a functor. A functor $G : \mathcal{D} \to \mathcal{C}$ is a *(left) adjoint to* F, denoted $G \dashv F$, if:

$$\mathrm{Mor}_{\mathcal{C}}(G(\_), \_) \cong \mathrm{Mor}_{\mathcal{D}}(\_, F(\_)) \text{ are naturally isomorphic.}$$

1.1.1 Fusion Categories

Definition 1.2 ([EGNO16, Chapter 2], [Rie16, Section E.2])**.** A *monoidal category* is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, \iota, \alpha)$ of a category $\mathcal{C}$ together with:

- bilinear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal product*,
- *unit* $(\mathbb{1}, \iota)$ for which $\iota : \mathbb{1} \times \mathbb{1} \xrightarrow{\sim} \mathbb{1}$ and $\_ \mapsto \mathbb{1} \otimes \_$ and $\_ \mapsto \_ \otimes \mathbb{1}$ are autoequivalences,
- and *associator* natural isomorphism $\alpha : (\_ \otimes \_) \otimes \_ \xrightarrow{\sim} \_ \otimes (\_ \otimes \_)$ satisfying the pentagon axiom:

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) \otimes W & \\ {\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} \swarrow & & \searrow {\scriptstyle \alpha_{X \otimes Y, Z, W}} \\ (X \otimes (Y \otimes Z)) \otimes W & \curvearrowright & (X \otimes Y) \otimes (Z \otimes W) \\ {\scriptstyle \alpha_{X, Y \otimes Z, W}} \searrow & & \swarrow {\scriptstyle \alpha_{X,Y,Z \otimes W}} \\ X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow[\mathrm{id}_X \otimes \alpha_{Y,Z,W}]{} & X \otimes (Y \otimes (Z \otimes W)) \end{array} \cdot \tag{1.14}$$

Example 1.3. The category **Set** of sets is monoidal. The monoidal product is the Cartesian product of sets. The associator is given by the set map:

$$((a, b), c) \mapsto (a, (b, c)). \tag{1.15}$$

The unit object is the distinguished singleton $\{\cdot\}$, and the unitor identifies $(\cdot, \cdot) \to (\cdot)$.

A *monoidal functor* is a tuple $(F, J) : \mathcal{C} \to \mathcal{D}$ of a functor F between and a natural isomorphism:

$$J = \left\{ J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \right\}$$

for which the following diagram commutes for every $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{array}{ccc} (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\ {\scriptstyle J_{X,Y} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes J_{Y,Z}} \\ F(X \otimes Y) \otimes F(Z) & \curvearrowright & F(X) \otimes F(Y \otimes Z) \\ {\scriptstyle F_{X \otimes Y, Z}} \downarrow & & \downarrow {\scriptstyle F_{X, Y \otimes Z}} \\ F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z)) \end{array} \cdot \tag{1.16}$$

A *monoidal natural transformation* $\eta : (F, J) \to (G, K)$ is a natural transformation $\eta : F \to G$ such that $\eta_{\mathbb{1}}$ is an isomorphism and for which the following diagram commutes:

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{J_{X,Y}} & F(X\otimes Y) \\ \downarrow{\scriptstyle \eta_X\otimes\eta_Y} & \curvearrowright & \downarrow{\scriptstyle \eta_{X\otimes Y}} \\ G(X)\otimes G(Y) & \xrightarrow{K_{X,Y}} & G(X\otimes Y) \end{array} . \tag{1.17}$$

A *monoidal natural isomorphism* is a monoidal natural transformation that is a natural isomorphism.

Example 1.4. The category **Grp** of groups is monoidal, and there is a canonical forgetful monoidal functor from **Grp** into **Set**.

Example 1.5. Let M a monoid. Then, the category $\mathbf{Set}(M)$ of M-graded sets with monoidal product:

$$\left(\bigoplus_{i\in M} X_i\right) \bigotimes \left(\bigoplus_{j\in M} Y_j\right) := \bigoplus_{ij=k} (X_i \times Y_j) \tag{1.18}$$

is monoidal. There is a canonical forgetful monoidal functor into **Set**.

Remark 1.6. A monoidal category is the *vertical categorification* of a monoid; objects have a monoid-like product and equality has been replaced with isomorphism. Conversely, by taking the decategorification of a small monoidal category (a set theoretic interpretation of isomorphism classes), we obtain a monoid. For example, **Set** is a categorification of the monoid of ordinals together with multiplication.

Let X an object in a monoidal category $\mathcal{C}$. A *left dual to* X [EGNO16, Section 2.10] is an object X^* for which there exist morphisms $\mathrm{ev}_X : X^*\otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X\otimes X^*$ for which the following compositions are identity:

$$X \xlongequal{\sim} \mathbb{1}\otimes X \xrightarrow{\mathrm{coev}_X\otimes \mathrm{id}} (X\otimes X^*)\otimes X \xrightarrow{\alpha} X\otimes(X^*\otimes X) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}_X} X\otimes\mathbb{1} \xlongequal{\sim} X \ , \tag{1.19}$$

$$X^* \xlongequal{\sim} X^*\otimes\mathbb{1} \xrightarrow{\mathrm{id}\otimes \mathrm{coev}_X} X^*\otimes(X\otimes X^*) \xrightarrow{\alpha^{-1}} (X^*\otimes X)\otimes X^* \xrightarrow{\mathrm{ev}_X\otimes \mathrm{id}} \mathbb{1}\otimes X^* \xlongequal{\sim} X^* \ . \tag{1.20}$$

Similarly, one can define a *right dual* object. The requirements that compositions (1.19) and (1.20) are identity are called the *triangle* or *zig-zag axioms*. A monoidal category is called *rigid* if every object has left and right duals.

Definition 1.7. A *tensor category over field* $\Bbbk$ is a $\Bbbk$-linear Abelian monoidal category with simple unit and for which the monoidal product is $\Bbbk$-bilinear.

Example 1.8. The categories $\mathbf{Vec}_\Bbbk$ of $\Bbbk$-vector spaces and $\mathbf{Rep}_\Bbbk(G)$ of $\Bbbk$-representations of group G are tensor. For each, there is a canonical forgetful monoidal functor into **Set**.

Definition 1.9 ([EGNO16, Definition 4.1.1])**.** A *fusion category* is a finite rigid finitely semisimple tensor category. That is, it is a finite finitely semisimple linear Abelian rigid monoidal category with simple unit.

Example 1.10. The category $\mathrm{Vec}_\Bbbk$ of finite dimensional $\Bbbk$-vector spaces is fusion with unit $\mathbb{1} = \Bbbk$. Direct sums, kernels, cokernels, tensor products, and duals are defined in the usual sense. There is one simple object up to isomorphism, $\Bbbk$. The associator is trivial.

Example 1.11. Let G a finite group and $\omega \in Z^3(G, \Bbbk^*)$ a 3-cochain, i.e:

$$\omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) = \omega(g_1, g_2, g_3g_4)\omega(g_1g_2, g_3, g_4). \tag{1.21}$$

The category $\mathrm{Vec}_\Bbbk(G, \omega)$ of G-graded vector spaces with G-graded linear transformations is fusion. There is one simple object for each graded component, denoted δ_g. The monoidal product is defined by extending $\delta_g \otimes \delta_h = \delta_{gh}$ linearly. The associator is given by:

$$\omega(g, h, k)\,\mathrm{id}_{\delta_{ghk}} : (\delta_g \otimes \delta_h) \otimes \delta_k \xrightarrow{\sim} \delta_g \otimes (\delta_h \otimes \delta_k). \tag{1.22}$$

Example 1.12. Let X be either a finite group, a matrix algebra, or a finite dimensional simple Lie algebra. Let $\Bbbk$ be an algebraically closed field of characteristic zero. The category $\mathrm{Rep}_\Bbbk(X)$ of finite dimensional representations of X is fusion. The associator is trivial.

Definition 1.13. Let $\mathcal{C}$ a fusion category and $X \in \mathrm{Ob}(\mathcal{C})$. The *Frobenius-Perron dimension of* X is defined by:

$$\mathrm{FPdim}(X)^2 := \dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes X^*). \tag{1.23}$$

Definition 1.14 ([JP17, Section 2])**.** A *unitary fusion category* is a C*-fusion category. That is:

- it is a fusion category: a finite rigid finitely-semisimple tensor category,
- it is a *dagger tensor category*, meaning:
  - it is a tensor category equipped with a *dagger structure*

$$\dagger : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

    which is an involutive anti-isomorphism, i.e. $f^{\dagger\dagger} = f$, $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$,
  - whose monoidal product is a dagger functor $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and
  - whose associator and unitor natural isomorphisms are unitary, and
- it is *C*-tensor*, or equivalently $\mathrm{Hom}_{\mathcal{C}}(X, X)$ is a C*-algebra for each $X \in \mathrm{Ob}(\mathcal{C})$, [GLR85].

Remark 1.15. One should think of a C*-category as the horizontal categorification of a C*-algebra; that fusion categories over $\mathbb{C}$ are finite implies that each morphism space is a finite C*-algebra, i.e. of the form $\mathbb{C}^d$.

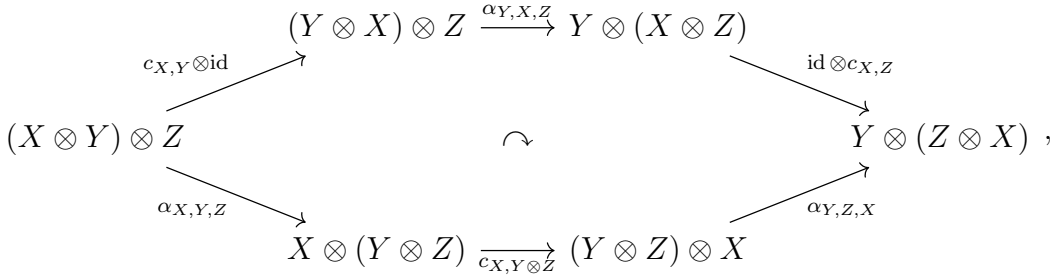
Example 1.16. The category Hilb of finite dimensional Hilbert spaces over $\mathbb{C}$ is a unitary fusion category.

1.1.2 Modular Tensor Categories

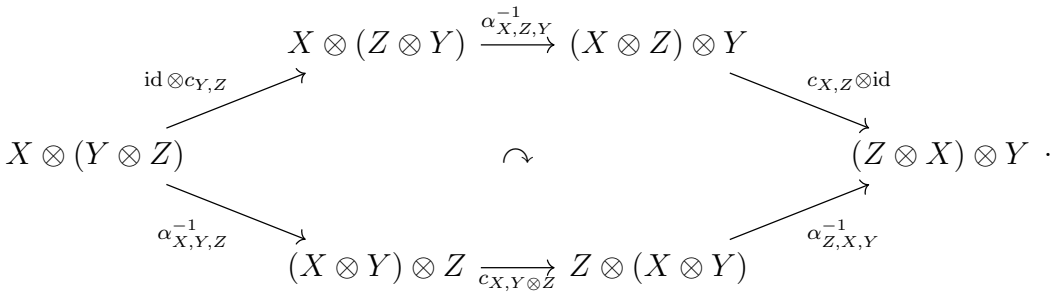
Definition 1.17 ([EGNO16, Section 8.1])**.** A *braided* monoidal category is a monoidal category equipped with natural isomorphism:

$$c = \{c_{X,Y} : X \otimes Y \to Y \otimes X\}$$

called a *braiding*, and for which the *hexagon identities* are true:



(1.24)



(1.25)

A braided monoidal category is called *symmetric* if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Example 1.18. The categories **Set** and **Grp** are braided monoidal when equipped with the trivial braiding.

Example 1.19. When M is a commutative monoid, $\mathbf{Set}(M)$ is braided with the trivial braiding.

Definition 1.20. A *braided tensor category* is both a tensor category and a braided monoidal category whose braiding is bilinear. A *braided fusion category* is a finite rigid finitely-semisimple braided tensor category.

Example 1.21. $\mathbf{Vec}_{\Bbbk}$ and $\mathbf{Rep}_{\Bbbk}(G)$ are braided tensor when equipped with the trivial braiding.

Example 1.22. Let G a finite Abelian group. $\mathrm{Vec}_{\Bbbk}(G)$ is a symmetric braided tensor category when equipped with the trivial braiding

Example 1.23. Let G a finite Abelian group, and $(\omega, c) \in Z^3_{ab}(G, \Bbbk^*)$ an Abelian 3-cocycle,

i.e.:

$$\omega(g_2,g_3,g_4)\omega(g_1,g_2g_3,g_4)\omega(g_1,g_2,g_3)=\omega(g_1,g_2,g_3g_4)\omega(g_1g_2,g_3,g_4),$$
$$\omega(g_2,g_3,g_1)c(g_1,g_2g_3)\omega(g_1,g_2,g_3)=c(g_1,g_3)\omega(g_2,g_1,g_3)c(g_1,g_2),$$
$$\omega^{-1}(g_3,g_1,g_2)c(g_1,g_2g_3)\omega^{-1}(g_1,g_2,g_3)=c(g_1,g_3)\omega^{-1}(g_1,g_3,g_2)c(g_2,g_3).$$

Then, $\mathrm{Vec}_{\Bbbk}(G,(\omega,c))$ is braided fusion. The braiding is given by:

$$c(g,h)\,\mathrm{id}_{\delta_{gh}}:\delta_h\otimes\delta_h\xrightarrow{\sim}\delta_h\otimes\delta_g. \tag{1.26}$$

Definition 1.24 ([EGNO16, Sections 4.7, 7.21])**.** A *pivotal fusion category* is a fusion category equipped with a *pivotal structure*; that is, a tensor autoequivalence $\delta:\_\xrightarrow{\sim}(\_)^{**}$ for which:

$$\delta_{X^*}=(\delta_X^*)^{-1}. \tag{1.27}$$

For $\mathcal{C}$ a pivotal fusion category and $f\in\mathrm{Mor}_{\mathcal{C}}(X,X)$, the *categorical trace of* f is defined to be:

$$\mathrm{tr}_\delta f:=\mathrm{ev}_{X^*}\circ(\delta_X\otimes\mathrm{id})\circ(f\otimes\mathrm{id})\circ\mathrm{coev}_X, \tag{1.28}$$

interpreted as a scalar in the base field. The *categorical dimension* X, $\dim_{\mathcal{C}}(X)$, is defined by:

$$\dim_\delta X:=\mathrm{tr}_\delta\,\mathrm{id}_X, \tag{1.29}$$

interpreted as a scalar in the base field. The pivotal structure/category is instead called *spherical* if:

$$\dim_\delta X=\dim_\delta X^*. \tag{1.30}$$

The *categorical dimension of* $\mathcal{C}$ is:

$$\dim_\delta \mathcal{C} = \sum_{X \in \mathrm{Irr}\mathcal{C}} \dim_\delta X \dim_\delta X^*. \tag{1.31}$$

Every fusion category over an algebraically closed field admits a spherical structure. Further, every known example fusion category admits a spherical structure. However, it is unknown whether every fusion category admits a spherical or even pivotal structure.

Definition 1.25 ([BK00],[EGNO16, Chapter 8])**.** A *ribbon fusion category* is a spherical braided fusion category. A *modular tensor category* is a ribbon fusion category whose braiding is nondegenerate, i.e. the matrix:

$$\left(\mathrm{tr}(c_{X_j,X_i} \circ c_{X_i,X_j})\right)_{i,j} \tag{1.32}$$

is invertible.

Example 1.26. The category $\mathrm{Vec}_{\Bbbk}(G, (\omega, c))$ is modular if and only if c is nondegenerate, i.e. when its trace quadratic form:

$$q(x) = \mathrm{tr}(c)(x) := c(x, x) \tag{1.33}$$

is nondegenerate.

Example 1.27. Let $\mathfrak{g}$ a simple Lie algebra, and denote its Cartan subalgebra by $\mathfrak{h}$, its root system by Δ, and its weight lattice by P. Denote by $\langle \cdot, \cdot \rangle$ the unique nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ for which the shortest root has squared length 2. The *fundamental Weyl chamber for* $\mathfrak{g}$ is:

$$\Lambda := \{\lambda \in P : \langle \lambda, \alpha \rangle > 0 \text{ for all positive } \alpha \in \Delta\}.$$

Now, let k a positive integer and denote the longest root by θ. The *Weyl alcove for* $\mathfrak{g}$ *at*

level k is:

$$\Lambda_0 := \{\lambda \in \Lambda : \langle \lambda, \theta \rangle \leq k\}.$$

Define the *category of tilting modules of* $\mathfrak{g}$ *at positive integer level* k:

$$\mathcal{C}(\mathfrak{g}, k) := \{\text{Weyl modules with weights inside } \Lambda_0\}/\{\text{Negligible morphisms}\}.$$

These categories $\mathcal{C}(\mathfrak{g}, k)$ are modular for every positive integer k.

Definition 1.28 ([Müg03, Section 3])**.** Let $\mathcal{C}$ a tensor category. The *Drinfeld center of* $\mathcal{C}$, $\mathcal{Z}(\mathcal{C})$, is the category with objects (X, ψ) where $X \in \mathrm{Ob}(\mathcal{C})$ and:

$$\psi = \left\{\psi_Y : X \otimes Y \xrightarrow{\sim} Y \otimes X\right\}$$

is a natural isomorphism satisfying the *hexagon equations* (1.24) and (1.25) called a *half braiding*, and with morphisms:

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \psi), (Y, \varphi)) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) : (f \otimes \mathrm{id}) \circ \psi = \varphi \circ (\mathrm{id} \otimes f)\}. \tag{1.34}$$

Remark 1.29. In contrast to a braiding, the half-braiding ψ need only be natural in one coordinate. However, the restriction on morphisms gives $\mathcal{Z}(\mathcal{C})$ the natural structure of a braided tensor category.

Theorem 1.30 ([Müg03, Theorem 1.2], [EGNO16, Corollary 8.20.14])**.** *Let* $\mathcal{C}$ *a spherical fusion category of nonzero dimension over an algebraically closed field of characteristic zero. Then, its center* $\mathcal{Z}(\mathcal{C})$ *is modular.*

The center of a fusion category is accessible through *induction*. That is, usually the image of the adjoint to the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is enough to understand the structure of $\mathcal{Z}(\mathcal{C})$.

1.1.3 Categorical Algebra

Definition 1.31 ([EGNO16, Definition 7.8.1], [BN11, Section 5.1])**.** Let $\mathcal{C}$ a tensor category. An *algebra object in* $\mathcal{C}$ is a pair (A, ι, μ) of an object $A \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\iota : \mathbb{1} \to A$, $\mu : A \otimes A \to A$ known as the unit and multiplication, for which the following diagrams commute:

$$\begin{array}{ccccc} \mathbb{1} \otimes A & \xleftarrow{l_A} & A & \xrightarrow{r_A} & A \otimes \mathbb{1} \\ {\scriptstyle \iota \otimes \mathrm{id}}\downarrow & \curvearrowright & \downarrow{\scriptstyle \mathrm{id}} & \curvearrowright & \downarrow{\scriptstyle \mathrm{id} \otimes \iota} \\ A \otimes A & \xrightarrow[\mu]{} & A & \xleftarrow[\mu]{} & A \otimes A \end{array}, \qquad \begin{array}{ccccc} (A \otimes A) \otimes A & & \xrightarrow{\alpha_{A,A,A}} & & A \otimes (A \otimes A) \\ {\scriptstyle \mu \otimes \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id} \otimes \mu} \\ A \otimes A & & \curvearrowright & & A \otimes A \\ & {\scriptstyle \mu}\searrow & & \swarrow{\scriptstyle \mu} & \\ & & A & & \end{array} . \tag{1.35}$$

Suppose $\mathcal{C}$ is additionally a braided tensor category with braiding c. The algebra (A, ι, μ) is called *commutative* if (ι and μ are morphisms in the braided category and):

$$\mu = \mu \circ c_{A,A}. \tag{1.36}$$

Definition 1.32 ([EGNO16, Definition 7.8.5], [BN11, Section 5.1])**.** Let (A, ι, μ) an algebra object in $\mathcal{C}$. A *right A-module* is an object (X, σ) where $\sigma : X \otimes A \to X$ for which the following diagram commutes:

$$\begin{array}{ccccc} (X \otimes A) \otimes A & & \xrightarrow{\alpha} & & X \otimes (A \otimes A) \\ {\scriptstyle \sigma \otimes \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id} \otimes \mu} \\ X \otimes A & & \curvearrowright & & X \otimes A \\ & {\scriptstyle \sigma}\searrow & & \swarrow{\scriptstyle \sigma} & \\ & & X & & \end{array} . \tag{1.37}$$

The *category of right A-modules*, $\mathcal{C}_A$, has A-modules as objects and morphisms:

$$\mathrm{Hom}_{\mathcal{C}_A}((X, \sigma), (Y, \tau)) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) : f \circ \sigma = \tau \circ (f \otimes \mathrm{id}) \} . \tag{1.38}$$

When $\mathcal{C}$ is braided with braiding c, define the tensor product of modules as an object:

$$(X \otimes_A Y, \pi) := \mathrm{Coker}(\sigma \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes (\tau \circ c_{A,Y})). \tag{1.39}$$

The tensor product is a module with product τ' defined by the commutativity of the diagram:

$$\begin{array}{ccccc} (X \otimes Y) \otimes A & \xrightarrow{\alpha_{X,Y,A}} & X \otimes (Y \otimes A) & \xrightarrow{\mathrm{id} \otimes \tau} & X \otimes Y \\ & \searrow^{\pi \otimes \mathrm{id}} & \curvearrowright & \swarrow_{\pi} & \\ & (X \otimes_A Y) \otimes A & \xrightarrow[\tau']{} & X \otimes_A Y & \end{array} \quad . \tag{1.40}$$

1.1.4 Categorical Groups

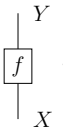
Definition 1.33 ([EGNO16, Definition 2.11.4])**.** A *categorical group* is a rigid monoidal category in which every morphism is isomorphism.

