STRUCTURAL DYNAMIC STATE ESTIMATION USING PROBABILISTIC SUBSTRUCTURING STRATEGIES

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ABSTRACT

This paper addresses problems of state estimation of structural dynamical systems using probabilistic substructuring methods known as Rao-Blackwellized particle filters. These are semi-analytical filters offering a partly exact and partly approximate solution to the Bayesian filtering problem for which only approximate solutions are available. The existence of a conditionally linear substructure in the governing equations of motion of a dynamical system gives a possibility of applying the substructuring based methods. In the field of structural mechanics, the problems of sensitivity model updation, structural system identification and force identification of nonlinear systems are found to have such a substructure making it amenable for application of Rao-Blackwellized filters. The estimates of these filters are found to be efficient by having a lower variance in comparison to their counterparts obtained using the tradition particle filters. In this paper we explore the updation of response sensitivity of an inelastic nonlinear dynamical system with memory dependent characteristics using the Rao-Blackwellized particle filters.

INTRODUCTION

The focus of this study is on problems of structural dynamical state estimation and system identification using Bayesian filtering methods. These problems are of interest in the context of health monitoring and condition assessment of existing engineering structures. The basic premises of Bayesian fitlering methods are that a process equation, based on physical and mathematical principles, is derivable and the imperfections in deducing this equation can be represented by a vector of white noise processes. Typically, in structural mechanics applications, the finite element model for the structural behavior serves as the process equation. Furthermore, the measured quantities are related to the system states, again, through an approximate mathematical model, with the errors in this representation getting represented as yet another vector of white noise processes. The system responses or parameters of interest are typically taken to be a set of states and the problem on hand consists of determining the joint probability density function of these states conditioned on measurements made. The problem of structural state estimation subsequently is posed as a problem of Bayesian estimation. It has been recognized in the existing literature that, posed this way, the problem of state estimation becomes invariably nonlinear for nonlinear dynamical systems. This calls for application of linearization based Kalman filtering tools or Monte Carlo simulation based filtering.

The present study investigates the idea of partitioning the system states such that a part of the problem can be solved exactly and the remaining numerically using simulation tools with suitable provisions for correct interfacing of the two solutions. The basis of this idea lies in the facts that for linear process and measurement models with additive Gaussian noises, the state estimation method can be solved exactly using the well known Kalman filter [1] while for nonlinear systems, one typically needs to employ Monte Carlo simulations based filtering tools [2,3]. By solving a part of the given problem analytically (using Kalman filters) the sampling variance associated with simulation based estimates essentially gets reduced. This idea is well studied in statistics literature and the principle is encapsulated in a theorem due to Rao and Blackwell [3]. A review of literature reveals that this type of statistical substructuring schemes has been studied only in the area of navigation and control [4,5] but to a very limited extent in structural mechanics. On the other hand, it is interesting to note that substructuring schemes, such as, component mode synthesis, have indeed attracted the attention of researchers but schemes that aim to reduce the computational burden (for a given level of accuracy) have not received as much attention.

The present authors have used Bayesian filtering methods in their recent work to address problems of reliability model updation [6] and sensitivity model updation [7]. It is observed that problems of structural mechanics, like sensitivity model updation, structural system identification, and force identification of nonlinear systems, have a probabilistic substructure making it possible to explore the application of Rao-Blackwellized filters.

In this paper the authors illustrate application of the substructuring based methods on one application namely the response sensitivity model updation of nonlinear dynamical systems.

PROBLEM STATEMENT

In this study we consider structural dynamical systems governed by the equation of the form

$$M(\theta)\ddot{U} + F\left[U(t), \dot{U}(t), t, \theta\right] = p(t) + \varsigma_U(t); U(0) = U_0; \dot{U}(0) = \dot{U}_0 \quad (1)$$

Here a dot represents differentiation with respect to the time variable *t*; $U(t), U_0, F, p(t)$ and $\varsigma_U(t)$ are $L \times 1$ vectors; M is the $L \times L$ mass matrix; θ is a $L_\theta \times 1$ vector of system parameters and $p(t)$ is a deterministic excitation vector. The components of *p(t)* could, in general, be aperiodic and contain multiple frequencies; typically they could be samples of nonstationary random processes. The quantity $\varsigma_U(t)$ accounts for modeling error in arriving at Eq.(1) and is represented by a vector Gaussian white noise process with zero mean and covariance $\Sigma_{\varsigma_U}(t)$. The gradient of the system response $U(t)$ with respect to the system parameters $\theta_k; k = 1, 2, \cdots, L_\theta$ is governed by

$$\frac{\partial M}{\partial \theta_k}\ddot{U} + M\frac{\partial U}{\partial \theta_k} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial \theta_k} + \frac{\partial F}{\partial \dot{U}}\frac{\partial \dot{U}}{\partial \theta_k} + \frac{\partial F}{\partial \theta_k} = \varsigma_k(t); k = 1, 2, \cdots, L_\theta \quad (2)$$

