EFFICIENT FORMULATION OF THE FINITE ELEMENT METHOD FOR HEAT CONDUCTION IN SOLIDS

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SUMMARY

The purpose of the paper is to describe efficient methods and computer programs for analysis of heat conduction problems related to design and control of components of nuclear power plants and similar structures where thermal problems are of interest.

A short presentation of basic equations and the finite element formulation of three-dimensional stationary and transient heat conduction is given. The finite element types that are used are isoparametric hexahedrons with eight or twenty nodes. The use of consistent as well as diagonal capacity matrices is discussed.

Reduction of the transient heat conduction problem may be accomplished by means of the "master-slave" technique. Furthermore, the superelement technique is discussed for both stationary and transient heat conduction. For the solution of transient problems, the trapezoidal time integration scheme is used.

The methods and principles outlined in the paper are materialized in a computer program, NV615, which is one of the application programs in the program system SESAM-69. A brief description is given of NV615.

Furthermore, attention is given to combined heat conduction and subsequent thermal stress analysis. Data representing geometry, calculated temperature distribution etc. may be transferred automatically from the heat conduction program to stress analysis programs (other application programs in SESAM-69).

In order to investigate the accuracy and the efficiency of the methods, some heat conduction problems for which analytical solutions are known are analysed. In the paper the results from the following calculations are considered:

a) Semi-infinite solid with specified surface temperature, varying periodically with time.
b) Semi-infinite solid with line heat source. Transient analysis.

As an example of practical application the temperature distribution versus time in a turbine wheel during start up is analysed. Thermal stresses are calculated at selected time instants.

It is demonstrated that the numerical solutions converge to analytical solutions. Finite element models of twenty-node elements converge faster than models in which eight-node elements are used. However, if the heat conduction analysis is part of a thermal stress problem, a model composed of eight-node elements for the heat conduction may prove advantageous in terms of economy of the total analysis. In this case the same finite element model of twenty-node elements is used for the stress calculation.

The examples also show that the use of superelements may reduce the computer time in solving heat conduction problems. Further, it is demonstrated that the superelement technique is a very flexible and user-orientated tool in analysing practical thermal problems. In particular, this is the case for combined heat conduction and thermal stress analysis.

1. Introduction

The finite element method has become an indispensable tool in design and control analysis of nuclear reactor components and associated equipment. Originally, the finite element method was used primarily in linear static stress analysis. Further development has led to recognition of its applicability also to analysis of a wide range of other problems in mathematical physics. Examples of such problems are structural dynamics, heat conduction, elasto-plastic analysis, fracture mechanics, etc.

The treatment of a given type of problem by the finite element method consists of two major aspects:

- Formulation of element relationships
- Solution of the complete system

Consequently, it is of great importance that both suitable element types and efficient solution techniques are available.

Finite element calculations are known to be expensive, computer time being a major cost factor. However, the manual efforts in preparing input data and checking results are also considerable. In thermal stress analyses, an automatic procedure for coupling the heat conduction calculation and the stress analysis is therefore highly desirable.

The present paper will concentrate on practical procedures for solving heat conduction problems using the finite element method.

2. Basic equations

Heat conduction, distribution of electric potential, diffusion of vapour and many other physical problems are described by the same type of differential equations. Accordingly, methods and computer programs developed may be used in solving any of these problems.

The differential equation for the heat conduction problem is:

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right) + q - c\rho\frac{\partial u}{\partial t} = 0 \qquad (1)$$

where

u	is the temperature
x, y, z	are the Cartesian coordinates
t	is the time
k	is the thermal conductivity
q	is the rate of internal heat generation
$c\rho$	is the heat capacity

For stationary heat conduction $\partial u/\partial t = 0$ and eq. (1) is reduced to Poisson's equation.

The boundary conditions to be satisfied are:

- Prescribed surface temperature
- Prescribed heat flux through the surface
- Convection at the surface

For transient heat conduction the initial temperature distribution has to be specified.

3. Finite element formulation of the heat conduction problem

Finite element formulations of the heat conduction problem are given in many textbooks, e.g. [1, 2]. In the initial applications of the method the tetrahedral elements were used [3]. In recent years the isoparametric elements has become popular and the power of these elements is well documented both for heat conduction and stress analysis, [4 - 6].

Efficient solution of the "system equations" for stationary heat conduction is accomplished by using Choleski's method. For transient heat conduction numerical integration algorithms are used for direct integration of the system of differential equations [7].

In order to reduce the number of unknowns (degrees of freedom) in the final system of equations, the superelement formulation of the finite element method [8] is adapted. For transient heat conduction the superelement reduction of the capacity matrix (see [1]) introduces an additional approximation. Several procedures have been suggested for this reduction [9]. Among these the "master-slave" technique has gained high popularity.

