

**APPLICATION OF FINITE ELEMENT METHOD  
TO THE THREE-DIMENSIONAL STRESS AND VIBRATION ANALYSIS  
OF BWR PRIMARY PLANT SYSTEMS**

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ABSTRACT

This paper presents the material non-linear stress analysis and the vibration analysis for the boiling water reactor (BWR) plant by a three-dimensional finite element method. This method is very powerful for both analyses. However, the big problem is to solve the large matrix whose order is, say 10,000 x 10,000 to 100,000 x 100,000 (The author calls it the Jumbo Matrix). It becomes so large because not only the shapes of the structures but the load and boundary conditions in the BWR plant are very complicated.

Following two methods have been used in order to analyse such complex system.

- (1) Direct method from element matrices using the conjugate gradient method
- (2) Band width method

The three-dimensional finite element method gives good results by these methods. The band width method is omitted in this paper because it is well-known.

1. Introduction

The reliability and the safety under severe conditions have been demanded with the rapid advancement of science. Therefore, more precise analyses have been required. The finite element method has made a remarkable progress by such strong demand and by the improvement of a computer.

The temperature and the pressure in structures of the boiling water reactor plant are so high that stresses are very severe. Besides, there is the problem of

earthquakes in Japan. We have to secure the safety of the plant in case of the big earthquake. Thus, the aseismic design has become important.

This fact means that not only the stress but the vibration analyses are required. In this paper emphasis will be given to the stress analysis for a complex system such as the EWR plant.

## 2. Finite Element Method

The finite element method has been applied to several fields, for example, many non-linear problems, field problems, and the diffusion equation of neutrons in the reactor core. Non-linear problems are not very successful from a mathematical point of view. Recently it has become more comfortable by exploiting the finite element method to solve non-linear problems.

As far as the structural mechanics is concerned, this method is applicable to material non-linearity problems and geometrical non-linearity problems.

As a future problem it is expected to get solutions for mixed phenomena such as the viscous flow and the non-linear heat transfer, though it has been not too much successful so far.

The finite element method is basically a special form of Ritz analysis and as such, it provides a means for discretization of continuum problems by expressing their displacements as a finite series of displacement functions. The major advantage of this method of discretization lies in the fact that continua of arbitrary shapes and variations of properties can be approximated as a system of finite elements having simple shapes and uniform (or simply varying) properties. Therefore, a standard set of interpolation functions appropriate to the chosen finite element shapes will serve to represent displacements in completely arbitrary structures.

Stiffness matrix and mass matrix are given as follows by the finite element method.

$$[K]_e = [A^{-1}]^T [ \int \int \int [B]^T [D] [B] dV ] [A^{-1}] \quad (2.1)$$

$$[M]_e = [A^{-1}]^T [ \int \int \int [S]^T \rho [S] dV ] [A^{-1}] \quad (2.2)$$

Where

- [K]<sub>e</sub> : Element Stiffness Matrix
- [M]<sub>e</sub> : Element Mass Matrix
- [A] : Co-ordinate Matrix
- [B] : Strain Coefficient Matrix
- [D] : Stress Strain Matrix
- [S] : Deflection Coefficient Matrix
- ρ : Mass Density

### 3. Stress Analysis

The stress analysis could be done by eq. (3.1) when the strain is solved by the deflection. The deflection is solved by eq. (3.2).

$$\{d\sigma\} = [D] \{d\varepsilon\} \quad (3.1)$$

Where

{σ} : Stress

{ε} : Strain

$$\{u\} = [K]^{-1} \{F\} \quad (3.2)$$

Where

{u} : Deflection (Unknown)

{F} : Load Vector

The problem is to solve eq. (3.2) because the order of [K] is very large.