Here $\varsigma_k(t)$ is a $L \times 1$ vector of Gaussian white noise processes with zero mean and covariance $\Sigma_{\varsigma_k}(t)$ that accounts for the error in modeling the sensitivities. It is assumed that $\varsigma_k(t); k = 1, 2, \cdots, L_\theta$ are mutually independent and independent of $\varsigma_U(t)$. We denote $N_x = 2L(1 + L_\theta)$ and introduce the $N_x \times 1$ vector

$$x(t) = \left[U(t) \quad \dot{U}(t) \quad \frac{\partial U}{\partial \theta_1} \quad \frac{\partial \dot{U}}{\partial \theta_1} \quad \cdots \quad \frac{\partial U}{\partial \theta_{L_\theta}} \quad \frac{\partial \dot{U}}{\partial \theta_{L_\theta}}\right]^t \quad (3)$$

Here superscript *t* denotes matrix transposition. Using this vector Eq.(1) and Eq.(2) are represented in the state space and consequently recast in the form of an Ito's stochastic differential equation (SDE) as

$$dx(t) = a\left[x(t), p(t), t\right]dt + b\left[x(t), t\right]dB(t);\ x(0) = x_0 \quad (4)$$

Here $a\left[x(t), p(t), t\right]$ is the drift vector of size $N_x \times 1$ in which the contribution of the deterministic excitation vector $p(t)$ is also included and $b\left[x(t), t\right]$ is the diffusion matrix of size $N_x \times m$. The Gaussian white noise processes in Eq.(1) and Eq.(2) [$\varsigma_U(t)$ and $\varsigma_k(t)$] are modeled as an $m \times 1$ vector of increments of Brownian motion processes $dB(t)$ with zero mean and $E\left[dB(t)dB^t(t)\right] = \Sigma_{W_c}(t)dt$; the subscript *c* emphasizes the fact that time is continuous; and, $E[\cdot]$ is the mathematical expectation operator. It may be emphasized that representation in Eq.(4) is valid only when the processes $\varsigma_U(t)$ and $\varsigma_k(t)$ can be interpreted as white noise processes.

It is observed that Eq.(4) is continuous in time. But in practice measurements are available only at discrete time instants. For Eq.(4) to be amenable for solution using the Bayesian filtering techniques it is discretized at time instants $\left\{t_k \in (0, T]\right\}_{k=1}^{N}$ using the order 1.5 strong Taylor's scheme [8] and the resulting equation can be shown to have the form

$$x_{k+1} = f_k(x_k) + F_k + G_k(x_k)w_k \quad (5)$$

Here $x_{k+1}, f_k\left(x_k\right), F_k$ are vectors of size $N_x \times 1$, $G_k\left(x_k\right)$ is a matrix of size $N_x \times N_w$ and w_k is $N_w \times 1$ vector of normally distributed random numbers with zero mean and covariance Q_k .

Furthermore, we assume that a set of measurements, denoted by $y\left(t_k\right)$, have been made at discrete time instants $\left\{t_k \in \left(0,T\right]\right\}_{k=1}^{N}$ and the measured quantities are taken to be related to the system states x_k through the relation

$$y\left(t_k\right) = H_k\left(x_k\right) + \nu_k ; k = 1, 2, \cdots, N \tag{6}$$

Here $y\left(t_k\right)$ and $H_k\left(x_k\right)$ are $N_y \times 1$ vectors and ν_k is a $r \times 1$ vector of sequence of independent normal random variables with zero mean and covariance Σ_ν . The quantity ν_k accounts for the measurement noise and imperfections involved in relating the measurement $y\left(t_k\right), k = 1, 2, \cdots N$ with the states x_k .Given the governing equation for a dynamical system of the form Eq.(5) and measurement equation of the form Eq.(6) with notation $y_{1:k} = \left\{y_1 \quad y_2 \quad \cdots \quad y_k\right\}^t$ and $x_{0:k} = \left\{x_0 \quad x_1 \quad \cdots \quad x_k\right\}^t$ the problem of Bayesian filtering is to compute the multi-dimensional conditional probability density function (pdf) $p\left(x_{0:k} \mid y_{1:k}\right)$, the marginal pdf $p\left(x_k \mid y_{1:k}\right)$ (also known as the filtering density function) and the corresponding mean $a_{k|k} = E\left[x_k \mid y_{1:k}\right]$ and covariance matrix $\Sigma_{k|k} = E\left[\left(x_k - a_{k|k}\right)^t \left(x_k - a_{k|k}\right) \mid y_{1:k}\right]$.

For linear state space models and additive Gaussian noises, the discrete time Kalman filter [1] provides an exact set of recursive relations for the evolution of the mean vector $a_{k|k}$ and covariance matrix $\Sigma_{k|k}$. For more general class of problems, involving nonlinear mechanics and non-Gaussian noises, approximate solution based on Monte Carlo simulations lead to a class of methods known as particle filters. In the present study we consider the application of particle filtering method namely, the sequential importance sampling (SIS) filter [3], which employs importance sampling to achieve variance reduction. It may be noted that the SIS filter is applicable for nonlinear state space equations and linear measurement models. There exists yet another type of filtering methods known as Rao Blackwellised particle filters which offer a partly exact and partly approximate solution to the filtering problem on hand. These filters make use of the existence of a inherent probabilistic substructure present in the governing equation to achieve variance reduction. The objective of this paper is to explore the application of these semi-analytical filters to problems of interest in structural mechanics where such a probabilistic substructure is often found to exist.

