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JOINT MEASURES AND CROSS-COVARIANCE OPERATORS

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Abstract

Let H_1 (resp., H_2) be a real and separable Hilbert space with Borel σ-field Γ_1 (resp., Γ_2), and let $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$ be the product measurable space generated by the measurable rectangles. This paper develops relations between probability measures on $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$, i.e., joint measures, and the projections of such measures on (H_1, Γ_1) and (H_2, Γ_2). In particular, the class of all joint Gaussian measures having two specified Gaussian measures as projections is characterized, and conditions are obtained for two joint Gaussian measures to be mutually absolutely continuous. The cross-covariance operator of a joint measure plays a major role in these results, and these operators are characterized.

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INTRODUCTION

Let H_1 (resp., H_2) be a real and separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$ (resp., $\langle \cdot, \cdot \rangle_2$) and Borel σ-field Γ_1 (resp., Γ_2). Let $\Gamma_1 \times \Gamma_2$ denote the σ-field generated by the measurable rectangles $A \times B$, $A \in \Gamma_1$, $B \in \Gamma_2$. Define $H_1 \times H_2 = \{(\underset{\sim}{u}, \underset{\sim}{v}): \underset{\sim}{u} \text{ in } H_1, \underset{\sim}{v} \text{ in } H_2\}$. $H_1 \times H_2$ is a real linear space, with addition and scalar multiplication defined by $(\underset{\sim}{u}, \underset{\sim}{v}) + (\underset{\sim}{z}, \underset{\sim}{y}) = (\underset{\sim}{u}+\underset{\sim}{z}, \underset{\sim}{v}+\underset{\sim}{y})$ and $k(\underset{\sim}{u}, \underset{\sim}{v}) = (k\underset{\sim}{u}, k\underset{\sim}{v})$. $H_1 \times H_2$ is a separable Hilbert space under the inner product $[\cdot, \cdot]$ defined by $[(\underset{\sim}{u}, \underset{\sim}{v}), (\underset{\sim}{t}, \underset{\sim}{z})] = \langle \underset{\sim}{u}, \underset{\sim}{t} \rangle_1 + \langle \underset{\sim}{v}, \underset{\sim}{z} \rangle_2$; moreover, the open sets under the norm obtained from this inner product generate $\Gamma_1 \times \Gamma_2$ [1]. Let $||\cdot||_1$ (resp., $||\cdot||_2$) denote the norm in H_1 (resp., H_2) obtained from the inner product, and let $|||\cdot|||$ denote the norm in $H_1 \times H_2$ obtained from the inner product. A probability measure on $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$ will be called a *joint measure*.

A probability measure μ_i on (H_i, Γ_i) $(i = 1 \text{ or } 2)$ that satisfies

$$\int_{H_i} ||\underset{\sim}{x}||_i^2 \, d\mu_i(\underset{\sim}{x}) < \infty \qquad (*)$$

defines an operator $\underset{\sim}{R}_i$ in H_i and a mean element $\underset{\sim}{m}_i$ of H_i by $\langle \underset{\sim}{m}_i, \underset{\sim}{u} \rangle_i = \int_{H_i} \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle_i d\mu_i(\underset{\sim}{x})$ and $\langle \underset{\sim}{R}_i \underset{\sim}{u}, \underset{\sim}{v} \rangle_i = \int_{H_i} \langle \underset{\sim}{x} - \underset{\sim}{m}_i, \underset{\sim}{u} \rangle_i \langle \underset{\sim}{x} - \underset{\sim}{m}_i, \underset{\sim}{v} \rangle_i d\mu_i(\underset{\sim}{x})$; R_i is a *covariance operator*; i.e., it is linear, bounded, non-negative, self-adjoint, and trace-class. μ_i is Gaussian if the probability distribution on the Borel sets

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of the real line induced from μ_i by every bounded linear functional on H_i is Gaussian. If μ_i is Gaussian, (*) is satisfied; moreover, to every covariance operator $\underset{\sim}{R}$ and element $\underset{\sim}{m}$ in H_i there corresponds a unique Gaussian measure [2]. All measures on (H_i,Γ_i) considered in this paper are probability measures that satisfy (*).

We are interested in determining relations between joint measures and their projections on (H_i,Γ_i). In particular, the following questions are answered: (1) What is the relation between the covariance operator of a joint measure and the covariance operators of its projections? (2) Given two Gaussian measures μ_i on (H_i,Γ_i), $i = 1$ and 2, how can one characterize the set of all joint Gaussian measures having μ_1 and μ_2 as projections? (3) What are conditions for equivalence of two joint Gaussian measures, given in terms of operators on H_1 and H_2? The answers to all three questions involve *cross-covariance* operators, and a characterization of such operators is given.

Joint Measures

A probability measure on $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$ will be called a joint measure. Suppose that μ_{XY} is a joint measure; the projection μ_X is the probability measure on (H_1, Γ_1) induced from μ_{XY} by the $\Gamma_1 \times \Gamma_2 / \Gamma_1$ measurable map P_1, $P_1(\underset{\sim}{x}, \underset{\sim}{y}) \equiv \underset{\sim}{x}$. Similarly, the projection μ_Y is the measure on (H_2, Γ_2) induced from μ_{XY} by the map P_2, $P_2(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{y}$. Note that there will, in general, not be a unique joint measure having μ_X and μ_Y as projections; the notation μ_{XY} is used to relate the joint measure to its (unique) projections.

A joint measure μ_{XY} is Gaussian if the probability distribution on the Borel sets of the real line defined by

$$P^{(\underset{\sim}{u}, \underset{\sim}{v})}[A] = \mu_{XY}\{(\underset{\sim}{x}, \underset{\sim}{y}): [(\underset{\sim}{x}, \underset{\sim}{y}), (\underset{\sim}{u}, \underset{\sim}{v})] \in A\}$$

is Gaussian for all $(\underset{\sim}{u}, \underset{\sim}{v})$ in $H_1 \times H_2$. $P^{(\underset{\sim}{u}, \underset{\sim}{v})}$ is clearly Gaussian for all $(\underset{\sim}{u}, \underset{\sim}{v})$ in $H_1 \times H_2$ if and only if the distribution $P_0^{(\underset{\sim}{u}, \underset{\sim}{v})}$ on $B[R^2]$ defined by

$$P_0^{(\underset{\sim}{u}, \underset{\sim}{v})}[A \times B] = \mu_{XY}\{(\underset{\sim}{x}, \underset{\sim}{y}): \langle \underset{\sim}{x}, \underset{\sim}{u} \rangle_1 \in A,\ \langle \underset{\sim}{y}, \underset{\sim}{v} \rangle_2 \in B\}$$

is Gaussian for all $(\underset{\sim}{u}, \underset{\sim}{v})$ in $H_1 \times H_2$.

μ_{XY} will have a covariance operator R_{XY} and a mean m_{XY} in $H_1 \times H_2$ if

$$\int_{H_1 \times H_2} |||(\underset{\sim}{x}, \underset{\sim}{y})|||^2 \, d\mu_{XY}(\underset{\sim}{x}, \underset{\sim}{y}) < \infty: \tag{**}$$

this is always the case if μ_{XY} is Gaussian [2]. We will assume that (**) is satisfied for all joint measures considered in this paper. This is consistent with the assumptions for the measures on (H_i, Γ_i), since if μ_{XY} is a joint measure with projections μ_X and μ_Y, then

$$\int_{H_1 \times H_2} |||(\underset{\sim}{u},\underset{\sim}{v})|||^2 \, d\mu_{XY}(\underset{\sim}{u},\underset{\sim}{v}) = \int_{H_1 \times H_2} \{||\underset{\sim}{u}||_1^2 + ||\underset{\sim}{v}||_1^2\} \, d\mu_{XY}(\underset{\sim}{u},\underset{\sim}{v})$$

$$= \int_{H_1} ||\underset{\sim}{u}||_1^2 \, d\mu_X(\underset{\sim}{u}) + \int_{H_2} ||\underset{\sim}{v}||_2^2 \, d\mu_Y(\underset{\sim}{v})$$

so that μ_{XY} satisfies (**) if and only if both μ_X and μ_Y satisfy (*). Given a joint measure μ_{XY}, we will use $\underset{\sim}{R}_X$ and $\underset{\sim}{m}_X$ (resp., $\underset{\sim}{R}_Y$ and $\underset{\sim}{m}_Y$) to denote the covariance operator and mean element of the projection μ_X (resp., μ_Y).