#### (1) Elastic Problem

Hooke's law is applied in the tensor form.

$$\varepsilon_{ij} = \frac{\sigma_{ij}'}{2G} + \delta_{ij} (1-2\nu) \frac{\sigma_{ii}}{3E} \quad (3.3)$$

$\delta_{ij}$  : Kronecker Delta

$E$  : Young Modulus

$G$  : Modulus of Transverse Elasticity

$\nu$  : Poisson Ratio

(2) Plastic Problem (Material Non-Linearity)

Prandtl-Reuss equations are applied instead of Hooke's law.

$$d\varepsilon_{ij}' = \sigma_{ij}' d\lambda + \frac{d\sigma_{ij}'}{2G} \quad (3.4)$$

Where

$$d\lambda = \frac{3 d\bar{\sigma}}{2 \bar{\sigma} H'} \quad (3.5)$$

$$H' = \frac{d\bar{\sigma}}{d\varepsilon^P} \quad (3.6)$$

: Rate of Strain-Hardening

$$\sigma_{ij}' = \sigma_{ij} - \frac{1}{3} \sigma_{ii} \delta_{ij} \quad , \quad \sigma_{ii} = \sigma_x + \sigma_y + \sigma_z \quad (3.7)$$

: Deviatoric Stress

$$\bar{\sigma} = \sqrt{\frac{3}{2}} (\sigma_{ij}' \sigma_{ij}')^{\frac{1}{2}} \quad (3.8)$$

: Generalized Stress

$$\frac{d\varepsilon^P}{d\varepsilon} = \sqrt{\frac{2}{3}} (d\varepsilon_{ij}^P d\varepsilon_{ij}^P)^{\frac{1}{2}} \quad (3.9)$$

: Generalized Plastic Strain Increment

Then the plastic stress strain matrix  $[D^P]$  is the explicit inverse of Prandtl-Reuss equations.

$$d\sigma_{ij} = 2G (d\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} d\varepsilon_{kk} - \sigma_{ij} \frac{1}{S} d\varepsilon_{kk}) \quad (3.10)$$

Where

$$S = \frac{2}{3} \bar{\sigma}^2 \left(1 + \frac{H'}{3G}\right) \quad (3.11)$$

Then in the matrix form

$$[D^P] = \frac{E}{1+\nu} \left[ \begin{array}{cccccc} \frac{1-\nu}{1-2\nu} - \frac{\sigma_x^2}{S} & & & & & \\ & \frac{\nu}{1-2\nu} - \frac{\sigma_x \sigma_y}{S} & \frac{1-\nu}{1-2\nu} - \frac{\sigma_y^2}{S} & & & \\ & \frac{\nu}{1-2\nu} - \frac{\sigma_x \sigma_z}{S} & \frac{\nu}{1-2\nu} - \frac{\sigma_y \sigma_z}{S} & \frac{1-\nu}{1-2\nu} - \frac{\sigma_z^2}{S} & & \\ & -\frac{\sigma_x \tau_{xy}}{S} & -\frac{\sigma_y \tau_{xy}}{S} & -\frac{\sigma_z \tau_{xy}}{S} & \frac{1}{2} - \frac{\tau_{xy}^2}{S} & \\ & -\frac{\sigma_x \tau_{yz}}{S} & -\frac{\sigma_y \tau_{yz}}{S} & -\frac{\sigma_z \tau_{yz}}{S} & -\frac{\tau_{xy} \tau_{yz}}{S} & \frac{1}{2} - \frac{\tau_{yz}^2}{S} \\ & -\frac{\sigma_x \tau_{zx}}{S} & -\frac{\sigma_y \tau_{zx}}{S} & -\frac{\sigma_z \tau_{zx}}{S} & -\frac{\tau_{zx} \tau_{xy}}{S} & -\frac{\tau_{yz} \tau_{zx}}{S} & \frac{1}{2} - \frac{\tau_{zx}^2}{S} \end{array} \right] \quad \text{SYMMETRY} \quad (3.12)$$

(3) Method to solve the large matrix

$$\{u\} = [K]^{-1} \{F\} \quad (3.13)$$

The stiffness matrix  $[K]$  has the order of 10,000 to 100,000 to analyse the complex system such as the BWR plant. The order becomes so large because the element matrices (eq. (2.1)) are assembled.

In order to solve eq. (3.13) the author has used the method by which the stress analysis can be done directly from the element stiffness matrices (eq. (2.1)) using the conjugate gradient method without having the assembled matrix.