We consider the problems where the structure of governing equation is such that Eq.(5) can be partitioned as

$$\begin{aligned} x_{k+1}^l &= f_k^l\left(x_k^n\right) + A_k^l\left(x_k^n\right)x_k^l + F_k^l + G_k^l\left(x_k^n\right)w_k^l \\ x_{k+1}^n &= f_k^n\left(x_k^n\right) + A_k^n\left(x_k^n\right)x_k^l + F_k^n + G_k^n\left(x_k^n\right)w_k^n \end{aligned} \tag{7,8}$$

where $x_k = \begin{Bmatrix} x_k^l \\ x_k^n \end{Bmatrix}, F_k = \begin{Bmatrix} F_k^l \\ F_k^n \end{Bmatrix}, w_k = \begin{Bmatrix} w_k^l \\ w_k^n \end{Bmatrix}, G_k\left(x_k\right) = \begin{bmatrix} G_k^l\left(x_k^n\right) & 0_{N_{x^l} \times N_{w^n}} \\ 0_{N_{x^n} \times N_{w^l}} & G_k^n\left(x_k^n\right) \end{bmatrix}$ and $Q_k = \begin{bmatrix} Q_k^l & Q_k^{\ln} \\ Q_k^{nl} & Q_k^n \end{bmatrix}$ with $Q_k^l = E\left[w_k^l w_k^{lT}\right], Q_k^n = E\left[w_k^n w_k^{nT}\right], Q_k^{\ln} = E\left[w_k^l w_k^{nT}\right], Q_k^{nl} = E\left[w_k^n w_k^{lT}\right] = \left(Q_k^{\ln}\right)^T$ and sizes of the various matrices appearing in Eq.(7) and Eq.(8) are as follows $x_k^l : N_{x^l} \times 1, x_k^n : N_{x^n} \times 1, f_k^l\left(x_k^n\right) : N_{x^l} \times 1, f_k^n\left(x_k^n\right) : N_{x^n} \times 1,$ $A_k^l\left(x_k^n\right) : N_{x^l} \times N_{x^l}, A_k^n\left(x_k^n\right) : N_{x^n} \times N_{x^n}, F_k^l : N_{x^l} \times 1, F_k^n : N_{x^n} \times 1, G_k^l\left(x_k^n\right) : N_{x^l} \times N_{w^l}, G_k^n\left(x_k^n\right) : N_{x^n} \times N_{w^n}, w_k^l : N_{w^l} \times 1,$ $w_k^n : N_{w^n} \times 1, Q_k^l : N_{w^l} \times N_{w^l}, Q_k^n : N_{w^n} \times N_{w^n}, Q_k^{\ln} : N_{w^l} \times N_{w^n}, Q_k^{nl} : N_{w^n} \times N_{w^l}$ and $N_x = N_{x^l} + N_{x^n}, N_w = N_{w^l} + N_{w^n}$. It is observed that Eq.(7) and Eq.(8) are linear in the state x_k^l and nonlinear in x_k^n and, therefore, x_k^l and x_k^n are known as linear and nonlinear states respectively.

We also rewrite the measurement equation [Eq.(6)] in terms of the partitioned states x_k^l and x_k^n as

$$y_k = h_k\left(x_k^n\right) + C_k\left(x_k^n\right)x_k^l + \nu_k \tag{9}$$

where $h_k\left(x_k^n\right): N_y \times 1$ and $C_k\left(x_k^n\right): N_y \times N_{x^l}$.

The problem of Bayesian filtering with respect to the partitioned set of governing equations [Eq.(7) and Eq.(8)] and measurement equation [Eq.(9)] is to find the quantity $p\left(x_k^l, x_{0:k}^n \mid y_{1:k}\right)$. Using the definition of conditional density functions

$$p\left(x_k^l, x_{0:k}^n \mid y_{1:k}\right) = p\left(x_k^l \mid x_{0:k}^n, y_{1:k}\right) p\left(x_{0:k}^n \mid y_{1:k}\right) \tag{10}$$

where $p\left(x_k^l \mid x_{0:k}^n, y_{1:k}\right)$ is exactly evaluated using the Kalman filter [considering Eq.(7) and Eq.(9) as the state and measurement equations respectively] and $p\left(x_{0:k}^n \mid y_{1:k}\right)$ is numerically evaluated using the SIS particle filter [considering Eq.(8) and Eq.(9) as state and measurement equation respectively]. Therefore the required pdf will have the form

$$p\left(x_k^l, x_{0:k}^n \mid y_{1:k}\right) = \mathcal{N}\left(x_k^l : x_{k|k}^l, P_{k|k}^l\right) \sum_{i=1}^{\tilde{N}} w\left(x_{i,0:k}^n\right) \delta\left(x_{0:k}^n - x_{i,0:k}^n\right) \tag{11}$$