Cross-Covariance Operators

Suppose μ_{XY} is a joint measure satisfying (**). Define a functional G on $H_1 \times H_2$ by

$$G(u,v) \;\equiv\; \int_{H_1 \times H_2} \langle x - m_X, u\rangle_1 \langle y - m_Y, v\rangle_2 \, d\mu_{XY}(x,y).$$

For fixed u (resp., v), G is a linear functional on H_2 (resp., H_1). Moreover, $|G(u,v)|^2 \le ||R_X^{\frac{1}{2}}u||_1^{\,2} ||R_Y^{\frac{1}{2}}v||_2^{\,2}$, where R_X and R_Y are the covariance operators of μ_X and μ_Y. Hence, for fixed u, there exists by Riesz' theorem a unique element g_u in H_2 such that $G(u,v) = \langle g_u, v\rangle_2$ for every v in H_2. Similarly, for fixed $v \in H_2$ there exists a unique element $g_v \in H_1$ such that $G(u,v) = \langle g_v, u\rangle_1$ for all $u \in H_1$. Define a map $R_{XY}\colon H_2 \to H_1$ by $R_{XY}v = g_v$. R_{XY} is single-valued, by the fact that g_v is unique. R_{XY} is defined everywhere in H_2, is clearly linear, and is bounded since

$$||R_{XY}v||_1^{\,2} \;=\; ||g_v||_1^{\,2} \;=\; \sup_{u \in H_1} \frac{\langle g_v, u\rangle_1^{\,2}}{||u||_1^{\,2}} \;=\; \sup_{u \in H_1} \frac{|G(v,u)|^2}{||u||_1^{\,2}}$$

$$\le \sup_{u \in H_1} \frac{||R_X^{\frac{1}{2}}u||_1^{\,2}}{||u||_1^{\,2}} \, ||R_Y^{\frac{1}{2}}v||_2^{\,2} \;\le\; ||R_X||_1 \, ||R_Y||_2 \, ||v||_2^{\,2} \, .$$

Clearly $R_{XY}^*\colon H_1 \to H_2$ is defined by $R_{XY}^* u = g_u$. Thus $G(u,v) = \langle R_{XY}v, u\rangle_1 = \langle v, R_{XY}^* u\rangle_2$ for all $u\ H_1$ and v in H_2. We define $R_{XY}^* = R_{YX}$. The operator R_{XY} will be called the *cross-covariance* operator of μ_{XY}. A partial characterization of the cross-covariance operator was given in [3] for the case where R_X and R_Y are both strictly positive, $H_1 = H_2$, and μ_{XY} was induced by a map from a probability space into $(H \times H, \Gamma \times \Gamma)$. Here the characterization will be extended, and without these restrictions.

Let P_X (resp., P_Y) be the projection operator mapping H_1 onto $\overline{\text{range}(R_X)}$ (resp., H_2 onto $\overline{\text{range}(R_Y)}$), where $\overline{\text{range}(R)}$ denotes the closure of range(R). We then have the following result.

THEOREM 1. (A) If μ_{XY} is a joint measure with a covariance operator and mean element, then the cross-covariance operator R_{XY} has a representation as $R_{XY} = R_X^{1/2} V R_Y^{1/2}$, where V is a unique bounded linear operator such that $V: H_2 \to H_1$, $||V|| \le 1$, and $V = P_X V P_Y$.

(B) If $R: H_2 \to H_1$ is a bounded linear operator of trace class, then there exists a joint Gaussian measure μ_{XY} such that R is the cross-covariance operator of μ_{XY}.

PROOF: (A) Let s be any fixed element in range($R_Y^{1/2}$), with z any element of H_2 satisfying $R_Y^{1/2} z = s$. Define a linear functional f_s on range($R_X^{1/2}$) by

$$f_s(R_X^{1/2} u) = \int_{H_1 \times H_2} \langle x - m_X, u \rangle_1 \, \langle y - m_Y, z \rangle_2 \, d\mu_{XY}(x,y)$$

$$= \langle R_{XY} z, u \rangle_1, \quad \text{all} \quad u \in H_1.$$

Since $|f_s(R_X^{1/2} u)| \le ||s||_2 \, ||R_X^{1/2} u||_1$, f_s is bounded on range($R_Y^{1/2}$) and thus can be extended by continuity to a bounded linear functional on $\overline{\text{range}(R_X)}$ ($= P_X[H_1]$). Note that the extension has norm $\le ||s||_2$. By Riesz' theorem, there exists a unique element h in $P_X[H_1]$ such that $f_s(w) = \langle h, w \rangle_1$ for all w in $P_X[H_1]$ and $||h||_1 \le ||s||_2$. Define a map $V': H_2 \to H_1$ by $V's = h$. V' is defined for all s in range($R_Y^{1/2}$), is clearly linear and single-valued, and is bounded because $||V's||_1 \le ||s||_2$. V' can thus be extended by continuity to a bounded linear operator V defined on $P_Y[H_2]$; note that $Vs = P_X V P_Y s$ for s in $P_Y[H_2]$, $||V|| \le 1$, and $f_s(w) = \langle Vs, w \rangle_1$. We extend the domain of V to all of H_2 by defining $Vu = 0$ for u in $(P_Y[H_2])^{\perp}$.

Thus, for any $\underset{\sim}{z}$ in H_2, for $\underset{\sim}{s} \equiv \underset{\sim}{R}_Y^{1/2}\underset{\sim}{z}$, and for any $\underset{\sim}{u}$ in H_1, one has $f_{\underset{\sim}{s}}(\underset{\sim}{R}_X^{1/2}\underset{\sim}{u}) = \langle \underset{\sim}{R}_{XY}\underset{\sim}{z},\underset{\sim}{u}\rangle_1 = \langle \underset{\sim}{V}\underset{\sim}{R}_Y^{1/2}\underset{\sim}{z},\underset{\sim}{R}_X^{1/2}\underset{\sim}{u}\rangle_1$, so that $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{1/2}\underset{\sim}{V}\underset{\sim}{R}_Y^{1/2}$, $||\underset{\sim}{V}|| \leq 1$, and $\underset{\sim}{V} = \underset{\sim}{P}_X\underset{\sim}{V}\underset{\sim}{P}_Y$.

To see that $\underset{\sim}{V}$ is unique, suppose that $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{1/2}\underset{\sim}{G}\underset{\sim}{R}_Y^{1/2}$, with $||\underset{\sim}{G}|| \leq 1$ and $\underset{\sim}{G} = \underset{\sim}{P}_X\underset{\sim}{G}\underset{\sim}{P}_Y$. Then $(\underset{\sim}{V}-\underset{\sim}{G})\underset{\sim}{R}_Y\,\underset{\sim}{u} = \underset{\sim}{R}_X^{1/2}(\underset{\sim}{V}-\underset{\sim}{G})\underset{\sim}{R}_Y^{1/2}\underset{\sim}{u} = \underset{\sim}{0}$, all $\underset{\sim}{u} \in H_2$, so that $\underset{\sim}{V}\underset{\sim}{u} = \underset{\sim}{G}\underset{\sim}{u}$, all $\underset{\sim}{u}$ in $\underset{\sim}{P}_Y[H_2]$. Since $\underset{\sim}{V}\underset{\sim}{u} = \underset{\sim}{G}\underset{\sim}{u} = \underset{\sim}{0}$ for $\underset{\sim}{u} \perp \underset{\sim}{P}_Y[H_2]$, $\underset{\sim}{V} = \underset{\sim}{G}$ on H_2.

(B) By the polar decomposition theorem [4], $\underset{\sim}{R} = \underset{\sim}{U}\underset{\sim}{T}^2$, where $\underset{\sim}{T}: H_2 \to H_2$, $\underset{\sim}{T}^2 = (\underset{\sim}{R}^*\underset{\sim}{R})^{1/2}$, and $\underset{\sim}{U}: H_2 \to H_1$ is partially isometric, isometric on $\underset{\sim}{P}_T[H_2]$ and zero on $(\underset{\sim}{P}_T[H_2])^{\perp}$, with $\text{range}(\underset{\sim}{U}) = \overline{\text{range}(\underset{\sim}{R})}$ ($\underset{\sim}{P}_T \equiv$ the projection operator in H_2 with range equal to $\overline{\text{range}(\underset{\sim}{T})}$). Since $\underset{\sim}{R}$ is trace-class, $\underset{\sim}{T}$ and $\underset{\sim}{U}\underset{\sim}{T}$ are Hilbert-Schmidt. Further, $\underset{\sim}{T}\underset{\sim}{U}^* = \underset{\sim}{W}(\underset{\sim}{U}\underset{\sim}{T}^2\underset{\sim}{U}^*)^{1/2}$ for a partially isometric $\underset{\sim}{W}: H_1 \to H_2$, $\underset{\sim}{W}$ isometric on $\overline{\text{range}(\underset{\sim}{U}\underset{\sim}{T}^2\underset{\sim}{U}^*)}$, and with $\text{range}(\underset{\sim}{W}) \subset \overline{\text{range}(\underset{\sim}{T})} = \overline{\text{range}(\underset{\sim}{R}^*\underset{\sim}{R})}$. Thus $\underset{\sim}{R} = \underset{\sim}{U}\underset{\sim}{T}^2 = (\underset{\sim}{U}\underset{\sim}{T}^2\underset{\sim}{U}^*)^{1/2}\underset{\sim}{W}^*\underset{\sim}{T}$.

