VARIANCES OF REGRESSION COEFFICIENTS FOR SPLIT PLOT MULTI-FACTOR EXPERIMENTS

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1. INTRODUCTION

Previous technical reports in this series (Numbers 1, 3, 4, 5 and 6) have considered the design and analysis of multi-factor experiments. For such experiments it has been postulated that a response variable η depends on several independent continuous variables x_1, x_2, ..., x_k according to an unknown functional relationship $\eta = \varphi(x_1, x_2, \ldots, x_k)$. It is convenient to regard this relationship as the equation of a hyper-surface over the k dimensional space whose coordinates are x_1, x_2, ..., x_k. In order to investigate the nature of this surface a number, N, of experiments are performed in which the response variable is measured at selected points in the k space, the over-all configuration of such points being called the <u>experimental design</u>.

Though the functional form, $\varphi(x_1, x_2, \ldots, x_k)$ is unknown, in practice it may often be replaced by a Taylor Series to terms of some order. For example, in three variables x_1, x_2 and x_3, and to terms of second order only, such a series is written

$$(1) \quad \eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2$$

The problem then reduces to determining the unknown coefficients in this series. In the simplest case it is next assumed that each measured response $y_\alpha = \eta_\alpha + \varepsilon_\alpha$ where ε_α represents a random experimental error having expected value zero and variance σ^2. Each such error is assumed completely independent of all other errors. The Least Squares criterion is then used to obtain estimates of the β's.

Let $\underline{Y}$ be the N x 1 matrix of observed values of y_α and let $\underline{X}$ be an N x L matrix of the values taken in the N experiments by the independent "variables" in the series. The term "variables" is here used for the basic factors, x_1, x_2, x_3

and also for products and powers thereof. It is convenient to include a dummy variable, $x_0 \equiv 1$, with the constant term β_0. The Least Squares estimates of the β's are given by the L x 1 matrix

$$(2) \quad \underline{B} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Y}$$

It is further easy to show that the variance-covariance matrix for the b's, that is, the L x L matrix whose elements are $E(b_i - \beta_i)(b_j - \beta_j)$, is given by

$$(3) \quad E(\underline{B} - \underline{\beta})(\underline{B} - \underline{\beta})' = \sigma^2(\underline{X}'\underline{X})^{-1}$$

Equation (3) has been derived on the specific assumption that the random experimental errors, ε_α, associated with the measured values of the response variable have average value zero, common variance, σ^2, and are independent of each other. One common type of multi-factor experimentation for which these assumptions must be modified is the so-called <u>split plot design</u>. This report will be concerned with the variance-covariance pattern of the regression coefficients (i.e. $E(\underline{B} - \underline{\beta})(\underline{B} - \underline{\beta})'$) when the data have been obtained by a split plot type experiment. It is felt that the matrix techniques by which the appropriate formulae are developed can readily be applied to a considerable number of designs differing from those specifically considered but having other types of split plot aspects.

2. SPLIT PLOT EXPERIMENTS

For the present discussion the essential characteristic of the split plot type experiment is its error pattern. Consider an experiment to study the effects of x_1 = soaking temperature, x_2 = knife angle and x_3 = pressure bar setting in cutting veneer on a rotary lathe. The response variable might be some measure of surface smoothness. A number of veneer "bolts" are soaked at a series of different temperatures. A bolt is then put on the lathe and veneer is cut under as many of the knife angle-pressure bar settings as possible before changing to a new bolt. The error pattern for such an experimental procedure is that shown in Figure 1.

