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TIME-VARIANT RELIABILITY OF MECHANICAL COMPONENTS

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ABSTRACT

The close integration of the reliability module in a finite element system facilitates the access to reliability analysis for the design engineer. Therefore a general purpose finite element code has been extended for the calculation of the reliability of mechanical components [1] [2]. A first and second order reliability method together with an importance sampling scheme have been implemented. The code has been extended including the influence of eigenvalues and eigenfrequencies as well as time-dependent problems.

1 INTRODUCTION

Reliabilities and sensitivities for linear dynamic systems can be computed by use of the modal approach. The sensitivities of eigenvalues and eigenfrequencies are calculated and the results of a linear dynamic analysis are used in the calculation of the reliability index. This approach is implemented because for large finite element models the modal time-history respectively frequency response are the only feasible methods.

A fundamental extension is related to time-dependent problems. Loads modeled as stationary and non-stationary Gaussian processes and rectangular wave renewal processes can be included. The parameters of those processes can in turn be modeled by ergodic sequences. Problems of load combinations, material fatigue and transient loading can thus be dealt with.

2 RELIABILITY ANALYSIS OF VIBRATING SYSTEMS

Most vibrations in machines and structures are undesirable because of the increased stresses and energy losses which accompany them. They should therefore be eliminated or reduced as much as possible by appropriate design.

The most catastrophic response phenomena occur, if a component is excited at a frequency which is near to one of its natural frequencies. It is therefore necessary to include the natural frequencies in the vector of FEM results that are input to the limit state function in a reliability analysis. Failure is then usually defined if the natural frequency reaches a predefined limit such as a known excitation frequency. Of course, the natural frequencies will be only influenced by those basic variables that have an influence on the stiffness or mass matrices. The sensitivities of the eigenfrequencies are implemented in the following way, starting from the equation of motion of the undamped vibrating system:

$$\boldsymbol{K r} + \boldsymbol{M \ddot{r}} = 0 \tag{1}$$

with the symmetric stiffness and mass matrices **K**, **M** and the displacement and acceleration vectors $\boldsymbol{r}$, $\boldsymbol{\ddot{r}}$. There exists a periodic solution of the form:

$$\boldsymbol{r} = \bar{\boldsymbol{y}} e^{i\omega t} \tag{2}$$

where $\bar{y}$ is an eigenvector of the system. Inserting eq. 1 into eq. 2 yields the eigenvalue problem:

$$\boldsymbol{K \bar{Y}} = \boldsymbol{M \bar{Y} \Omega}^2 \tag{3}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix containing the eigenvalues and $\bar{\boldsymbol{Y}}$ is the matrix of the eigenvectors.

It can be shown, that the modal matrix transforms stiffness and mass simultaneously to a diagonal form:

$$\bar{\boldsymbol{Y}}^T \boldsymbol{K} \bar{\boldsymbol{Y}} = \boldsymbol{\Omega}^2 \tag{4}$$

$$\bar{\boldsymbol{Y}}^T \boldsymbol{M} \bar{\boldsymbol{Y}} = \boldsymbol{I} \tag{5}$$

However, since numerical methods usually yield the largest eigenvalues with highest precision but in mechanical problems the lowest ones are of interest, in PERMAS the alternate normalisation is used:

$$\boldsymbol{Y}^T \boldsymbol{K Y} = \boldsymbol{I} \tag{6}$$

$$\boldsymbol{Y}^T \boldsymbol{M Y} = \boldsymbol{\Lambda} \tag{7}$$

with the associated eigenvalue problem:

$$\boldsymbol{M Y} = \boldsymbol{K Y \Lambda} \tag{8}$$

which we can write for each eigenvector $\boldsymbol{y}$ and its associated eigenvalue λ as

$$\boldsymbol{M y} = \lambda \boldsymbol{K y} \tag{9}$$

Taking the partial derivative with respect to a scalar basic variable x we derive from equation 9

$$\frac{\partial \boldsymbol{M}}{\partial x}\boldsymbol{y} + \boldsymbol{M}\frac{\partial \boldsymbol{y}}{\partial x} = \frac{\partial \lambda}{\partial x}\boldsymbol{K y} + \lambda \frac{\partial \boldsymbol{K}}{\partial x}\boldsymbol{y} + \lambda \boldsymbol{K}\frac{\partial \boldsymbol{y}}{\partial x} \tag{10}$$

from which finally the following system of equations is derived:

$$(\lambda \boldsymbol{K} - \boldsymbol{M})\frac{\partial \boldsymbol{y}}{\partial x} = \boldsymbol{R}(\boldsymbol{y}, x) \tag{11}$$

$$\boldsymbol{R}(\boldsymbol{y}, x) = \left(\frac{\partial \boldsymbol{M}}{\partial x} - \lambda \frac{\partial \boldsymbol{K}}{\partial x}\right)\boldsymbol{y} - \frac{\partial \lambda}{\partial x}\boldsymbol{K y}. \tag{12}$$