Since $\underset{\sim}{T}$ is self-adjoint and Hilbert-Schmidt, there exists a Gaussian measure μ_Y on (H_2,Γ_2) with covariance operator $\underset{\sim}{T}^2$ and null mean element. Define $Y: H_2 \to H_2$ as the identity map, and $X: H_2 \to H_1$ by $X\underset{\sim}{v} = \underset{\sim}{U}\underset{\sim}{v}$. X is Γ_2/Γ_1 measurable as a continuous map, and thus induces from μ_Y a probability measure μ_X on (H_1,Γ_1), $\mu_X(A) \equiv \mu_Y\{v: \underset{\sim}{U}\underset{\sim}{v} \in A\}$, $A \in \Gamma_1$. μ_X is Gaussian, with null mean element and covariance operator $\underset{\sim}{R}_X = \underset{\sim}{U}\underset{\sim}{T}^2\underset{\sim}{U}^*$.

The map $(X,Y): H_2 \to H_1 \times H_2$, $(X,Y)(\underset{\sim}{v}) = (\underset{\sim}{U}\underset{\sim}{v},\underset{\sim}{v})$, is $\Gamma_2/\Gamma_1\times\Gamma_2$ measurable, thus induces from μ_Y a measure μ_{XY} on $(H_1\times H_2,\Gamma_1\times\Gamma_2)$, defined by $\mu_{XY}(C) = \mu_Y\{u: (\underset{\sim}{U}\underset{\sim}{u},\underset{\sim}{u}) \in C\}$, $C \in \Gamma_1\times\Gamma_2$. Moreover,

$$\langle \underset{\sim}{R}_{XY}\underset{\sim}{u},\underset{\sim}{v}\rangle_1 = \int_{H_1\times H_2} \langle \underset{\sim}{x},\underset{\sim}{v}\rangle_1 \langle \underset{\sim}{y},\underset{\sim}{u}\rangle_2 \, d\mu_{XY}(\underset{\sim}{x},\underset{\sim}{y})$$

$$= \int_{H_2} \langle \underset{\sim}{y},\underset{\sim}{U}^*\underset{\sim}{v}\rangle_2 \langle \underset{\sim}{y},\underset{\sim}{u}\rangle_2 \, d\mu_Y(\underset{\sim}{y})$$

$$= \langle \underset{\sim}{R}_Y \underset{\sim}{U}^* \underset{\sim}{v}, \underset{\sim}{u} \rangle_2 \quad \text{for all } \underset{\sim}{u} \text{ in } H_2, \; \underset{\sim}{v} \text{ in } H_1.$$

Hence $\underset{\sim}{R}_{XY} = \underset{\sim}{U}\underset{\sim}{R}_Y = \underset{\sim}{U}\underset{\sim}{T}^2 = \underset{\sim}{R}$. Finally, it is clear from the definitions that μ_{XY} is Gaussian. This completes the proof.

Covariance Operators for Joint Measures

Suppose that μ_{XY} is a joint measure satisfying (**). We proceed to determine the relations between the covariance operator and mean element of μ_{XY}, and the covariance operators and mean elements of the projections μ_X and μ_Y.

Prop. 1: Let μ_{XY} be a joint measure such that $\int_{H_1\times H_2} |||(\underset{\sim}{u},\underset{\sim}{v})|||^2 d\mu_{XY}(\underset{\sim}{u},\underset{\sim}{v}) < \infty$. Let $\mathcal{R}_{XY}$ and m_{XY} be the covariance operator and mean element of μ_{XY}, and denote by $\underset{\sim}{R}_X$ and $\underset{\sim}{m}_X$ (resp., $\underset{\sim}{R}_Y$ and $\underset{\sim}{m}_Y$) the covariance operator and mean element of the projection μ_X (resp., μ_Y). Then, $m_{XY} = (\underset{\sim}{m}_X,\underset{\sim}{m}_Y)$, and $\mathcal{R}_{XY}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X\underset{\sim}{u} + \underset{\sim}{R}_{XY}\underset{\sim}{v},\ \underset{\sim}{R}_Y\underset{\sim}{v} + \underset{\sim}{R}_{YX}\underset{\sim}{u})$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $H_1\times H_2$.

Proof: It is clear that $m_{XY} = (\underset{\sim}{m}_X,\underset{\sim}{m}_Y)$; for example,

$$\begin{aligned}\langle \underset{\sim}{m}_X,\underset{\sim}{u}\rangle_1 &= \int_{H_1} \langle \underset{\sim}{x},\underset{\sim}{u}\rangle_1\, d\mu_X(\underset{\sim}{x}) \\ &= \int_{H_1\times H_2} [(\underset{\sim}{x},\underset{\sim}{y}),\ (\underset{\sim}{u},\underset{\sim}{0})]\, d\mu_{XY}(\underset{\sim}{x},\underset{\sim}{y}) \\ &= [m_{XY},\ (\underset{\sim}{u},\underset{\sim}{0})].\end{aligned}$$

To describe the covariance operator $\mathcal{R}_{XY}$, we can assume that μ_{XY} has null mean element. Then

$[\mathcal{R}_{XY}(\underset{\sim}{u},\underset{\sim}{v}),\ (\underset{\sim}{t},\underset{\sim}{z})] = \int_{H_1\times H_2}[(\underset{\sim}{x},\underset{\sim}{y}),\ (\underset{\sim}{u},\underset{\sim}{v})][(\underset{\sim}{x},\underset{\sim}{y}),\ (\underset{\sim}{t},\underset{\sim}{z})\ d\mu_{XY}(\underset{\sim}{x},\underset{\sim}{y})$
$= \int_{H_1\times H_2}\left\{\langle \underset{\sim}{x},\underset{\sim}{u}\rangle_1 + \langle \underset{\sim}{y},\underset{\sim}{v}\rangle_2\right\}\left\{\langle \underset{\sim}{x},\underset{\sim}{t}\rangle_1 + \langle \underset{\sim}{y},\underset{\sim}{z}\rangle_2\right\}d\mu_{XY}(\underset{\sim}{x},\underset{\sim}{y}) = \langle \underset{\sim}{R}_X\underset{\sim}{u},\underset{\sim}{t}\rangle_1 +$
$\langle \underset{\sim}{R}_{XY}\underset{\sim}{v},\underset{\sim}{t}\rangle_1 + \langle \underset{\sim}{R}_{XY}\underset{\sim}{z},\underset{\sim}{u}\rangle_1 + \langle \underset{\sim}{R}_Y\underset{\sim}{z},\underset{\sim}{v}\rangle_2 = [(\underset{\sim}{R}_X\underset{\sim}{u} + \underset{\sim}{R}_{XY}\underset{\sim}{v},\ \underset{\sim}{R}_Y\underset{\sim}{v} + \underset{\sim}{R}_{YX}\underset{\sim}{u}),\ (\underset{\sim}{t},\underset{\sim}{z})]$.

One thus sees that $\underset{\sim}{R}_X$, $\underset{\sim}{R}_Y$ and $\underset{\sim}{R}_{XY}$ completely characterize $\mathcal{R}_{XY}$.

Characterization of Joint Gaussian Measures

One can now characterize the set of all Gaussian joint measures having given Gaussian measures μ_X and μ_Y as projections.

Theorem 2: Suppose that μ_X and μ_Y are Gaussian measures on (H_1,Γ_1) and (H_2,Γ_2), respectively. Let $\underset{\sim}{R}_X$ and $\underset{\sim}{m}_X$ (resp., $\underset{\sim}{R}_Y$ and $\underset{\sim}{m}_Y$) denote the covariance operator and mean element of μ_X (resp., μ_Y).

(A) A joint Gaussian measure μ, having R as covariance operator and m as mean element, has μ_X and μ_Y as projections if and only if $m = (\underset{\sim}{m}_X,\underset{\sim}{m}_Y)$ and $R(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X\underset{\sim}{u} + \underset{\sim}{R}_{XY}\underset{\sim}{v}, \underset{\sim}{R}_Y\underset{\sim}{v} + \underset{\sim}{R}_{XY}^*\underset{\sim}{u})$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $H_1 \times H_2$, where $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{V}\underset{\sim}{R}_Y^{\frac{1}{2}}$ for a bounded linear operator $\underset{\sim}{V}: H_2 \to H_1$ with $||\underset{\sim}{V}|| \leq 1$.