	Soaking Temperature			
Knife Angle Pressure Bar	$X_1 = X_{11}$ (Bolt 1)	$X_1 = X_{12}$ (Bolt 2)	- - - -	$X_1 = X_{1p}$ (Bolt p)
$X_2 = X_{21}$ $X_3 = X_{31}$	$b_1 + w_{11}$	$b_2 + w_{21}$	- - - -	$b_p + w_{p1}$
$X_2 = X_{22}$ $X_3 = X_{32}$	$b_1 + w_{12}$	$b_2 + w_{22}$	- - - -	$b_p + w_{p2}$
$X_2 = X_{23}$ $X_3 = X_{33}$	$b_1 + w_{13}$	$b_2 + w_{23}$	- - - -	$b_p + w_{p3}$
⋮	⋮	⋮		⋮
$X_2 = X_{2q}$ $X_3 = X_{3q}$	$b_1 + w_{1q}$	$b_2 + w_{2q}$		$b_p + w_{pq}$

Figure 1: Error Pattern for Simple Split Plot Experiment

Each cell of the table contains the random error for the observation made at the factor levels indicated. The important point to note is that the errors in any column have a common component, b_i, associated with bolt to bolt variation. In addition they have a second component, w_{ij}, which differs from one observation to the next. Following terminology carried over from field experimentation the bolts would often be referred to as "whole plots" and the sections of veneer on which the individual measurements were taken would be called "sub-plots".

The above example has been chosen for simplicity. In general the distinguishing characteristics of these experiments are

(1) The error affecting each observation is regarded as the sum of two or more random components.

(2) One component varies from one observation to the next.

(3) The other components vary only from one group of observations to another group, the groups being determined by the experimental procedure followed.

An actual example from the field of ceramic engineering involving three random components in a rather unusual pattern will be included in this report.

3. SINGLE RANDOM COMPONENT

For completeness we begin by considering the simple experiment in which there is only a single random component. The N x 1 matrix of observations for the response variable, y, is postulated to be of the form

$$(4) \quad \underline{Y} = \underline{X}\,\underline{\beta} + \underline{\varepsilon}$$

Where $\underline{X}$ is the N x L matrix made up of the values taken by the "independent variables" in the Taylor Series equation and $\underline{\beta}$ the L x 1 matrix of unknown coefficients in this equation (e.g. equation (1)). The N x 1 matrix $\underline{\varepsilon}$ consists of the random errors. Its elements are assumed to have expected value zero, variance σ^2 and to be independent of each other. In matrix notation

$$(5) \quad E\,\underline{\varepsilon}\,\underline{\varepsilon}' = \sigma^2\underline{I}$$

where $\underline{I}$ is the N x N identity matrix.

As indicated in (2)

$$\underline{B} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Y}$$

whence on substitution for $\underline{Y}$ we get

$$(6) \quad \underline{B} = (\underline{X}'\underline{X})^{-1} \underline{X}'(\underline{X}\,\underline{\beta} + \underline{\varepsilon})$$
$$= \underline{\beta} + (\underline{X}'\underline{X})^{-1} \underline{X}'\underline{\varepsilon}$$

The variance-covariance matrix of the b's (elements of $\underline{B}$) is therefore given by

$$(7) \quad E(\underline{B} - \underline{\beta})(\underline{B} - \underline{\beta})' = (\underline{X}'\underline{X})^{-1} \underline{X}'E\ \underline{\varepsilon}\ \underline{\varepsilon}'\underline{X}(\underline{X}'\underline{X})^{-1}$$
$$= \sigma^2(\underline{X}'\underline{X})^{-1}$$

4. SIMPLE SPLIT PLOT (TWO RANDOM COMPONENTS)

Consider next an experiment of the type whose error pattern is illustrated in Figure 1. The k variables may be divided into two groups, those which require "whole plots" and those which require "sub-plots". We shall assume a second order series is sufficient and for convenience will write this series with sub-plot variable terms, the whole plot variable terms and the mixed terms each grouped together. It will further be convenient to write the pure quadratic terms in their orthogonal form, i.e. $x_i^2 - C_i$, where C_i is chosen so that $\overset{N}{\underset{j=1}{S}} x_0(x_{ij}^2 - C_i) = 0$. Also each of the x variables will be measured from the centroid of the design so that $\overset{N}{\underset{j=1}{S}} x_{ij} = 0$.