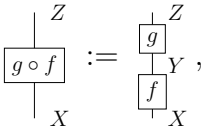
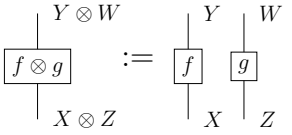
Example 1.34. Let $\mathcal{C}$ a monoidal category. The subcategory of objects with left and right duals for whom evaluation and coevaluation are isomorphism together with the subset of invertible morphisms is a categorical group, sometimes called the *monoidal core*.

Example 1.35. Let $\mathcal{C}$ a category. The category $\underline{\mathrm{Aut}}(\mathcal{C})$ whose objects are autoequivalences of $\mathcal{C}$ and whose morphisms are natural isomorphisms is a categorical group. Similarly, one can define the categorical groups $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$ and $\underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{C})$ of monoidal and braided monoidal autoequivalences.

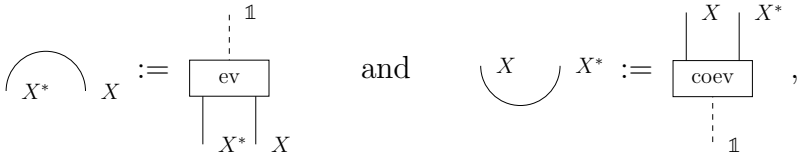
1.2 The Graphical Calculus

In a monoidal category, we can use graphical methods to represent and perform computations on morphisms. One such representation is the *(bottom-up/optimistic) graphical calculus* [Coe10; BK00]: objects are represented by points, and morphisms by rectangular coupons on strands between points. In a sufficiently nice category, strands are considered up to isotopy:

Assign $X \in \mathrm{Ob}(\mathcal{C}) \rightsquigarrow$, $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y) \rightsquigarrow$  , (1.41)

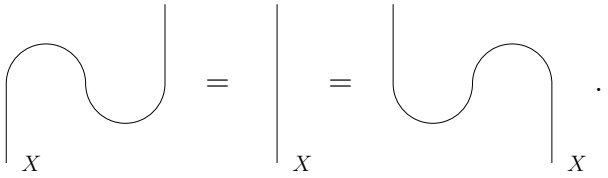
and define  and  . (1.42)

Special properties of a category are usually represented graphically with special morphisms. For example, in a rigid category, we may equivalently ask that, for each object X there exists a dual object X^* and maps:



(1.43)

such that the *triangle/zigzag identities* are satisfied:



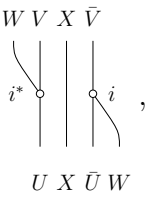
. (1.44)

1.2.1 Drinfeld Centers

The Drinfeld center can be examined by proxy through induction. The *induction functor* $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is the (left) adjoint to the forgetful functor $\mathrm{Forg} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. We use the concrete model:

$$I(X) = \left(\bigoplus_{U \in \operatorname{Irr} \mathcal{C}} U \otimes X \otimes \bar{U},\, \psi_{I(X)} \right), \qquad I(f) = \bigoplus_{U \in \operatorname{Irr} \mathcal{C}} \mathrm{id}_U \otimes f \otimes \mathrm{id}_{\bar{U}}, \tag{1.45}$$

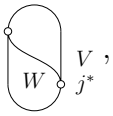
$$\psi_{I(X),W} = \bigoplus_{U,V \in \operatorname{Irr} \mathcal{C}} \sum_i \sqrt{\dim U} \sqrt{\dim V}$$



, (1.46)

where direct sums are over some consistent choice of representatives of isomorphism classes of simple objects and $\{i^*\} \subseteq \operatorname{Hom}(U, W \otimes V)$, $\{i\} \subseteq \operatorname{Hom}(\bar{U} \otimes W, \bar{V})$ are dual bases with respect to the pairing:

$$\delta_{i,j} =$$



, (1.47)

using the canonical unitary spherical structure.

Chapter 2

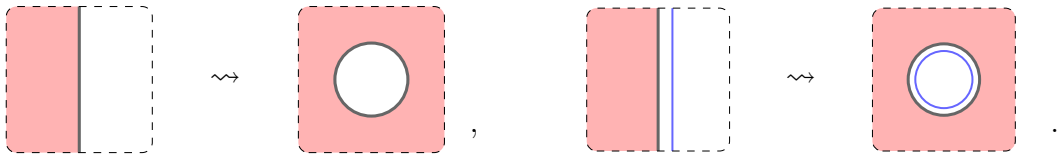
Boundary Symmetries of (2+1)D Topological Orders

The study of symmetry enriched topological order (SETO) is an important component of the field of topological phases of matter, with applications in both condensed matter physics and quantum information [BBCW19; Del19]. A (2+1)D SETO is a G-crossed braided extension of a unitary modular tensor category (UMTC) $\mathcal{C}$, such as one arising from a (2+1)D gapped topological phase of matter equipped with global on-site group G-symmetry.

An interesting class of topological phase transitions between gapped (2+1)D phases are those which arise from *anyon condensation* [FSV13; Kon14; Bur18]. Mathematically, condensable anyons are characterized by commutative algebra objects in the bulk MTC. Understanding the phase transitions which preserve symmetry is a topic of additional import [BJLP19]. Obstructions to the preservation of symmetry are equivalent to lifts of the underlying group action to the 2-group of boundary symmetries. Further, equivariant structure on the corresponding algebra object is known to be sufficient for preservation of group symmetry.

The boundary theory has its own internal topological symmetries - namely 'string operators' - which are characterized by matrix product operators (MPOs) [BMW+17; WBV17]. These symmetries holographically represent the 'dual' string net model, and their algebraic structure describes a *hypergroup* [SW03]. It is natural to ask when extensions of bulk symmetry to the boundary theory are coherent with internal symmetries of the boundary.

For a separable algebra object A in a unitary braided fusion category one may endow its endomorphism ring $\mathrm{End}(A)$ with a *convolution product* – a generalization of the convo-



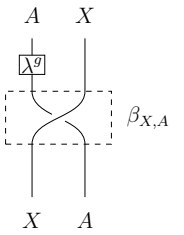
(a) Gapped boundary to vacuum becomes canonical Lagrangian algebra.

(b) String operator symmetry becomes Lagrangian algebra endomorphism.

Figure 2.1: 'Rolling up' the boundary.

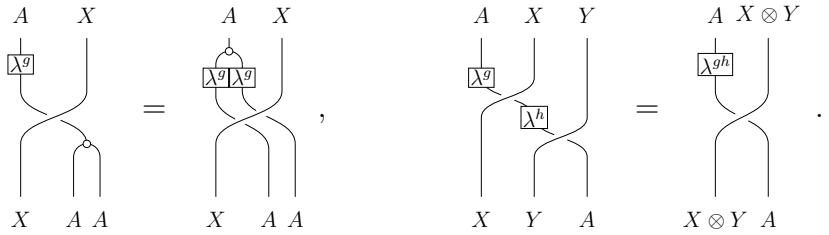
lution of finite group characters – to obtain the *convolution algebra*. If A is commutative, its convolution algebra is commutative semisimple with respect to this convolution product [BD20]. From the data of the convolution algebra of a commutative separable algebra A we may define a hypergroup called $\mathsf{HyperAut}(A)$ with extreme points given by orthogonal convolution idempotents of $\mathrm{End}(A)$. There is a natural identification of the hypergroup of string operators on a boundary spatially representing condensation by A with $\mathsf{HyperAut}(A)$. When $A \in \mathcal{C}$ is *Lagrangian*, $\mathsf{HyperAut}(A)$ can be identified with the fusion ring of $\mathcal{C}_A$ in a canonical way [BJ22].

We briefly outline the physical reasoning leading to an obstruction. Let $\mathcal{F}$ a G-crossed braided extension of unitary modular tensor category $\mathcal{E}$ with braided action $G \curvearrowright \mathcal{E}$. The equivariant structure on the algebra object A provides for each g-graded defect $X \in \mathcal{F}_g$ an isomorphism [BJLP19]:



$$= (\lambda^g \otimes \mathrm{id}) \circ \beta_{X,A} : X \otimes A \to A \otimes X.$$

which is compatible with multiplication and coherent with the G-crossed braiding:



A string operator symmetry e should commute with λ up to the G-action, that is:

$$e \circ \lambda^g = \lambda^g \circ g(e).$$

Further, if a braided action does not admit an equivariant structure on the algebra A which is compatible with string operator symmetries, then internal topological symmetries are incompatible with the phase transition represented by the algebra A in any microscopic realization of the associated SETO. Given that the string operator symmetries assemble into a hypergroup H, we may define a categorical group representing the obstruction:

$$\underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{E}|A, \mathsf{H}) := \left\{(\alpha, \eta^\alpha, \lambda^\alpha) \in \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{E}|A) : e \circ \lambda^\alpha = \lambda^\alpha \circ \alpha(e), \quad \forall e \in \mathsf{H}\right\}.$$

Given a unitary fusion category $\mathcal{C}$ - i.e. one describing the excitations of a gapped boundary to vacuum - and a full unitary fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$, we may define the categorical group of autoequivalences which are trivial (as linear functors) on $\mathcal{D}$:

$$\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}|_{\mathcal{D}}) := \left\{(\alpha, \eta^\alpha) \in \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}) : \alpha|_{\mathcal{D}} \cong \mathrm{Id}_{\mathcal{D}} \text{ as } \textit{linear} \text{ functors}\right\}.$$

In the case where $A \in \mathcal{E} \cong \mathcal{Z}(\mathcal{C})$ is Lagrangian and $\mathsf{H} = \mathsf{K}_0(\mathcal{D}) \subseteq \mathsf{K}_0(\mathcal{C}) \cong \mathsf{HyperAut}(\mathsf{A})$ is some canonical hypersubgroup, we obtain the following characterization of the obstruction:

Theorem 2.1. *Let $\mathcal{C}$ unitary fusion, $L = I(\mathbb{1})$ its canonical Lagrangian algebra, and $\mathcal{D} \subseteq \mathcal{C}$ a full fusion subcategory. Then, there is an equivalence of 2-groups:*

$$\underline{I^L_{\mathcal{D}}} : \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}|_{\mathcal{D}}) \xrightarrow{\sim} \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{Z}(\mathcal{C})|L, \mathsf{K}_0(\mathcal{D})).$$

Indeed, as hypersubgroups are exactly those which are closed under the monoid product (fusion) and weak inverses (duals), every symmetry on the boundary is described by the action of some hypersubgroup $\mathsf{K}_0(\mathcal{D}) \subseteq \mathsf{K}_0(\mathcal{C})$ indexed by fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$.

Using Theorem 2.1, we give a description of isomorphism classes of $\underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{E}|A, \mathsf{H})$ in the case where $\mathcal{E}$ is pointed, Tambara-Yamagami, and when $\mathcal{E}$ is modular of type C. Interpreting the data of a hypergroup as an MPO symmetry on a spin chain, Theorem 2.1 gives a description of the spatial realizations of equivalences of symmetric Hamiltonians [JSW24].

2.1 Preliminaries

Recall that a *monoidal functor* [EGNO16, Section 2.4] is a tuple (α, η^α) of a functor α between monoidal categories and a natural isomorphism:

$$\eta^\alpha : \alpha(\_) \otimes \alpha(\_) \to \alpha(\_ \otimes \_), \tag{2.1}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (\alpha(X) \otimes \alpha(Y)) \otimes \alpha(Z) & \xrightarrow{\sim} & \alpha(X) \otimes (\alpha(Y) \otimes \alpha(Z)) \\ \downarrow{\scriptstyle \eta^\alpha_{X,Y} \otimes \mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_X \otimes \eta^\alpha_{Y,Z}} \\ \alpha(X \otimes Y) \otimes \alpha(Z) & & \alpha(X) \otimes \alpha(Y \otimes Z) \\ \downarrow{\scriptstyle \eta^\alpha_{X\otimes Y,Z}} & & \downarrow{\scriptstyle \eta^\alpha_{X,Y\otimes Z}} \\ \alpha((X \otimes Y) \otimes Z) & \xrightarrow{\sim} & \alpha(X \otimes (Y \otimes Z)) \end{array} . \tag{2.2}$$

Recall that a *monoidal natural transformation* $\pi : (\alpha, \eta^\alpha) \to (\beta, \eta^\beta)$ is a natural transformation $\pi : \alpha \to \beta$ such that the following diagram commutes:

$$\begin{array}{ccc} \alpha(X) \otimes \alpha(Y) & \xrightarrow{\eta^\alpha_{X,Y}} & \alpha(X \otimes Y) \\ \downarrow{\scriptstyle \pi_X \otimes \pi_Y} & & \downarrow{\scriptstyle \pi_{X\otimes Y}} \\ \beta(X) \otimes \beta(Y) & \xrightarrow{\eta^\beta_{X,Y}} & \beta(X \otimes Y) \end{array} .$$

We will refer to naturality in the sense of equation (2.1) as *distributive* and naturality in the sense of the diagram (2.2) as *associative*.

Recall that a *unitary fusion category* [Pen18; EGNO16] is a finitely semisimple rigid C^*-tensor category with simple unit. We will denote the object dual to X by $\bar{X}$ and the dagger structure by $\dagger : \mathrm{Hom}(-_{(1)}, -_{(2)}) \xrightarrow{\sim} \mathrm{Hom}(-_{(2)}, -_{(1)})$. Recall that a *unitary monoidal functor* is a monoidal functor (α, η^α) between unitary fusion categories for which α is a $\dagger$-functor and η^α is a unitary natural isomorphism.

For $\mathcal{C}$ unitary fusion, recall that the *Drinfeld center of* $\mathcal{C}$ [Müg03, Definition 3.4] denoted $\mathcal{Z}(\mathcal{C})$ has objects (X, ψ) where $X \in \mathrm{Ob}(\mathcal{C})$ and $\psi : X \otimes - \to - \otimes X$ is an isomorphism such that for any $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$:

$$\psi^\alpha_Z \circ (\mathrm{id} \otimes f) = (f \otimes \mathrm{id}) \circ \psi^\alpha_Y, \quad \text{and} \tag{2.3}$$

$$\psi^\alpha_{Y \otimes Z} = (\mathrm{id} \otimes \psi^\alpha_Z) \circ (\psi^\alpha_Y \otimes \mathrm{id}). \tag{2.4}$$

In other words, ψ is a natural isomorphism satisfying the braid relation (2.4). Morphisms in the center are compatible with half-braidings, that is:

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \psi), (Y, \varphi) = \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) : (\mathrm{id} \otimes f) \circ \psi = \varphi \circ (f \otimes \mathrm{id})\}. \tag{2.5}$$

The Drinfeld center is monoidal with product:

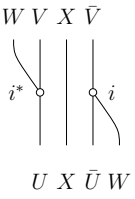
$$(X, \psi) \otimes (Y, \varphi) = (X \otimes Y, (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi)),$$

and is braided with the braiding $C_{(X,\psi),(Y,\varphi)} = \psi_Y$. The Drinfeld center of a fusion category is modular [Müg03; BV13]. All half-braidings in the Drinfeld center of a unitary fusion category are automatically unitary [Gal14], meaning the Drinfeld center of a unitary fusion category is unitary modular. There is a natural unitary monoidal forgetful functor $\mathrm{Forg} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, denote its (left) adjoint $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ the *induction functor*. We will use the following concrete model [KB10; BJ22; HP16], described in the (bottom-up/optimistic) graphical

calculus [Coe10]:

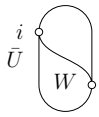
$$I(X) = \left(\bigoplus_{U \in \operatorname{Irr} \mathcal{C}} U \otimes X \otimes \bar{U},\ \psi_{I(X)} \right), \qquad I(f) = \bigoplus_{U \in \operatorname{Irr} \mathcal{C}} \mathrm{id}_U \otimes f \otimes \mathrm{id}_{\bar{U}},$$

$$\psi_{I(X),W} = \bigoplus_{U,V \in \operatorname{Irr} \mathcal{C}} \sum_i \sqrt{\dim U}\sqrt{\dim V}$$



where direct sums are over some consistent choice of representatives of isomorphism classes of simple objects and $\{i^*\} \subseteq \operatorname{Hom}(U, W \otimes V)$, $\{i\} \subseteq \operatorname{Hom}(\bar{U} \otimes W, \bar{V})$ are dual bases with respect to the pairing:

$$\delta_{i,j} =$$



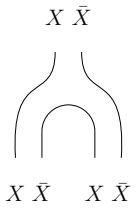
(2.6)

using the canonical unitary spherical structure.

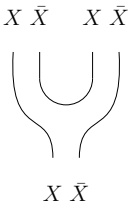
2.1.1 Algebra Objects

A *Q-system object* [CPJP22, Definition 3.1] in a unitary fusion category $\mathcal{C}$ is a separable C^*-Frobenius algebra $(A, \Delta, \iota, \nabla, \epsilon)$ whose comultiplication and counit are the dagger of the multiplication and unit respectively. The *canonical Lagrangian algebra* $L = I(\mathbb{1})$ has the structure of a commutative Q-system object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ with structure maps:

$$\Delta_L = \bigoplus_X \frac{1}{\sqrt{\dim X}}$$

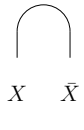


$$, \qquad \nabla_L = \Delta_L^\dagger = \bigoplus_X \frac{1}{\sqrt{\dim X}}$$



$$, \qquad \iota_L = \bigoplus_X \sqrt{\dim X}$$

$$, \qquad \epsilon_L = \iota_L^\dagger = \bigoplus_X \sqrt{\dim X}$$



.

Note that we do note require our Q-systems to be standard in the sense of [BKLR15], rather:

$$\Delta_L \circ \nabla_L = \mathrm{id}_L, \qquad \text{and} \qquad \epsilon_L \circ \iota_L = \sum (\dim X)^2 = \dim L.$$

Let $A \in \mathrm{Ob}(\mathcal{C})$ a Q-system object. Denote by $\mathcal{C}_A$ the *category of right A-modules over $\mathcal{C}$* [EGNO16, Section 7.8], whose objects are pairs (X, μ) where $X \in \mathrm{Ob}(\mathcal{C})$ and $\mu : X \otimes A \to X$ is a morphism such that the following diagram commutes:

$$\begin{array}{ccc} (X \otimes A) \otimes A & \xrightarrow{\sim} & X \otimes (A \otimes A) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \Delta} & & {\scriptstyle \mu \otimes \mathrm{id}}\downarrow \\ X \otimes A & & X \otimes A \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & X & \end{array} , \tag{2.7}$$

and whose morphisms are compatible with the module action, that is:

$$\mathrm{Hom}_{\mathcal{C}_A}((X, \mu), (Y, \nu)) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) : f \circ \mu = \nu \circ (f \otimes \mathrm{id}) \}. \tag{2.8}$$

This category is again unitary, finite, and semisimple [CPJP22; Ost03].

When A is commutative and central, $\mathcal{C}_A$ has the structure of a unitary fusion category [BN11, Proposition 5.1]. When $L = I(\mathbb{1})$ the canonical Lagrangian, $I_L : \mathcal{C} \cong \mathcal{Z}(\mathcal{C})_L$ where $I_L(X) = (I(X), \eta^I_{X,\mathbb{1}})$ is an equivalence of unitary fusion categories [BN11, Corollary 5.6].

2.1.2 Categorical Groups

A *categorical group (2-group)* [EGNO16, Definition 2.11.4] is a 2-category with one object whose morphisms are invertible up to 2-morphism and whose 2-morphisms are invertible. Equivalently, a categorical group is a monoidal category whose objects are all invertible up to isomorphism and whose morphisms are all isomorphism.

We denote categorical groups and functors of categorical groups with underline. A *2-*

subgroup is the one object together with morphisms given by a full monoidal subcategory.

Let $\mathcal{C}$ a unitary fusion category. Define $\underline{\mathrm{Aut}}_\otimes(\mathcal{C})$ to be the categorical group with one object whose morphisms are unitary monoidal autoequivalences of $\mathcal{C}$ and whose 2-morphisms are unitary monoidal natural isomorphisms.

Definition 2.2 ([BJLP19, Definition 4.1])**.** Let $\mathcal{C}$ be a unitary modular category and $A \in \mathcal{C}$ a commutative Q-system object. Define $\underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A)$ to be the categorical group whose morphisms are tuples $(\alpha, \eta^\alpha, \lambda^\alpha)$, where (α, η^α) is a unitary braided monoidal autoequivalence of $\mathcal{C}$ and $\lambda^\alpha : A \to \alpha(A)$ an algebra isomorphism; and 2-morphisms are:

$$\mathrm{Hom}((\alpha, \eta^\alpha, \lambda^\alpha), (\beta, \eta^\beta, \lambda^\beta)) = \left\{\pi \in \mathrm{Hom}((\alpha, \eta^\alpha), (\beta, \eta^\beta)) : \pi_A \circ \lambda^\alpha = \lambda^\beta\right\}.$$

We denote by $\mathrm{Aut}_\otimes(\mathcal{C})$ and $\mathrm{Aut}^{br}_\otimes(\mathcal{C}|A)$ the group truncations – i.e. the groups of isomorphism classes – of $\underline{\mathrm{Aut}}_\otimes(\mathcal{C})$ and $\underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A)$ respectively.