(B) Let $\underset{\sim}{V}$ be any bounded linear operator mapping H_2 into H_1, with $||\underset{\sim}{V}|| \leq 1$. Define an operator $\underset{\sim}{R}_{XY}: H_2 \to H_1$ by $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{V}\underset{\sim}{R}_Y^{\frac{1}{2}}$, and define $R: H_1 \times H_2 \to H_1 \times H_2$ by $R(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X\underset{\sim}{u} + \underset{\sim}{R}_{XY}\underset{\sim}{v}, \underset{\sim}{R}_Y\underset{\sim}{v} + \underset{\sim}{R}_{XY}^*\underset{\sim}{u})$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $H_1 \times H_2$. R is then a covariance operator, and the Gaussian joint measure having R as covariance operator and $(\underset{\sim}{m}_X,\underset{\sim}{m}_Y)$ as mean element has μ_X and μ_Y as projections.

Proof: (A) If μ is a joint Gaussian measure with projections μ_X and μ_Y, then the assertion on the form of R and m follows from Prop. 1.

Conversely, suppose that μ is a joint Gaussian measure with covariance operator and mean element having the form given in the statement of (A). Suppose also that μ has projections μ_Z and μ_W, and that these measures have covariance operators $\underset{\sim}{R}_Z$ and $\underset{\sim}{R}_W$ and mean elements $\underset{\sim}{m}_Z$ and $\underset{\sim}{m}_W$. The projections must be Gaussian, and by Prop. 1, $m = (\underset{\sim}{m}_Z,\underset{\sim}{m}_W)$ and also $R(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_Z\underset{\sim}{u} + \underset{\sim}{R}_{ZW}\underset{\sim}{v}, \underset{\sim}{R}_W\underset{\sim}{v} + \underset{\sim}{R}_{WZ}\underset{\sim}{u})$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $H_1 \times H_2$, where $\underset{\sim}{R}_{ZW}$ is the cross-covariance operator of μ. It is clear that $\underset{\sim}{m}_X = \underset{\sim}{m}_Z$ and $\underset{\sim}{m}_Y = \underset{\sim}{m}_W$. Now let $\{\underset{\sim}{u}_n\}$ be a c.o.n. set in H_1; for any $\underset{\sim}{u}$ in H_1, one has that

$[\mathcal{R}(\underset{\sim}{u},\underset{\sim}{0}),\ (\underset{\sim}{u}_k,\underset{\sim}{0})] = \langle \underset{\sim}{R}_X\underset{\sim}{u},\underset{\sim}{u}_k\rangle_1 = \langle \underset{\sim}{R}_Z\underset{\sim}{u},\underset{\sim}{u}_k\rangle_1$ for each $\underset{\sim}{u}_k$ in $\{\underset{\sim}{u}_n\}$, so that $\underset{\sim}{R}_X = \underset{\sim}{R}_Z$. Similarly, $\underset{\sim}{R}_Y = \underset{\sim}{R}_W$. Hence, $\mu_X = \mu_Z$ and $\mu_Y = \mu_W$, by the unique correspondence between a Gaussian measure and its covariance operator and mean element.

(B) It is sufficient to show that the operator $\mathcal{R}$ is a covariance operator, with $\mathcal{R}$ as defined in the theorem, and with $\underset{\sim}{V}$ any bounded linear operator mapping H_2 into H_1 with $||\underset{\sim}{V}|| \le 1$.

It is straightforward to verify that $\mathcal{R}$ is linear and self-adjoint; since $\mathcal{R}$ is defined everywhere in $H_1\times H_2$, $\mathcal{R}$ is also bounded, by the closed-graph theorem. To see that $\mathcal{R}$ is non-negative, $[\mathcal{R}(\underset{\sim}{u},\underset{\sim}{v}),\ (\underset{\sim}{u},\underset{\sim}{v})] = \langle \underset{\sim}{R}_X\underset{\sim}{u},\underset{\sim}{u}\rangle_1 + \langle \underset{\sim}{R}_Y\underset{\sim}{v},\underset{\sim}{v}\rangle_2 + 2\langle \underset{\sim}{R}_{XY}\underset{\sim}{v},\underset{\sim}{u}\rangle_1 = ||\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u}||_1^2 + ||\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v}||_2^2 + 2\langle \underset{\sim}{V}\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v},\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u}\rangle_1$
$\ge ||\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u}||_1^2 + ||\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v}||_2^2 - 2||\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u}||_1||\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v}||_2 = (||\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u}||_1 - ||\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v}||)^2 \ge 0.$

It remains to show that $\mathcal{R}$ is trace-class. Let $\{\underset{\sim}{u}_n\}$, $n = 1,2,\ldots$ be c.o.n. in H_1, and let $\{\underset{\sim}{v}_n\}$, $n = 1,2,\ldots$ be c.o.n. in H_2. Then the set $\{(\underset{\sim}{u}_n,\underset{\sim}{0}),\ n=1,2,\ldots\} \cup \{(\underset{\sim}{0},\underset{\sim}{v}_n),\ n=1,2,\ldots\}$ is c.o.n. in $H_1\times H_2$, and

$$\sum_n [\mathcal{R}(\underset{\sim}{u}_n,\underset{\sim}{0}),\ (\underset{\sim}{u}_n,\underset{\sim}{0})] + \sum_n [\mathcal{R}(\underset{\sim}{0},\underset{\sim}{v}_n,\ (\underset{\sim}{0},\underset{\sim}{v}_n)] = \sum_n \langle \underset{\sim}{R}_X\underset{\sim}{u}_n,\underset{\sim}{u}_n\rangle_1 + \sum_n \langle \underset{\sim}{R}_Y\underset{\sim}{v}_n,\underset{\sim}{v}_n\rangle_2$$

$$= \mathit{Trace}\ \underset{\sim}{R}_X + \mathit{Trace}\ \underset{\sim}{R}_Y.$$

Thus, $\mathcal{R}$ is a covariance operator, and by (A) the joint Gaussian measure having $\mathcal{R}$ as covariance operator and $(\underset{\sim}{m}_X,\underset{\sim}{m}_Y)$ as mean element has μ_X and μ_Y as projections.

COROLLARY. Let $\mathcal{R}$ be a bounded linear operator in $H_1\times H_2$. If there exist bounded linear operators $\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$ and $\underset{\sim}{R}_3$, with $\underset{\sim}{R}_1\colon H_1\to H_1$, $\underset{\sim}{R}_2\colon H_2\to H_2$, and $\underset{\sim}{R}_3\colon H_2\to H_1$, such that $\mathcal{R}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_1\underset{\sim}{u} + \underset{\sim}{R}_3\underset{\sim}{v},\ \underset{\sim}{R}_2\underset{\sim}{v} + \underset{\sim}{R}_3{}^*\underset{\sim}{u})$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $H_1\times H_2$, then these operators are unique.

PROOF: Follows directly from the second part of the proof of (A).

Further Properties of the Covariance Operator

This section contains more details on the covariance operator R_{XY} for a joint measure μ_{XY}. Included is an explicit expression for the square root of R_{XY}, and a description of spectral properties of operators related to the cross-covariance operator.

The following result will be used in this and succeeding sections.

Lemma 2 [5]. Suppose that H is a real Hilbert space, and that R_1 and R_2 are bounded linear operators, $R_1: H_1 \to H$, $R_2: H_2 \to H$. Let P_1 (resp., P_2) be the projection operator mapping H_1 (resp., H_2) onto $\overline{\text{range}(R_1^*)}$ (resp., $\overline{\text{range}(R_2^*)}$). Then $\text{range}(R_1) \subset \text{range}(R_2)$ if and only if there exists a bounded linear operator $G: H_1 \to H_2$ such that $R_1 = R_2 G$, $G = P_2 G P_1 = G P_1$.

Proof: The "if" part is obvious. Suppose that $\text{range}(R_1) \subset \text{range}(R_2)$, and define $G: H_1 \to H_2$ by $Gx = GP_1 x = P_2 u$ when $R_1 x = R_2 u$. It is straightforward to show that G is single-valued; G is obviously linear; and one can show G is closed. By the closed-graph theorem, G is bounded.

Corollary [5].

(a) $\text{Range}(R_1) \subset \text{range}(R_2)$ <=> there exists a scalar $k < \infty$ such that $||R_1^* u||^2 \leq k||R_2^* u||^2$ for all u in H <=> there exists a bounded linear operator $Q: H_2 \to H_2$ such that $R_1 R_1^* = R_2 Q R_2^*$.

(b) $\text{Range}(R_1) = \text{range}(R_2)$ <=> there exists a bounded linear operator Q having bounded inverse with $(R_1 R_1^*)^{1/2} = (R_2 R_2^*)^{1/2} Q$ <=> there exists a bounded linear operator T having bounded inverse with $R_1 R_1^* = (R_2 R_2^*)^{1/2} T (R_2 R_2^*)^{1/2}$.

(c) $(R_1 R_1^*)^{1/2} = R_1 A^*$, where A is partially isometric, isometric on $\overline{\text{range}(R_1^*)}$ and $Au = 0$ for $u \perp \overline{\text{range}(R_1^*)}$.