In partitioned form then the $\underline{X}$ matrix is (see next page)

$$\underline{X} = \begin{array}{c|c|c|c|c}
 & & \text{Sub-Plot Variables} & \text{Whole Plot Variables} & \text{Mixed Variables} \\
 & x_0 & x_1, x_2, \text{etc} & x_5, x_6, \text{etc} & x_1 x_5, \text{ etc} \\
\hline
\text{Whole Plot} & \underline{1} & \underline{X}_S & \underline{X}_{W1} & \underline{X}_{SW1} \\
\hline
\text{Whole Plot} & \underline{1} & \underline{X}_S & \underline{X}_{W2} & \underline{X}_{SW2} \\
\hline
\text{Whole Plot} & \underline{1} & \underline{X}_S & \underline{X}_{W3} & \underline{X}_{SW3} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\text{Whole Plot} & \underline{1} & \underline{X}_S & \underline{X}_{Wp} & X_{SWp} \\
\hline
\end{array}$$

where $\underline{1}$ is a q x 1 column vector of 1's; $\underline{X}_S$ is the matrix of values taken on by the sub-plot variables within each whole plot; $\underline{X}_{Wi}$ is the matrix of values taken on by the whole plot variables in the i^{th} whole plot and $\underline{X}_{SWi}$ the matrix of interaction variables between the whole plot and sub-plot variables. Note that $\underline{X}_S$ has been assumed to be the same in each whole plot. This is ordinarily the case, though it will not always be true. It will also be assumed that the sub-plot design and the whole plot design are such that the interaction variables are orthogonal to the mean, i.e. $\overset{N}{\underset{j=1}{S}} x_{ij} x_{i'j} = 0$ $(i \neq i')$. The conventional factorials, the central composite designs and the rotatable designs all have this property. With these assumptions and conventions the following matrix relations hold.

$$(8) \quad \underline{1}' \, \underline{X}_S = 0$$

$$(9) \quad \underline{X}_{W1} + \underline{X}_{W2} + \ldots + \underline{X}_{Wp} = 0$$

$$(10) \quad \underline{1}' \, \underline{X}_{SWi} = 0$$

(11) $\underline{X}_{SW1} + \underline{X}_{SW2} + \dots + \underline{X}_{SWp} = 0$

(12) $\underline{X}_S' \, \underline{X}_{Wi} = 0$

(13) $\underline{X}_{SW1} + \underline{X}_{SW2} + \dots + \underline{X}_{SWp} = 0$

(14) $\underline{X}_{Wi}' \, \underline{X}_{SWi} = 0$

The four columns of the partitioned matrix $\underline{X}$ are therefore orthogonal to each other and we have

$$(15) \quad \underline{X}'\underline{X} = \begin{bmatrix} p\,\underline{1}'\underline{1} & 0 & 0 & 0 \\ 0 & p\,\underline{X}_S'\,\underline{X}_S & 0 & 0 \\ 0 & 0 & \underline{X}_W'\,\underline{X}_W & 0 \\ 0 & 0 & 0 & \underline{X}_{SW}'\,\underline{X}_{SW} \end{bmatrix}$$

where $\underline{X}_W$ and $\underline{X}_{SW}$, without subscripts, represent the "whole plot" and "mixed" columns of $\underline{X}$ respectively.

The inverse matrix is therefore

$$(16) \quad (\underline{X}'\underline{X})^{-1} = \begin{bmatrix} \frac{1}{p}(\underline{1}'\underline{1})^{-1} & 0 & 0 & 0 \\ 0 & \frac{1}{p}(\underline{X}_S'\,\underline{X}_S)^{-1} & 0 & 0 \\ 0 & 0 & (\underline{X}_W'\,\underline{X}_W)^{-1} & 0 \\ 0 & 0 & 0 & (\underline{X}_{SW}\,\underline{X}_{SW})^{-1} \end{bmatrix}$$