Assuming that the initial vector is the force vector.

$$\{u\}_i = \{F\} \quad (3.14)$$

Direction of move

$$\{G\}_{i+1} = (\{F\} - [K] \{u\}_i) \quad (3.15)$$

$$\beta_i = \frac{\{G\}_{i+1}^T \{G\}_{i+1}}{\{G\}_i^T \{G\}_i} \quad (3.16)$$

$$\{S\}_{i+1} = \{G\}_{i+1} + \beta_i \{S\}_i \quad (3.17)$$

Step length along the direction

$$\alpha_i = \frac{\{S\}_i^T \{G\}_i}{\{S\}_i^T [K] \{S\}_i} \quad (3.18)$$

Iteration

$$\{u\}_{i+1} = \{u\}_i + \alpha_i \{s\}_i \quad (3.19)$$

Convergence condition

$$\frac{\sqrt{\sum_{j=1}^n ((u_j)_{i+1} - (u_j)_i)^2}}{\sqrt{\sum_{j=1}^n (u_j)_i^2}} < \epsilon \quad (3.20)$$

Where

i : Iteration Times

This convergence value  $\epsilon$  is recommended to be around 0.001.

Fig. 3.1 shows  $\epsilon$ , namely the convergence of each iteration time in case of the stress analysis for the drywell (the primary containment). The order of the matrix is 9180. The calculation converges at 9 iterations and at 201 iterations to solve the stress due to the pressure and due to lateral loads respectively. The iteration times namely the calculation time completely depends on the initial vector. Therefore, one should select the reasonable initial vector as far as possible.

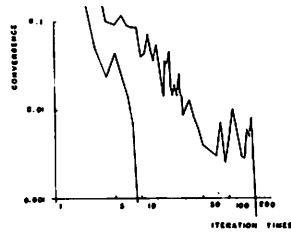


Fig 3.1  
Convergence

#### 4. Eigenvalue Analysis

In case of the aseismic design the eigenvalues should be solved at first. The multi-mass system is normally used to know the eigenvalues of the atomic power plant from the reactor building to the uranium fuel.

The order of the stiffness matrix is around 100. Therefore, we use the Jacobi method, Householder method and Hessenberg method.

Recently the finite element method has become popular to analyse the eigenvalue. However, the difficulty is to solve the large eigenvalue equation. The eigenvalue analysis can be done by the same method as the stress analysis. It is solved directly from the element stiffness and mass matrices without having the large assembled matrices using the conjugate gradient (or steepest descent) method.

Normal eigenvalue equation is

$$[ (K) - \lambda (M) ] (x) = \{ 0 \} \quad (4.1)$$

Where

$\lambda$  : Eigenvalue

$(x)$  : Eigenvector

The large sparse matrices should be required to solve this equation. The energy method is applied in order to avoid the large assembled matrices.

Rayleigh quotient is

$$\lambda = \frac{\{x\}^T (K) \{x\}}{\{x\}^T (M) \{x\}} \quad (4.2)$$

Namely

$$\lambda = \frac{\sum_{k=1}^N \{x\}_k^T (K)_k \{x\}_k}{\sum_{k=1}^N \{x\}_k^T (M)_k \{x\}_k} \quad (4.3)$$

Where

$k$  : Element No.

$N$  : Total Element No.

Then apply the conjugate gradient method.

From now on all matrices and vectors mean those of the element.

$$R(x) = \frac{\{x\}^T (K) \{x\}}{\{x\}^T (M) \{x\}} \quad (4.4)$$

$R(x)$  becomes the eigenvalue when  $\{x\}$  gives the minimum of the function  $R(x)$ .

Gradient of Function

$$\{\Delta R\} = \frac{2(\{K\}\{x\} - R(x)\{M\}\{x\})}{\{x\}^T\{M\}\{x\}} \quad (4.5)$$

Direction of Move

$$\{G\} = \{\Delta R(x)\} \quad (4.6)$$

$$\{S\} = -\{G\} \quad (4.7)$$

Step Length along the Vector

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (4.8)$$

$$\begin{aligned} A &= (\{S\}^T\{K\}\{S\}) (\{x\}^T\{M\}\{S\}) \\ &\quad - (\{x\}^T\{K\}\{S\}) (\{S\}^T\{M\}\{S\}) \\ B &= (\{S\}^T\{K\}\{S\}) (\{x\}^T\{M\}\{x\}) \\ &\quad - (\{x\}^T\{K\}\{x\}) (\{S\}^T\{M\}\{S\}) \\ C &= (\{x\}^T\{K\}\{S\}) (\{x\}^T\{M\}\{x\}) \\ &\quad - (\{x\}^T\{K\}\{x\}) (\{x\}^T\{M\}\{S\}) \end{aligned} \quad (4.9)$$