where $\mathcal{N}(x : a, b)$ represents a Gaussian distribution of x with mean a and covariance matrix b , $\delta(\bullet)$ is the Dirac delta function and $\left\{w\left(x_{i,0:k}^n\right)\right\}_{i=1}^{\tilde{N}}$ are the importance weights obtained using SIS particle filtering. Using Eq.(11) the moments of the linear state $x_{k|k}^L = E\left[x_k^l \mid y_{1:k}\right]$ and $P_{k|k}^L = E\left[\left(x_k^l - x_{k|k}^L\right)\left(x_k^l - x_{k|k}^L\right)^T \mid y_{1:k}\right]$ and moments of the nonlinear state $x_{k|k}^N = E\left[x_k^n \mid y_{1:k}\right]$ and $P_{k|k}^N = E\left[\left(x_k^n - x_{k|k}^N\right)\left(x_k^n - x_{k|k}^N\right)^T \mid y_{1:k}\right]$ can be obtained.

The partitioning of the state vector and computation of the corresponding pdf in the way described above is found to be a variance reduction technique which has its bearing in the Rao-Blackwell theorem. For this reason this method of Bayesian filtering is also known as Rao-Blackwellized particle filtering. The details of how the algorithm given in this paper relates to the Rao-Blackwell theorem and the steps in deriving the Rao-Blackwellized SIS filter are provided in [9]. In the next section we give the steps for implementation of the Rao-Blackwellized SIS filter and then illustrate its application in obtaining updated response sensitivities of a cubic-hysteretic dynamical system.

RAO-BLACKWELLIZED SIS FILTER (RBSIS FILTER)

This is a semi-analytical filter where part of the problem is exactly solved using the Kalman filter and part of the problem is solved numerically using the particle filter (SIS filter in this paper). Due to constraint in space we provide here the steps in the implementation of this filter. For details about the derivation the reader is referred to [9].Following are the steps adopted for implementation of the RBSIS filter.

1. Set $k = 0$; draw samples $\left\{x_{i,0}^n\right\}_{i=1}^{\widehat{N}}$ from an assumed initial density function $p\left(x_0^n\right)$ and assume initial values for $\left\{x_{i,0|0}^l, P_{i,0|0}^l\right\}_{i=1}^{\widehat{N}}$.

2. Set $k = k + 1$.

(i) Nonlinear state prediction

Sample $x_{i,k}^n \sim p\left(x_k^n \mid x_{i,0:k-1}^n, y_{1:k-1}\right) = \mathcal{N}\left(x_k^n : f_{k-1}^n\left(x_{i,k-1}^n\right) + A_{k-1}^n x_{i,k-1|k-1}^l + F_{k-1}^n, A_{k-1}^n P_{i,k-1|k-1}^l \left(A_{k-1}^n\right)^t + G_{k-1}^n Q_{k-1}^n \left(G_{k-1}^n\right)^t\right)$ for $i \in [1, \widehat{N}]$

(ii) Linear state prediction

for $i \in [1, \widehat{N}]$

a) Pseudo measurement update $p\left(x_{k-1}^{l} \mid x_{i,0:k}^{n}, y_{1:k-1}\right)$ – estimates of nonlinear state act as pseudo measurements

$$x_{i,k-1|k-1}^{l}{}^{*} = x_{i,k-1|k-1}^{l} + K_{k-1}^{l}{}^{*}\left[x_{i,k}^{n} - f_{k-1}^{n}\left(x_{i,k-1}^{n}\right) - A_{k-1}^{n} x_{i,k-1|k-1}^{l} - F_{k-1}^{n}\right]$$

$$P_{i,k-1|k-1}^{l}{}^{*} = P_{i,k-1|k-1}^{l} - K_{k-1}^{l}{}^{*} A_{k-1}^{n} P_{i,k-1|k-1}^{l}$$

$$K_{k-1}^{l}{}^{*} = P_{i,k-1|k-1}^{l}\left(A_{k-1}^{n}\right)^{t}\left[A_{k-1}^{n} P_{i,k-1|k-1}^{l}\left(A_{k-1}^{n}\right)^{t} + G_{k-1}^{n} Q_{k-1}^{n}\left(G_{k-1}^{n}\right)^{t}\right]^{-1}$$

b) Linear state prediction $p\left(x_{k}^{l} \mid x_{i,0:k}^{n}, y_{1:k-1}\right)$

$$x_{i,k|k-1}^{l} = f_{k-1}^{l}\left(x_{i,k-1}^{n}\right) + \overline{A}_{k-1}^{l} x_{i,k-1|k-1}^{l}{}^{*} + F_{k-1}^{l} + G_{k-1}^{l} Q_{k-1}^{\mathrm{ln}}\left(G_{k-1}^{n} Q_{k-1}^{\mathrm{n}}\right)^{-1}\left(x_{i,k}^{n} - f_{k-1}^{n}\left(x_{i,k-1}^{n}\right) - F_{k-1}^{n}\right)$$

$$P_{i,k|k-1}^{l} = \overline{A}_{k-1}^{l} P_{i,k-1|k-1}^{l}{}^{*}\left(\overline{A}_{k-1}^{l}\right)^{t} + G_{k-1}^{l} \overline{Q}_{k-1}^{l}\left(G_{k-1}^{l}\right)^{t}$$