Together with the orthogonality condition for the eigenvectors we finally can derive the sensitivity of the eigenvalue as:

$$\frac{\partial \lambda}{\partial x} = \boldsymbol{y}^T \left(\frac{\partial \boldsymbol{M}}{\partial x} - \lambda \frac{\partial \boldsymbol{K}}{\partial x}\right)\boldsymbol{y} \tag{13}$$

The implementation is somewhat more difficult in the case of multiple eigenvalues, but it follows the same principle. The availability of these sensitivities allows to implement eigenfrequencies as constraints into the limit function within the standard algorithm searching for the β point.

3 TIME-VARIANT RELIABILITY

Theory and concepts for the computation of time-invariant reliability are now well-known and can be performed efficiently and reliably. The theory of time-variant reliability is much less developed. Computationally feasible approaches to time-variant reliability problems at present are all of asymptotic nature. They rest on the construction of a Poisson process for the exits of the structural state function into the failure domain. The intensity parameter of this Poisson process is determined from the outcrossing rate of the load effect process through the possibly time-variant limit state function. The mean number of exits into the failure domain has to be determined by time integration of the outcrossing rate. The calculation of the outcrossing rates is a non-trivial task. At present solutions for differentiable Gaussian vector processes and rectangular wave renewal processes are available [3, 4]. A second difficulty usually arises when assuring the Poissonian nature (lack of memory) of the outcrossings under the presence of time invariant or at least non-ergodic basic variables.

Consider the general task of estimating the probability $P_f(t)$ that a realization $z(\tau)$ of a random state vector $\boldsymbol{Z}(\tau)$ representative for a given problem, enters the failure domain $V = \{z(\tau)|g(z(\tau),\tau) \leq 0,\ 0 \leq \tau \leq t\}$. $g(\cdot)$ is the limit state function. $\boldsymbol{Z}(\tau)$ may conveniently be separated into three components as

$$\boldsymbol{Z}^T(\tau) = [\boldsymbol{R}^T \quad \boldsymbol{Q}^T(\tau) \quad \boldsymbol{S}^T(\tau)] \tag{14}$$

where $\boldsymbol{R}$ is a vector of random variables independent of time t, $\boldsymbol{Q}(\tau)$ is a slowly varying random vector sequence and $S(\tau)$ is a vector of not necessarily stationary but sufficiently mixing random process variables having fast fluctuations as compared to $\boldsymbol{Q}(\tau)$.

In [5] the following formula has been established in part by making use of the ergodicity theorem

$$P_f(t) \approx 1 - E_{\boldsymbol{R}}[exp(-E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{R},\boldsymbol{Q})]])] \leq E_{\boldsymbol{R}}[E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{R},\boldsymbol{Q})]]] \tag{15}$$

Herein

$$E[N_{\boldsymbol{S}}^+(t|\boldsymbol{R},\boldsymbol{Q})] = \int_0^t \nu^+(\tau|\boldsymbol{R},\boldsymbol{Q})d\tau \tag{16}$$

is the mean value of exits into the failure domain and

$$\nu^+(\tau|\boldsymbol{R},\boldsymbol{Q}) = \lim_{\Theta\to 0}\frac{1}{\Theta}P(\{\boldsymbol{S}(\tau) \in V(\boldsymbol{R},\boldsymbol{Q},\tau)\} \cap \{\boldsymbol{S}(\tau+\Theta) \in \bar{V}(\boldsymbol{R},\boldsymbol{Q},\tau+\Theta)\}) \tag{17}$$

the outcrossing rate. Eq.15 is a rather good approximation for the stationary case but must be considered as a first approximation whenever $S(\tau)$ is non-stationary or the limit state function exhibits strong dependence on τ. The bound given in eq.15 is strict but close to the exact result only for very small failure probabilities.