Now the error pattern associated with the experiment under consideration is that indicated earlier in Figure 1, where the error for the j^{th} sub-plot in the i^{th} whole plot is $\epsilon_{ij} = b_i + w_{ij}$. We assume b_i has expectation zero, variance σ_b^2 and that the b_i are independent of each other. Similarly we assume the w_{ij} have expectation zero, variance σ_w^2 and are independent of each other and further the w_{ij}

are independent of the b_i. If then we let $\underline{V}$ be the matrix of variances and covariances of all the observations, i.e. the matrix corresponding to $E\,\underline{\varepsilon}\,\underline{\varepsilon}'$, we have

$$(17)\quad \underline{V} = \begin{bmatrix} \underline{A} & 0 & & \cdots & 0 \\ 0 & \underline{A} & & \cdots & 0 \\ 0 & 0 & \underline{A} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \underline{A} \end{bmatrix}$$

where

$$(18)\quad \underline{A} = \begin{bmatrix} \sigma_b^2 + \sigma_w^2 & \sigma_b^2 & \sigma_b^2 & \cdots & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 + \sigma_w^2 & \sigma_b^2 & \cdots & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 & \sigma_b^2 + \sigma_w^2 & \cdots & \sigma_b^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \sigma_b^2 & \sigma_b^2 & \sigma_b^2 & \cdots & \sigma_b^2 + \sigma_w^2 \end{bmatrix}$$

that is, each observation has variance $\sigma_b^2 + \sigma_w^2$, any two observations in the same whole plot have covariance σ_b^2, and any two observations in different whole plots have covariance zero. It will be convenient to write

$$(19)\quad \underline{A} = \sigma_b^2\,\underline{1}\,\underline{1}' + \sigma_w^2\,\underline{I}$$

Now analogous to equation (7) we have the variance-covariance matrix of the coefficients b_i given by

$$(20)\quad E(\underline{B} - \underline{\beta})(\underline{B} - \underline{\beta})' = (\underline{X}'\underline{X})^{-1}\,\underline{X}'\,\underline{V}\,\underline{X}(\underline{X}'\underline{X})^{-1}$$

Performing the matrix multiplication $\underline{X}'\ \underline{V}\ \underline{X}$ we get

$$(21)\quad \underline{X}'\ \underline{V}\ \underline{X} = \begin{bmatrix} p\ \underline{1}'\underline{A}\ \underline{1} & p\ \underline{1}'\underline{A}\ \underline{X}_S & \underline{1}'\underline{A}\sum \underline{X}_{Wi} & \underline{1}'\underline{A}\sum \underline{X}_{SWi} \\ p\ X_S'\ \underline{A}\ \underline{1} & p\ \underline{X}_S'\ \underline{A}\ \underline{X}_S & X_S'\ \underline{A}\sum X_{Wi} & \underline{X}_S'\ \underline{A}\sum \underline{X}_{SWi} \\ (\sum \underline{X}_{Wi}'\underline{A}\ \underline{1} & (\sum \underline{X}_{Wi}')\underline{AX}_S & \sum \underline{X}_{Wi}'\underline{A}\ \underline{X}_{Wi} & \sum \underline{X}_{Wi}'\underline{A}\ \underline{X}_{SWi} \\ (\sum \underline{X}_{SWi}')\underline{A1} & (\sum \underline{X}_{SWi})\underline{AX}_S & \sum \underline{X}_{SWi}'\underline{AX}_{Wi} & \sum \underline{X}_{SWi}'\underline{A}\ \underline{X}_{SWi} \end{bmatrix}$$

By virtue of equations (8) to (14) all the off-diagonal terms in $\underline{X}'\ \underline{V}\ \underline{X}$ are zero. Substituting for A we get

$$(22)\quad \underline{X}'\ \underline{V}\ \underline{X} = \begin{bmatrix} pq^2\sigma_b^2 + pq\sigma_w^2 & \cdot & \cdot & \cdot \\ & p\sigma_w^2\, \underline{X}_S'\ \underline{X}_S & \cdot & \cdot \\ & & (q\sigma_b^2 + \sigma_w^2)\underline{X}_W'\ \underline{X}_W & \cdot \\ & & & \sigma_w^2\ \underline{X}_{SW}'\ \underline{X}_{SW} \end{bmatrix}$$

In getting the third element advantage is taken of the fact that within any given whole plot the q elements of any column of X_{Wi} are constant, therefore

$$\sum \underline{X}_{Wi}'\ \underline{1}\ \underline{1}'\ \underline{X}_{Wi} = q\ \underline{X}_W'\ \underline{X}_W$$