(iii) Nonlinear state updation

Sample $\hat{x}_{i,k}^{n} \sim \pi\left(x_{k}^{n} \mid x_{i,0:k-1}^{n}, y_{1:k-1}\right) = \mathcal{N}\left(x_{k}^{n} : m_{i,k}, \Sigma_{i,k}\right)$

where

$$m_{i,k} = \Sigma_{i,k}\left\{\left[A_{k-1}^{n} P_{i,k-1|k-1}^{l}\left(A_{k-1}^{n}\right)^{t} + G_{k-1}^{n} Q_{k-1}^{n}\left(G_{k-1}^{n}\right)^{t}\right]^{-1}\left[f_{k-1}^{n}\left(x_{i,k-1}^{n}\right) + A_{k-1}^{n} x_{i,k-1|k-1}^{l} + F_{k-1}^{n}\right] h_{k}^{t}\left[C_{k} P_{i,k|k-1}^{l} C_{k}^{t} + \Sigma_{v_{k}}\right]^{-1}\left[y_{k} - C_{k} x_{i,k|k-1}^{l}\right]\right\}$$

$$\Sigma_{i,k}^{-1} = \left[A_{k-1}^{n} P_{i,k-1|k-1}^{l}\left(A_{k-1}^{n}\right)^{t} + G_{k-1}^{n} Q_{k-1}^{n}\left(G_{k-1}^{n}\right)^{t}\right]^{-1} + h_{k}^{t}\left[C_{k} P_{i,k|k-1}^{l} C_{k}^{t} + \Sigma_{v_{k}}\right]^{-1} h_{k}$$

with importance weights $\tilde{w}\left(x_{i,0:k}^{n}\right)$ given by

$$\tilde{w}\left(x_{i,0:k}^{n}\right) = \frac{w\left(x_{i,0:k}^{n}\right)}{\sum_{j=1}^{\widehat{N}} w\left(x_{j,0:k}^{n}\right)}$$

$$w\left(x_{i,0:k}^{n}\right) = w\left(x_{i,0:k-1}^{n}\right)\frac{p\left(y_{k} \mid x_{i,0:k}^{n}, y_{1:k-1}\right) p\left(x_{k}^{n} \mid x_{i,0:k-1}^{n}, y_{1:k-1}\right)}{\pi\left(x_{k}^{n} \mid x_{i,0:k-1}^{n}, y_{1:k-1}\right)}$$

$$p\left(y_{k} \mid x_{i,0:k}^{n}, y_{1:k-1}\right) = \mathcal{N}\left(y_{k} : h_{k}\hat{x}_{i,k}^{n} + C_{k} x_{i,k|k-1}^{l}, R_{k} + C_{k} P_{i,k|k-1}^{l} C_{k}^{t}\right)$$

$$p\left(x_{k}^{n} \mid x_{i,0:k-1}^{n}, y_{1:k-1}\right) = \mathcal{N}\left(x_{k}^{n} : f_{k-1}^{n}\left(x_{i,k-1}^{n}\right) + A_{k-1}^{n} x_{i,k-1|k-1}^{l} + F_{k-1}^{n}, A_{k-1}^{n} P_{i,k-1|k-1}^{l}\left(A_{k-1}^{n}\right)^{t} + G_{k-1}^{n} Q_{k-1}^{n}\left(G_{k-1}^{n}\right)^{t}\right)$$

$$\pi\left(x_{k}^{n} \mid x_{i,0:k-1}^{n}, y_{1:k-1}\right) = \mathcal{N}\left(x_{k}^{n} : m_{i,k}, \Sigma_{i,k}\right)$$

and set $\hat{x}_{i,0:k}^{n} = \left\{x_{i,0:k-1}^{n}, \hat{x}_{i,k}^{n}\right\}$ for $i \in [1, \widehat{N}]$.

(iv) Linear state updation $p\left(x_{k}^{l} \mid x_{i,0:k}^{n}, y_{1:k}\right)$

$$x_{i,k|k}^{l} = x_{i,k|k-1}^{l} + K_{k-1}^{l}\left[y_{k} - h_{k}\left(\hat{x}_{i,k}^{n}\right) - C_{k} x_{i,k|k-1}^{l}\right]$$

$$P_{i,k|k}^{l} = P_{i,k|k-1}^{l} - K_{k}^{l} C_{k} P_{i,k|k-1}^{l}$$

$$K_{k}^{l} = P_{i,k|k-1}^{l} C_{k}^{t}\left[C_{k} P_{i,k|k-1}^{l} C_{k}^{t} + \Sigma_{v_{k}}\right]^{-1}$$