The outcrossing rate can be computed by $FORM/SORM$. The same methodology is also applied for the time integration in eq.16. Substantially more difficult is the expectation operation with respect to the non ergodic R-variables. This expectation can be performed either by crude Monte Carlo integration or with importance sampling. Provided the important region for integration is known by $\boldsymbol{r}^*$ it is

$$E_{\boldsymbol{R}}[1 - exp(-E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{Q},\boldsymbol{R})]])] = \int_{\boldsymbol{R}^n} \left(1 - exp(-E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{Q},\boldsymbol{r})]])\right)\frac{\varphi_{\boldsymbol{R}}(\boldsymbol{r})}{h_{\boldsymbol{R}}(\boldsymbol{r})}h_{\boldsymbol{R}}(\boldsymbol{r})d\boldsymbol{r} \tag{18}$$

where $h_{\boldsymbol{R}}(r)$ is the sampling density. Then,

$$E_{\boldsymbol{R}}[1 - exp(-E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{Q},\boldsymbol{R})]])] = \approx \frac{1}{N}\sum_{i=1}^{N}\left(1 - exp(-E_{\boldsymbol{Q}}[E[N_{\boldsymbol{S}}^+(t|\boldsymbol{Q},\boldsymbol{r}_i)]])\right)\frac{\varphi_{\boldsymbol{R}}(\boldsymbol{r}_i)}{h_{\boldsymbol{R}}(\boldsymbol{r}_i)} \tag{19}$$

The sampling density (standard space) can conveniently be chosen as the standard normal density with mean r^* from the upper bound solution and covariance matrix **I**. A crude, usually conservative estimate is already obtained for $r_i = r^*$.

4 EXAMPLES

Simple examples are chosen in order to show the basic principle. In actual projects, the finite element model may exist of several hundred thousand unknowns taking into account the power provided by the hardware systems available today. This implies that for realistic problems all the CPU time is spent in the finite element calculation including the evaluation of the sensitivity information for the β-point search algorithm.

The size and complexity of the finite element mesh are only limited by the available hardware, the same holds true for the number of basic variables. Even small workstation exhibit considerable power in this respect, as could be observed on different industrial projects. For example, a finite element model with about 10000 unknowns and 24 uncertain variables needs a total CPU time for the time-variant reliability analysis of 801 seconds on an HP 9000/730 workstation [6].

4.1 Example 1: Time-Variant Reliability Analysis

The basic principle of load combination problems can be shown using the simple structure in figure 1:

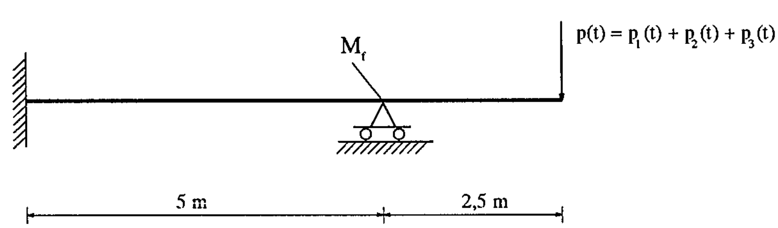

Figure 1: Simple beam structure with time-variant loading

The example is taken from [7], chapter 10.4 and there exists a semi-analytic solution making it well suited for system verification. The agreement between the theoretical results and the PERMAS solution is very good in all cases studied so far.

Structural failure is defined by reaching a limit moment M_f which is assumed to be a normally distributed random variable with a mean of 20 [kNm] and a standard deviation of 2[kNm]. The reference period is one year.

The loading is a combination of three independent rectangular random processes with the parameters shown in the following table:

Load Process	Jump Rate	mean $\mu_{P_i(t)}$	deviation $\sigma_{P_i(t)}$	Distribution
$P_1(t)$	1/year	0.5 [kN]	0.2 [kN]	Gaussian
$P_2(t)$	12/year	-0.2 [kN]	0.4 [kN]	Gaussian
$P_3(t)$	360/yea	-2.0 [kN]	1.0 [kN]	Gaussian

This loading follows the Ferry Borghes-Castanheta model which assumes that the processes are statistically independent, the repetition numbers are integers and the quotients of subsequent repetitions are natural numbers. The reliability index is calculated as $\beta = 3.23$ with a corresponding probability that failure will occur during the reference period of 6.0 x 10^{-4}.