In many applications, such as determining equivalence of two joint Gaussian measures, one must verify conditions involving the square root of a covariance operator. In this section, we give an explicit representation for the square root of the covariance operator of a joint measure μ_{XY}.

Let $R_{X\otimes Y}$ denote the covariance operator of $\mu_X \otimes \mu_Y$, the product measure for μ_X and μ_Y on $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$. For $\mu_X \otimes \mu_Y$, the cross-covariance operator is the null operator, so that

$$R_{X\otimes Y}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X\underset{\sim}{u},\underset{\sim}{R}_Y\underset{\sim}{v}).$$

Hence $R_{X\otimes Y}^{\frac{1}{2}}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u},\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v})$, and one can write R_{XY} as $R_{XY}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{R}_X\underset{\sim}{u},\underset{\sim}{R}_Y\underset{\sim}{v}) + (\underset{\sim}{R}_{XY}\underset{\sim}{v}, \underset{\sim}{R}_{YX}\underset{\sim}{v}) = R_{X\otimes Y}(\underset{\sim}{u},\underset{\sim}{v}) + (\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{V}\underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{v}, \underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{V}^*\underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{u})$ (where $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{V}\underset{\sim}{R}_Y^{\frac{1}{2}}$, $\underset{\sim}{V}: H_2 \to H_1$, $||\underset{\sim}{V}|| \le 1) = R_{X\otimes Y}(\underset{\sim}{u},\underset{\sim}{v}) + R_{X\otimes Y}^{\frac{1}{2}} V R_{X\otimes Y}^{\frac{1}{2}}(\underset{\sim}{u},\underset{\sim}{v})$, with $V(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{V}\underset{\sim}{v}, \underset{\sim}{V}^*\underset{\sim}{u})$. V is a self-adjoint bounded linear operator with $|||V||| = ||\underset{\sim}{V}||$, as can be easily verified.

We have established the following result.

Prop. 2: $R_{XY} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]R_{X\otimes Y}^{\frac{1}{2}}$, and $R_{XY}^{\frac{1}{2}} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]^{\frac{1}{2}}A^*$, where A is a partially isometric operator, isometric on $\overline{\text{range}(R_{XY})}$, and zero on the null space of R_{XY}, and I is the identity operator in $H_1 \times H_2$.

The second part of this result follows from the corollary to Lemma 1, and the fact that $\overline{\text{range}(R_{XY})} = \overline{\text{range}(R_{XY}^{\frac{1}{2}})}$.

Note that Prop. 2 and Lemma 1 imply that $\text{range}(R_{XY}^{\frac{1}{2}}) \subset \text{range}(R_{X\otimes Y}^{\frac{1}{2}})$, with equality if and only if $I + V$ has bounded inverse.

The operators $I+V$ and V, defined above, play an important role in the study of equivalence of joint Gaussian measures. We proceed to examine the spectral properties of these operators. Recall that the set of limit points of the spectrum of a self-adjoint bounded linear operator consists of all points of the continuous spectrum, all limit points of the point spectrum, and all eigenvalues of infinite multiplicity. The identity operator in H_1 and the identity operator in H_2 will both be denoted by the symbol $\mathbf{I}$; the appropriate space will be clear from the context.

THEOREM 3. (a) $\mathbf{VV}^*$ has a c.o.n. (in H_1) set of eigenvectors if and only if $\mathbf{V}^*\mathbf{V}$ has a c.o.n. (in H_2) set of eigenvectors.

(b) $\mathcal{V}(\mathbf{u},\mathbf{v}) = \lambda(\mathbf{u},\mathbf{v}) \iff \mathbf{VV}^*\mathbf{u} = \lambda^2\mathbf{u}$, $\mathbf{V}^*\mathbf{V}\mathbf{v} = \lambda^2\mathbf{v}$, and $\mathbf{V}^*\mathbf{u} = \lambda\mathbf{v} \iff \mathcal{V}(\mathbf{u},-\mathbf{v}) = -\lambda(u,-v)$. $\mathcal{V}$ has a c.o.n. set of eigenvectors if and only if $\mathbf{VV}^*$ has a c.o.n. set of eigenvectors.

(c) $\mathcal{V}$ is compact if and only if $\mathbf{V}$ is compact.

(d) $\mathcal{I} + \mathcal{V}$ is compact if and only if both H_1 and H_2 are finite-dimensional spaces.

(e) If there exists a scalar $\alpha < 1$ such that $||\mathbf{V}^*\mathbf{u}||_2^{\,2} \le \alpha||\mathbf{u}||_1^{\,2}$ for all $\mathbf{u}$ in $\overline{\text{range}(\mathbf{I}-\mathbf{VV}^*)}$, then $\mathcal{I}+\mathcal{V}$, $\mathbf{I}-\mathbf{V}^*\mathbf{V}$, and $\mathbf{I}-\mathbf{VV}^*$ each has closed range, and $\mathcal{I}+\mathcal{V} \ge [1-\alpha^{\frac{1}{2}}]\mathcal{I}$ on range$(\mathcal{I}+\mathcal{V})$.

(f) α is a limit point of the spectrum of $\mathcal{I}+\mathcal{V}$ if and only if $1-(1-\alpha)^2$ is a limit point of the spectrum of $\mathbf{I}-\mathbf{VV}^*$.

PROOF: (a) Suppose $\mathbf{VV}^*\mathbf{u}_n = \lambda^2{}_n\mathbf{u}_n$, with $\{\mathbf{u}_n\}$ c.o.n. in H_1. Define $\lambda_n\mathbf{v}_n = \mathbf{V}^*\mathbf{u}_n$ for all n such that $\lambda_n \neq 0$. Then $\mathbf{V}^*\mathbf{V}\mathbf{v}_n = \lambda^2{}_n\mathbf{v}_n$. Now suppose that there exists $\mathbf{v}$ in H_2 such that $\langle\mathbf{v},\mathbf{v}_n\rangle_2 = 0$ for all $\mathbf{v}_n$. Then $\mathbf{V}\mathbf{v}$ is in the null space of $\mathbf{V}^*$, so that $\mathbf{V}^*\mathbf{V}\mathbf{v} = \mathbf{0}$. Hence $\{\mathbf{v}_n\} \cup$ {null space of $\mathbf{V}^*\mathbf{V}$} contains a set that is c.o.n. in H_2. The converse follows by symmetry.

(b) $\mathcal{V}(\mathbf{u},\mathbf{v}) = \lambda(\mathbf{u},\mathbf{v}) \iff \mathbf{V}^*\mathbf{u} = \lambda\mathbf{v}$ and $\mathbf{V}\mathbf{v} = \lambda\mathbf{u} \iff \mathbf{VV}^*\mathbf{u} = \lambda^2\mathbf{u}$ and $\lambda\mathbf{v} = \mathbf{V}^*\mathbf{u} \iff \mathbf{VV}^*\mathbf{u} = \lambda^2\mathbf{u}$ and $-\lambda(-\mathbf{v}) = \mathbf{V}^*\mathbf{u} \iff \mathcal{V}(\mathbf{u},-\mathbf{v}) = -\lambda(\mathbf{u},-\mathbf{v})$. If $\{(\mathbf{u}_n,\mathbf{v}_n)\}$ is a complete set in $H_1 \times H_2$, then $\{\mathbf{u}_n\}$ must be complete in H_1, so that $\mathbf{VV}^*$ has a complete set of eigenvectors if $\mathcal{V}$ has a complete set. Conversely, if $\mathbf{VV}^*\mathbf{u} = \lambda^2\mathbf{u}$, define $\mathbf{v}$ by $\lambda\mathbf{v} = \mathbf{V}^*\mathbf{u}$ if $\lambda \neq 0$, and $\mathbf{v} = \mathbf{0}$ if $\lambda = 0$. Then $\mathcal{V}(\mathbf{u},\mathbf{v}) = \lambda(\mathbf{u},\mathbf{v})$. If $[(\mathbf{m},\mathbf{z}), (\mathbf{u},\mathbf{v})] = 0$ for all such eigenvectors

$(\underset{\sim}{u},\underset{\sim}{v})$, then $\langle \underset{\sim}{m},\underset{\sim}{u}\rangle_1 + \langle \underset{\sim}{z},\underset{\sim}{v}\rangle_2 = 0$ and also $\langle \underset{\sim}{m},\underset{\sim}{u}\rangle_1 - \langle \underset{\sim}{z},\underset{\sim}{v}\rangle_2 = 0$ (by the first part of (b)), so that $\underset{\sim}{m} = \underset{\sim}{0}$ if the eigenvectors of $\underset{\sim}{V}^*\underset{\sim}{V}$ are complete in H_1. In this case, the eigenvectors of $\underset{\sim}{V}^*\underset{\sim}{V}$ are complete in H_2, by (a), and the proof of (a) shows that the non-zero point spectrum of $\underset{\sim}{V}^*\underset{\sim}{V}$ is identical to the non-zero point spectrum of $\underset{\sim}{V}\underset{\sim}{V}^*$. Thus, the element $\underset{\sim}{z}$ above must belong to the null space of $\underset{\sim}{V}^*\underset{\sim}{V}$. If $\{\underset{\sim}{x}_n\}$ is c.o.n. in the null space of $\underset{\sim}{V}^*\underset{\sim}{V}$, the set $\{(\underset{\sim}{0},\underset{\sim}{x}_n)\}$ are eigenvectors of V corresponding to the eigenvalue zero. Hence, the union of this set with the eigenvectors $\{(\underset{\sim}{u},\underset{\sim}{v})\}$ derived as above from the eigenvectors of $\underset{\sim}{V}\underset{\sim}{V}^*$ constitutes a complete set in $H_1 \times H_2$.