Remark 2.3. Let $(F, J) : \mathcal{C} \to \mathcal{D}$ be a unitary monoidal equivalence of unitary fusion categories. Then, (F, J) extends to a canonical unitary monoidal equivalence of 2-groups:

$$\begin{aligned} & \underline{F} : \underline{\mathrm{Aut}}_\otimes(\mathcal{C}) \to \underline{\mathrm{Aut}}_\otimes(\mathcal{D}), \qquad \underline{F}(\alpha, \eta^\alpha) := (\tilde{\alpha}, \tilde{\eta}^\alpha), \\ \text{where} \quad & \tilde{\alpha} \circ F(X) := F \circ \alpha(X), \qquad \tilde{\alpha} \circ F(f) := F \circ \alpha(f), \\ \text{and} \quad & \tilde{\eta}^\alpha_{FX,FY} := \tilde{\alpha}(J_{X,Y})^{-1} \circ F(\eta^\alpha_{X,Y}) \circ J_{\alpha X, \alpha Y}, \end{aligned}$$

which acts on 2-morphisms $\pi \in \mathrm{Hom}((\alpha, \eta^\alpha), (\beta, \eta^\beta))$ by:

$$\underline{F}(\pi)_{FX} = F(\pi_X).$$

2.2 Symmetries of Gapped Boundaries to Vacuum

Well known to experts is the correspondence between modular tensor categories containing a Lagrangian algebra object and fusion categories. Define two 2-groupoids:

- $\underline{\mathrm{UFC}}$, whose objects are unitary fusion categories, morphisms are unitary monoidal equivalences, and 2-morphisms are unitary monoidal natural isomorphisms.

- $\underline{\mathrm{QM}}$, whose objects are pairs (C, L) of a modular tensor category and a Lagrangian algebra, whose morphisms $(F, \varphi) : (C_1, L_1) \to (C_2, L_2)$ are unitary braided monoidal autoequivalences $F : C_1 \xrightarrow{\sim} C_2$ together with algebra isomorphisms $\varphi : F(L_1) \xrightarrow{\sim} L_2$, and whose 2-morphisms are unitary monoidal natural isomorphisms $\eta : (F_1, \varphi_1) \to (F_2, \varphi_2)$ such that $\varphi_2 \circ \eta_L = \varphi_1$.

There is an expected equivalence $\underline{\mathrm{UFC}} \xrightarrow{\sim} \underline{\mathrm{QM}}$ [DMNO13], which relates a unitary fusion category to its Drinfeld center together with its canonical Lagrangian algebra. We state and prove a specification of this result for the benefit of the reader:

Corollary 2.4 ([DMNO13, Proposition 4.4])**.** *Let $\mathcal{C}$ a fusion category, $\mathcal{Z}(\mathcal{C})$ its Drinfeld center, and $I(\mathbb{1})$ its canonical Lagrangian algebra. There is an equivalence of 2-groups $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{Z}(\mathcal{C})|I(\mathbb{1}))$.*

Suppose $\mathcal{C}$ is a unitary fusion category of rank n and (α, η^α) is a monoidal autoequivalence of $\mathcal{C}$. Then, the autoequivalence α must descend to a set permutation of the isomorphism classes of simple objects $\sigma \in S([\operatorname{Irr}\mathcal{C}]) \cong S(n)$. As $\mathcal{C}$ is an Abelian category, there is a natural isomorphism:

$$\begin{aligned} &S^\alpha : F_{\mathrm{id}} \to F_\sigma, \qquad \text{where} \qquad F_{\mathrm{id}}, F_\sigma : \mathcal{C}^{\oplus n} \to \mathcal{C}^{\oplus n} \qquad \text{by} \\ &F_{\mathrm{id}}(X_1 \oplus X_2 \oplus \cdots \oplus X_n) = X_1 \oplus X_2 \oplus \cdots \oplus X_n, \\ &F_\sigma(X_1 \oplus X_2 \oplus \cdots \oplus X_n) = X_{\sigma(1)} \oplus X_{\sigma(2)} \oplus \cdots \oplus X_{\sigma(n)}. \end{aligned} \tag{2.9}$$

For some fixed choice of representatives of isomorphism classes of simple objects $\operatorname{Irr}\mathcal{C}$, choose for each representative a unitary isomorphism $\lambda^\alpha_U : \alpha(U) \xrightarrow{\sim} V$. Define a map:

$$\tilde{\lambda}^\alpha = \left(\bigoplus_{U \in \operatorname{Irr}\mathcal{C}} \eta^\alpha_{U,\overline{U}} \right) \circ \left(\bigoplus_{U \in \operatorname{Irr}\mathcal{C}} (\lambda^\alpha_U)^{-1} \otimes (\lambda^\alpha_U)^* \right) \circ S^\alpha_L : L \to \alpha(L). \tag{2.10}$$

The isomorphism S^α_L puts the simple object $V \cong \alpha(U)$ into the summand originally corresponding to U. The isomorphisms $(\lambda^\alpha_U)^{-1}$ and $(\lambda^\alpha_U)^*$ then identify V and $\overline{V}$ with $\alpha(U)$ and $\overline{\alpha(U)} \cong \alpha(\overline{U})$ respectively. Lastly, the tensorator gives us summands which are alpha

applied to the tensor product of simple objects, as desired. We are implicitly using the additive nature of α to identify $\bigoplus \alpha(U \otimes \overline{U})$ and $\alpha\left(\bigoplus U \otimes \bar{U}\right)$.

This isomorphism $\tilde{\lambda}^\alpha : L \to \alpha(L)$ gives $\alpha(L)$ the canonical structure of an algebra. Choose a unitary monoidal natural isomorphism $\xi^\alpha : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} \alpha\alpha^{-1}$. Given an object $(X, \psi) \in \mathcal{Z}(\mathcal{C})$, define a collection of maps $\psi^\alpha = \{\psi^\alpha_Y : \alpha(X) \otimes Y \xrightarrow{\sim} Y \otimes \alpha(X)\}$ by:

$$\psi^\alpha_Y = ((\xi^\alpha_Y)^{-1} \otimes \mathrm{id}_{\alpha X}) \circ (\eta^\alpha_{\alpha^{-1}Y,X})^{-1} \circ \alpha(\psi_{\alpha^{-1}Y}) \circ \eta^\alpha_{X,\alpha^{-1}Y} \circ (\mathrm{id}_{\alpha X} \otimes \xi^\alpha_Y). \tag{2.11}$$

Given a monoidal autoequivalence $(\alpha, \eta^\alpha) \in \underline{\mathrm{Aut}}_\otimes(\mathcal{C})$, define $(\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha)$, by:

$$\tilde{\alpha}(X, \psi) := (\alpha X, \psi^\alpha), \qquad \tilde{\alpha}(f) = \alpha \circ \mathrm{Forg}(f), \qquad \tilde{\eta}^\alpha = \eta^\alpha.$$

where $\tilde{\lambda}^\alpha$ as in (2.10) and ψ^α as in (2.11). Further, for monoidal natural isomorphism $\pi \in \mathrm{Hom}((\alpha, \eta^\alpha), (\beta, \eta^\beta))$, define:

$$\tilde{\pi}_X := \pi_{\mathrm{Forg}\, X}.$$

Proposition 2.5. *$\tilde{\alpha}$ is an endofunctor on $\mathcal{Z}(\mathcal{C})$.*

Proof. Let (α, η^α) a monoidal autoequivalence of $\mathcal{C}$. We first show that $\underline{I}^L$ sends morphisms to morphisms by demonstrating ψ^α is a half braiding for αX. Consider the diagram in Figure 2.2. Cells 1 and 5 commute by the naturality of ξ^α. Cells 2 and 4 commute by the distributive naturality of the tensorator. Cell 3 commutes by the functoriality of α and the naturality of ψ. Cells 6 and 7 commute by definition of the half-braiding ψ^α. Thus, the diagram is commutative and ψ^α satisfies equation (2.3). Similarly, we may show that $\tilde{\alpha}$ is well defined on morphisms in $\mathcal{Z}(\mathcal{C})$. Consider now the diagram in Figure 2.3:

Cells 1, 3, 7, 9, 10, 11, 15, 17, and 21 commute by the naturality of the monoidal product. Cells 2 and 22 commute as ξ is a monoidal natural isomorphism. Cells 4, 12 and 18 commute by the associative naturality of the tensorator. Cells 5 and 19 commute by the definition of the composition of monoidal functors. Cells 6, 8, 16 and 20 commute by the distributive naturality of the tensorator. Cell 13 commutes by the functoriality of α and since ψ satisfies (2.4). Cell 14 commutes by the functoriality of α and since ψ satisfies (2.3). Cells 23, 24 and

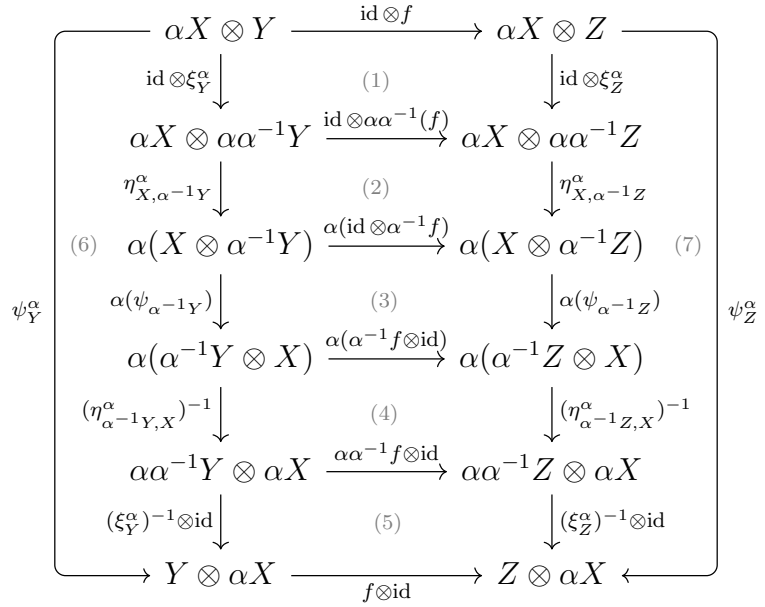


Figure 2.2: A diagram verifying the naturality of ψ^α.

25 commute by the definition of ψ^α. Thus, the diagram commutes and ψ^α satisfies equation (2.4). Therefore, ψ^α is a half-braiding and $\tilde{\alpha}$ is well-defined on objects of $\mathcal{Z}(\mathcal{C})$. □

Proposition 2.6. *$(\tilde{\alpha}, \tilde{\eta}^\alpha)$ is a monoidal endofunctor on $\mathcal{Z}(\mathcal{C})$.*

Proof. We now demonstrate that $(\tilde{\alpha}, \tilde{\eta}^\alpha)$ is a monoidal functor. As (α, η^α) is a monoidal functor, we need only show that the tensorator is compatible with the monoidal product in the center. Let $(X, \psi), (Y, \varphi) \in \mathcal{Z}(\mathcal{C})$ and denote the braiding of their product $\overline{\psi} := (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi)$. We thus need that that:

$$(\mathrm{id}_Z \otimes \eta^\alpha_{X,Y}) \circ (\psi^\alpha_Z \otimes \mathrm{id}_{\alpha Y}) \circ (\mathrm{id}_{\alpha X} \otimes \varphi^\alpha_Z) = \overline{\psi}^\alpha_Z \circ (\eta^\alpha_{X,Y} \otimes \mathrm{id}_Z), \qquad \text{for all } Z \in \mathcal{C}. \quad (2.12)$$

Consider the diagram in Figure 2.4. Cell 1 and 7 commute by the naturality of the monoidal product. Cells 2, 4, and 6 commute by the associative naturality of the tensorator. Cells 3 and 5 commute by the distributive naturality of the tensorator. Cell 8 commutes by the functoriality of α. Cells 9, 10 and 11 commute by the definition of the half-braidings. □

Proposition 2.7. *$(\tilde{\alpha}, \tilde{\eta}^\alpha)$ is a braided monoidal endofunctor on $\mathcal{Z}(\mathcal{C})$.*

Proof. We now show that $(\tilde{\alpha}, \eta^\alpha)$ is *braided* monoidal endofunctor. For $\tilde{\alpha}$ to be braided, we

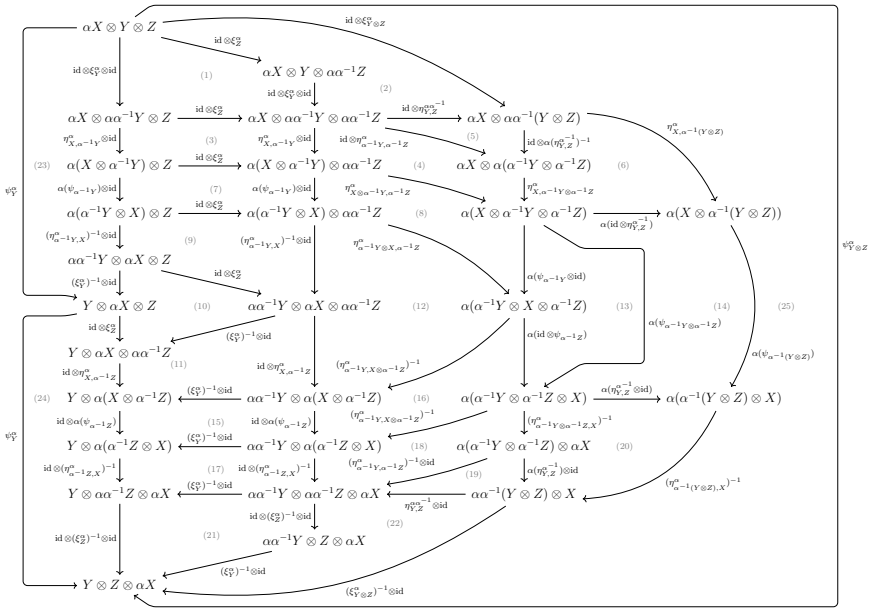


Figure 2.3: A diagram verifying ψ^α satisfies the hexagon equations.

need that:

$$\alpha(C_{(X,\psi),(Y,\varphi)}) \circ \eta^\alpha_{X,Y} = \eta^\alpha_{Y,X} \circ C_{\tilde{\alpha}(X,\psi),\tilde{\alpha}(Y,\varphi)}.$$

and their composition are morphisms $Y \to Y$. As $C_{\tilde{\alpha}(X,\psi),\tilde{\alpha}(Y,\varphi)} = \psi^\alpha_{\alpha Y}$ is a half-braiding, we have:

$$\begin{aligned}
&C_{\tilde{\alpha}(X,\psi),\tilde{\alpha}(Y,\varphi)} = \psi^\alpha_{\alpha Y} = \psi^\alpha_{\alpha Y} \circ (\mathrm{id} \otimes \mathrm{id}) \\
&= \psi^\alpha_{\alpha Y} \circ (\mathrm{id} \otimes (\alpha(\xi^{\alpha^{-1}}_Y)^{-1} \circ \xi^\alpha_{\alpha Y} \circ (\xi^\alpha_{\alpha Y})^{-1} \circ \alpha(\xi^{\alpha^{-1}}_Y))) \\
&= ((\alpha(\xi^{\alpha^{-1}}_Y)^{-1} \circ \xi^\alpha_{\alpha Y}) \otimes \mathrm{id}) \circ \psi^\alpha_{\alpha Y} \circ (\mathrm{id} \otimes ((\xi^\alpha_{\alpha Y})^{-1} \circ \alpha(\xi^{\alpha^{-1}}_Y))) \\
&= ((\alpha(\xi^{\alpha^{-1}}_Y)^{-1} \circ \xi^\alpha_{\alpha Y}) \otimes \mathrm{id}) \circ \\
&\qquad ((\xi^\alpha_{\alpha Y})^{-1} \otimes \mathrm{id}) \circ (\eta^\alpha_{\alpha^{-1}\alpha Y, X})^{-1} \circ \alpha(\psi_{\alpha\alpha^{-1}Y}) \circ \eta^\alpha_{X,\alpha^{-1}\alpha Y} \circ (\mathrm{id} \otimes \xi^\alpha_{\alpha Y}) \circ \\
&\qquad\qquad (\mathrm{id} \otimes ((\xi^\alpha_{\alpha Y})^{-1} \circ \alpha(\xi^{\alpha^{-1}}_Y))) \\
&= (\alpha(\xi^{\alpha^{-1}}_Y)^{-1} \otimes \mathrm{id}) \circ (\eta^\alpha_{\alpha^{-1}\alpha Y, X})^{-1} \circ \alpha(\psi_{\alpha\alpha^{-1}Y}) \circ \eta^\alpha_{X,\alpha^{-1}\alpha Y} \circ (\mathrm{id} \otimes \alpha(\xi^{\alpha^{-1}}_Y)).
\end{aligned}$$

However, by the distributive naturality of the tensorator, the functoriality of α, and the

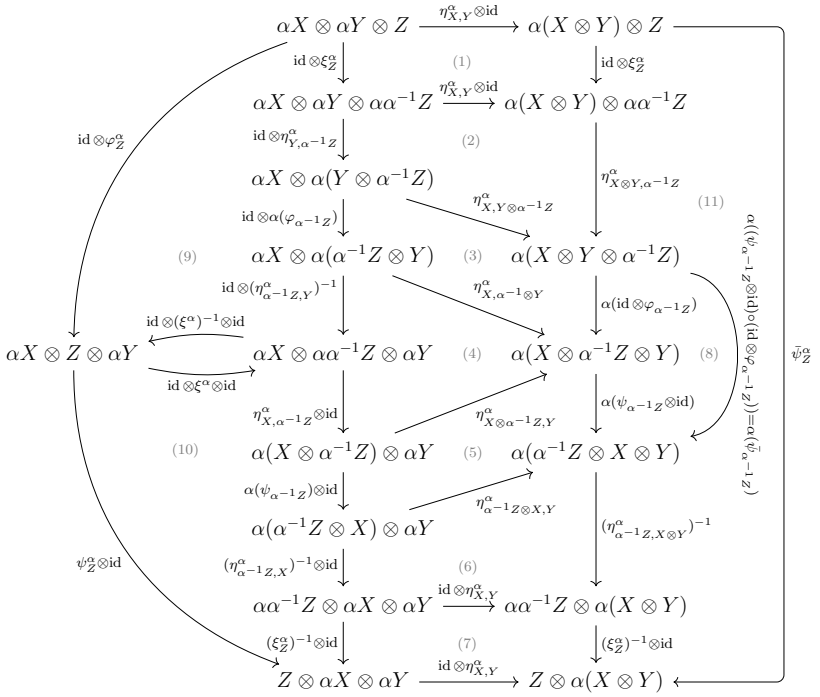


Figure 2.4: A diagram verifying that $(\tilde{\alpha}, \tilde{\eta}^\alpha)$ is monoidal.

naturality of the half-braiding ψ, the $\alpha(\xi_Y^{\alpha^{-1}})$ terms pass over the braiding and we have that:

$$C_{\tilde{\alpha}(X,\psi),\tilde{\alpha}(Y,\varphi)} = (\eta^\alpha_{Y,X})^{-1} \circ \alpha(\psi_Y) \circ \eta^\alpha_{X,Y} = (\eta^\alpha_{Y,X})^{-1} \circ \alpha(C_{(X,\psi),(Y,\varphi)}) \circ \eta^\alpha_{X,Y},$$

as desired. Thus, $(\tilde{\alpha}, \eta^\alpha)$ is a braided monoidal endofunctor. □

Proposition 2.8. $(\tilde{\alpha}, \tilde{\eta}^\alpha)$ *is a braided monoidal autoequivalence on* $\mathcal{Z}(\mathcal{C})$.

Proof. We now show that $(\tilde{\alpha}, \tilde{\eta}^\alpha)$ is an *autoequivalence*. As (α, η^α) is an autoequivalence, we need only show that $(\psi^\beta)^\alpha = \psi^{\alpha\beta}$. Consider the diagram in Figure 2.5: Cells 1, 7, 8 and 9 commute by definition. Cells 2 and 6 commute by the distributive naturality of the tensorator. Cells 3 and 5 commute by the definition of the composition of monoidal functors. Cell 4 commutes by the functoriality of α and the definition of the half-braiding. Thus, the

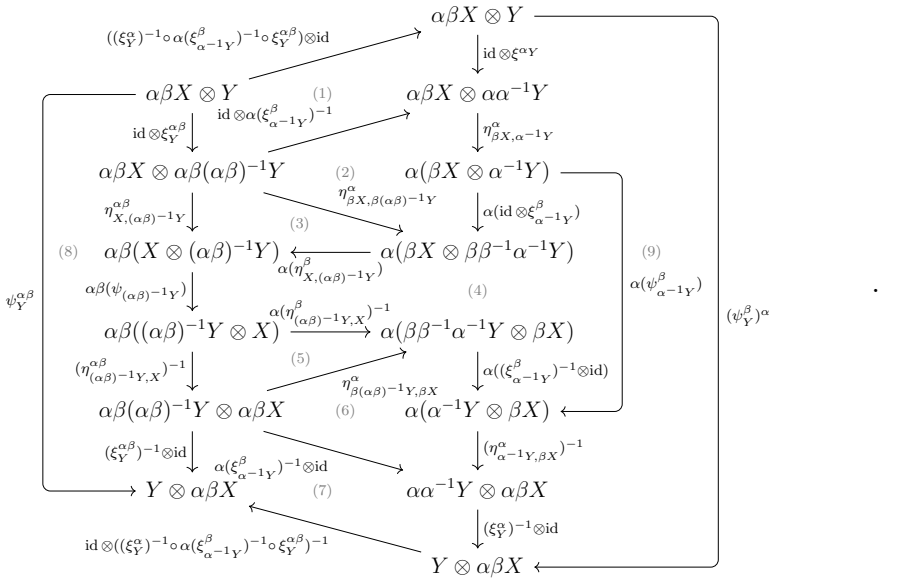


Figure 2.5: Verifying that the half-braiding behaves with composition.

diagram commutes and we obtain:

$$\begin{aligned}
\psi^{\alpha\beta}_{Y} &= (((\xi^{\alpha\beta})^{-1} \circ \alpha(\xi^{\beta}) \circ \xi^{\alpha})^{-1} \circ \otimes \mathrm{id}) \circ (\psi^{\beta}_{Y})^{\alpha} \circ (\mathrm{id} \otimes ((\xi^{\alpha\beta})^{-1} \circ \alpha(\xi^{\beta}) \circ \xi^{\alpha})) \\
&= ((((\xi^{\alpha\beta})^{-1} \circ \alpha(\xi^{\beta}) \circ \xi^{\alpha})^{-1} \circ ((\xi^{\alpha\beta})^{-1} \circ \alpha(\xi^{\beta}) \circ \xi^{\alpha})) \otimes \mathrm{id}) \circ (\psi^{\beta}_{Y})^{\alpha} \\
&= (\mathrm{id} \otimes \mathrm{id}) \circ (\psi^{\beta}_{Y})^{\alpha} = (\psi^{\beta}_{Y})^{\alpha},
\end{aligned}$$

as $(\psi^{\beta}_{Y})^{\alpha}$ is a half-braiding. Thus, $(\tilde{\alpha}, \tilde{\eta}^{\alpha})$ is a braided monoidal autoequivalence. □

Proposition 2.9. *2-morphisms in* $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$ *lift to 2-morphisms in* $\underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{Z}(\mathcal{C}))$.