The computational effort involved in the reduction of a consistent (banded) capacitance matrix is considerable. However, this may be very much reduced by using a diagonal capacity matrix.

The concepts related to the superelement technique for stationary as well as transient heat conduction have been presented in [5].

For nonlinear problems in which the nonlinear behaviour is confined to a small part of the component, a substantial amount of computer time may be saved by reducing the total size of the problem to merely include the nonlinear part of the component. Mathematically this can conveniently be carried out by means of a modified superelement formulation [10, 5].

The methods and principles outlined above are materialized in a computer program, NV615, which is one of the application programs in the SESAM-69 program system. Eight-node and twenty-node isoparametric hexahedron elements with consistent or diagonal capacity matrices are available. The trapezoidal scheme is adapted in the program.

In order to automate the analyses of thermal stresses, data representing geometry, calculated temperature distributions etc. may be transferred automatically from the heat conduction program to stress analysis programs, (other application programs in SESAM-69).

4. Numerical examples

4.1 Semi-infinite solid with specified surface temperature, varying periodically with time

Consider a semi-infinite solid extending from the surface at x = 0 to infinity, and where the surface temperature is the cosine function

$$u_o = u_{om} \cos\frac{2\pi nt}{t_o} \tag{2}$$

where u_{om} is the maximum absolute value of the surface temperature variation, n is the positive integer numbers and t_o is the period. Then the temperature distribution is given by:

$$u = u_{om} e^{-\sqrt{\frac{n\pi}{at_o}}\, x} \cos\left(\sqrt{\frac{n\pi}{at_o}}\, x - \frac{2\pi nt}{t_o}\right) \tag{3}$$

where a is the thermal diffusivity ($a = k/\rho c$).

Models utilizing eight-node as well as twenty-node isoparametric elements (one degree of freedom per node) were analyzed using different element idealizations as illustrated in Fig. 1.

Fig. 1 shows the results of four calculations. The integration time-steps was $t_o/16$ for all calculations.

The results show that if the number of degrees of freedom in the direction normal to the free surface is the same, twenty-node element models are more accurate than eight-node models. Furthermore, it is seen that results obtained with both element types converge to the analytical solution with increased fineness of the element mesh.

4.2 Semi-infinite solid with line heat source

Consider a semi-infinite solid with zero initial temperature, the surface temperature kept at zero temperature and heated by a line source with constant intensity q_o at a distance d from the surface. The temperature distribution is then given by:

$$u = -\frac{q_o}{4\pi a}\left[\mathrm{Ei}\left(-\frac{r_1^2}{4at}\right) - \mathrm{Ei}\left(-\frac{r_2^2}{4at}\right)\right] \tag{4}$$

where r_1 and r_2 are as defined in Fig. 2a. Fig. 2b illustrates the temperature distribution defined by eq. (4).

Different procedures for reducing the number of degrees of freedom of the system during time integration were considered. Only eight-node elements were used in this investigation.

For the finite element models shown in Figs. 3a and 3b the element meshes are identical. The number of master degrees of freedom is, however, different. For the model shown in Fig. 3c the fineness of the element mesh is increased significantly. The number of master degrees of freedom is, however, identical for the models shown in Figs. 3a and 3c.

The calculations are carried out for the case d = 4 m (see Fig. 2) $q_o = 1$ W/m, the thermal conductivity $k = 1.0 \cdot 10^{-2}$ W/mK and the capacity $\rho c = 1.0 \cdot 10^{-3}$ T/m^{3}K. The results from the different calculations (temperature versus time at the point x = 4, y = 1) are plotted in Figs. 8 and 9. Based on the results the following conclusions can be drawn:

- If the number of master degrees of freedom is constant, the accuracy of the calculation is increased if the total number of degrees of freedom is increased.
- If the total number of degrees of freedom is constant, the accuracy of the calculation is increased if the number of master degrees of freedom is increased.
- If the total number of degrees of freedom and the number of master degrees of freedom is constant, consistent capacity matrices are more accurate than lumped capacity matrices.
- Computer time is reduced considerably using the master-slave technique.

4.3 Turbine wheel

The purpose of this analyses was to predict temperature distribution and thermal stresses in the turbine wheel of the KG5 radial gasturbine designed by Kongsberg Våpenfabrikk A/S. The turbine wheel is shown in Fig.5. Due to symmetry only one half of one turbine blade with a sector of the hub was considered in the analysis.

As shown in Fig. 8 two first level superelements were assembled to one second level superelement giving the complete model. For this analysis the stresses in the transition region between the hub and blade was of main interest. The mesh fineness is therefore increased in this region. Outside this region the element mesh is rather coarse.