Now multiplying $(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{V}\ \underline{X}(\underline{X}'\underline{X})^{-1}$ we get

$$(23)\quad \begin{bmatrix} (q\sigma_b^2 + \sigma_w^2)/pq & & & \\ & \frac{\sigma_w^2}{p}(\underline{X}_S'\ \underline{X}_S)^{-1} & & \\ & & (q\sigma_b^2 + \sigma_w^2)(\underline{X}_W'\ \underline{X}_W)^{-1} & \\ & & & \sigma_w^2(\underline{X}_{SW}'\ \underline{X}_{SW})^{-1} \end{bmatrix}$$

The elements are respectively the variance of the mean (i.e. b_0), the variance-covariance matrices of the coefficients of the sub-plot variables, the whole plot variables and the mixed variables

5. AN EXAMPLE (TWO RANDOM COMPONENTS)

Suppose the veneer cutting experiment described earlier is set up as a 3 x 3 x 3 factorial and that for each of the temperature levels we use two bolts making a total of 54 observations. Let the three levels of each variable be coded to -1, 0, +1 and write the second order series to be fitted in the form

$$(24)\quad \begin{aligned} y = b_0 + b_1x_1 + b_2x_2 + b_{11}(x_1^2 - C_1) = b_{22}(x_2^2 - C_2) + b_{12}x_1x_2 + b_3x_3 \\ + b_{33}(x_3^2 - C_3) + b_{13}x_1x_3 + b_{23}x_2x_3 \end{aligned}$$

where x_1 = knife angle, x_2 = pressure bar setting, x_3 = temperature and for orthogonality $C_1 = C_2 = C_3 = \frac{SX_2^2}{N} = 2/3$.

For this experiment p = 6, q = 9 hence the variance of b_0 is given by $(9\sigma_b^2 + \sigma_w^2)/54$. The sub-plot design is a 3 x 3 factorial hence the variance of the coefficients of the sub-plot variables is given by

$$(25)\quad \frac{\sigma_w^2}{6} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_{11} & b_{22} & b_{12} \end{array} \\ \left[\begin{array}{ccccc} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]^{-1} \end{array}$$

The variances of the whole plot coefficients are

$$(9\sigma_b^2 + \sigma_w^2) \begin{array}{c} \begin{array}{cc} b_3 & b_{33} \end{array} \\ \left[\begin{array}{cc} 36 & 0 \\ 0 & 12 \end{array}\right]^{-1} \end{array} \tag{26}$$

The variances of the mixed coefficients are

$$\sigma_w^2 \begin{array}{c} \begin{array}{cc} b_{13} & b_{23} \end{array} \\ \left[\begin{array}{cc} 24 & 0 \\ 0 & 24 \end{array}\right]^{-1} \end{array} \tag{27}$$

Estimates of σ_w^2 and of $9\sigma_b^2 + \sigma_w^2$ can be found from the following analysis of variance:

Source of Variation		d.f.
Mean		1
Whole Plots		5
Temp. Coefficients	2	
Bolts within Temps	3	
Sub-Plots		48
Pressure and Knife Angle Coefficients	5	
Mixed Coefficients	2	
Higher Order Coefficients	17	
Sub-Plot Error*	24	
Total		54

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* The sub-plot error is made up of Bolts x P(6 d.f.), Bolts x K(6 d.f.) and Bolts x K x P(12 d.f.).

In this analysis of variance the mean square for "Bolts within Temps" is a direct estimate of $9\sigma_b^2 + \sigma_w^2$ and the mean square for "Sub-Plot Error" is an estimate of σ_w^2.