for $i \in [1, \widehat{N}]$

(v) Moments

$$x_{k|k}^{L} = \sum_{i=1}^{\tilde{N}} x_{i,k|k}^{l}\, w\left(x_{i,0:k}^{n}\right); P_{k|k}^{L} = \sum_{i=1}^{\tilde{N}} w\left(x_{i,0:k}^{n}\right)\left[P_{i,k|k}^{l} + \left(x_{i,k|k}^{l} - x_{k|k}^{L}\right)\left(x_{i,k|k}^{l} - x_{k|k}^{L}\right)^{t}\right]$$

$$x_{k|k}^{N} = \sum_{i=1}^{\tilde{N}} x_{i,k}^{n}\, w\left(x_{i,0:k}^{n}\right); P_{k|k}^{N} = \sum_{i=1}^{\tilde{N}} w\left(x_{i,0:k}^{n}\right)\left[\left(x_{i,k|k}^{n} - x_{k|k}^{N}\right)\left(x_{i,k|k}^{n} - x_{k|k}^{N}\right)^{t}\right]$$

(vi) Resampling (for the nonlinear states)

a) Evaluate the effective sample size $N_{eff} = 1\Big/\sum_{i=1}^{\widehat{N}}\left\{\tilde{w}\left(x_{i,0:k}^{n}\right)\right\}^{2}$.

b) If $N_{eff} > N_{thres}$, where N_{thres} is the threshold of the sample size, set $x^n_{i,0:k} = \hat{x}^n_{i,0:k}$ go to step (i) if $k < N$, otherwise end.

c) If $N_{eff} \le N_{thres}$, implement the following:

- For $i \in [1, \widehat{N}]$, sample an index $j(i)$ distributed according to the discrete distribution with $\widehat{N}$ elements satisfying $P\left[j(i) = l\right] = \tilde{w}\left(x^n_{l,0:k}\right)$ for $l = 1, 2, ..., \widehat{N}$.
- For $i \in [1, \widehat{N}]$, set $x^n_{i,0:k} = \hat{x}^n_{j(i),0:k}$ and $\tilde{w}\left(x^n_{i,0:k}\right) = 1/\widehat{N}$.
- Go to step (i) if $k < N$, otherwise end.

NUMERICAL ILLUSTRATION

To illustrate the application of RBSIS filter, we consider the 2-degrees of freedom (dof) dynamical system shown in Fig. 1. The springs k_1 and k_2 are assumed to have cubic force displacement characteristics while the spring k_3 is a hysteretic spring with memory dependent nonlinear characteristics modeled following the Bouc-Wen model [10]. The governing equations are obtained as

$$\begin{aligned}
&m_1\ddot{u}_1 + c_1\dot{u}_1 + c_2\left(\dot{u}_1 - \dot{u}_2\right) + k_1u_1 + \alpha_1u_1^3 + k_2\left(u_1 - u_2\right) + \alpha_2\left(u_1 - u_2\right)^3 = p_1\left(t\right) + w_1\left(t\right)\\
&m_2\ddot{u}_2 + c_2\left(\dot{u}_2 - \dot{u}_1\right) + c_3\dot{u}_2 + k_2\left(u_2 - u_1\right) + \alpha_2\left(u_2 - u_1\right)^3 + k_3\lambda u_2 + k_3z\left(1 - \lambda\right) = p_2\left(t\right) + w_2\left(t\right)\\
&\dot{z} = -\gamma\,|\dot{u}_2|\,z\,|z|^{\bar{n}-1} - \beta\dot{u}_2\,|z|^{\bar{n}} + A\dot{u}_2 + w_3\left(t\right)
\end{aligned} \tag{12}$$

Here z is an internal state variable which imparts memory dependence to the force displacement characteristic of spring k_3 and, by assigning different values to the model parameters $\lambda, \gamma, \bar{n}, \beta$ and A, different shapes of the hysteresis loops could be obtained [10]. The noise terms $\left\{w_i\left(t\right)\right\}_{i=1}^{3}$ are taken to be mutually independent and to have zero mean and $E\left[w_i(t)w_i(t+\tau)\right] = \bar{\sigma}_i^2\delta\left(\tau\right); i = 1, 2, 3$. The equations governing $v_i(t); i = 1, 2, 3$ with $v_1(t) = \partial u_1/\partial k_2, v_2(t) = \partial u_2/\partial k_2$ and $v_3(t) = \partial z/\partial k_2$ are obtained as

$$\begin{aligned}
&m_1\ddot{v}_1 + c_1\dot{v}_1 + c_2\left(\dot{v}_1 - \dot{v}_2\right) + k_1v_1 + 3\alpha_1v_1u_1^2 + k_2\left(v_1 - v_2\right) + \left(u_1 - u_2\right) + 3\alpha_2\left(u_1 - u_2\right)^2\left(v_1 - v_2\right) = w_4\left(t\right)\\
&m_2\ddot{v}_2 + +c_2\left(\dot{v}_2 - \dot{v}_1\right) + c_3\dot{v}_2 + k_2\left(v_2 - v_1\right) + \left(u_2 - u_1\right) + 3\alpha_2\left(u_2 - u_1\right)^2\left(v_2 - v_1\right) + k_3\lambda v_2 + k_3v_3\left(1 - \lambda\right) = w_5\left(t\right)\\
&\dot{v}_3 = -\gamma z\,|z|^{\bar{n}-1}\,\mathrm{sgn}\left(\dot{u}_2\right)\dot{v}_2 - \gamma\,|\dot{u}_2|\,|z|^{\bar{n}-1}\,v_3 - \gamma\,|\dot{u}_2|\,z\left(\bar{n} - 1\right)|z|^{\bar{n}-2}\,\mathrm{sgn}\left(z\right)v_3 - \beta\dot{v}_2\,|z|^{\bar{n}}\\
&\qquad - \beta\dot{u}_2\bar{n}\,|z|^{\bar{n}-1}\,\mathrm{sgn}\left(z\right)v_3 + A\dot{v}_2 + w_6\left(t\right)
\end{aligned} \tag{13}$$