Iteration

$$\{x_{i+1}\} = \{x_i\} + \alpha_i \{S_i\} \quad (4.10)$$

$$\{G_{i+1}\} = \{\Delta R(x_{i+1})\} \quad (4.11)$$

$$\beta_i = \frac{\{G_{i+1}\}^2}{\{G_i\}^2} \quad (4.12)$$

$$\{S_{i+1}\} = -\{G_{i+1}\} + \beta_i \{S_i\} \quad (4.13)$$

$i$  : Iteration No.

Convergence Condition

$$\frac{\sqrt{\sum_{j=1}^n ((x_j)_{i+1} - (x_j)_i)^2}}{\sqrt{\sum_{j=1}^n (x_j)_i^2}} < \epsilon \quad (4.14)$$

Function  $R(x)$  in equation (4.4) becomes the eigenvalue when the vector  $(x)$  satisfies the convergence condition (4.14).

But Rayleigh Quotient gives only the lowest eigenvalue. In order to solve the higher eigenvalues, Lagrange-Rayleigh Function is applied.

Higher Eigenvalues

Lagrange-Rayleigh Function

$$L_\theta = R(x) - U_1 ((x)^T (e) - 1) - \sum_{i=2}^{\theta} U_i (x)_i^T [M](x)_{i-1} \quad (4.15)$$

Where

$U$ : Lagrange Multiplier

$$\{U\} = ([N_\theta]^T [N_\theta])^{-1} [N_\theta]^T \{G\} \quad (4.16)$$

$$[N_\theta] = [(e), \{[M](x)_I\}, \{[M](x)_II\}, \dots, \{[M](x)_{\theta-1}\}] \quad (4.17)$$

$(e)$ : Unit Vector

$(x)_\theta$ :  $\theta$  - th Eigenvalue

Then the higher eigenvalues are solved by the "Lagrange-Rayleigh Function" when the vectors satisfy the convergence equation (4.14) by the same procedure.

Plotter outputs solved by the finite element method are shown as examples of numerical calculations in the next chapter.

5. Examples

(1) Photoelastic Experiment (Dove Tail of Turbine Blade)

The centrifugal force reaches 200 Ton for the recent plant. The dove tail of the turbine blade should stand for the load and the thermal load. The stress becomes very severe even though the material is 12% chrome steel. The stress analysis by the finite element method (Fig 5.2) has given good results as compared to the photoelastic experiment results (Fig 5.1).



Fig 5.1

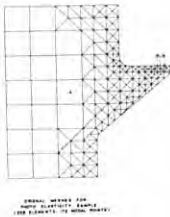
Photoelastic Fringe Patterns



PHOTO ELASTICITY  
STRESS ANALYSIS

Fig 5.2

Stress Analysis by F.E.M.



ORIGINAL MESHES FOR  
PHOTO ELASTICITY STRESS ANALYSIS  
(FOR COMPARISON OF THE RESULTS)

Fig 5.3

Original Meshes for F.E.M.

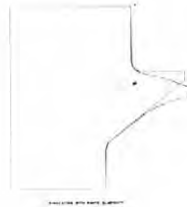


Fig 5.4

Deflection of Dove Tail

(2) Turbine Nozzle Box

The Turbine nozzle box has a complicated shape.

The pressure and temperature are so high that the stress is very severe.



Fig 5.5

Upper Part of Nozzle Box



Fig 5.6

Deflection due to Pressure and Temperature

(3) Cylinder with Uniform Load

This problem has the theoretical solution by Timoshenko.