If the load process $P_2(t)$ is assumed to be inactive, i.e. P_2=0 during the reference period, the reliability index becomes $\beta = 3.734$ (P_f=9.4 x 10^{-5}).

The number of necessary finite element calculations is typically between 10 and 20 for a time-variant reliability analysis. However, if the integration in the time domain only involves loading functions, there is no need to recalculate the stiffness matrix in each iteration. This fact is automatically recognized by the system and leads to considerable savings in computer time since the mere substitution of a right-hand side is not very time-consuming.

4.2 Example 2: Vibration Analysis of an Elastically Supported Beam

The simple beam structure shown in figure 2 is supported on two springs whose stiffnesses c_1 and c_2 are uncertain variables. The first natural bending mode has a frequency of 4.75 Hz.

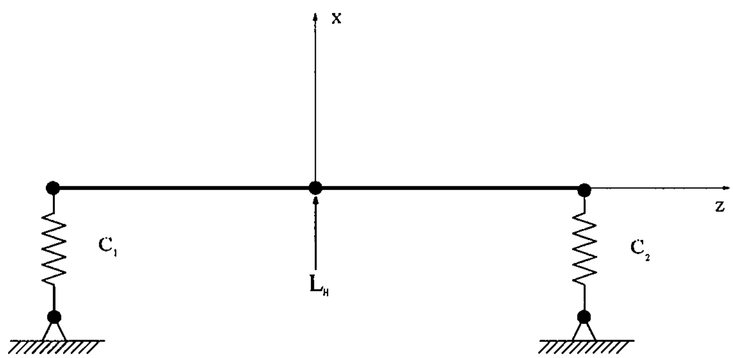

Figure 2: Elastically Supported Beam Structure

If the structure is subject to a harmonic loading L_H at a frequency of 4.1 Hz, the failure of the structure will occur if the first natural frequency f_1 reaches this value, i.e. the limit state function has the form

$$g(Z) = f_1 - 4.1. \qquad (20)$$

Let us first consider the case of **one** uncertain variable, namely the stiffness of both springs c_1 and c_2. The reliability index β in this case is found to be 3.9 and the corresponding probability of failure is 3.4 x 10^{-5}.

If we now define the stiffnesses of the springs as two independent uncertain variables, the reliability index β becomes 5.63 with a corresponding probability of failure of 8.7 x 10^{-9}.

As a next step we can define a coefficient of correlation between the two basic variables. The following table shows the influence of this correlation coefficient on the reliability index and the corresponding probabilities of failure:

Case	Number of Basic Variables	Coefficient of Correlation	Reliability Index β	Probability of Failure
1	1	—	3.9	3.4 x 10^{-5}
2	2	0.0	5.6	8.7 x 10^{-9}
3	2	0.1	5.4	3.9 x 10^{-8}
4	2	0.3	5.0	3.9 x 10^{-7}
5	2	0.5	4.6	2.1 x 10^{-6}
6	2	0.8	4.2	1.3 x 10^{-5}
7	2	0.99	4.0	3.2 x 10^{-5}

Typically about 10 dynamic and sensitivity analyses are necessary for one computation, if a first order β-point search is performed. Hence, the price of a reliability analysis is about 15 times the computer time of a conventional dynamic analysis.

Performing a second order approximation or improving the β-value using an importance sampling technique will considerably increase the usage of computer resources.

5 CONCLUSION

The integration of the established FORM/SORM algorithm in the general purpose finite element code PERMAS offers the simulation of finite element models with stochastic properties and loads. The combination of the two codes in a single program yields an efficient and easy to use solution since the design sensitivity information from the finite element model can directly be utilized by the algorithm for the β-point search.

The necessary additional input to complement the finite element model with the definition of the stochastic model is very small. Hence, reliability analysis is made available to engineers being familiar to simulation methods (i.e finite elements) without having to acquire a detailed knowledge on the probabilistic methods involved with the FORM/SORM algorithms.

The nature of the problems to be dealt with can be from a wide range. The efficient handling of eigenfrequencies as input to the failure function has extended the applicability to the analysis of vibrating structures.

The extension to time-variant problems such as load combinations or deteriorating strength includes Gaussian vector processes and rectangular wave renewal processes thus covering many problems of practical interest.

The hardware platforms available today and their price-performance ratio make such analyses feasible not only on supercomputers but even on ordinary workstations.

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