Proof. We now verify that $\underline{I}^{L}$ is well-defined on 2-morphisms. Given $\pi \in \mathrm{Hom}((\alpha, \eta^{\alpha}), (\beta, \eta^{\alpha}))$ a monoidal natural transformation, we define $\underline{I}^{L}(\pi)_{X}$ to be the lift of $\pi_{\mathrm{Forg}\, X}$. By abuse of notation, we write $\pi_{\mathrm{Forg}\, X} = \pi_{X}$. For this lift to be well-defined, we need to show that π_{X} is a morphism in the center. That is, given $(X, \psi) \in \mathcal{Z}(\mathcal{C})$, for any $Y \in \mathcal{C}$:

$$\psi^{\beta}_{Y} \circ (\pi_{X} \otimes \mathrm{id}_{Y}) = (\mathrm{id}_{Y} \otimes \pi_{X}) \circ \psi^{\alpha}_{Y}.$$

Consider the diagram in Figure 2.6. Cells 1, 8 and 9 commute by the naturality of the monoidal product. Cells 2 and 7 commute as π is a monoidal natural isomorphism. Cells 3 and 6 commute by the distributive naturality of the tensorator. Cells 4 and 5 commute by

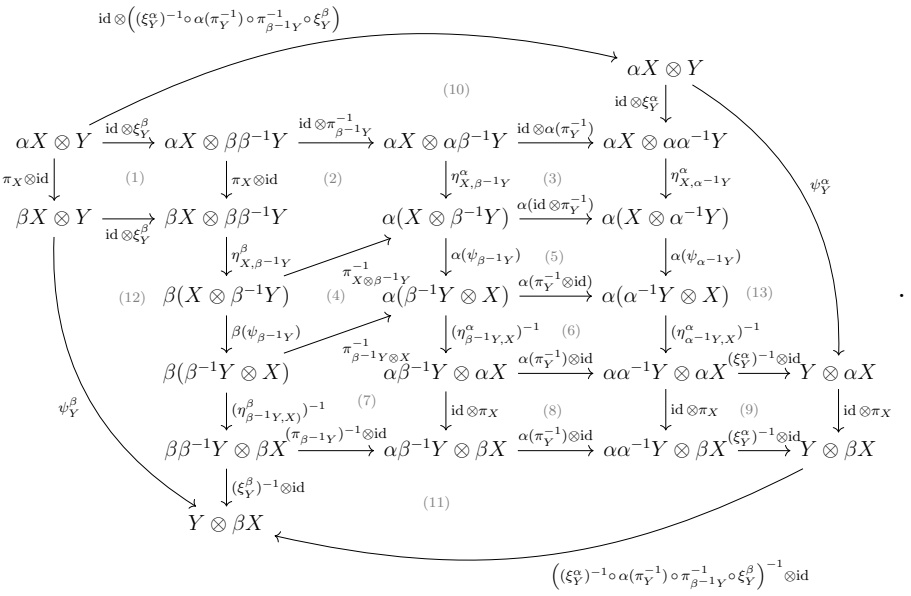


Figure 2.6: A diagram verifying the lift of a natural transformation is central.

the naturality of π. Cells 12 and 13 commute by the definition of the half-braiding. Cells 10 and 11 commute by definition.

Thus, the diagram is commutative and we obtain:

$$\begin{aligned}
&\psi^\beta_Y\circ(\pi_X\otimes\mathrm{id})\\
&=(((\xi^\alpha_Y)^{-1}\circ\alpha(\pi_Y^{-1})\circ\pi^{-1}_{\beta^{-1}Y}\circ\xi^\beta_Y)^{-1}\otimes\mathrm{id})\\
&\quad\circ(\mathrm{id}\otimes\pi_X)\circ\psi^\alpha_Y\circ(\mathrm{id}\otimes((\xi^\alpha_Y)^{-1}\circ\alpha(\pi_Y^{-1})\circ\pi^{-1}_{\beta^{-1}Y}\circ\xi^\beta_Y))\\
&=(((((\xi^\alpha_Y)^{-1}\circ\alpha(\pi_Y^{-1})\circ\pi^{-1}_{\beta^{-1}Y}\circ\xi^\beta_Y)^{-1}\\
&\quad\circ((\xi^\alpha_Y)^{-1}\circ\alpha(\pi_Y^{-1})\circ\pi^{-1}_{\beta^{-1}Y}\circ\xi^\beta_Y))\otimes\mathrm{id})\circ(\mathrm{id}\otimes\pi_X)\circ\psi^\alpha_Y\\
&=(\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\pi_X)\circ\psi^\alpha_Y=(\mathrm{id}\otimes\pi_X)\circ\psi^\alpha_Y,
\end{aligned}$$

as ψ^α_Y is a half-braiding. We have thus shown that π_X is a morphism in the center. That π_X is a natural isomorphism of braided monoidal functors comes from the naturality of the half-braiding. □

Proposition 2.10. $\tilde{\lambda}^\beta=\tilde{\pi}_L\tilde{\lambda}^\alpha$.

Proof. By Schur's lemma, $(\pi_U\circ(\lambda^\alpha_U)^{-1})=c\cdot(\lambda^\beta_U)^{-1}$. Recall that π is a monoidal natural

transformation, so $\pi \circ \eta^{\alpha}_{U,\overline{U}} = \eta^{\beta}_{U,\overline{U}} \circ (\pi \otimes \pi)$. Notice that:

$$\begin{aligned}
&\mathrm{ev} \circ ((\pi_U \circ (\lambda^{\alpha}_U)^{-1}) \otimes (\pi_{\overline{U}} \circ (\lambda^{\alpha}_U)^*)) \circ \mathrm{coev} \\
&= \mathrm{ev} \circ ((\pi_U \circ (\lambda^{\alpha}_U)^{-1}) \otimes (\pi_{\overline{U}} \circ (\lambda^{\alpha}_U)^*)) \circ \mathrm{coev} \\
&= \pi_{\mathbb{1}} \circ \alpha^{-1}(\mathrm{ev}) \circ ((\lambda^{\alpha}_U)^{-1} \otimes (\lambda^{\alpha}_U)^*) \circ \mathrm{coev} \\
&= \mathrm{ev} \circ \mathrm{coev} = \dim_U,
\end{aligned}$$

as well as:

$$\begin{aligned}
&\mathrm{ev} \circ ((\lambda^{\beta}_U)^{-1} \otimes (\lambda^{\beta}_U)^*) \circ \mathrm{coev} \\
&= \mathrm{ev} \circ \mathrm{coev} = \dim_U .
\end{aligned}$$

In particular, $(\pi_{\overline{U}} \circ (\lambda^{\alpha}_U)^*) = c^{-1} \cdot (\lambda^{\beta}_U)^*$. Composition of S^{α} and S^{β} is described by composition of permutations. By the bilinearity of the monoidal product, we are done. □

Corollary 2.11. *$\underline{I}^L : \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}) \to \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{Z}(\mathcal{C})|L)$ is a pseudofunctor.*

Recall that, for $L = I(\mathbb{1})$ the canonical Lagrangian algebra, the equivalence $\mathcal{C} \to \mathcal{Z}(\mathcal{C})_L$ induces a 2-equivalence $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}) \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{Z}(\mathcal{C})_L)$. Define a strict 2-functor $\underline{F}_L$ as in [BJLP19]: $\underline{F}_L(\tilde{\beta}, \tilde{\eta}^{\beta}, \tilde{\lambda}^{\beta}) = (\hat{\beta}, \hat{\eta}^{\beta})$, $\hat{\eta}^{\beta} = \tilde{\eta}^{\beta}$, and:

$$\hat{\beta}(X, \psi, \mu) = \left(\tilde{\beta}(X, \psi), \tilde{\mu}^{\beta}\right), \qquad \text{and} \qquad \tilde{\mu}^{\beta} := \tilde{\beta}(\mu) \circ \tilde{\eta}^{\beta}_{X,L} \circ (\mathrm{id} \otimes \tilde{\lambda}^{\beta}).$$

For natural isomorphism $\tilde{\pi} \in \mathrm{Hom}((\tilde{\alpha}, \tilde{\eta}^{\alpha}, \tilde{\lambda}^{\alpha}), (\tilde{\beta}, \tilde{\eta}^{\beta}, \tilde{\lambda}^{\beta}))$, we define $\underline{F}_L(\tilde{\pi})_X = \underline{F}_L(\tilde{\pi}_X)$.

Proposition 2.12. *The following diagram commutes up to monoidal equivalence:*

$$\begin{array}{ccc}
\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C}) & \xrightarrow{\underline{I}^L} & \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{Z}(\mathcal{C})|L) \\
& \searrow^{\underline{I}_L} & \downarrow{\scriptstyle \underline{F}_L} \\
& & \underline{\mathrm{Aut}}_{\otimes}(\mathcal{Z}(\mathcal{C})_L)
\end{array} .$$

Proof. Define $\omega : \underline{I}_L \to \underline{F}_L \circ \underline{I}^L$ by:

$$\omega_{\alpha,X} : \underline{I}_L(\alpha, \eta^\alpha)(I_L(X)) \to (\underline{F}_L \circ \underline{I}^L)(\alpha, \eta^\alpha) I_L(X),$$
$$\omega_{\alpha,X} = \left(\bigoplus_{U \in \operatorname{Irr} \mathcal{C}} \eta^\alpha_{UX,\overline{U}} \circ (\eta^\alpha_{U,X} \otimes \mathrm{id}) \right) \circ \left(\bigoplus_{U \in \operatorname{Irr} \mathcal{C}} (\lambda^\alpha_U)^{-1} \otimes \mathrm{id} \otimes (\lambda^\alpha_U)^* \right) \circ S^\alpha_{I(X)},$$

where S is the natural isomorphism (2.9). It is clear that ω is a natural isomorphism forgetting the braiding and module structure. Additionally, we have:

$$\omega_{(\alpha,\eta^\alpha)\circ(\beta,\eta^\beta)} = \omega_{(\alpha,\eta^\alpha)} \circ \omega_{(\beta,\eta^\beta)} \circ \left(\bigoplus ((\lambda^{\alpha\beta})^{-1} \circ \alpha(\lambda^\beta) \circ \lambda^\alpha) \otimes \mathrm{id} \otimes ((\lambda^{\alpha\beta})^{-1} \circ \alpha(\lambda^\beta) \circ \lambda^\alpha) \right).$$

However, $((\lambda^{\alpha\beta})^{-1} \circ \alpha(\lambda^\beta) \circ \lambda^\alpha)$ is an automorphism of a simple i.e. a scalar. By choice of component isomorphisms we obtain $\omega_{(\alpha,\eta^\alpha)} \otimes \omega_{(\beta,\eta^\beta)} = \omega_{(\alpha,\eta^\alpha)\otimes(\beta,\eta^\beta)}$.

We now aim to show that ω is well defined on objects. We have:

$$\operatorname{Hom}(\overline{U} \otimes W, \overline{V}) \cong \operatorname{Hom}(\alpha(\overline{U} \otimes W), \alpha\overline{V}) \underbrace{\cong}_{\eta^\alpha} \operatorname{Hom}(\alpha\overline{U} \otimes \alpha W, \alpha\overline{V}),$$

meaning we have the natural identification of bases:

$$\hat{j} := \alpha(j) \circ \eta^\alpha, \qquad \text{and} \qquad \hat{i^*} := (\eta^\alpha)^{-1} \circ \alpha(i^*).$$

Consider the diagram in Figure 2.7. Cells 1 and 5 commute as $\overline{\alpha(V)} \cong \alpha(\overline{V})$. Cells 2 and 4 commute by the distributive naturality of the tensorator. Cell 3 commutes by the associative naturality of the tensorator. Cells 6 and 7 commute by the definition of $\hat{i^*}, \hat{j}$. Cell 8 commutes by the functoriality of α and the normalization of i^*, j. Thus, the diagram commutes. So, $\{\hat{i^*}\}$ and $\{\hat{j}\}$ are bases satisfying the desired normalization.

Consequently, for $\lambda^\alpha_U : \alpha(U) \cong Y$ and $\lambda^\alpha_V : \alpha(V) \cong Z$, by the choice of isomorphisms λ^α the following bases satisfy the normalization (2.6):

$$\{\bar{j} := ((\lambda^\alpha_V)^*)^{-1} \circ \alpha(j) \circ \eta^\alpha_{\overline{U},\alpha^{-1}W} \circ ((\lambda^\alpha_U)^* \otimes \xi^\alpha_W)\} \subseteq \operatorname{Hom}(\overline{Y} \otimes W, \overline{Z}) \tag{2.13}$$
$$\{\bar{i}^* := ((\xi^\alpha_W)^{-1} \otimes \lambda^\alpha_V) \circ (\eta^\alpha_{\alpha^{-1}W,V})^{-1} \circ \alpha(i^*) \circ (\lambda^\alpha_U)^{-1}\} \subseteq \operatorname{Hom}(Y, W \otimes Z).$$

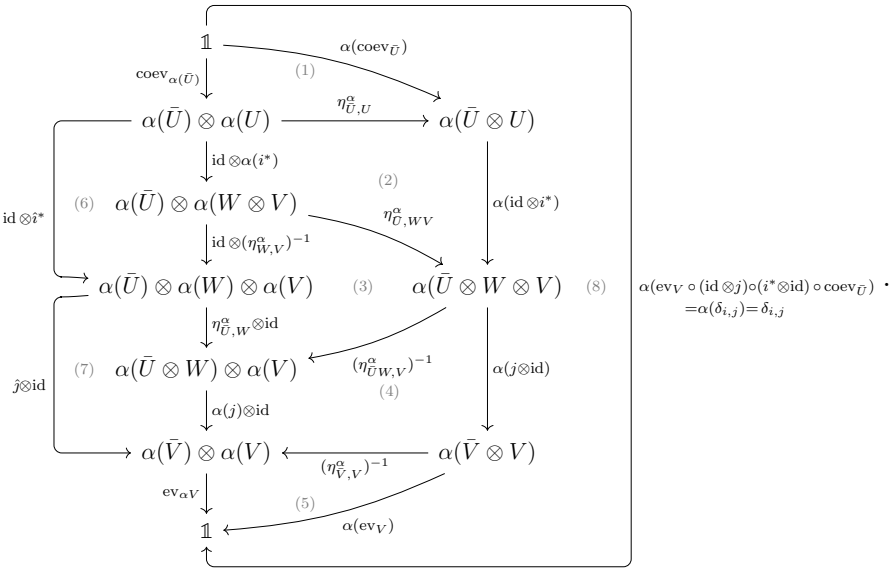


Figure 2.7: A diagram verifying the normalization of a new basis.

We now aim to show that $\omega_{\alpha,X}$ is a morphism in the center. That is, that:

$$(\mathrm{id}\otimes\omega_{\alpha,X})\circ\psi_{I(\alpha X),W}=\psi^{\alpha}_{I(X),W}\circ(\omega_{\alpha,X}\otimes\mathrm{id}).$$

Consider the diagram in Figure 2.8. Cells 1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 16, 17, 19,

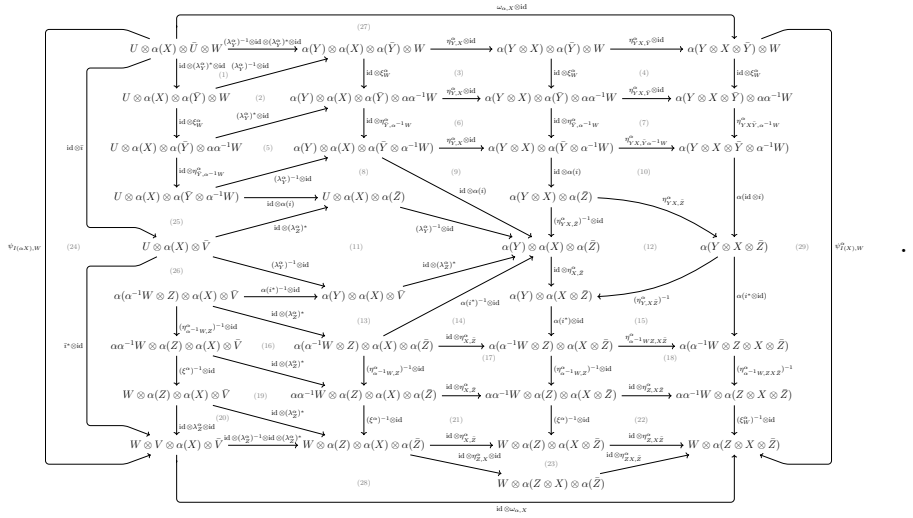


Figure 2.8: A diagram verifying that ω_α is central.

20, 21 and 22 commute by the naturality of the monoidal product. Cells 7, 12, 18 and 23 commute by the associative naturality of the tensorator. Cells 10 and 15 commute by the

distributive naturality of the tensorator. Cells 25 and 26 commute by the definitions of $\overline{\jmath}, \overline{\imath}^*$ as in equation (2.13). Cells 24 and 29 commute by the definition of the half-braidings $\psi_{I(\alpha X)}$, $\psi^{\alpha}_{I(X)}$. Cells 27 and 28 commute by the definition of the ω_α. Thus, the diagram commutes and $\omega_{\alpha,X}$ is a morphism in the center.

We now aim to show that $\omega_{\alpha,X}$ is a morphism of L-modules over $\mathcal{Z}(\mathcal{C})$. Recall from an earlier remark that $\omega_{\alpha,\mathbb{1}} = \tilde{\lambda}^{\alpha}$. So, we aim to show that:

$$\alpha(\eta^I_{X,\mathbb{1}}) \circ \eta^{\alpha}_{I(X),L} \circ (\omega_{\alpha,X} \otimes \omega_{\alpha,\mathbb{1}}) = \omega_{\alpha,X} \circ \eta^I_{\alpha X,\mathbb{1}}.$$

However, by the definition of the composition of monoidal functors, $\alpha(\eta^I_{X,\mathbb{1}}) \circ \eta^{\alpha}_{I(X),L} = \eta^{\alpha I}_{X,\mathbb{1}}$. Additionally, we note that $I(\eta^{\alpha}_{X,\mathbb{1}}) = I(r_X)$ which we do not write by abuse of notation. So, if we may show that each ω_α is a *monoidal* natural isomorphism, we are done.

Consider the diagram in Figure 2.9. Cell 1 commutes by the naturality of the monoidal

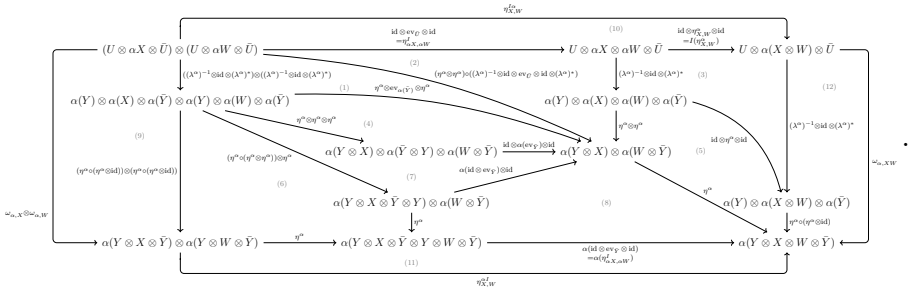


Figure 2.9: A diagram verifying that ω_α is a monoidal natural transformation.

product and the unitarity of λ^{α}. Cells 2 and 3 commute by the naturality of the monoidal product. Cell 4 commutes by as $\overline{\alpha(Y)} \cong \alpha(\overline{Y})$. Cells 5 and 6 commute by the associative naturality of the monoidal product. Cells 7 and 8 commute by the distributive naturality of the monoidal product. Cells 9 and 12 commute by the definition of ω_α. Cells 10 and 11 commute by the definition of the composition of monoidal functors. Thus, the diagram commutes and we obtain that ω_α is a monoidal natural transformation. Therefore, $\omega : \underline{I}_L \xrightarrow{\sim} \underline{F}_L \circ \underline{I}^L$ is a monoidal natural isomorphism. □

Proposition 2.13. *$\underline{I}^L$ is a 2-equivalence.*

Proof. As $\underline{I}_L$ is an equivalence, $\underline{F}_L$ is full and essentially surjective. We now show that $\underline{F}_L$ is

faithful and thus an equivalence. Given natural transformation $\pi \in \mathrm{Hom}(\underline{F}_L(\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha), \underline{F}_L(\tilde{\beta}, \tilde{\eta}^\beta, \tilde{\lambda}^\beta))$, consider the diagram in Figure 2.10.