It was assumed that the accuracy of computed stresses using the twenty-node isoparametric hexahedron element (three degress of freedom per node) was of the same order as the temperature in the eight-node hexahedron. Therefore, since an automatic transfer of data from the heat conduction program to the stress analysis program requires the same finite element subdivision (but not the same element types), it was found economical to use the eight-node element for the heat conduction analysis and the twenty-node element for the stress analysis. Number of degrees of freedom, computer time, etc. for both the heat conduction and the stress calculations are given in Table 1.

Convection applied at the shaded surfaces of Fig. 6 was used as boundary conditions for the heat conduction calculations. By specifying the relative stagnation temperature and the appropriate heat transfer coefficient at the nodes, the variation of these quantities over the surface is taken care of. They are, on the other hand, specified independently of time and surface temperature.

The conductivity and the capacity are assumed constant but slightly different for hub and blade. These values correspond to the average temperature for each of the two first level superelements during the heating up period.

The calculated temperature distributions at 60 sec. and 360 sec. after starting up are shown in Figs. 9 and 10 respectively. The isotherms are plotted for the shaded surfaces indicated in Fig. 7.

Immediately after starting up the heat flow is normal to the surface subjected to convection. After 60 sec. the heat flows more radially towards the axis of rotation, and this radial flow is still more dominating 360 sec. after the starting up.

The accuracy of the calculations has been evaluated by means of alternative models of critical details. Except for the first few seconds after start up, the accuracy of the calculation was found to be good (within 10 percent deviation from check results). The deviation immediately after the starting up is caused by the coarse element mesh in the blade. This element mesh is much too coarse for an accurate representation of the heat flow normal to the surface of the blade.

The discussion of the stress analysis falls outside the scope of this paper.

5. Conclusions

Various solution procedures for analysing heat conduction in solids using the finite element method have been considered. Numerical examples using eight-node and twenty-node isoparametric hexahedron elements as well as the master-slave technique (superelement technique) have been discussed.

It is demonstrated that the numerical solutions converge to analytical solutions. Finite element models of twenty-node elements converge faster than models in which eight-node elements are used. However, if the heat conduction analysis is part of a thermal stress problem, a model composed of eight-node elements for the heat conduction may prove advantageous in terms of economy of the total analysis. In this case the same finite element model of twenty-node elements is used for the stress calculation.

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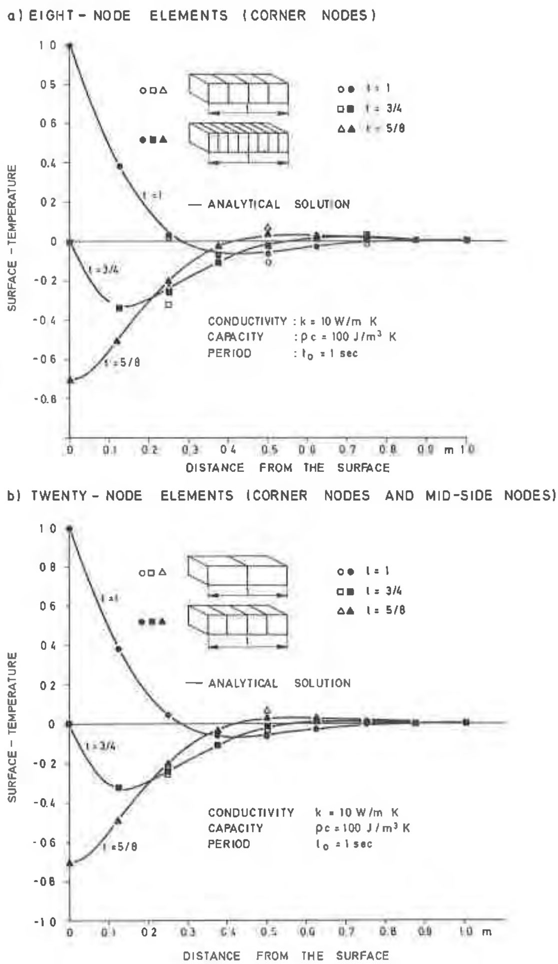

Fig. 1. Oscillating temperature in a semi-infinite solid. Comparison between different element types and different element idealizations.

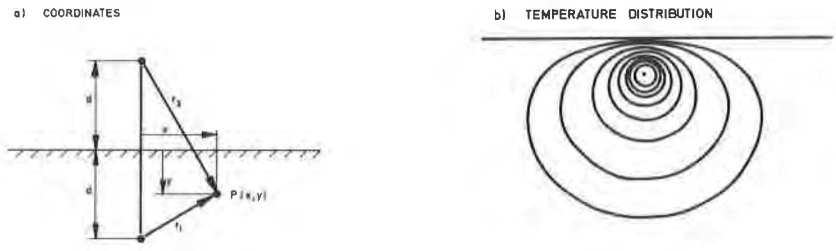

Fig. 2. Semi-infinite solid with line heat source.