(c) Follows directly from (b).

(d) From (b), $I+V$ cannot have zero as the only limit point of its non-zero eigenvalues, since λ is an eigenvalue of $I+V$ if and only if $2-\lambda$ is an eigenvalue. $I+V$ is thus compact if and only if range$(I+V)$ is finite-dimensional. The above eigenvalue relation also shows that the dimension of the null space of $I+V$ is equal to the multiplicity of the eigenvalue $\lambda = 2$. Hence, the dimension of the range of $I+V$ will be finite-dimensional if and only if $H_1 \times H_2$ is finite-dimensional. This occurs if and only if both H_1 and H_2 are finite-dimensional.

(e) Suppose there exists $\alpha < 1$ such that $||\underset{\sim}{V}^*\underset{\sim}{u}||_2{}^2 \le \alpha ||\underset{\sim}{u}||_1{}^2$ on $\overline{\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)}$. Then, $\underset{\sim}{V}\underset{\sim}{V}^* \le \alpha \underset{\sim}{I}$ on $\overline{\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)}$, $\underset{\sim}{V}^*\underset{\sim}{V} \le \alpha \underset{\sim}{I}$ on $\overline{\text{range}(\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V})}$, and hence $V^2 \le \alpha I$ on $\overline{\text{range}(I-V^2)}$. If u is in $\overline{\text{range}(I-V^2)}$, the above inequalities yield $[(I+V)u,u] \ge (1-\alpha^{\frac{1}{2}})|||u|||^2$. Suppose that $u \perp \overline{\text{range}(I-V^2)}$; then either $u \perp \overline{\text{range}(I+V)}$ or else $(I+V)u = 2u$. Noting that the null space of $I+V$ is contained in the null space of $I-V^2$, one sees that $I+V \ge (1-\alpha^{\frac{1}{2}})I$ on $\overline{\text{range}(I+V)}$. This implies that $I+V$ has closed range. The fact that $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$ and $\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}$ each has closed range follows from the preceding inequalities.

(f) α is a limit point of the spectrum of $I+V$ if and only if there exists a normalized sequence $(\underset{\sim}{u}_n, \underset{\sim}{v}_n)$ in $H_1 \times H_2$ which is weakly convergent to zero, and such that $|||(I+V-\alpha I)(\underset{\sim}{u}_n, \underset{\sim}{v}_n)||| \to 0$ [6]. $(I+V-\alpha I)(\underset{\sim}{u}_n, \underset{\sim}{v}_n) \to (\underset{\sim}{0}, \underset{\sim}{0})$ if and only if $(1-\alpha)\underset{\sim}{u}_n + \underset{\sim}{V}\underset{\sim}{v}_n \to \underset{\sim}{0}$, $(1-\alpha)\underset{\sim}{v}_n + \underset{\sim}{V}^*\underset{\sim}{u}_n \to \underset{\sim}{0}$, and $(1-\alpha)^2\underset{\sim}{u}_n - \underset{\sim}{V}\underset{\sim}{V}^*\underset{\sim}{u}_n \to \underset{\sim}{0}$. If $||\underset{\sim}{u}_n||_1$ is bounded away from zero, it is clear that $1 - (1-\alpha)^2$ is a limit point for $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$, since $\{\underset{\sim}{u}_n\}$ must be weakly convergent to zero. Thus, suppose that $\underset{\sim}{u}_{n_k} \to \underset{\sim}{0}$ for some subsequence $\{\underset{\sim}{u}_{n_k}\}$. Then $\underset{\sim}{V}^*\underset{\sim}{u}_{n_k} \to \underset{\sim}{0}$ and hence $\underset{\sim}{v}_{n_k} \to \underset{\sim}{0}$; this contradicts the assumption that $\{(\underset{\sim}{u}_n, \underset{\sim}{v}_n)\}$ is normalized in $H_1 \times H_2$. Hence, $||\underset{\sim}{u}_n||_1$ must be bounded away from zero, and thus $1 - (1-\alpha)^2$ is a limit point for $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$.

Conversely, suppose that $\{\underset{\sim}{u}_n\}$ is a normalized sequence in H_1, weakly convergent to zero, with $(1-\alpha)^2\underset{\sim}{u}_n - \underset{\sim}{V}\underset{\sim}{V}^*\underset{\sim}{u}_n \to \underset{\sim}{0}$. Define positive scalars $\{k_n\}$ by $k_n{}^2 = 1 + (1-\alpha)^{-2}\,||\underset{\sim}{V}^*\underset{\sim}{u}_n||_2{}^2$. $\{k_n{}^{-1}(\underset{\sim}{u}_n, (\alpha-1)^{-1}\underset{\sim}{V}^*\underset{\sim}{u}_n)\}$ is a normalized sequence in $H_1 \times H_2$, weakly convergent to zero, with $[I+V-\alpha I](k_n{}^{-1}\underset{\sim}{u}_n,\ k_n{}^{-1}(\alpha-1)^{-1}\underset{\sim}{V}^*\underset{\sim}{u}_n) \to (\underset{\sim}{0}, \underset{\sim}{0})$. α is thus a limit point of $I+V$ when $1 - (1-\alpha)^2$ is a limit point of $I-VV^*$ This completes the proof of the theorem.

Equivalence of Joint Gaussian Measures

For two Gaussian measures on a real and separable Hilbert space, necessary and sufficient conditions for equivalence (mutual absolute continuity) have been the subject of extensive investigations. It is known that the two measures are either equivalent (denoted by $\sim$) or orthogonal [7][8]; the necessary and sufficient conditions for equivalence can be expressed in terms of the covariance operators and mean elements of the two measures (see, e.g. [9], Theorem 5.1, or [10]). Since these results apply to any two Gaussian measures on any real and separable Hilbert space, they can be used to determine equivalence or orthogonality of two joint Gaussian measures on $(H_1 \times H_2, \Gamma_1 \times \Gamma_2)$. Our objective here is to formulate conditions for equivalence in terms of operators and elements in the individual spaces H_1 and H_2.

Throughout this section, we will consider two Gaussian joint measures, μ_{XY} and μ_{ZW}. The cross-covariance operators will have the form $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{\frac{1}{2}} \underset{\sim}{V} \underset{\sim}{R}_Y^{\frac{1}{2}}$ and $\underset{\sim}{R}_{ZW} = \underset{\sim}{R}_Z^{\frac{1}{2}} \underset{\sim}{T} \underset{\sim}{R}_W^{\frac{1}{2}}$, where $\underset{\sim}{V}$ and $\underset{\sim}{T}$ are linear, $\|\underset{\sim}{V}\| \le 1$ and $\|\underset{\sim}{T}\| \le 1$. μ_{XY} has mean $m_{XY} = (\underset{\sim}{m}_X, \underset{\sim}{m}_Y)$, and μ_{ZW} has mean $m_{ZW} = (\underset{\sim}{m}_Z, \underset{\sim}{m}_W)$. We will also assume that $\overline{\mathrm{range}(R_{X\otimes Y})} = \overline{\mathrm{range}(R_{Z\otimes W})} = H_1 \times H_2$, where $R_{X\otimes Y}$ is the covariance operator of $\mu_X \otimes \mu_Y$, similarly $R_{Z\otimes W}$ for $\mu_Z \otimes \mu_W$. This is no restriction, for the following reasons: (1) $\mathrm{range}(R_{XY}^{\frac{1}{2}}) \subset \mathrm{range}(R_{X\otimes Y}^{\frac{1}{2}})$, as shown previously; (2) conditions for equivalence of μ_{ZW} and μ_{XY} depend only on elements in, and operators defined on, $\overline{\mathrm{range}(R_{XY}^{\frac{1}{2}})}$ and $\overline{\mathrm{range}(R_{ZW}^{\frac{1}{2}})}$, by Lemma 2 below; (3) $\mu_X \otimes \mu_Y \perp \mu_Z \otimes \mu_W$ if $\overline{\mathrm{range}(R_{X\otimes Y}^{\frac{1}{2}})} \ne \overline{\mathrm{range}(R_{Z\otimes W}^{\frac{1}{2}})}$, as one can easily show, and we will see that $\mu_{XY} \perp \mu_{ZW}$ if $\mu_X \otimes \mu_Y \perp \mu_Z \otimes \mu_W$, independently of the assumption that $\overline{\mathrm{range}(R_{X\otimes Y}^{\frac{1}{2}})} = \overline{\mathrm{range}(R_{Z\otimes W}^{\frac{1}{2}})} = H_1 \times H_2$.