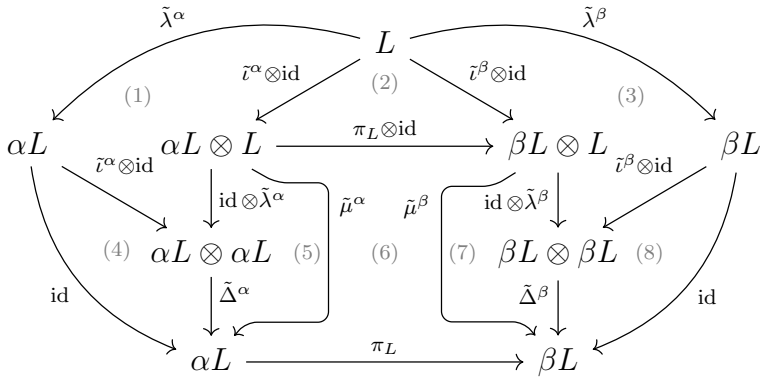


Figure 2.10: A diagram verifying that natural transformations of modules lift.

Cells 1 and 3 commute by the naturality of the monoidal product. Cell 2 commutes as the units reduce to $\bigoplus \alpha(\mathrm{coev})$, $\bigoplus \beta(\mathrm{coev})$ and as π_L is a natural transformation. Cells 4 and 8 commute as αL, βL are Frobenius. Cells 5 and 7 commute by the definition of the module actions $\tilde{\mu}^\alpha$, $\tilde{\mu}^\beta$. Cell 6 commutes as π_L is a morphism of modules. In particular, $\pi_l \circ \tilde{\lambda}^\alpha = \tilde{\lambda}^\beta$ meaning there is a canonical lift of π to $\tilde{\pi} \in \mathrm{Hom}((\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha), (\tilde{\beta}, \tilde{\eta}^\beta, \tilde{\lambda}^\beta))$. $\underline{F}_L$ is thus faithful. As $\underline{I}_L$, $\underline{F}_L$ are both monoidal equivalences, then $\underline{I}^L \cong \underline{F}_L^{-1} \circ \underline{I}_L$ is an equivalence. □

2.2.1 Hypergroups

Definition 2.14 ([SW03, Definition 1.1])**.** Given a distinguished finite set $\mathcal{B} = \{e_0, \ldots, e_{n-1}\}$, a *finite hypergroup* G *with convex basis* $\mathcal{B}$ is the simplex:

$$\mathsf{G} := \left\{ \sum_{\mathsf{i}=0}^{\mathsf{n}-1} \lambda_{\mathsf{i}} \mathsf{e}_{\mathsf{i}} : 0 \leq \lambda_{\mathsf{i}} \in \mathbb{R} \text{ and } \sum_{\mathsf{i}=0}^{\mathsf{n}-1} \lambda_{\mathsf{i}} = 1 \right\}, \tag{2.14}$$

equipped with an affine monoid product $*$ for which e_0 is the unit, together with an involutive adjunction $e_i \mapsto e_{\bar{\imath}}$ taking each element to its unique weak inverse. I.e.:

$$e_i * e_j = \sum c_{i,j}^k e_k, \qquad c_{i,j}^k \geq 0, \quad \sum c_{i,j}^k = 1, \qquad \text{and} \tag{2.15}$$

$$\text{for each } i, \text{ there is unique } \bar{\imath} \text{ such that } c_{i,j}^0 \neq 0 \iff j = \bar{\imath}. \tag{2.16}$$

A morphism of hypergroups is an affine morphism of monoids. We denote hypergroups and their morphisms using sans serif. The elements $e_i \in \mathsf{G}$ are called *extreme*, the number $c_{i,j}^k$ is called the *weight of* e_k *in* $e_i * e_j$, and the positive integer n is called the *rank of* G.

Example 2.15. For field $\Bbbk/\mathbb{R}$, the convex hull of G in the group ring $\Bbbk[G]$ is a hypergroup. The rank of this hypergroup is $|G|$, weak inverses are $\overline{g} = g^{-1}$, and weights are $c_{g,h}^k = \delta_{gh=k}$.

Let $\mathcal{C}$ a unitary fusion category. As $\mathcal{C}$ is rigid, each simple object X has a unique up to isomorphism dual object $\overline{X}$, and $\mathbb{1}$ is a subobject of $X \otimes \overline{X}$.

Definition 2.16. For $\mathcal{C}$ unitary fusion, the *Grothendieck hypergroup of* $\mathcal{C}$ $\mathsf{K}_0(\mathcal{C})$ has basis

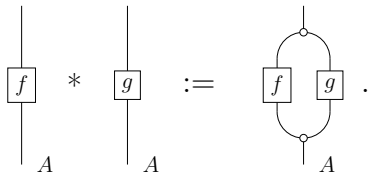
$$\left\{ \frac{[X]}{\dim X} : X \in \operatorname{Irr} \mathcal{C} \right\},$$

for some choice of representatives of isomorphism classes of simple objects, and monoid product given by the tensor product:

$$\frac{[X]}{\dim X} * \frac{[Y]}{\dim Y} = \frac{[X \otimes Y]}{\dim X \dim Y} = \sum_{Z \in \operatorname{Irr} \mathcal{C}} \frac{N_{X,Y}^Z \dim Z}{\dim X \dim Y} \frac{[Z]}{\dim Z}.$$

The unit is $[1]$, the adjunction sends $[X]/\dim X \mapsto [\overline{X}]/\dim X$, and the rank is $\operatorname{rank} \mathcal{C}$.

Let A a commutative Q-system object in unitary modular category $\mathcal{C}$. One can define a *convolution product* on its endomorphism space:



Further, $Q(A) := (\operatorname{End}(A), *, \circ)$ is a commutative semisimple algebra [BD20] and admits a basis of orthogonal convolution idempotents.

Definition 2.17 ([BD20])**.** For A a commutative Q-system object in a unitary modular tensor category, define its *hyperautomorphism hypergroup* $\mathsf{HyperAut}(A)$ to be the hypergroup with convex basis given by orthogonal convolution idempotents of $Q(A)$ with product given

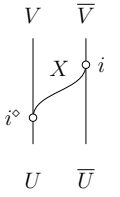
by composition of morphisms. For G a hypergroup, an *action of* G *on* A is a morphism:

$$\mathsf{G} \to \mathsf{HyperAut}(A).$$

Other authors have studied $\mathsf{HyperAut}(A)$ under the name *the symmetry hypergroup of* A in the context of actions on completely rational conformal nets [Bis17].

Theorem 2.18 ([BJ22, Corollary 3.4])**.** *Let* $\mathcal{C}$ *a unitary fusion category, and* $L = I(\mathbb{1})$ *its canonical Lagrangian algebra. Then, there is an isomorphism of hypergroups:*

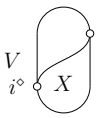
$$\mathsf{K}_0(\mathcal{C}) \to \mathsf{HyperAut}(L), \qquad \textit{by} \qquad \frac{[X]}{\dim X} \mapsto \chi_X := \bigoplus_{U,V} \sum_i \frac{\sqrt{\dim U}\sqrt{\dim V}}{\dim X}$$



,

where $\{i\} \subseteq \mathrm{Hom}(X \otimes \overline{U}, \overline{V})$, $\{i^\diamond\} \subseteq \mathrm{Hom}(U, V \otimes X)$ *are dual with respect to the pairing:*

$$\delta_{i,j} =$$



. (2.17)

Remark 2.19. The above isomorphism gives a canonical hypergroup action of $\mathsf{K}_0(\mathcal{C})$ on L.

2.2.2 A Classification

The author plans to further elaborate the physical details of these symmetries in upcoming work, in the context of symmetric quantum cellular automata [JSW24].

For some G-crossed braided extension of a UMTC – as in one representing a (2+1)D SETO – the action of a hypergroup on a condensable anyon gives rise to a spatial interpretation of such hypergroup actions as *boundary symmetries*. It is thus natural to inquire about the coherence of these boundary symmetries and the bulk on-site G-symmetries under condensation by the associated anyon.

Recall that the categorical group $\underline{\mathrm{Aut}}^{br}_{\otimes}(\mathcal{E}|A)$ represents the obstruction to preservation of symmetry under condensation by the anyon corresponding to connected separable Q-system object $A \in \mathcal{E}$.

Proposition 2.20. *Let $\mathcal{C}$ a unitary fusion category, $L = I(\mathbb{1}) \in \mathcal{Z}(\mathcal{C})$ and denote by* $\operatorname{Fus}\mathcal{C}$ *the fusion ring of $\mathcal{C}$. The following diagram of groups is commutative:*

$$\begin{array}{ccc} \operatorname{Aut}_\otimes(\mathcal{C}) & \xrightarrow{I^L} & \operatorname{Aut}^{br}_\otimes(\mathcal{Z}(\mathcal{C})|L) \\ {\scriptstyle \text{Forg}}\downarrow & & \downarrow{\scriptstyle \text{ad}} \\ \operatorname{Aut}(\operatorname{Fus}\mathcal{C}) & \xrightarrow[\text{can}]{} & \operatorname{Aut}(\operatorname{End} L) \end{array},$$

where $\operatorname{ad}[\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha](f) = (\tilde{\lambda}^\alpha)^{-1} \circ \tilde{\alpha}(f) \circ \tilde{\lambda}^\alpha$.

Proof. Notice that, for $\pi : (\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha) \cong (\tilde{\beta}, \tilde{\eta}^\beta, \tilde{\lambda}^\beta)$ two naturally isomorphism autoequivalences, we have $\tilde{\lambda}^\beta = \pi_L \circ \tilde{\lambda}^\alpha$. Thus:

$$\begin{aligned} (\tilde{\lambda}^\beta)^{-1} \circ \beta(f) \circ \tilde{\lambda}^\beta &= (\tilde{\lambda}^\alpha)^{-1} \circ \pi_L^{-1} \circ \beta(f) \circ \pi_L \circ \tilde{\lambda}^\alpha \\ &= (\tilde{\lambda}^\alpha)^{-1} \circ \pi_L^{-1} \circ \pi_L \circ \alpha(f) \circ \tilde{\lambda}^\alpha = (\tilde{\lambda}^\alpha)^{-1} \circ \alpha(f) \circ \tilde{\lambda}^\alpha, \end{aligned}$$

so ad is indeed well defined. By [BJ22, Corollary 3.4], it is sufficient to show that $\operatorname{ad} \circ I^L = \text{can} \circ \text{Forg}$ on the image of simples under the canonical action of $\operatorname{Fus} C$ on L. Let $(\alpha, \eta^\alpha) \in \underline{\operatorname{Aut}_\otimes}(\mathcal{C})$.

Consider the diagram in Figure 2.11. Cells 2 and 8 commute by the definition of $\tilde{\lambda}^\alpha$.

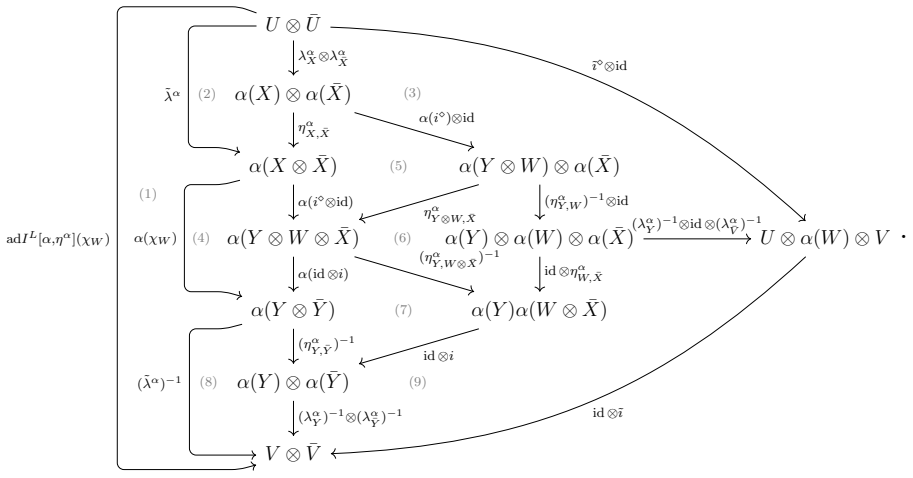


Figure 2.11: A diagram verifying that $\operatorname{ad} \circ I^L = \text{can} \circ \text{Forg}$.

Cell 4 commutes by the functoriality of α and the definition of χ_W. Cell 1 commutes by the definition of ad. Cells 5 and 7 commute by the distributive naturality of the monoidal product. Cell 6 commutes by the associative naturality of the monoidal product. We take

cells 3 and 9 to be definitions of the maps $\tilde{\imath}^{\diamond}$, $\tilde{\imath}$. Through an analogous computation to that in (2.7) and (2.13), we have that $\{\tilde{\imath}^{\diamond}\}$ and $\{\tilde{\imath}\}$ are bases satisfying the normalization (2.17). Thus, we may reindex and obtain that $\operatorname{ad} I^L[\alpha, \eta^\alpha](\omega_W) = \omega_{\alpha W} = \operatorname{can}\operatorname{Forg}[\alpha, \eta^\alpha](\omega_W)$ as desired. □

Definition 2.21. Let $\mathcal{C}$ unitary fusion and $\mathcal{D} \subseteq \mathcal{C}$ a full unitary fusion subcategory. Define $\underline{\mathrm{Aut}}_\otimes(\mathcal{C}|_\mathcal{D})$ to be the 2-subgroup of $\underline{\mathrm{Aut}}_\otimes(\mathcal{C})$ such that:

$$\underline{\mathrm{Aut}}_\otimes(\mathcal{C}|_\mathcal{D}) = \{(\alpha, \eta^\alpha) \in \underline{\mathrm{Aut}}_\otimes(\mathcal{C}) : \alpha|_\mathcal{D} \cong \mathrm{Id}_\mathcal{D} \text{ as } \textit{linear} \text{ functors}\}.$$

Other authors have studied the group $\mathrm{Aut}_\otimes(\mathcal{C}|_\mathcal{C})$ under the name *soft tensor autoequivalences*, and a characterization is known for $\mathcal{C} = \mathcal{D} = \mathrm{Hilb}(G)$ [Dav14].

Definition 2.22. Let A a commutative Q-system object in a unitary modular category $\mathcal{C}$ and H a hypergroup acting on A. Define $\underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A, \mathsf{H})$ to be the 2-subgroup of $\underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A)$:

$$\underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A, \mathsf{H}) = \{(\alpha, \eta^\alpha, \lambda^\alpha) \in \underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{C}|A) : \lambda^\alpha \circ h = \alpha(h) \circ \lambda^\alpha, \text{ for all } h \in \mathsf{H}\}.$$

Theorem 2.23. *For $\mathcal{C}$ a unitary fusion category and $\mathcal{D} \subseteq \mathcal{C}$ a full fusion subcategory, there is an equivalence of 2-groups:*

$$\underline{I}^L|_\mathcal{D} : \underline{\mathrm{Aut}}_\otimes(\mathcal{C}|_\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Aut}}^{br}_\otimes(\mathcal{Z}(\mathcal{C})|L, \operatorname{Fus}\mathcal{D})$$

Proof. We remark that there is a well defined notion of $\operatorname{Fus}\mathcal{D}$ acting on L by restricting the canonical action. Consider the subset of automorphisms of $\mathrm{End}(L)$ fixing $\{\chi_W : W \in \operatorname{Irr}\mathcal{D}\}$. The preimage of these automorphisms under ad will be $[\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha]$ such that:

$$\operatorname{ad}[\tilde{\alpha}, \tilde{\eta}^\alpha, \tilde{\lambda}^\alpha](\chi_W) = \chi_W \iff \tilde{\alpha}(\chi_W) \circ \tilde{\lambda}^\alpha = \tilde{\lambda}^\alpha \circ \chi_W,$$

which is exactly $\mathrm{Aut}^{br}_\otimes(\mathcal{Z}(\mathcal{C})|L, \operatorname{Fus}\mathcal{D})$. Similarly, the preimage of such automorphisms under $\operatorname{can} \circ \operatorname{Forg}$ are the autoequivalences fixing simples in $\mathcal{D}$, which is exactly $\mathrm{Aut}_\otimes(\mathcal{C}|_\mathcal{D})$. By Proposition 2.20, $I^L : \mathrm{Aut}_\otimes(\mathcal{C}|_\mathcal{D}) \xrightarrow{\sim} \mathrm{Aut}^{br}_\otimes(\mathcal{Z}(\mathcal{C})|L, \operatorname{Fus}\mathcal{D})$ are isomorphic as groups. As 2-subgroups are 'full,' we obtain the categorified result. □

2.3 Examples

While a general description of the canonical Lagrangian algebra and its braiding are given, a decomposition into simple objects is often difficult. In the following examples we do not give the associator for the automorphism 2-group, and will only describe the simple objects of $\mathcal{Z}(\mathcal{C})$ when a general description is known.

Example 2.24. Let G a finite group and consider the category $\mathrm{Hilb}(G)$ of finite dimensional G-graded Hilbert spaces. Full fusion subcategories of $\mathrm{Hilb}(G)$ are in one-to-one correspondence with subgroups H of G. It is well known [EGNO16, Proposition 2.6.1] that:

$$\mathrm{Aut}_{\otimes}(\mathrm{Hilb}(G)) \cong \mathrm{Aut}(G) \rtimes H^2(G, \mathbb{C}^{\times}).$$

It follows that:

$$\mathrm{Aut}_{\otimes}(\mathrm{Hilb}(G)|_{\mathrm{Hilb}(H)}) \cong \mathrm{Stab}(N) \rtimes H^2(G, \mathbb{C}^{\times}).$$

The simple objects of $\mathcal{Z}(\mathrm{Hilb}(G))$ are indexed by conjugacy classes together with representations of centralizers of orbits [EGNO16, Example 8.5.4].

Let $G = S_4$. It is known that every automorphism of S_4 is inner, so stabilizers under the inner action are exactly centralizers. Additionally, $H^2(S_4, \mathbb{C}^{\times}) \cong \mathbb{Z}_2$ has no nontrivial automorphism. There are 30 subgroups of S_4, of which there are 11 distinct types. The number of each type, a representative of each class, and a representative stabilizer are presented in the Table 2.1.

Example 2.25. Let A a finite Abelian group, χ a nondegenerate bicharacter on A, and τ a choice of sign. Denote the associated Tambara-Yamagami category [TY98] by $\mathcal{TY}(A, \chi, \tau)$. It is well known [Edi22, Lemma 2.16] that the group of isomorphism classes of monoidal autoequivalences of $\mathcal{TY}(A, \chi, \tau)$ is isomorphic to the group of automorphisms of A preserving the bicharacter χ, i.e.:

$$\mathrm{Aut}_{\otimes}(\mathcal{TY}(A, \chi, \tau)) \cong \mathrm{Aut}(A, \chi).$$

Table 2.1: Subgroup types, number, representatives, and stabilizers for S_4.

Subgroup	**Number**	**Representative**	**Stabilizer**
Trivial	1	–	S_4
$\mathbb{Z}_2$ (one transp.)	6	$\langle(12)\rangle$	$\langle(12),(34)\rangle$
$\mathbb{Z}_2$ (two transp.)	3	$\langle(12)(34)\rangle$	$\langle(1324),(12)\rangle$
$A_3 \cong \mathbb{Z}_3$	4	$\langle(123)\rangle$	0
$\mathbb{Z}_4$	3	$\langle(1234)\rangle$	$\langle(1234)\rangle$
Normal K_4	1	$\langle(12)(34),(13)(24)\rangle$	$\langle(12)(34),(13)(24)\rangle$
Non-normal K_4	3	$\langle(12),(34)\rangle$	$\langle(12),(34)\rangle$
D_8	3	$\langle(1234),(13)\rangle$	$\langle(13)(24)\rangle$
S_3	4	$\langle(12),(23)\rangle$	0
A_4	1	–	0
S_4	1	–	0

Further, it is known that $\mathrm{Aut}(\mathrm{Id}_{\mathcal{C}}) = \mathrm{Hom}(\mathcal{U}_{\mathcal{C}}, \mathbb{C}^\times)$ for $\mathcal{U}_{\mathcal{C}}$ the universal grading group [EGNO16, Proposition 4.14.3] and that $\mathcal{U}_{\mathcal{C}} = \mathbb{Z}_2$ [Nat13, Proposition 5.3]. The associator for $\underline{\mathrm{Aut}}_\otimes(\mathcal{C})$ is given by a 2-cocycle $\omega \in H^2(\mathrm{Aut}(A,\chi), \mathbb{Z}_2)$.

The strict full fusion subcategories of $\mathcal{C}$ are in one-to-one correspondence with subgroups $H \subseteq A$, which we denote $\langle H\rangle \subseteq \mathcal{C}$. It follows that:

$$\mathrm{Aut}_\otimes(\mathcal{C}|_{\langle H\rangle}) \cong \mathrm{Stab}(H,\chi) \subseteq \mathrm{Aut}(A,\chi),$$

the subgroup of autoequivalences preserving the bicharacter χ that fix the subgroup H.

A description of the simple objects of $\mathcal{Z}(\mathcal{TY}(A,\chi,\tau))$ is known [GNN09, Section 4], of which there $2\,|A|$ invertible simple objects and $4\,|A| + \binom{|A|}{2}$ total simples. The author of this note was unable to find a description of the simple objects of the center of arbitrary $\mathcal{TY}(A,\chi,\tau)$ nor of its Brauer-Picard group.