6. A FURTHER EXAMPLE (THREE RANDOM COMPONENTS)

As a further and somewhat more complex example we consider an experiment in the field of ceramic engineering*. The experimenter wished to study the effect on y = absorption, of the following variables:

x_1 = ratio of clay to nepheline syenite plus feldspar
x_2 = ratio of clay to flint
x_3 = ratio of nepheline syenite to feldspar
x_4 = firing temperature

It was decided to set the three composition variables up in a central composite design (see references 1, 3, 6) and to repeat this design at three firing temperatures (equally spaced). Two firings were made at each of the three temperatures. For each firing the kiln contained duplicate test specimens for all of the fifteen different compositions required by the three variable central composite design. Two batches of the mixed raw materials were prepared and the duplicate specimens actually consisted of a representative from each batch. It was felt that three distinct components of random error would be involved, namely $w_{ijk\ell}$ the specimen to specimen variation, $b_{k\ell}$ the batch to batch variation (for fixed composition) and f_{ij} the firing to firing variation (for nominally fixed temperature). These are postulated to have zero expectation, variance σ_w^2, σ_b^2 and σ_f^2 respectively and to be independent of each other. The experimental arrangement is shown in Figure 2.

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* W. C. Hackler "The Effect of Raw Material Ratios on the Absorption of Whiteware Compositions", Ph.D. Thesis, N. C. State College (1954).

		Compositions						
		1		2			15	
Temperature	Firing	1a	1b	2a	2b		15a	15b
1	I							
	II							
2	I							
	II							
3	I							
	II							

Figure 2: Arrangement of Ceramic Experiment

In Figure 2 the firing component, f_{ij}, is constant in a given row; the batch component, $b_{k\ell}$, is constant in a given column; the specimen component $w_{ijk\ell}$ varies from one observation to the next.

The variance-covariance matrix of the observations has the form

$$(28)\quad V = \begin{bmatrix} (\underline{A}+\underline{D}) & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & (\underline{A}+\underline{D}) & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & (\underline{A}+\underline{D}) & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & (\underline{A}+\underline{D}) & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & (\underline{A}+\underline{D}) & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & (\underline{A}+\underline{D}) \end{bmatrix}$$

where

$$(29)\quad \underline{A} = \begin{bmatrix} \sigma_f^2 + \sigma_w^2 & \sigma_f^2 & & \sigma_f^2 \\ & \sigma_f^2 + \sigma_w^2 & \cdots\cdots & \sigma_f^2 \\ \vdots & & & \\ \sigma_f^2 & \sigma_f^2 & & \sigma_f^2 + \sigma_w^2 \end{bmatrix}$$

and

$$(30)\quad \underline{D} = \begin{bmatrix} \sigma_b^2 & 0 & \cdots\cdots & 0 \\ 0 & \sigma_b^2 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & \sigma_b^2 \end{bmatrix}$$

The variance of each observation is $\sigma_f^2 + \sigma_b^2 + \sigma_w^2$. Two observations in the same firing have covariance σ_f^2; two observations in the same batch have covariance σ_b^2; observations are otherwise independent.

We note that $\underline{V}$ may be written as

$$(31)\quad \underline{V}_a + \underline{V}_b = \begin{bmatrix} \underline{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & \underline{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{A} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{A} & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{A} & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{A} \end{bmatrix} + \begin{bmatrix} \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} & \underline{D} \end{bmatrix}$$

The matrix $(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{V}_a\ \underline{X}'(\underline{X}'\underline{X})^{-1}$ is that already evaluated in the two random component problem (though with σ_f^2 substituted for σ_b^2). We therefore merely evaluate $(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{V}_b\ \underline{X}(\underline{X}'\underline{X})^{-1}$ to be added to the earlier results. We find first

$$(32)\quad \underline{X}'\underline{V}_b\underline{X} = \begin{bmatrix} 36\ \underline{1}'\underline{D}\ \underline{1} & 36\ \underline{1}\ \underline{D}\ \underline{X}_S & 6\ \underline{1}\ \underline{D}\ \sum \underline{X}_{Wi} & 6\ \underline{1}'\underline{D}\ \sum \underline{X}_{SWi} \\ & 36\ \underline{X}_S'\underline{D}\ \underline{X}_S & 6\ \underline{X}_S'\ \underline{D}\ \sum \underline{X}_{Wi} & 6\ \underline{X}_S'\underline{D}\ \sum \underline{X}_{SWi} \\ & & \sum(\sum \underline{X}_{Wi}'\underline{D})\underline{X}_{Wi} & \sum(\sum \underline{X}_{Wi}'\underline{D})\underline{X}_{SWi} \\ \text{symmetric} & & & \sum(\sum \underline{X}_{SWi}'\underline{D})\underline{X}_{SWi} \end{bmatrix}$$