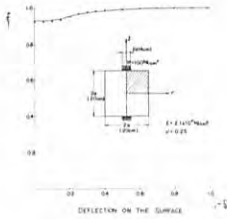


Fig. 5.7

Deflection

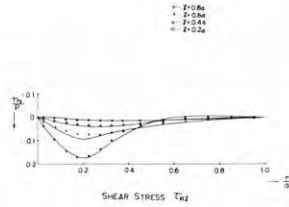


Fig. 5.8

Shear Stress

(4) Vibration Analysis

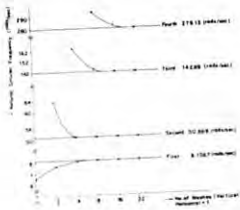


Fig. 5.9

Convergence by No. of Elements

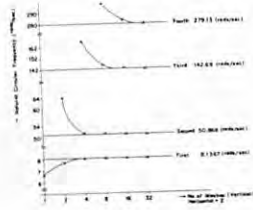


Fig 5.10

Convergence by Element Sizes

(5) Concrete Containment

The new type of the boiling water reactor adopts the concrete containment instead of the drywell made of steel.

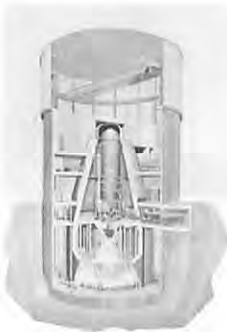


Fig 5.11

New Type Reactor Building  
of B.W.R. Plant

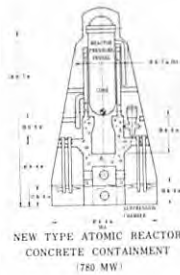
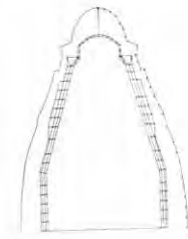


Fig 5.12

Concrete Containment



STRESS ANALYSIS  
-PRESSURE, THERMAL, GRAVITY,  
ATOMIC REACTOR CONCRETE CONTAINMENT

Fig 5.13

Stress Analysis

(6) Drywell and Reactor Pressure Vessel

The stresses of the drywell, the reactor pressure vessel, the skirt, the pedestal, the shield wall and the stabilizer are solved simultaneously.



Fig 5.14  
Drywell (JAPCO-1 TSURUGA)



Fig 5.15  
Reactor Pressure Vessel

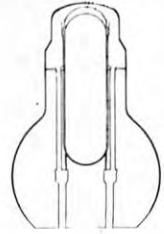


Fig 5.16  
Stress Analysis

(7) Reactor Building



Fig 5.17  
R/B (TEPCO-1,2,3 FUKUSHIMA)



Fig 5.18  
R/B (JAPCO-1 TSURUGA)

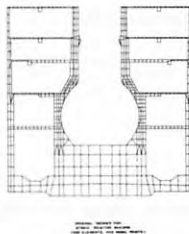


Fig 5.19  
Original Meshes for F.E.M.

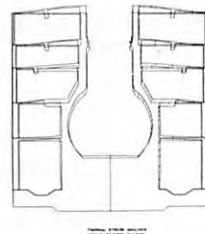


Fig 5.20  
Deflection of R/B

(8) Drywell

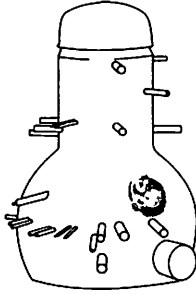


Fig 5.21  
Stress Analysis of Drywell  
by F.E.M.

6. Conclusion

It is important in the nuclear power plant to analyse the elastic-plastic stresses due to external loads, thermal loads, and loads by flows and vibrations. The finite element method is powerful to the precise analysis for structures with arbitrary shapes. And also field problems like the non-linear heat transfer etc. can be solved by this method. But the trouble is to solve the large matrix. The direct method from element stiffness (and/or mass) matrices, stated here has the advantage when we have to analyse the complex system. Because one can solve it directly from the element matrices without having the assembled matrix by exploiting this method. It should be emphasized that it is not better to divide the model into much more finite elements at random even though the large matrix has not become uncomfortable. We better select the point where the error of the geometrical shape and that of the numerical calculation balance. As the proverb runs "Don't use the elephant gun to shoot a rabbit".

By exploiting the direct method from element matrices the computer core is minimized and the computational time does not take so long compared with other methods. However, it still takes a couple of hours when the author has solved the matrix with the order of around 50,000.

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