Example 2.26. Consider the category $\mathcal{C}(\mathfrak{so}_{2r+1}, 2)$ of $\mathcal{U}_q(\mathfrak{so}_{2r+1})$ tilting modules at level $k = 2$ [Sch20]. This category has rank $4+r$ and Frobenius-Perron dimension $4+8r$. The simple roots and fundamental weights of $\mathfrak{so}_{2r+1}$ are:

$$\text{(simple roots)} = \{\alpha_i = \epsilon_i - \epsilon_{i+1} : i < r\} \cup \{\alpha_r = \epsilon_r\}$$
$$\text{(fundamental weights)} = \left\{\lambda_i = \sum_{j\le i} \epsilon_j : i < r\right\} \cup \left\{\lambda_r = \frac{1}{2}\sum_{j\le r} \epsilon_j\right\}$$

The unique long root in the Weyl chamber is $\theta := \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$. The simple objects of $\mathcal{C}(\mathfrak{so}_{2r+1}, 2)$ correspond to weights in the Weyl alcove, which consists of weights λ in the Weyl chamber for which $\langle \lambda, \theta \rangle \leq k = 2$. Explicitly, these are the weights:

$$\Lambda_0 = \{0, \lambda_1, \lambda_2, \ldots, \lambda_r, 2\lambda_1, 2\lambda_r, \lambda_1 + \lambda_r\}.$$

The simply connected Lie group corresponding to $\mathfrak{so}_{2r+1}$ is the spin group $\mathrm{Spin}(2r+1)$, whose center is $\mathbb{Z}_2$ [Var04, Section 5.3]. Full fusion subcategories of $\mathcal{C}(\mathfrak{so}_{2r+1}, 2)$ correspond to closed subsets of the Weyl alcove Λ_0. There are 4 closed subsets geometrically arising from subgroups of $Z(\mathrm{Spin}(2r+1))$ and possibly many anomalous subsets arising from divisors of $2r+1$ [Saw06]. Explicitly these are:

$$\begin{aligned}
&\Delta_0 = \{0\}, \qquad \Delta_{\mathbb{Z}_2} = \{0, 2\lambda_1\}, \qquad \Gamma_0 = \Lambda_0, \\
&\Gamma_{\mathbb{Z}_2} = \{0, \lambda_1, \lambda_2, \ldots, \lambda_{r-1}, 2\lambda_1, 2\lambda_r\} = \Lambda_0 \setminus \{\lambda_r, \lambda_1 + \lambda_r\}, \\
&\Xi_j := \left\{0, 2\lambda_1, \lambda_j, \lambda_{2j}, \ldots, \lambda_{(2r+1-j)/2}\right\}, \quad \text{for} \quad 2 < j | 2r+1.
\end{aligned}$$

Define $\omega(n)$ to be the number of distinct prime factors of n. The group structure of the autoequivalences of $\mathcal{C}(\mathfrak{so}_{2r+1}, 2)$ is known [Edi22]:

$$\mathrm{Aut}_{\otimes}(\mathcal{C}(\mathfrak{so}_{2r+1}, 2)) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\omega(2r+1)-1} & p \equiv 1 \mod 4 \text{ for all prime } p | 2r+1 \\ \mathbb{Z}_2 \times \mathbb{Z}_2^{\omega(2r+1)-1} & \text{otherwise.} \end{cases}$$

The first $\mathbb{Z}_2$ corresponds to the autoequivalence which swaps λ_r and $\lambda_1 + \lambda_r$ but fixes every other simple. The second and third factor corresponds to exotic fusion ring automorphisms, which act by sending:

$$\lambda_i \mapsto \lambda_{\min(mi \mod 2r+1, -mi \mod 2r+1)}, \quad \text{for} \quad m \in \mathbb{Z}_{2r+1}^{\times}, \quad m^2 \equiv \pm 1 \mod 2r+1.$$

Write $2r+1 = \prod p_i^{\alpha_i}$ a decomposition into distinct prime powers. In particular:

$$n^2 \equiv 1 \mod 2r+1 \iff n^2 \equiv 1 \mod p_i^{\alpha_i}, \ \forall i \iff n \equiv \pm 1 \mod p_i^{\alpha_i}, \ \forall i.$$

There are $2^{\omega(2r+1)-1}$-many choices of n modulo sign. Let j a divisor of $2r+1$ and define:

$$\theta(j; 2r+1) := \#\left\{i \in \omega(2r+1) : p_i^{\alpha_i} \text{ divides } j\right\}.$$

One sees that Ξ_j is fixed by automorphisms for which $n \equiv 1 \mod (2r+1/j)$, of which there are $2^{\theta(j;2r+1)}$-many possible choices modulo sign. It follows that:

- $\mathrm{Stab}(\Delta_0) = \mathrm{Stab}(\Delta_{\mathbb{Z}_2}) = \mathrm{Aut}_{\otimes}(\mathcal{C}(\mathfrak{so}_{2r+1}, 2))$,
- $\mathrm{Stab}(\Gamma_0) = 0$,
- $\mathrm{Stab}(\Gamma_{\mathbb{Z}_2}) = \mathbb{Z}_2$,
- $\mathrm{Stab}(\Xi_j) = \mathbb{Z}_2 \times \mathbb{Z}_2^{\theta(j;2r+1)}$.

Chapter 3

Symmetric Quantum Cellular Automata and Categorical Dualities

Quantum cellular automata (QCA) are a class of quantum operations that capture the fundamental properties of unitarity and locality [Wat95; Far20; Arr19]. The problem of classifying QCA up to finite depth unitary circuits has been the focus of much attention [GNVW12; SW04; FHH22; FH20; Haa21; Haa22], due to both the fundamental nature of QCA and their wide ranging applications. These include invertible quantum phases of matter [FHH20; JM21], Floquet quantum dynamics [GNP22; PFM+16; PFVP17; PVF18; PM17; ZL21; AHLM23], many-body localization [LE24], unitary tensor network operators [CPSV17; ŞSBC18; PC20; PTSC21; RWW22], and simulations of quantum field theories [ANW11; FS20; BM20].

Thus far, most work has focused on QCAs embodied by bounded-spread automorphisms on the standard quasi-local operator algebra of a spin system, $A = \otimes_{\mathbb{Z}^n} \mathrm{Mat}_{d\times d}(\mathbb{C})$, formed by taking the tensor product of finite qudit degrees of freedom on the sites of a lattice.

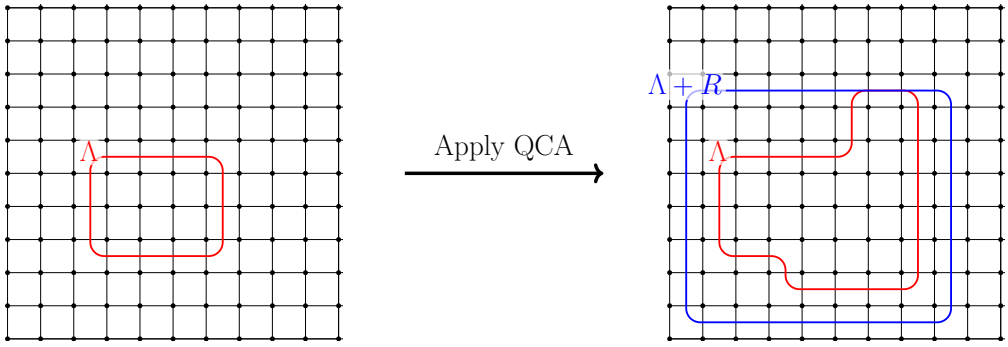


Figure 3.1: Tracking a state on the lattice $\mathbb{Z}^2$ after QCA.

This has led to a full classification of QCAs in one spatial dimension up to finite-depth circuits [GNVW12] and partial classifications in higher spatial dimensions [FH20; Haa21; Haa22]. There is a closely related problem of understanding *dualities*, which in this context are mathematically described by bounded-spread isomorphisms between *subalgebras* of the usual quasi-local algebra that interchange quantum phases of matter. Questions in this direction have attracted much recent interest [AMF16; TW24; CCH+22; KOZ23; LDOV23; Jon24; JL24; BDSY24; MLC24].

Dualities can arise naturally in the presence of a global symmetry $\mathcal{C} \curvearrowright A$, which may correspond to an action of a group, a (weak) Hopf algebra, or a fusion category. If a pair of Hamiltonians has symmetric local terms, then these lie in the algebra $A^{\mathcal{C}} \subseteq A$ of symmetric operators. In this work, we use the term *duality* to formally denote a bounded-spread isomorphism between subalgebras of symmetric operators $A^{\mathcal{C}}$ under some global symmetry which exchange Hamiltonians *but may not extend* to bounded-spread automorphisms of the whole quasi-local algebra A (QCA). The motivating example for this class of dualities is due to Kramers and Wannier, originally introduced in the context of classical statistical mechanics [KW41].

In recent years, there has been significant interest in extending results from the setting of conventional global symmetries, which are represented by tensor products of single-site unitary operators, to categorical global symmetries, which are represented by matrix product operators (MPOs) [FFRS04; BMW+17; VBW+18; AMF16; TW24]. In this work, we study this flavor of duality in one spatial dimension. The resulting algebra generated by symmetric local operators (preserving the unit MPO sector) is called a *fusion spin chain*. Recently, the idea of Kramers-Wannier (KW) duality has been generalized in this direction to categorical symmetries, where matrix-product operators have been found to implement dualities based on any invertible bimodule categories between fusion categories [AMF16; WBV17; AFM20; LDOV23]. This suggests a deep connection between categorical dualities and the algebraic structure of fusion categories.

A key feature of non-trivial dualities following the KW paradigm is that bounded-spread isomorphisms of the symmetric operators *should not be* locally extendable to the whole spin chain (or in the case of non-unital MPOs, to the local edge-restricted sector). We call such an

extension a *spatial implementation*. Spatial implementations are themselves ordinary (1+1)D QCAs in the case of group or Hopf algebra symmetries. More generally, spatial implementations are *edge-restricted* QCA. These extensions lead naturally to the following question, whose answer sheds light on the relationship between QCA and categorical dualities:

Question 3.1. When do bounded-spread isomorphisms between symmetric local operator algebras (i.e. categorical dualities) extend to QCA defined on the full (or edge-restricted) local operator algebras, and how are the possible extensions characterized?

To answer this question, we first establish a precise reformulation in terms of *abstract fusion spin chains* $A(\mathcal{E}, X)$, built from a fusion category $\mathcal{E}$ and an object $X \in \mathcal{E}$, by defining local algebras as endomorphism algebras of tensor powers of X. A fusion category $\mathcal{C}$ acting on a spin chain via MPOs is specified by the data of an indecomposable right $\mathcal{C}$-module category $\mathcal{M}$, and an object $X \in \mathcal{C}^*_{\mathcal{M}}$. Then, the symmetric operator algebra is isomorphic to the fusion spin chain $A(\mathcal{C}^*_{\mathcal{M}}, X)$. If we view $\mathcal{M}$ as a left $\mathcal{C}^*_{\mathcal{M}}$-module category, then rewriting $\mathcal{E} := \mathcal{C}^*_{\mathcal{M}}$ we can reformulate the data of a categorical symmetry as an abstract fusion spin chain $A(\mathcal{C}, X)$ together with an indecomposable left $\mathcal{C}$-module category $\mathcal{M}$. The inclusion $A(\mathcal{E}, X) \hookrightarrow A(\mathcal{E}, X)_{\mathcal{M}}$ corresponds to the inclusion of symmetric operators into all (edge-restricted) local operators, which we call a *spatial realization* of the chain $A(\mathcal{E}, X)$. Question 3.1 can be given a precise mathematical formulation as follows:

Question 3.2. Given a bounded-spread isomorphism $\alpha : A(\mathcal{C}, X) \to A(\mathcal{D}, Y)$ between abstract fusion spin chains, when does it extend to a QCA between spatial realizations $A(\mathcal{C}, X)_{\mathcal{M}} \to A(\mathcal{D}, Y)_{\mathcal{N}}$, and how are these extensions characterized?

In this form, we can address the problem using the machinery of Doplicher-Haag-Roberts (DHR) bimodules, introduced in [Jon24] and generalizing the earlier work of [NS97]. Associated to an abstract spin system A is a braided C*-tensor category $\mathrm{DHR}(A)$, and associated to any bounded-spread isomorphism $\alpha : A \to B$ is a braided equivalence $\mathrm{DHR}(\alpha) : \mathrm{DHR}(A) \cong \mathrm{DHR}(B)$. Intuitively, if we view an abstract spin chain as operators living at a cut boundary of a (2+1)D topologically ordered spin system (as in [JNPW23]) then DHR bimodules capture the structure of the bulk topological order (see also [CW23; IW23]). For a fusion

spin chain $A(\mathcal{C}, X)$, there is a braided equivalence $\mathrm{DHR}(A(\mathcal{C}, X)) \cong \mathcal{Z}(\mathcal{C})$, where the latter denotes the Drinfeld center of $\mathcal{C}$.

We now set up the statement of our main theorem: For any indecomposable $\mathcal{C}$-module category $\mathcal{M}$, let $Z(\mathcal{M}) \in \mathcal{Z}(\mathcal{C})$ denote the canonically associated Lagrangian algebra [Dav10]. Then we have the following (which is a special case of Theorem 3.24):

Theorem 3.3. *Let $\alpha : A(\mathcal{C}, X) \to A(\mathcal{D}, Y)$ be a bounded-spread isomorphism between abstract fusion spin chains. Then for any indecomposable module categories $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{C}$ and $\mathcal{D}$ respectively, spatial implementations $A(\mathcal{C}, X)_{\mathcal{M}} \to A(\mathcal{D}, Y)_{\mathcal{N}}$ of α are in bijective correspondence with algebra isomorphisms $\mathrm{DHR}(\alpha)(Z(\mathcal{M})) \cong Z(\mathcal{N})$ in $\mathcal{Z}(\mathcal{D})$. As a consequence:*

1. *If $\mathrm{DHR}(\alpha)(Z(\mathcal{M}))$ is not isomorphic to $Z(\mathcal{N})$, then α has no spatial implementation.*
2. *If $\mathrm{DHR}(\alpha)(Z(\mathcal{M})) \cong Z(\mathcal{N})$, then the spatial implementations of α form a torsor over $\mathrm{Aut}(Z(\mathcal{M})) \cong \mathrm{Inv}(\mathcal{C}^*_{\mathcal{M}})$.*

We can use this formalism to make contact with the literature concerning symmetric QCA (sQCA). In this setting, we consider a global, on-site finite group G acting faithfully by unitaries in a self-dual representation. Then the symmetric operators A^G have DHR category equivalent to the Drinfeld center $\mathcal{Z}(\mathrm{Rep}(G))$. We can immediately recover an $H^2(G, U(1))$ index (c.f. [GSSC20]) as a consequence of our framework (see Theorem 3.28).

Our results point to a number of directions for future work. First, what is the full classification of bounded-spread isomorphisms on local algebras for fusion spin chains? Using our results, we expect to be able to reduce this problem to the classification of edge-restricted QCAs in one spatial dimension, which is an open problem. Second, can our results be applied to characterize symmetries of topological orders in two dimensions via their action by dualities on the holographic boundary (for example, in the sense of [JNPW23])? Third, can techniques from the theory of operator algebras, subfactors, and fusion categories be applied to classify dualities and QCAs in higher spatial dimensions?

3.1 Dualities for Spin Chains

First, we briefly recall the algebraic approach to spin chains [BR81]. Consider the lattice $\mathbb{Z} \subseteq \mathbb{R}$ as a discrete metric space. A *(concrete) spin chain* is specified by choosing an on-site Hilbert space $\mathbb{C}^d$: for every interval $I \subseteq \mathbb{Z}$, we have the local Hilbert space $\otimes_{x \in I} \mathbb{C}^d$ and the algebra of operators localized in I:

$$A_I := \otimes_{x \in I} \mathrm{Mat}_{d \times d}(\mathbb{C}).$$

For any subinterval $J \subseteq I$, we have natural inclusion of Hilbert spaces $A_J \hookrightarrow A_I$ defined by:

$$\varphi \mapsto \left(\bigotimes_{I<J} \mathrm{Id}_d \right) \otimes \varphi \otimes \left(\bigotimes_{I>J} \mathrm{Id}_d \right), \qquad \text{where}$$

$I < J := \{i \in I : i < j \text{ for all } j \in J\}$ and similarly $I > J := \{i \in I : i > j \text{ for all } j \in J\}$.

We call the union of all A_I the *local algebra*. We call the colimit of all A_I in the category of *-algebras the *quasi-local algebra*. States of the system in the thermodynamic limit are normalized, positive linear functionals on A, while Hamiltonians can be expressed as (unbounded) derivations on A (see [BR87; BR81]). A *local Hamiltonian* is an assignment of a Hermitian (self-adjoint) matrix $H_I \in A_I$ to every interval $I \subseteq \mathbb{Z}$ satisfying: there exists uniform $N \geq 0$ for which $\mathrm{length}(I) > N$ implies $H_I = 0$. This defines an (unbounded) derivation on the local algebra:

$$\delta_H(a) := \sum_I [H_I, a], \qquad \text{where } [H_I, a] := H_I a - a H_I \text{ is the normal commutator.}$$

This derivation is well-defined since the above summation contains only finitely many non-zero terms.

Definition 3.4. Let $A = \cup_I A_I$ and $B = \cup_I B_I$ be the local algebras of two spin chains on $\mathbb{Z}$. A *bounded-spread isomorphism* is a $*$-isomorphism $\alpha : A \to B$ for which there exists a uniform $R \geq 0$ so that $\alpha(A_I) \subseteq B_{I \pm R}$ for every interval $I \subseteq \mathbb{Z}$, where we define

$I \pm R := \{j \in \mathbb{Z} : \exists i \in I \text{ for which } |j - i| \leq R\}$. We call R the *spread of* α. A *quantum cellular automaton (QCA)* is a bounded-spread automorphism $\alpha : A \to A$.

Physically, a bounded-spread isomorphism is a natural mapping between the observables of the two spin chains that *uniformly* preserves locality.

Definition 3.5. Let H and K be local Hamiltonians for two spin chains with local algebras $A = \cup_I A_I$ and $B = \cup B_I$ respectively. An *equivalence of the Hamiltonians* H and K is a bounded spread isomorphism $\alpha : A \to B$ such that $\alpha(\delta_H(\alpha^{-1}(b))) = \delta_K(b)$ for all operators $b \in B$. In this case, we denote $\alpha(H) = K$.

Let $H = \{H_I\}$ a local Hamiltonian on A and $\alpha : A \to B$ a bounded spread isomorphism with spread R. Then, notice that:

$$\alpha(H) := \{\alpha(H)_{I \pm R} := \alpha(H_I) \text{ and } \alpha(H)_J = 0 \text{ for all length}(J) \leq 2R\}$$

is a local Hamiltonian on B and α implements an equivalence between A and $\alpha(A)$. In particular, given any local Hamiltonian H on A and any bounded-spread isomorphism $\alpha : A \to B$, α always implements an equivalence between H and *some* local Hamiltonian on B. This suggests that it could be useful to shift our focus to the *groupoid* of local algebras of spin chains and bounded-spread isomorphisms between them.

Definition 3.6. An *abstract spin chain (over* $\mathbb{Z}$*)* is a choice of unital C*-algebra A_I for every interval $I \subseteq \mathbb{Z}$ satisfying:

1. (Isotony) For any subinterval $J \subseteq I$, $A'_J \subseteq A'_I$.
2. (Locality) If $I \cap J = \varnothing$, then $[A'_I, A'_J] = 0$.
3. (Weak algebraic Haag duality) There exists uniform $R \geq 0$ such that for every interval $I \subseteq \mathbb{Z}$:

$$\{x \in A \ : [x, y] = 0 \ \text{ for all } y \in A'_J \text{ and all } J \subseteq \mathbb{Z} \setminus I\} \subseteq A'_{I \pm R}.$$

Above, A'_I denotes the image of A_I in $A := \text{colim}\{A_I\}$.

If the last condition is satisfied with $R = 0$, it is called *algebraic Haag duality*. The abstract spin chains of interest in this paper always satisfy this stronger version of the last axiom. For two abstract spin chains, the definition of *bounded-spread isomorphism* used above makes perfect sense unchanged. We note that if α is bounded spread, then it follows from weak algebraic Haag duality that α^{-1} is as well.

Definition 3.7. Let A be a spin system with local Hilbert space $\mathbb{C}^d$ and let $\mathcal{C}$ be a unitary fusion category. An *action of* $\mathcal{C}$ *on* A consists of

1. An indecomposable, semisimple right module category $\mathcal{M}$ of $\mathcal{C}$.
2. An object $X \in \mathcal{C}^*_{\mathcal{M}}$ which is *strongly tensor generating* for $\mathcal{C}^*_m$.
3. A unitary isomorphism $\bigoplus_{i,j\in \mathrm{Irr}(\mathcal{M})} \mathcal{M}(i, X \triangleright j) \cong \mathbb{C}^d$.

The action is called *unital* if $\mathcal{M}$ is equivalent to Hilb as C*-categories.

3.1.1 Edge-Restricted Algebra

For non-unital fusion categorical symmetries, it is natural to consider a non-unital subalgebra of A which coheres with the functor X. For each $i, j \in \mathrm{Irr}\,\mathcal{M}$, pick an orthonormal basis $E_{i\to j}$ for the space $\mathrm{Hom}_{\mathcal{M}}(i, X \triangleright j)$. That is, we have the matrix representation $X = [\mathbb{C}[E_{i\to j}]]_{i,j}$. Define a finite directed multigraph $\mathcal{G}$ whose:

- vertices are indexed by objects in $\mathrm{Irr}\,\mathcal{M}$, and
- edges from i to j are labeled by morphisms in $E_{i\to j}$.