Again by virtue of equations (8) to (14) only the first two elements of the diagonal are non-zero. Upon substitution for $\underline{D}$ these become

$$36\,\underline{1}'(\sigma_b^2\,\underline{I})\underline{1} = (36)(30)\sigma_b^2$$

and $$36\,\underline{X}_S'(\sigma_b^2\,\underline{I})\underline{X}_S = 36\sigma_b^2\,\underline{X}_S'\,\underline{X}_S$$

Hence $(\underline{X}'\underline{X})^{-1}\underline{X}'\,\underline{V}_b\,\underline{X}(\underline{X}'\underline{X})^{-1}$ is given by

$$(33)\quad \begin{bmatrix} \sigma_b^2/30 & 0 & 0 & 0 \\ 0 & \sigma_b^2(X_S'\,X_S)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally adding this to the previously obtained $(\underline{X}'\underline{X})^{-1}\underline{X}'\,\underline{V}_a\,\underline{X}(\underline{X}'\underline{X})^{-1}$ we get

$$(34)\quad E(\underline{B} - \underline{\beta})(\underline{B} - \underline{\beta})' = (\underline{X}'\underline{X})^{-1}\underline{X}'\,\underline{V}\,\underline{X}(\underline{X}'\underline{X})^{-1} =$$

$$= \begin{bmatrix} (30\sigma_f^2 + 6\sigma_b^2 + \sigma_W^2)/180 & & & \\ & \dfrac{6\sigma_b^2 + \sigma_W^2}{6}(\underline{X}_S'\,\underline{X}_S)^{-1} & & \\ & & (30\sigma_f^2 + \sigma_W^2)(\underline{X}_W'\,\underline{X}_W)^{-1} & \\ & & & \sigma_W^2(\underline{X}_{SW}'\,\underline{X}_{SW})^{-1} \end{bmatrix}$$

The elements are respectively the variance of the mean, b_0, the variance and covariance of the coefficients of the composition variables, the variance of temperature coefficients and the variance of the mixed coefficients. To estimate the variance components involved we use the following analysis of variance:

Source	d.f.	m.s.	E(m.s.)
Temperatures	2	1179.9889	$\sigma_w^2+30\sigma_f^2+30\sum t_i^2$
Firings within Temps	3	.1521	$\sigma_w^2 + 30\sigma_f^2$
Compositions	14	10.3360	$\sigma_w^2+6\sigma_b^2+12\sum C_k^2/14$
Batches within Compositions	15	.7405	$\sigma_w^2 + 6\sigma_b^2$
Temps x Comps.	28	1.1130	$\sigma_w^2+4\sum\sum(tC)_{ik}^2/28$
(Comps) x (Firings within Temps)	42	.0818	σ_w^2
(Temps) x (Batches within Comps)	30	.0857	σ_w^2
(Batches w/n Comps)(Firings w/n Temps)	45	.0631	σ_w^2
Total	179		

The expected values of the mean squares follow from the model

$$y_{ijk\ell} = \mu + t_i + C_k + (tC)_{ik} + f_{ij} + b_{k\ell} + w_{ijk\ell}$$

where t_i, C_k and $(tC)_{ik}$ are parameters for temperature, composition and temperature-composition interaction respectively. This model differs somewhat from the second degree Taylor Series model but the difference is confined to the non-random portion of the model. The analysis of variance based on it furnishes the necessary estimates of σ_w^2, $\sigma_w^2 + 6\sigma_b^2$ and $\sigma_W^2 + 30\sigma_f^2$ as indicated. The last three lines may be pooled to estimate σ_w^2.

7. SUMMARY

In this report we have presented a matrix development of formulae for variances and covariances of regression coefficients in split plot multi-factor experiments. We have considered the simplest type of split plot experiment plus one more complex example which arose in the field of ceramic engineering. It is felt, however, that the techniques by which these formulae are derived can readily be applied to many other designs having the characteristics outlined on page four of this report.

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