Here $w_i(t); i = 4, 5, 6$ are noise terms accounting for the error in modeling the sensitivity. They are modeled as mutually independent Gaussian white noise processes with zero mean and $E\left[w_i\left(t\right)w_i\left(t+\tau\right)\right] = \bar{\sigma}_i^2\delta\left(\tau\right); i = 4, 5, 6$. Theses noises are also taken to be independent of $\left\{w_i\left(t\right)\right\}_{i=1}^{3}$. In the numerical work we take $m_1 = 10\,\mathrm{kg}, m_2 = 15\,\mathrm{kg}, k_1 = 0.1\,\mathrm{kN/m}, k_2 = 0.2\,\mathrm{kN/m}, k_3 = 0.15\,\mathrm{kN/m}, \eta_1 = \eta_2 = 0.05, \alpha_1 = 1, \alpha_2 = 2, \lambda = 0.05, \gamma = 0.5, \beta = -0.5,$ $A = 1, \bar{n} = 4, T = 71.5863\,s, t_0 = 9.5448\,s, \Delta = 0.0191\,s, \sigma_1 = 10\,\mathrm{N}, \sigma_2 = 20\,\mathrm{N}, \bar{\sigma}_1 = 0.100\,\mathrm{N}, \bar{\sigma}_2 = 0.200\,\mathrm{N}, \bar{\sigma}_3 = 0.100\,\mathrm{N},$ $\bar{\sigma}_4 = 0.001\,\mathrm{N}$ and $\sigma_5 = 0.002\,\mathrm{N}$. The measurements are assumed to be made on $u_1\left(t\right)$ leading to

$$y\left(t_k\right) = x_k^1 + \nu_k; k = 1, 2, \cdots, N \tag{14}$$

Here ν_k is a zero mean identical and independent Gaussian sequence of random variables (with variance = 0.0017 m^2) which are assumed to be independent of process noise terms. We now interpret Eq.(12) and Eq.(13) as a set of

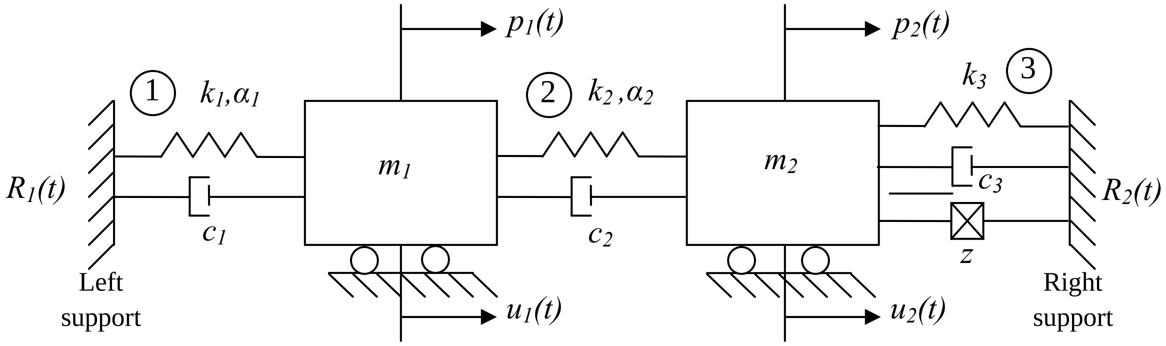
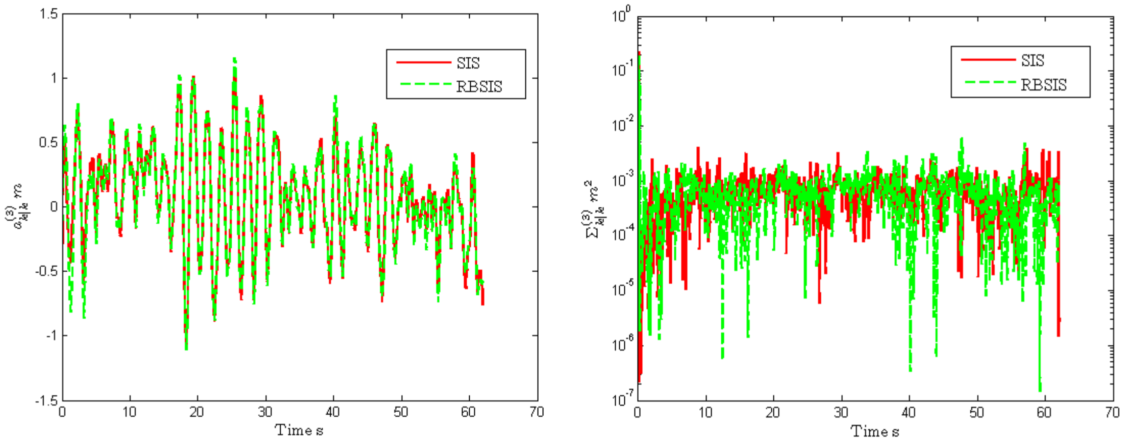
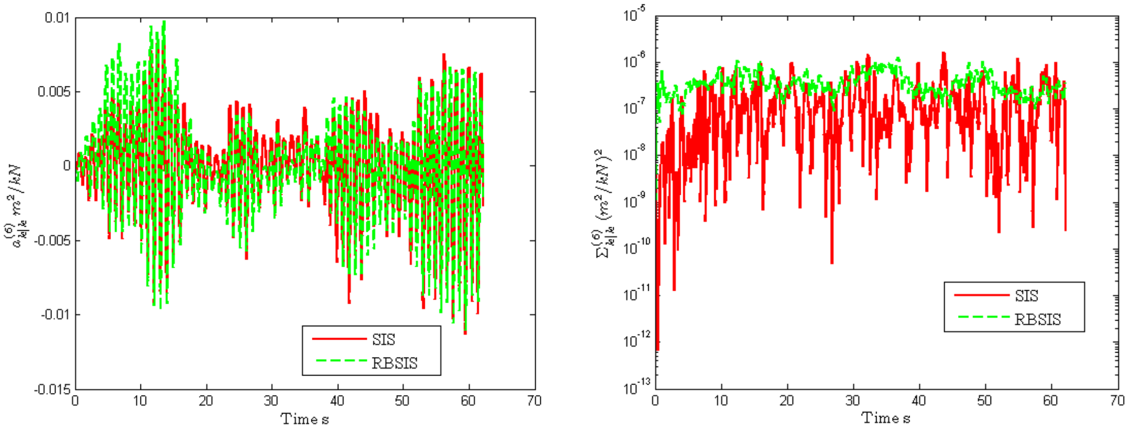