Notice that, since X is a strong tensor generator for $\mathcal{C}^*_{\mathcal{M}}$, the graph $\mathcal{G}$ has no sinks or sources. Further, $X \otimes X = [\sum_j \mathbb{C}[E_{i\to j}] \otimes \mathbb{C}[E_{j\to k}]]_{i,k} = [\mathbb{C}[E_{i\xrightarrow{2}k}]]_{i,k}$, where $E_{i\xrightarrow{2}k}$ is an orthonormal basis for the "paths" of length two from i to k. Define by $E_{i\xrightarrow{n}k}$ the (i,k)th entry of $\otimes_n X$. For interval $I = [n, m]$, define the local algebra:

$$A^\circ_I := \bigoplus_{i,j\in V(\mathcal{G})} \mathbb{C}\left[E_{i\xrightarrow{m-n+1}j}\right].$$

For subinterval $J = [k, l] \subseteq I$, there is a natural inclusion sending a length $l - k + 1$ path to a length $m - n + 1$ path which aligns on the $k - n + 1$ to $l - n + 1$ steps. Denote by A° the colimit of all A_I°.

Note that from this description, A_I° is isomorphic to a (non-unital) subalgebra of A_I, but not coherently. Nevertheless, A° has the structure of an abstract spin chain called the *edge-restricted algebra*. It is straightforward to verify all axioms other than weak algebraic Haag duality, which follows from Lemma 3.21 below. Denote by P the projection $A \to A^\circ$. Then, $H^\circ := \{PH_I\}$ defines a local Hamiltonian on the chain A°.

3.1.2 Symmetric Algebra

The operators of interest to us are the symmetric local operators. Recall that $X \in \mathcal{C}_{\mathcal{M}}^*$ is $\mathcal{C}$-linear. Consider the directed multisubgraph $\mathcal{H} \subseteq \mathcal{G}$ whose vertices are specified by $i \in \operatorname{Irr} \mathcal{M}$ such that $Y \triangleright i \cong i$ for every $Y \in \operatorname{Irr} \mathcal{C}$. Denote by $E^{\mathcal{C}}_{i \xrightarrow{n} k}$ an orthonormal basis for the paths of length n from i to k within $\mathcal{H}$. Similarly, for an interval $I = [n, m]$ define the local algebra:

$$A_I^{\mathcal{C}} := \bigoplus_{i,j \in V(\mathcal{H})} \mathbb{C}\left[E^{\mathcal{C}}_{i \xrightarrow{m-n+1} j}\right],$$

for which there is again natural inclusion $A_J^{\mathcal{C}} \hookrightarrow A_I^{\mathcal{C}}$ for $J \subseteq I$. $A_I^{\mathcal{C}}$ is a subalgebra of A_I° and is isomorphic to a subalgebra of A_I, again in a way which is not necessarily coherent. Again, $A^{\mathcal{C}}$ has the structure of an abstract spin chain called the *symmetric algebra*. We can now state the problem that motivates this work.

Definition 3.8. Let $A = \{A_I\}$ a spin system with local Hamiltonian $H = \{H_I\}$. Let $\mathcal{C}$ be a fusion category acting on A. We say the Hamiltonian H is *$\mathcal{C}$-symmetric* if for every interval $I \subseteq \mathbb{Z}$ with $H_I \neq 0$: $H_I P = P H_I \neq 0$ and $H_I P \in A_I^{\mathcal{C}}$ (interpreted as operators on A_I).

In this case the collection $H^\circ := \{PH_I\} = \{H_I P\}$ defines a local Hamiltonian internal to the symmetric chain $A^{\mathcal{C}}$, which we denote $H^{\mathcal{C}} := \{H_I^{\mathcal{C}}\}$

Definition 3.9. Let A and B be spin chains with local Hamiltonians H and K respectively. A *categorical duality* between (A, H) and (B, K) consists of:

1. Categorical symmetries $\mathcal{C} \curvearrowright A$ and $\mathcal{D} \curvearrowright B$ for which H is $\mathcal{C}$-symmetric and K is $\mathcal{D}$-symmetric, and

2. a bounded-spread isomorphism $\alpha : A^{\mathcal{C}} \to B^{\mathcal{D}}$ such that $\alpha(H^{\mathcal{C}}) = K^{\mathcal{D}}$ are equivalent Hamiltonians.

Recall that $A^{\mathcal{C}}$ has the natural structure of a subchain of A°. This leads to the following definition:

Definition 3.10. A bounded-spread isomorphism $\alpha : A^{\mathcal{C}} \to B^{\mathcal{D}}$ is called *spatially implemented* if it admits an extension to a bounded-spread isomorphism $\widetilde{\alpha} : A^{\circ} \to B^{\circ}$.

Spatially implemented bounded-spread isomorphisms are not proper dualities in the sense of Kramer-Wannier, but are actually more of an *emergent equivalence* [CON11]. Some may view an emergent equivalence as a kind of duality, but we regard it as closer to a proper equivalence. Indeed, if $A^{\circ} = A$ and $B^{\circ} = B$ (or in other words, the categorical action is unital – the module categories correspond to fiber functors), then the emergent equivalence reduces to an ordinary equivalence. In any case, we can now give a precise mathematical statement of Question 3.1:

Question 3.11. Given a bounded-spread isomorphism $\alpha : A^{\mathcal{C}} \to B^{\mathcal{D}}$, can we characterize when it is spatially implemented?

3.2 Abstract fusion spin chains

In the previous section, we set up the problem of studying bounded-spread isomorphisms between algebras of operators invariant under a categorical symmetry (i.e. $A^{\mathcal{C}}$), and asked when this can be extended to the edge-restricted algebra (i.e. A°). In this section, we present an abstract formulation of this problem in terms of fusion spin chains.

Definition 3.12. Let $\mathcal{E}$ be an indecomposible multi-fusion category and $X \in \mathrm{Ob}(\mathcal{E})$ a (not necessarily simple) object. Define the *fusion spin chain* $A(\mathcal{E}, X)$ by:

- For each interval I, define $A(\mathcal{C}, X)_I := \mathrm{End}_{\mathcal{E}}(\otimes_I X)$.

- For any subinterval $J \subseteq I$, define the inclusion $A(\mathcal{E}, X)_J \hookrightarrow A(\mathcal{E}, X)_I$ by:

$$\varphi \mapsto \left(\bigotimes_{I<J} \mathrm{id}_X \right) \otimes \varphi \otimes \left(\bigotimes_{I>J} \mathrm{id}_X \right).$$

- Define $A(\mathcal{E}, X) := \mathrm{colim}\{A(\mathcal{E}, X)_I\}$ in the category of $*$-algebras.

Locality follows from the bilinearity of the monoidal product. It turns out the three examples of abstract spin chains we saw in the previous section can be realized as fusion spin chains.

Example 3.13 (Concrete spin systems). Choose the fusion category $\mathcal{E} = \mathrm{Hilb}$ of finite-dimensional Hilbert spaces, and pick the object $X := \mathbb{C}^d$. Then, $A(\mathrm{Hilb}, X)$ can be identified with the ordinary spin chain A constructed from $\mathbb{C}^d$.

Recall that objects in $\mathrm{Mat}_n(\mathrm{Hilb})$ are matrices whose entries are vector spaces. Morphisms are matrices of linear transformations. The direct sum and monoidal product are given by "matrix addition" and "matrix multiplication," where sums are given by direct sums and products by the monoidal product in Hilb [FP20].

Example 3.14 (Edge-restricted algebras). Consider a fusion category action on an ordinary spin system $\mathcal{C} \curvearrowright A$, and let $X \in \mathcal{C}^*_{\mathcal{M}}$. The matrix representation of X gives an object $\widetilde{X} \in \mathrm{Mat}_n(\mathrm{Hilb}) := \mathcal{E}$, where n is the rank of $\mathcal{M}$. It is straightforward to see that the edge-restricted abstract spin chain A° can be identified with the fusion spin chain $A(\mathrm{Mat}_n(\mathrm{Hilb}), \widetilde{X})$.

Example 3.15 (Symmetric algebras). Again, suppose we have an action of a fusion category $\mathcal{C} \curvearrowright A$ and denote $\mathcal{E} := \mathcal{C}^*_{\mathcal{M}}$. Then, we can identify $A^{\mathcal{C}}$ with the fusion spin chain $A(\mathcal{C}^*_{\mathcal{M}}, X)$ [Kaw22; Kaw21].

This leads us to define the following groupoid:

Definition 3.16. Define Dua to be the groupoid whose:

- objects are fusion spin chains $A(\mathcal{E}, X)$, and whose
- morphisms are bounded-spread isomorphisms with composition.

Our discussion above has motivated the definition of Dua, whose objects are *abstract* spin systems of symmetric operators. However, we are interested in studying spin systems *together with embeddings into some edge-restricted algebra*. Here we give an abstract formulation.

Definition 3.17. Let $\mathcal{E}$ be an indecomposible multi-fusion category. A *quotient of* $\mathcal{E}$ is an indecomposible multifusion category $\mathcal{D}$ and a dominant tensor functor $F : \mathcal{E} \to \mathcal{D}$.

We typically consider quotients up to (unitary) monoidal equivalence. Now, given a dominant tensor functor $F : \mathcal{E} \to \mathcal{D}$, we have a natural inclusion of nets:

$$\iota_F : A(\mathcal{E}, X) \hookrightarrow A(\mathcal{D}, F(X))$$

where $f \in A(\mathcal{E}, X)_I$ is sent to the composition:

$$F(X)^{\otimes n} \xrightarrow{\text{can}} F(X^{\otimes n}) \xrightarrow{F(f)} F(X^{\otimes n}) \xrightarrow{\text{can}} F(X)^{\otimes n}$$

in $A(\mathcal{D}, F(X))_I$. Here, can denotes the canonical unitary isomorphism built from the monoidal structure of F. We call such an inclusion of nets a *categorical inclusion*.

Notice that ι_F is a unital inclusion of the local C*-algebras for each I, and is compatible with the connecting inclusions associated with $I \subseteq J$. Thus, ι_F extends to a unital inclusion of quasi-local algebras. Further, if we have two quotients $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, then the composition $G \circ F : \mathcal{C} \to \mathcal{E}$ is a quotient. It is straightforward to see by construction that $\iota_{G \circ F} := \iota_G \circ \iota_F$.

Given a finitely semisimple category $\mathcal{M}$, its category of endofunctors $\text{End}(\mathcal{M})$ is an indecomposable multi-fusion category equivalent to $\text{Mat}_n(\text{Hilb})$, where $n := \text{rank}(\mathcal{M})$. Recall that the data of a (left) $\mathcal{E}$-module category structure on a finitely semisimple category $\mathcal{M}$ can be expressed as a tensor functor $F : \mathcal{E} \to \text{End}(\mathcal{M})$, where $F(Y) := Y \triangleright \cdot$.

Definition 3.18. A *spatial realization* of a fusion categorical net $A(\mathcal{E}, X)$ associated to the indecomposable left $\mathcal{E}$-module category $\mathcal{M}$ is a categorical inclusion:

$$A(\mathcal{E}, X) \hookrightarrow A(\text{End}(\mathcal{M}), X \triangleright \cdot) := A(\mathcal{E}, X)_{\mathcal{M}}.$$

When $\text{rank}(\mathcal{M}) = 1$, $\text{End}(\mathcal{M}) \cong \text{Hilb}$, hence a spatial realization is an honest, unital inclusion of a fusion spin chain into the local operators of a concrete spin chain.

The motivation for this definition arises from our concept of fusion categorical symmetries. Above, we characterized these with the data of a fusion category $\mathcal{C}$, an indecomposable left module category $\mathcal{M}$, and a strongly tensor generating. object $X \in \mathcal{C}^*_{\mathcal{M}}$. These correspond precisely to spatial realizations of the fusion categorical net $A(\mathcal{C}^*_{\mathcal{M}}, X)$: if we change our perspective and denote $\mathcal{E} := \mathcal{C}^*_{\mathcal{M}}$, then we can view $\mathcal{M}$ as a left indecomposable $\mathcal{E}$-module category, and thus spatial realizations of the fusion spin chain $A(\mathcal{E}, X)$, parameterized by a left module category $\mathcal{M}$, correspond precisely to the algebra of symmetric operators inside A° for an action of the dual category $\mathcal{C}^*_{\mathcal{M}} \curvearrowright A$. This leads us to reformulate Question 3.11 as follows:

Question 3.19. Let $A(\mathcal{E}, X)$ and $A(\mathcal{D}, Y)$ be fusion spin chains, with spatial realizations parameterized by module categories $\mathcal{M}$ and $\mathcal{N}$ respectively. When does bounded-spread isomorphism $\alpha : A(\mathcal{E}, X) \to A(\mathcal{D}, Y)$ extend to a bounded-spread isomorphism $\widetilde{\alpha} : A(\mathcal{E}, X)_{\mathcal{M}} \to A(\mathcal{D}, Y)_{\mathcal{N}}$?

3.3 The algebra model for categorical inclusions

Let $L \in \mathcal{Z}(\mathcal{C})$ a connected, commutative Q-system object with multiplication μ, unit ι, and braiding σ. Consider the category $\mathcal{C}_L$ of right L-modules in $\mathcal{C}$. The commutative central structure on L equips $\mathcal{C}_L$ with the structure of a unitary multi-fusion category such that the free module functor $F_L : \mathcal{C} \to \mathcal{C}_L$, $x \mapsto x \otimes L$ is dominant. A Theorem of Bruguières and Natale ([BN11, Corollary 5.6]) shows that any quotient $F : \mathcal{C} \to \mathcal{D}$ is of the form described above, i.e. there exists a (unique up to isomorphism) connected commutative Q-system $L \in \mathcal{Z}(\mathcal{C})$ such that the digram of monoidal functors commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle F}\downarrow & \searrow^{F_L} & \\ \mathcal{D} & \xrightarrow[\cong]{} & \mathcal{C}_L \end{array} .$$

This gives an "internal" description of a quotient as opposed to the a-priori "external"

description requiring an indecomposable multi-fusion category $\mathcal{D}$. Now, we have a nice way to describe the net obtained from categorical quotients $F : \mathcal{C} \to \mathcal{D}$ directly in terms of $\mathcal{C}$. Let L be the central commutative Q-system corresponding to this quotient, and F_L the model for F as above. Then, for an interval $I \subseteq \mathbb{Z}$:

$$A(\mathcal{C}_L, X)_I = \mathrm{End}_{\mathcal{C}_L}(\otimes_I X) \cong \mathrm{Hom}_{\mathcal{C}}(\otimes_I X, (\otimes_I X) \otimes L).$$

Furthermore, for $J \subseteq I$, the inclusion $A(\mathcal{C}_L, X)_J \hookrightarrow A(\mathcal{C}_L, X)_I$ is described by:

$$\varphi \mapsto \left(\left(\otimes_{(J<I)\cup I} \mathrm{id}_X\right) \otimes \sigma_{L,\otimes_{J>I} X}\right) \circ \left(\left(\bigotimes_{J<I} \mathrm{id}_X\right) \otimes \varphi \otimes \left(\bigotimes_{J>I} \mathrm{id}_X\right)\right)$$

The inclusion $A(\mathcal{C}, X)_I \hookrightarrow A(\mathcal{C}_L, X)_I$ is given by $\varphi \mapsto \varphi \otimes \iota$.

Remark 3.20. By the above discussion, spatial realizations of $A(\mathcal{C}, X)$ are parameterized by choices of indecomposable $\mathcal{C}$-module categories. Every such module category defines a Lagrangian algebra $Z(\mathcal{M}) \in \mathcal{Z}(\mathcal{C})$ [Dav10]. In particular, we can always view the spatial realizations $A(\mathcal{C}, X) \hookrightarrow A(\mathcal{C}, X)_{\mathcal{M}}$ as $A(\mathcal{C}, X) \hookrightarrow A(\mathcal{C}_{Z(\mathcal{M})}, X)$.

3.3.1 Haag Duality for Spatial Realizations

In this section, we prove a lemma necessary for the proof of the main theorem, concerning relative commutants. As mentioned above, fusion spin chains satisfy weak algebraic Haag duality, and in fact, satisfy an even stronger version simply called algebraic Haag duality. In particular, if we choose n such that $X^{\otimes n}$ contains a copy of every simple object in $\mathcal{C}$ (the existence of such an n is the definition of strong tensor generating), then we have that for any interval I with $\mathrm{length}(I) > n$:

$$A(\mathcal{C}, X)_I = \{x \in A(\mathcal{C}, X) : [x, y] = 0 \text{ for all } y \in A(\mathcal{C}, X)_{\mathbb{Z}\setminus I}\}.$$

The following lemma shows that the nets $\mathfrak{A}(\mathcal{C}_L, X)$ satisfy an even stronger "relative" version of this property.

Lemma 3.21. *Let $A(\mathcal{C}, X)$ be a fusion spin chain and n as above. For any connected, commutative Q-system $L \in \mathcal{Z}(\mathcal{C})$, and for any interval I with* $\text{length}(I) > n$,

$$A(\mathcal{C}_L, X)_I = \{x \in A(\mathcal{C}_L, X) \ : \ [x, y] = 0 \ \ \textit{for all} \ \ y \in A(\mathcal{C}, X)_{\mathbb{Z}\setminus I}\}.$$

Proof. Note that we can view the quasi-local algebra $A(\mathcal{C}_L, X)$ as a finite index extension of $A(C, X)_{\mathbb{Z}\setminus I}$, supported on the canonical copy of $\mathcal{C} \boxtimes \mathcal{C}^{mp} \hookrightarrow \text{Bim}(A(C, X)_{\mathbb{Z}\setminus I})$ [Jon24]. Indeed, $A(\mathcal{C}_L, X)$ is the Q-system realization of the Q-system $K \in \mathcal{C} \boxtimes \mathcal{C}^{mp}$ obtained from considering $\mathcal{C}_L$ as a $\mathcal{C}$-$\mathcal{C}$ bimodule category, choosing the object $X^{\otimes n} \in \mathcal{C}_L$ and taking internal endomorphisms. In symbols:

$$A(\mathcal{C}_L, X) \cong |K|.$$

Since the embedding of $\mathcal{C} \boxtimes \mathcal{C}^{mp} \hookrightarrow \text{Bim}(A(C, X)_{\mathbb{Z}\setminus I})$ is fully faithful, by the usual Q-system arguments from subfactor theory [Jon24; CPJP22]:

$$A(\mathcal{C}, X)'_{\mathbb{Z}\setminus I} \cap A(\mathcal{C}_L, X) \cong 1_{A(\mathcal{C},X)_{\mathbb{Z}\setminus I}} \otimes \text{Hom}_{\mathcal{C}}(\mathbb{1}, K) \subseteq |K|.$$

By the definition of internal End and Q-system realization, under the identification of the realization subspace with our concrete net, we obtain:

$$A(\mathcal{C}_L, X) \cong 1_{A(\mathcal{C},X)_{\mathbb{Z}\setminus I}} = \text{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes n} \otimes L) = A(\mathcal{C}_L, X)_I. \qquad \square$$

3.4 Solving the extension problem with DHR bimodules

Here, we recall the definition of a DHR bimodule on a net of algebras (over $\mathbb{Z}$). First, if A is a (unital) C*-algebra, then a *bimodule* is an algebraic A-A bimodule X, together with a right A-valued inner product $\langle \cdot \mid \cdot \rangle : X \times X \to A$ (making X into a Hilbert A-module) such that the left A action is by adjointable operators. What we call bimodules are more typically called *correspondences* in the C*-algebra literature.

We denote the collection of all A-A bimodules by $\mathrm{Bim}(A)$. This forms a C*-tensor category, whose objects are bimodules, morphisms are adjointable bimodule intertwiners, and tensor product is given by the relative tensor product of bimodules, denoted $\boxtimes_A$ (see [CPJP22]).

If $X \in \mathrm{Bim}(A)$, a *projective basis* is a finite set $\{b_i\} \subseteq X$ such that for all $x \in X$:

$$x = \sum_i b_i \langle b_i | x \rangle.$$

Definition 3.22. Consider an abstract spin chain with quasi-local algebra $A = \mathrm{colim}_I \{A_I\}$. A bimodule over the quasilocal algebra A is called a *DHR bimodule* if there exists a uniform $n \geq 0$ such that: for every interval I with $\mathrm{length}(I) > n$, X admits a projective basis $\{b_i\}$ such that $ab_i = b_i a$ for all $x \in A_{\mathbb{Z}\setminus I}$.

The collection of DHR bimodules is closed under the relative product $\boxtimes_A$ and thus defines a full tensor subcategory $\mathrm{DHR}(A) \subseteq \mathrm{Bim}(A)$. If A also satisfies *weak algebraic Haag duality* (a property satisfied by fusion spin chains, see [Jon24]), then $\mathrm{DHR}(A)$ admits a *canonical braiding*.