The two lemmas below are fundamental to our results.

LEMMA 2. [9]. Suppose μ_1 and μ_2 are two Gaussian measures on the Borel σ-field of a real and separable Hilbert space, H. Let $\underset{\sim}{R}_i$ and $\underset{\sim}{m}_i$ be the covariance operator and mean element of μ_i. Then $\mu_1 \sim \mu_2$ if and only if

(a) $\underset{\sim}{m}_1 - \underset{\sim}{m}_2$ is in the range of $R_1^{\frac{1}{2}}$

(b) $\underset{\sim}{R}_1 = \underset{\sim}{R}_2 + \underset{\sim}{R}_2^{\frac{1}{2}}\underset{\sim}{W}\underset{\sim}{R}_2^{\frac{1}{2}}$, where W is a Hilbert-Schmidt operator that does not have -1 as a eigenvalue, and $\underset{\sim}{W}$ is identically zero on the null space of $\underset{\sim}{R}_2$.

This is simply a restatement of Rao-Varadarajan Theorem 5.1 [9], slightly modified to allow for $\overline{\text{range}(\underset{\sim}{R}_2^{\frac{1}{2}})} \neq H$. The extension follows easily from Theorem 4.1 of [9], and the fact that the support of a Gaussian measure is the closure of the range of its covariance operator [11] (note that $\overline{\text{range}(\underset{\sim}{R}_2^{\frac{1}{2}})} = \overline{\text{range}(\underset{\sim}{R}_2)}$). Since the values of $\underset{\sim}{W}'$ on the nullspace of $\underset{\sim}{R}_2$ do not affect the values of $\underset{\sim}{R}_1 = \underset{\sim}{R}_2 + \underset{\sim}{R}_2^{\frac{1}{2}}\underset{\sim}{W}'\underset{\sim}{R}_2^{\frac{1}{2}}$ on H, it is clear that if one defines $\underset{\sim}{W}$ by $\underset{\sim}{W} = \underset{\sim}{W}'$ on $\overline{\text{range}(\underset{\sim}{R}_2)}$, and $\underset{\sim}{W} \equiv \underset{\sim}{0}$ on $\overline{\text{range}(\underset{\sim}{R}_2)}^{\perp}$, then $\underset{\sim}{R}_1 = \underset{\sim}{R}_2 + \underset{\sim}{R}_2^{\frac{1}{2}}\underset{\sim}{W}\underset{\sim}{R}_2^{\frac{1}{2}}$.

LEMMA 3. $\mu_X \otimes \mu_Y \sim \mu_Z \otimes \mu_W$ if and only if $\mu_X \sim \mu_Z$ and $\mu_Y \sim \mu_W$.

PROOF: Suppose $\mu_X \sim \mu_Z$ and $\mu_Y \sim \mu_W$. Then $\mu_X \otimes \mu_Y[A] = \int_H \mu_X\{\underset{\sim}{x}: (\underset{\sim}{x},\underset{\sim}{y}) \in A\} d\mu_Y(\underset{\sim}{y})$, so that $\mu_X \otimes \mu_Y[A] = 0$ if and only if $\mu_X\{x: (\underset{\sim}{x},\underset{\sim}{y}) \in A\} = 0$ a.e. $d\mu_Y(\underset{\sim}{y})$ and $\mu_Y\{y: (\underset{\sim}{x},\underset{\sim}{y}) \in A\} = 0$ a.e. $d\mu_X(\underset{\sim}{x})$. It follows easily that $\mu_X \otimes \mu_Y \sim \mu_Z \otimes \mu_W$. For the converse, one notes that $\mu_X(B) = \mu_X \otimes \mu_Y(B \times H_2)$.

We now examine the equivalence or singularity of the joint Gaussian measures μ_{XY} and μ_{ZW}. We have seen that $R_{XY} = R_{X\otimes Y}^{\frac{1}{2}}(I+V)R_{X\otimes Y}^{\frac{1}{2}}$ and $R_{ZW} = R_{Z\otimes W}^{\frac{1}{2}}(I+T)R_{Z\otimes W}^{\frac{1}{2}}$, where $V(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{V}\underset{\sim}{v},\underset{\sim}{V}^*\underset{\sim}{u})$ and $T(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{T}\underset{\sim}{v},\underset{\sim}{T}^*\underset{\sim}{u})$. If $\text{range}(\underset{\sim}{R}_X^{\frac{1}{2}}) = \text{range}(\underset{\sim}{R}_Z^{\frac{1}{2}})$ and $\text{range}(\underset{\sim}{R}_Y^{\frac{1}{2}}) = \text{range}(\underset{\sim}{R}_W^{\frac{1}{2}})$, then by Lemma 1 there exist bounded linear operators $\underset{\sim}{B}_1: H_1 \to H_1$ and $\underset{\sim}{B}_2: H_2 \to H_2$ such that both $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ have bounded inverse, and $\underset{\sim}{R}_Z^{\frac{1}{2}} = \underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{B}_1$, $\underset{\sim}{R}_W^{\frac{1}{2}} = \underset{\sim}{R}_Y^{\frac{1}{2}}\underset{\sim}{B}_2$. We can then define an operator $\mathcal{B}$ in $H_1 \times H_2$ by $\mathcal{B}(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{B}_1\underset{\sim}{u},\underset{\sim}{B}_2\underset{\sim}{v})$. $\mathcal{B}$ is linear, bounded, and has bounded inverse.

THEOREM 4. $\mu_{XY} \sim \mu_{ZW}$ if and only if

(a) $\mu_X \sim \mu_Z$ and $\mu_Y \sim \mu_W$.

(b) There exists a Hilbert-Schmidt operator $\mathcal{W}$ in $H_1 \times H_2$ such that $\mathcal{W}$ does not have -1 as an eigenvalue, and

$$I + V = \mathcal{B}[I+T]\mathcal{B}^* + \mathcal{B}[I+T]^{\frac{1}{2}} \mathcal{W}[I+T]^{\frac{1}{2}}\mathcal{B}^*$$

(c) $m_{XY} - m_{ZW}$ belongs to the range of $R_{X\otimes Y}^{\frac{1}{2}}[I+V]^{\frac{1}{2}}$.

REMARK: Note that condition (a) implies (Lemma 2) that the operator $\mathcal{B}$ (defined above) exists and is linear, bounded, and has bounded inverse.

PROOF: (A) Suppose $\mu_{XY} \sim \mu_{ZW}$. Condition (a) is obviously satisfied, since, for example, $\mu_X(A) = \mu_{XY}(A \times H)$. This implies (Lemma 3) that $\mu_X \otimes \mu_Y \sim \mu_Z \otimes \mu_W$; since $R_{Z\otimes W} = R_{X\otimes Y}^{\frac{1}{2}}\mathcal{B}\mathcal{B}^* R_{X\otimes Y}^{\frac{1}{2}}$, the operator $I - \mathcal{B}\mathcal{B}^*$ is Hilbert-Schmidt and does not have -1 as an eigenvalue, by Lemma 2.

$\mu_{XY} \sim \mu_{ZW}$ also implies that $R_{XY} = R_{ZW}^{\frac{1}{2}}(I+K)R_{ZW}^{\frac{1}{2}}$, where K is Hilbert-Schmidt, K can be taken as zero on the null space of R_{ZW}, and -1 is not an eigenvalue of K. Hence $R_{XY} = R_{X\otimes Y}^{\frac{1}{2}}\mathcal{B}A(I+K)A^*\mathcal{B}^* R_{X\otimes Y}^{\frac{1}{2}}$, where $AA^* = I+T$. R_{XY} also can be written as $R_{XY} = R_{X\otimes Y}^{\frac{1}{2}}(I+V)R_{X\otimes Y}^{\frac{1}{2}}$. Hence

$$\begin{aligned} I + V &= \mathcal{B}AA^*\mathcal{B}^* + \mathcal{B}AKA^*\mathcal{B}^* \\ &= \mathcal{B}[I+T]\mathcal{B}^* + \mathcal{B}[I+T]^{\frac{1}{2}} \mathcal{W}[I+T]^{\frac{1}{2}}\mathcal{B}^* \end{aligned}$$

where $\mathcal{W} \equiv \mathcal{D}^*K\mathcal{D}$, $\mathcal{D}$ the partially isometric operator satisfying $R_{ZW}^{\frac{1}{2}} = R_{Z\otimes W}^{\frac{1}{2}}[I+T]^{\frac{1}{2}}\mathcal{D}^*$, $A = (I+T)^{\frac{1}{2}}\mathcal{D}^*$. Condition (b) is satisfied.