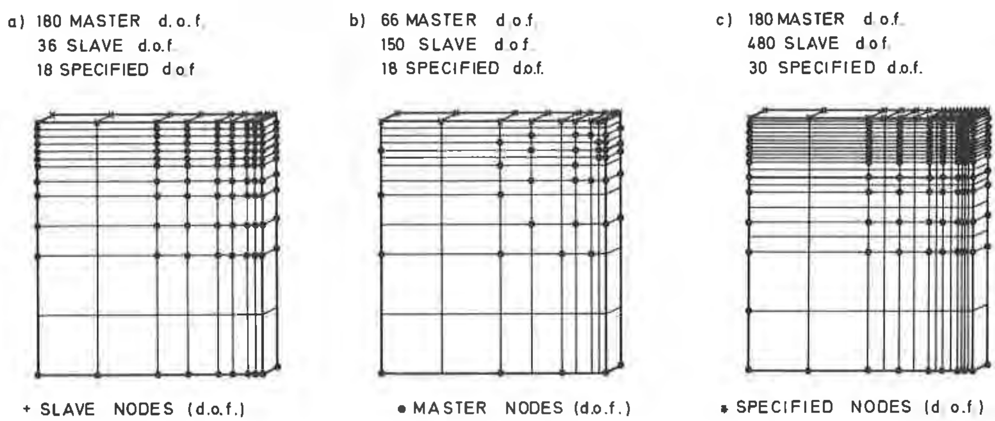

Fig. 3. Semi-infinite solid. Element meshes.

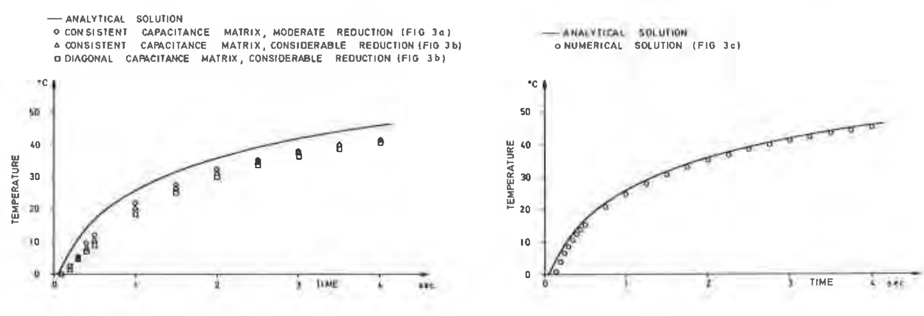

Fig. 4. Semi-infinite solid with line heat source,
analytical and numerical solutions.

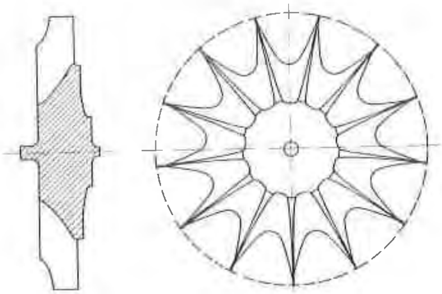

Fig. 5. Turbine wheel

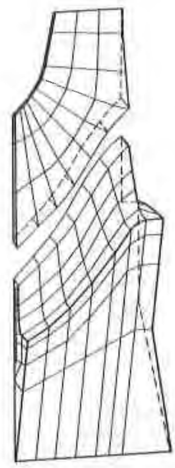

Fig. 8. Element mesh

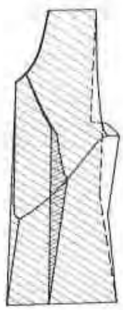

Fig. 6.
Boundary conditions

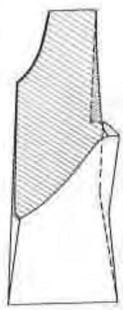

Fig. 7. Surfaces where results are presented

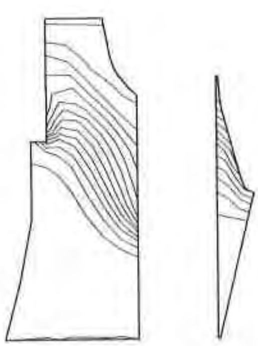

Fig. 9. Temperature distribution 60 sec. after starting up

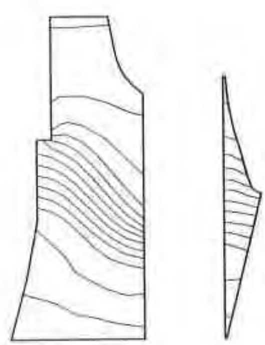

Fig. 10. Temperature distribution 360 sec. after starting up

TABLE I. COMPUTING INFORMATION

	Heat cond. analyses	Stress analyses
No. of elements	132	132
No. of nodes	278	933
No. of d.o.f.	278	2799
No. of master d.o.f.	223	-
No. of integration steps (variable length)	48	-
Computer time (CAU min. UNIVAC 1110)	3	24