Theorem 3.23 ([Jon24, Theorem C])**.** *If X is a strong, self-dual tensor generator of the unitary fusion category $\mathcal{C}$, then $\mathrm{DHR}(A(\mathcal{C}, X)) \cong \mathcal{Z}(\mathcal{C})$ as braided C*-tensor categories.*

One of the main motivations of [Jon24] is that the DHR construction defines a functor $\mathrm{DHR} : \mathrm{Net}_{\mathbb{Z}} \to$ C*-BrTens, where $\mathrm{Net}_{\mathbb{Z}}$ is the groupoid of abstract spin chains and bounded-spread isomorphisms. This associates to each abstract spin chain its braided tensor category of DHR bimodules, while:

$$\mathrm{DHR}(\alpha) : \mathrm{DHR}(A) \to \mathrm{DHR}(B),$$

acts on DHR bimodules by sending a DHR bimodule X of A to $\mathrm{DHR}(\alpha)(X)$, which is isomorphic to X as a vector space with left and right B actions given by:

$$b \triangleright x \triangleleft c := \alpha^{-1}(b) x \alpha^{-1}(c),$$

and B valued inner product:

$$\langle x \mid y \rangle_B := \alpha(\langle x \mid y \rangle_A).$$

In particular, for every bounded-spread isomorphism between fusion spin chains

$$\alpha : A(\mathcal{C}, X) \to A(\mathcal{D}, Y),$$

there is an induced unitary braided equivalence of categories

$$\mathrm{DHR}(\alpha) : \mathcal{Z}(\mathcal{C}) \cong \mathrm{DHR}(A(\mathcal{C}, X)) \to A(\mathcal{D}, Y) \cong \mathcal{Z}(\mathcal{D}).$$

Theorem 3.24. *Let $\alpha : A(\mathcal{C}, X) \to A(\mathcal{D}, Y)$ be a bounded-spread isomorphism. Then for any commutative, connected Q-systems $L \in \mathcal{Z}(\mathcal{C})$ and $K \in \mathcal{Z}(\mathcal{D})$, extensions of α to bounded-spread isomorphisms $\widetilde{\alpha} : A(\mathcal{C}_L, X) \to A(\mathcal{D}_K, Y)$ are in bijective correspondence with algebra isomorphisms $\gamma : \mathrm{DHR}(\alpha)(L) \cong K$ in $\mathcal{Z}(\mathcal{D})$. As a consequence:*

1. *If $\mathrm{DHR}(\alpha)(L)$ is not isomorphic to K, then α has no extension to a bounded-spread isomorphism $\widetilde{\alpha} : A(\mathcal{C}_L, X) \to A(\mathcal{D}_K, Y)$*
2. *If $\mathrm{DHR}(\alpha)(L) \cong K$, spatial implementations form a torsor over $\mathrm{Aut}(L)$.*

Proof. Let $L \in \mathcal{Z}(\mathcal{C})$ be a commutative, connected Q-system. Then $A(\mathcal{C}_L, X)$ is algebraically a bimodule over the quasi-local algebra $A(\mathcal{C}, X)$ obtained by including the latter as a subalgebra of the former. The *canonical condition expectation* equips $A(\mathcal{C}_L, X)$ with the structure of a right Hilbert $A(\mathcal{C}, X)$ module, hence $(A(\mathcal{C}_L, X), E) \in \mathrm{Bim}(A(\mathcal{C}, X))$. In fact, from the canonical embedding $\mathcal{Z}(\mathcal{C}) \cong \mathrm{DHR}(A(\mathcal{C}, X)) \subseteq \mathrm{Bim}(A(\mathcal{C}, X))$, we have that the quasi-local algebra with condition expectation $A(\mathcal{C}_L, X), E)$ is simply the Q-system realization $|L|$ of $L \in \mathcal{Z}(\mathcal{C})$, and in particular $(A(\mathcal{C}_L, X), E) \in \mathrm{DHR}(A(\mathcal{C}, X))$ is a connected, commutative Q-system.

First, suppose $\widetilde{\alpha} : A(\mathcal{C}_L, X) \to A(\mathcal{D}_K, Y)$ is a bounded-spread isomorphism, restricting to α on the subsystem $A(\mathcal{C}, X)$. We claim that $\widetilde{\alpha}$ *is* an algebra object isomorphism

$\mathrm{DHR}(\alpha)(A(\mathcal{C}_L, X)) \cong A(\mathcal{D}_K, X)$. Indeed, $\mathrm{DHR}(\alpha)(A(\mathcal{C}_L, X))$ is simply $A(\mathcal{C}_K, X)$ with left and right action twisted by α^{-1}. We check that for $a \in A(\mathcal{C}_L, X)$ and $b, c \in A(\mathcal{D}, Y)$:

$$\widetilde{\alpha}(b \triangleright a \triangleleft c) = \widetilde{\alpha}(\alpha^{-1}(b) a \alpha^{-1}(c)) = b\widetilde{\alpha}(a)c,$$

where we have used $\widetilde{\alpha}|_{A(\mathcal{C},X)} = \alpha$. Thus $\widetilde{\alpha} : \mathrm{DHR}(\alpha)(A(\mathcal{C}_L, X)) \cong A(\mathcal{D}_K, X)$ is an isomorphism of $A(\mathcal{D}, Y)$ bimodules. Since it is an algebra isomorphism, this bimodule isomorphism is an isomorphism of algebra objects.

Conversely, suppose we have an isomorphism of algebra objects $\widetilde{\alpha} : \mathrm{DHR}(\alpha)(A(\mathcal{C}_L, X)) \cong A(\mathcal{D}_K, X)$ in the category of $A(\mathcal{D}, Y)$ bimodules. Then reversing the above logic, $\widetilde{\alpha}$ is an algebra isomorphism $A(\mathcal{C}_L, X) \to A(\mathcal{D}_K, Y)$ which restricts to α on the algebra $A(\mathcal{C}, X)$. It remains to show that $\widetilde{\alpha}$ is bounded spread.

Suppose α has spread at most R. We claim $\widetilde{\alpha}$ has spread at most R. Indeed, let $a \in A(\mathcal{C}_L, X)_I$. Note that $A(\mathcal{D}, Y)_{\mathbb{Z}\setminus(I\pm R)} \subseteq \alpha(A(\mathcal{C}, X)_{\mathbb{Z}\setminus I})$, and since $[a, b] = 0$ for all $b \in A(\mathcal{C}, X)_{\mathbb{Z}\setminus I}$:

$$0 = [\widetilde{\alpha}(a), \widetilde{\alpha}(b)] = [\widetilde{\alpha}(a), \alpha(b)],$$

and hence:

$$[\widetilde{\alpha}(A(\mathcal{C}_L, X)_I), A(\mathcal{D}, Y)_{\mathbb{Z}\setminus(I\pm R)}] = 0.$$

By Lemma 3.21, this implies:

$$\widetilde{\alpha}(A(\mathcal{C}_L, X)_I) \subseteq A(\mathcal{C}_K, X)_{I\pm R},$$

so that $\widetilde{\alpha}$ is bounded spread. This concludes the proof of the main statement of the theorem.

The first consequence listed above follows immediately. The second follows from the fact that for any isomorphisms $\gamma, \gamma' : \mathrm{DHR}(\alpha(L)) \cong K$, $\gamma^1 \circ \gamma' \in \mathrm{Aut}(\mathrm{DHR}(\alpha(L)) \cong \mathrm{Aut}(L)$. $\square$

We recommend the reader to compare the above theorem with an analogous result concerning extending symmetries in (1+1)D algebraic quantum field theory, [BJLP19, Section

6].

Proof of Theorem 3.3. The Theorem 3.3 stated in the introduction follows as a special case of the above, and we believe it provides a satisfactory answer to Question **??**. Indeed, consider the case where $L \in \mathcal{Z}(\mathcal{C})$ and $K \in \mathcal{Z}(\mathcal{D})$ are $Z(\mathcal{M})$ and $Z(\mathcal{N})$ associated to the $\mathcal{C}$ and $\mathcal{D}$ module categories, respectively. Then we have identifications of the spatial realizations $A(\mathcal{C}, X)_{\mathcal{M}} \cong A(\mathcal{C}_{Z(\mathcal{M})}, X)$, whence the first part of the result follows from the previous theorem. That $\mathrm{Aut}(Z(\mathcal{M})) \cong (\mathcal{C}_{\mathcal{M}}^*)^\times$ follows, for example, from [BJ22]. □

For any algebra $L \in \mathcal{Z}(\mathcal{C})$, there is a group $\mathrm{Aut}_\otimes^{br}(\mathcal{Z}(\mathcal{C}) \mid L)$, which consists of equivalence classes of pairs (α, γ), where α is a braided tensor equivalence of $\mathcal{Z}(\mathcal{C})$ and $\gamma : \alpha(L) \cong L$ is an isomorphism of algebras. An equivalence between pairs (α, γ) and (β, δ) consists of a monoidal natural isomorphism $\nu : \alpha \cong \beta$ such that $\delta \circ \nu_L = \gamma$. By [DMNO13; Sch24], for a Lagrangian algebra L corresponding to an indecomposable $\mathcal{C}$-module category $\mathcal{M}$, we have:

$$\mathrm{Aut}_\otimes^{br}(\mathcal{Z}(\mathcal{C}) \mid L) \cong \mathrm{Aut}_\otimes(\mathcal{C}_{\mathcal{M}}^*),$$

where the latter is the group of tensor autoequivalences up to monoidal natural isomorphism.

Denote by $\mathrm{QCA}(\mathcal{C}, X, \mathcal{M})$ the group of quantum cellular automata on $A(\mathcal{C}, X)_{\mathcal{M}}$ which preserve the subalgebra $A(\mathcal{C}, X)$. We can think of this as the group of spatially implemented dualities, or as the group of "edge-restricted" QCA that preserve the symmetric subalgebra. Then the above theorem yields the following corollary:

Corollary 3.25. *There is a homomorphism* $\pi : \mathrm{QCA}(\mathcal{C}, X, \mathcal{M}) \to \mathrm{Aut}_\otimes(\mathcal{C}_{\mathcal{M}}^*)$.

We can easily choose examples so that this is surjective: consider the object $X = \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} Y$. Then $\pi : \mathrm{QCA}(\mathcal{C}, X, \mathcal{C}) \to \mathrm{Aut}_\otimes(\mathcal{C})$ is surjective (see [Jon24]). A family of examples of interest, which arise from onsite finite group symmetry, are when $\mathcal{C} = \mathrm{Rep}(G)$, $\mathcal{M} = \mathrm{Hilb}$, made into a $\mathcal{C}$-module category with the standard fiber functor structure on $\mathrm{Rep}(G)$. In this case, we obtain a homomorphism:

$$\pi : \mathrm{QCA}(\mathrm{Rep}(G), X, \mathrm{Hilb}) \to \mathrm{Aut}_\otimes(\mathrm{Hilb}(G)) \cong H^2(G, \mathrm{U}(1)) \rtimes \mathrm{Aut}(G).$$

3.5 Examples and Applications

In this section, we consider a family of examples generalizing Kramers-Wannier (KW) dualities [LDOV23], and we also discuss applications of our previous result.

3.5.1 Generalized shifts

Let $\mathcal{C}$ be a unitary fusion category, and $\alpha \in \mathrm{Aut}_{\otimes}^{br}(\mathcal{Z}(\mathcal{C}))$, such that autoequivalence α generates a categorical action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{Z}(\mathcal{C})$ for some n. Then the o_4 obstruction necessarily vanishes as $H^4(\mathbb{Z}/n\mathbb{Z}, U(1))$ is trivial, and thus by [ENO10] there exists a $\mathbb{Z}/n\mathbb{Z}$-graded extension:

$$\mathcal{D} := \bigoplus_{g \in \mathbb{Z}/n\mathbb{Z}} \mathcal{C}_g,$$

with $\mathcal{C}_e = \mathcal{C}$. For any $Y \in \mathcal{C}_\alpha$, $Y^{\otimes n} \in \mathcal{C}_e = \mathcal{C}$. We can always choose Y such that $X := Y^{\otimes n}$ is a strong tensor generator for $\mathcal{C}$, for example, $Y = \bigoplus_{Z \in \mathrm{Irr}(\mathcal{C}_\alpha)} Z$.

Now, consider the fusion spin chain $A(\mathcal{C}, X)$. We define a "generalized translation" QCA by mapping $a \in A(\mathcal{C}, X)_{[i,j]} \cong \mathrm{Hom}_{\mathcal{C}}(X^{\otimes(j-i+1)}, X^{\otimes(j-i+1)}) \cong \mathrm{Hom}_{\mathcal{D}}(Y^{\otimes n(j-i+1)}, Y^{\otimes n(j-i+1)})$ to:

$$\tau_\alpha(a) := 1_{Y^{n-1}} \otimes a \otimes 1_Y \in A(\mathcal{C}, X)_{[i-1,j]}.$$

Then, since for any $Z \in \mathcal{Z}(\mathcal{C})$ we have natural isomorphisms $Y \otimes Z \cong \alpha(Z) \otimes Y$, it is easy to verify that:

$$\mathrm{DHR}(\tau_\alpha) \cong \alpha.$$

Thus, any braided autoequivalence of $\mathcal{Z}(\mathcal{C})$ can be implemented by a QCA on the some fusion spin chain of the form $A(\mathcal{C}, X)$, although the choice of X depends on the autoequivalence.

3.5.2 Dualities from Tambara-Yamagami

The easiest example is the case $\mathcal{C} := \mathrm{Hilb}(A)$, where A is a finite abelian group, in which case $\mathcal{Z}(\mathcal{C}) \cong \mathrm{Hilb}(A \times \widehat{A})$. Any symmetric, non-degenerate bicharacter $\chi : A \times A \to U(1)$, yields an isomorphism $\widetilde{\chi} : A \to \widehat{A}$, given by $\widetilde{\chi}(x) = \chi(a, \cdot)$. This results in a braided autoequivalence $\alpha_\chi \in \mathrm{Aut}_\otimes^{br}(\mathrm{Hilb}(A \times \widehat{A}))$ defined by:

$$\alpha_\chi(a, f) = (\widetilde{\chi}^{-1}(f), \widetilde{\chi}(a)).$$

Further, this results in a $\mathbb{Z}/2\mathbb{Z}$ extension of $\mathrm{Hilb}(A)$ called a *Tambara-Yamagami* category [TY98], whose simple objects are simply $A \cup \{\rho\}$ with fusion rules:

$$\begin{aligned} a \otimes b &= ab, \\ a \otimes \rho &= \rho = \rho \otimes a, \\ \rho \otimes \rho &\cong \bigoplus_{a \in A} a. \end{aligned}$$

If we set $Y = \rho$, then $X := Y^2 = \bigoplus_{a \in A} a$ is a strong tensor generator for $\mathcal{C}$, and thus the above construction yields a QCA on the spin chain:

$$\tau_{\alpha_\chi} : A(\mathcal{C}, X) \to A(\mathcal{C}, X),$$

with $\mathrm{DHR}(\tau_{\alpha_\chi}) \cong \alpha_\chi$. For $A = \mathbb{Z}_2$, there is a unique symmetric, non-degenerate bicharacter χ, and τ_{α_χ} is exactly the Kramers-Wannier duality map.

There are two canonical spatial realizations of $A(\mathcal{C}, X)$ described by the Lagrangian algebras $L_1 := \mathbb{C}[A] \times 1$ and $L_2 := 1 \times \mathbb{C}[\widehat{A}]$, sometimes called electric and magnetic algebras. Then α_χ swaps L_1 and L_2. In particular, by Theorem 3.3, τ_{α_χ} cannot extend to a QCA on either spatial realization $A(\mathcal{C}_{L_1}, X)$ or $A(\mathcal{C}_{L_2}, X)$.

3.5.3 Q-system complete fusion categories

A unitary fusion category $\mathcal{C}$ is called *Q-system complete* (also called *torsion-free*) if there is exactly one indecomposable module category (namely, $\mathcal{C}$ itself) up to equivalence. This terminology comes from the fact that the fusion category $\mathcal{C}$, thought of as a 2-category with one object, is Q-system complete in the sense of [CPJP22]. Examples include Fib and more generally $\mathrm{PSU}(2)_{2k+1}$, and $\mathrm{Hilb}(\mathbb{Z}/p\mathbb{Z}, \omega)$, where $[\omega] \in H^3(\mathbb{Z}/p\mathbb{Z})$ is non-trivial.

For Q-system complete fusion categories $\mathcal{C}$, any fusion spin chain of the form $A(\mathcal{C}, X)$ has only one spatial realization $A(\mathcal{C}, X)_{\mathcal{C}}$. This unique spatial realization in turn corresponds to the one unique isomorphism class of Lagrangian algebra object in $\mathcal{Z}(\mathcal{C})$. This results in the following corollary of Theorem 3.3.

Proposition 3.26. *Let $\mathcal{C}$ be a Q-system complete fusion category. Suppose $\alpha : A(\mathcal{C}, X) \to A(\mathcal{D}, Y)$ is a bounded-spread isomorphism between fusion spin chains. Then:*

1. *$\mathcal{C} \cong \mathcal{D}$ as fusion categories and in particular $\mathcal{D}$ is Q-system complete.*
2. *There exists a spatial implementation $\widetilde{\alpha} : A(\mathcal{C}, X)_{\mathcal{C}} \to A(\mathcal{D}, Y)_{\mathcal{D}}$ of α.*
3. *The spatial implementations form a torsor over $\mathrm{Inv}(\mathcal{C})$. In particular, if $\mathcal{C}$ is has no non-trivial invertible objects (e.g $PSU(2)_{2k+1}$), then there is a unique spatial implementation.*

Proof. Let L denote the unique Lagrangian algebra object in $\mathrm{DHR}(A(\mathcal{C}, X)) \cong \mathcal{Z}(\mathcal{C})$, so that $\mathcal{C} \cong \mathcal{Z}(\mathcal{C})_L$. Then $\mathrm{DHR}(\alpha)(L) \in \mathrm{DHR}(A(\mathcal{D}, Y)) \cong \mathcal{Z}(\mathcal{D})$ is the unique Lagrangian algebra (since $\mathrm{DHR}(\alpha)$ is a braided equivalence). In particular, we have $K := \mathrm{DHR}(L)$ must be the canonical Lagrangian algebra in $\mathcal{Z}(\mathcal{D})$, hence $\mathcal{Z}(\mathcal{D})_K \cong \mathcal{D}$, and thus $\mathrm{DHR}(\alpha)$ induces a monoidal equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{C})_L \to \mathcal{Z}(\mathcal{D})_K \cong \mathcal{D}$.

Since the spatial realizations must correspond to the Lagrangian algebras L and K respectively, the existence of a spatial implementation of α, and the statement about the torsor, follows immediately from Theorem 3.3. □

3.5.4 Classification of QCA

In this section, apply our above analysis to give a better understanding of QCA which commute with a group symmetry, and connect our perspective to the literature.

Suppose we have a group G acting unitarily on the local Hilbert space $V := \mathbb{C}^d$, given by an assignment $g \mapsto U_g \in U(d)$. Let A denote the associated spin system $\{A_I = \otimes_I \mathrm{Mat}_{d\times d}(\mathcal{C})\}$. This action extends to the automorphism of each A_I by $\alpha_g(a) := (U_g)^{\otimes n} a (U_g^*)^{\otimes n}$, which extends to a homomorphism $G \to \mathrm{Aut}(A)$. We assume this homomorphism is injective.

In the literature, there have been several investigations into *symmetric QCA (sQCA)* [GSSC20], which in our language are bounded-spread isomorphisms $\alpha : A \to A$ such that $\alpha \circ \alpha_g = \alpha_g \circ \alpha$ for all $g \in G$. Our goal in this section is to make contact with these results from our perspective.

First, we recast this picture into the language of fusion spin chains by noting that A^g has the structure of an abstract spin chain with local algebras $A_I^g := \{x \in A_I \ : \alpha_g(x) = x \text{ for all } g \in G\}$.

Now if we view V as an object in $\mathrm{Rep}(G)$, and we further assume that V is self-dual and strongly tensor generating (e.g. $V \cong L^2(G)$ is the left regular representation), then

$$A^g \cong A(\mathrm{Rep}(G), V).$$

The inclusion $A^g \subseteq A$ is a spatial realization which corresponds to the standard fiber functor $\mathrm{Rep}(G) \to \mathrm{Hilb}$. This leads to the following natural question:

Question 3.27. Suppose $\alpha : A^g \to A^g$ is a duality and $\widetilde{\alpha} : A \to A$ is a spatial implementation. Under what conditions is the spatial implementation $\widetilde{\alpha}$ a symmetric QCA?

To answer this question, note that the Lagrangian algebra resulting from this spatial realization in $\mathcal{Z}(\mathrm{Rep}(G))$ can actually be viewed as an algebra object in $\mathrm{Rep}(G) \subseteq \mathcal{Z}(\mathrm{Rep}(G))$, where we identify $\mathrm{Rep}(G)$ with the full subcategory of its center using the standard symmetric braiding. The algebra is simply $L := \mathrm{Fun}(G)$, the commutative algebra of $\mathbb{C}$-valued functions, viewed as a G-representation where G acts by translation [EGNO16].

Theorem 3.28. *Suppose $\alpha : A^g \to A^g$ is a duality and $\widetilde{\alpha} : A \to A$ is a spatial implementation, and $\gamma : \mathrm{DHR}(\alpha)(L) \cong L$ is the associated algebra isomorphism. Then*

1. *Map $\widetilde{\alpha}$ is a symmetric QCA if and only if for every $\beta \in \mathrm{Aut}(L)$, $\gamma \circ \mathrm{DHR}(\alpha)(\beta) = \beta \circ \gamma$.*
2. *There is a homomorphism from the group of sQCA on A to $H^2(G, U(1))$.*

Proof. A, as an algebra object (Q-system) in $\mathrm{Bim}(A^g)$, is isomorphic to the Q-system realization $|L|$ of the function algebra $\mathrm{Fun}(G) \in \mathrm{Rep}(G) \subseteq \mathcal{Z}(\mathrm{Rep}(G))$, and thus we have that the group $\mathrm{Aut}(A|A^g) \leq \mathrm{Aut}_{\mathrm{Rep}(G)}(A)$ of automorphisms of A that are the identity on A^g, is isomorphic to $\mathrm{Aut}_{\mathrm{Rep}(G)}(L) = G$. But the group action $g \mapsto \alpha_g$ is an injective homomorphism $G \to \mathrm{Aut}(A|A^g) \leq G$, which therefore must also be surjective and we must have that $\mathrm{Aut}(A|A^g) = G$ since G is finite.

Furthermore, since $\mathrm{Hilb}(G)$ is the dual category of $\mathrm{Rep}(G)$ associated to the Lagrangian L, we have that the linear span $\mathbb{C}[\mathrm{Aut}_{\mathrm{Rep}(G)}(L)] = \mathrm{End}_{\mathrm{Rep}(G)}(L)$, by [BJ22]. This proves part (1). Part (2) follows from [Sch24]. □

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