Fig. 1: Cubic-hysteretic system considered for numerical illustration; *z* is the hysteretic spring element



(a) (b)

Fig. 2: Conditional moments of the system displacement response $u_2(t)$; (a) mean; (b) variance.



(a) (b)

Fig. 3: Conditional moments of the system response sensitivity $v_1(t)$; (a) mean; (b) variance.

Ito’s SDE-s with the state vector $x(t)=\{u_1(t)\ \ \dot{u}_1(t)\ \ u_2(t)\ \ \dot{u}_2(t)\ \ z(t)\ \ v_1(t)\ \ \dot{v}_1(t)\ \ v_2(t)\ \ \dot{v}_2(t)\ \ v_3(t)\}^t$. The resulting discrete map is obtained using order 1.5 strong Taylor’s scheme [8] is found to have the structure of equations given by Eq,(7) and Eq.(8) where the nonlinear state vector is

$x^n(t) = \{u_1(t) \quad \dot{u}_1(t) \quad u_2(t) \quad \dot{u}_2(t) \quad z(t) \quad \dot{v}_2(t) \quad v_3(t)\}^t$ and the linear state vector is $x^l(t) = \{v_1(t) \quad \dot{v}_1(t) \quad v_2(t)\}$. This form makes it possible to apply the RBSIS filter to obtain conditional estimates of the pdf and moments of the response sensitivities given by Eq.(13) conditioned on the measurement given by Eq.(14).We use 200 particles to estimate the desired moments using RBSIS filter. The estimate of conditional expectation and variance of displacement $u_2(t)$ are shown, respectively, in Fig. 2a and Fig. 2b. Similar results for response sensitivity $v_1(t)$ are shown in Fig. 3a and Fig. 3b. The same problem can be solved using the general SIS filter on the non-partitioned system of equations. The corresponding results using 200 particles are plotted for comparison along with the RBSIS results in Fig. 2 and Fig. 3. A reasonable match between results of the two methods shows promise in applying RBSIS to more complex problems where a probabilistic substructure required for its application exists, ensuring efficient estimation.

CLOSURE

In this paper the application of Rao-Blackwellized particle filter to update the response sensitivity of a nonlinear dynamical system has been demonstrated. The existence of a conditionally linear substructure in the governing equation of motion of the dynamical system makes it possible to apply this filter. The filter results in a partially exact solution which is found to reduce the variance of the estimated quantities in comparison to the estimates obtained using the full scale particle filter. The method can handle transient, nonstationay and non-Gaussian excitations and also uncertainties in modeling the governing equation and measurement equation. There are problems of significant interest in the field of structural mechanics where a substructure of the kind described in this paper exists: sensitivity model updation, system identification and force identification of nonlinear dynamical systems to name a few. The present authors are exploring the application of the RBSIS filter to these problems in their ongoing research work.

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