Finally, $\mu_{XY} \sim \mu_{ZW}$ implies that $m_{XY} - m_{ZW}$ belongs to range$(R_{XY}^{\frac{1}{2}})$.

Since $R_{XY}^{\frac{1}{2}} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]^{\frac{1}{2}}G^*$, G partially isometric and isometric on $\overline{\text{range}(R_{XY})}$, one sees that $m_{XY} - m_{ZW} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]^{\frac{1}{2}}m$ for some m in $H_1 \times H_2$.

(B) Suppose conditions (a), (b) and (c) are satisfied. Condition (a) implies the existence of the operator B, $R_{Z\otimes W} = R_{X\otimes Y}^{\frac{1}{2}}BB^*R_{X\otimes Y}^{\frac{1}{2}}$, $I-BB^*$ Hilbert-Schmidt, and BB^* having bounded inverse. Assumption (b) then implies that $R_{XY}-R_{ZW} = R_{ZW}^{\frac{1}{2}}KR_{ZW}^{\frac{1}{2}}$, where $K = DWD^*$, D partially isometric, isometric on $\overline{\text{range}(R_{ZW})}$ and zero on $\overline{\text{range}(R_{ZW})}^{\perp}$. K is thus Hilbert-Schmidt, vanishes on the null space of R_{ZW}, and does not have -1 as an eigenvalue. Assumption (c) implies $m_{XY}-m_{ZW}$ in $\text{range}(R_{XY}^{\frac{1}{2}})$, by Prop. 2. Conditions (a), (b) and (c) thus imply $\mu_{XY} \sim \mu_{ZW}$, by Lemma 2, and the theorem is proved.

The necessary and sufficient conditions given in Theorem 4 are completely general. However, they are stated in terms of operators in $H_1 \times H_2$, whereas one might prefer conditions given in terms of operators in H_1 and in H_2. The next result gives such conditions for equivalence for a wide class of joint Gaussian measures.

THEOREM 5. Suppose that $||\underset{\sim}{V}^*\underset{\sim}{x}||_2 \le \alpha||\underset{\sim}{x}||_1$ for all $\underset{\sim}{x}$ in $\overline{\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)}$, some $\alpha < 1$. Then $\mu_{XY} \sim \mu_{ZW}$ if and only if

(a) $\mu_X \sim \mu_Z$ and $\mu_Y \sim \mu_W$

(b) $\underset{\sim}{V}-\underset{\sim}{B}_1\underset{\sim}{T}\underset{\sim}{B}^*_2$ is Hilbert-Schmidt

(c) There exists a finite and strictly positive scalar λ such that

$$\lambda\underset{\sim}{B}_1(\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*)\underset{\sim}{B}^*_1 \le \underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^* \le \lambda^{-1}\underset{\sim}{B}_1(\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*)\underset{\sim}{B}^*_1 \quad \text{and}$$

$$\lambda\underset{\sim}{B}_2(\underset{\sim}{I}-\underset{\sim}{T}^*\underset{\sim}{T})\underset{\sim}{B}^*_2 \le \underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V} \le \lambda^{-1}\underset{\sim}{B}_2(\underset{\sim}{I}-\underset{\sim}{T}^*\underset{\sim}{T})\underset{\sim}{B}^*_2.$$

(d) There exist elements $\underset{\sim}{u}$ in H_1 and $\underset{\sim}{v}$ in H_2 such that

$$\underset{\sim}{m}_X-\underset{\sim}{m}_Z = \underset{\sim}{R}_X^{\frac{1}{2}}(\underset{\sim}{u}+\underset{\sim}{V}\underset{\sim}{v}) \quad \text{and} \quad \underset{\sim}{m}_Y-\underset{\sim}{m}_W = \underset{\sim}{R}_Y^{\frac{1}{2}}(\underset{\sim}{v}+\underset{\sim}{V}^*\underset{\sim}{u}).$$

PROOF: (A) Suppose $\mu_{XY} \sim \mu_{ZW}$. Condition (a) is necessary, from Theorem 4, and implies that $I-BB^*$ in Hilbert-Schmidt. Condition (b) is then implied by condition (b) of Theorem 4. To see that (c) holds, one notes that $\text{range}(R_{XY}^{\frac{1}{2}}) = \text{range}(R_{ZW}^{\frac{1}{2}})$ implies the existence of strictly positive and finite scalars β_1 and β_2 such that $\beta_1 R_{ZW} \le R_{XY} \le \beta_2 R_{ZW}$, from Lemma 1. This is equivalent to $\beta_1 B(I+T)B^* \le I+V \le \beta_2 B(I+T)B^*$, which is equivalent to $\text{range}([B(I+T)B^*]^{\frac{1}{2}}) = \text{range}[(I+V)^{\frac{1}{2}}]$. This implies that the null space of $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$ is identical to the null space of $\underset{\sim}{B}_1(\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*)\underset{\sim}{B}^*_1$ and that the null space of $\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}$ is identical to the null space of $\underset{\sim}{B}_2(\underset{\sim}{I}-\underset{\sim}{T}^*\underset{\sim}{T})\underset{\sim}{B}^*_2$. The assumption that $||\underset{\sim}{V}^*\underset{\sim}{x}||_2$ is bounded away from $||\underset{\sim}{x}||_1$ for $\underset{\sim}{x}$ in $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)$ implies, by Theorem 3, that $I+V$, $\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}$, and $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$ each has closed range. Hence, $(I+V)^{\frac{1}{2}}$ has closed range, and $\text{range}(I+V) = \text{range}(B[I+T]B^*)$, so that $\text{range}(B[I+T]B^*)$ is closed. This implies that the range spaces of $\underset{\sim}{B}_1[\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*]\underset{\sim}{B}^*_1$ and $\underset{\sim}{B}_2[\underset{\sim}{I}-\underset{\sim}{T}^*\underset{\sim}{T}]\underset{\sim}{B}^*_2$ are closed, so that $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*) = \text{range}\underset{\sim}{B}_1[\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*]\underset{\sim}{B}^*_1$ and $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}) = \text{range}\underset{\sim}{B}_2[\underset{\sim}{I}-\underset{\sim}{T}^*\underset{\sim}{T}]\underset{\sim}{B}^*_2$. Since these range spaces are all closed, condition (c) follows.

Finally, $\mu_{XY} \sim \mu_{ZW}$ implies that $m_{XY}-m_{ZW}$ belongs to the range of $R_{XY}^{\frac{1}{2}}$, so that $m_{XY}-m_{ZW} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]^{\frac{1}{2}}m$ for some m in $H_1 \times H_2$. This implies condition (d), since $\text{range}([I+V]^{\frac{1}{2}}) = \text{range}(I+V)$.

(B) Suppose conditions (a)-(d) are satisfied. (a) implies that $R_{X\otimes Y} = R_{Z\otimes W}^{\frac{1}{2}}BB\, R_{Z\otimes W}^{\frac{1}{2}}$, where $I-BB^*$ has bounded inverse. The assumption that $||\underset{\sim}{V}^*\underset{\sim}{x}||_2 \le \alpha||\underset{\sim}{x}||_1$ for all $\underset{\sim}{x}$ in $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)$, some $\alpha < 1$, implies that $\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*$, $\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}$, and $I+V$ have closed range. Condition (c) then implies $\text{range}[\underset{\sim}{B}_1(\underset{\sim}{I}-\underset{\sim}{T}\underset{\sim}{T}^*)\underset{\sim}{B}^*_1] = \text{range}(\underset{\sim}{I}-\underset{\sim}{V}\underset{\sim}{V}^*)$ and $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V}) = \text{range}(\underset{\sim}{B}_2[\underset{\sim}{I}+\underset{\sim}{T}^*\underset{\sim}{T}]\underset{\sim}{B}^*_2)$, so that $\text{range}(B[I+T]B^*) = \text{range}(I+V)$. Since $\text{range}([I+V]^{\frac{1}{2}})$ is closed, this implies that $\text{range}([I+V]^{\frac{1}{2}}) = \text{range}([B(I+T)B^*]^{\frac{1}{2}})$, so that by Lemma 1 there exist finite and strictly positive scalars β_1, β_2 such that

$$\beta_1 B[I+T]B^* \le I + V \le \beta_2 B[I+T]B^*.$$

This is equivalent to $\beta_1 R_{ZW} \le R_{XY} \le \beta_2 R_{ZW}$, which implies $\text{range}(R_{XY}^{\frac{1}{2}}) =$

$\text{range}(R_{ZW}^{1/2})$. Thus, there exists a bounded linear operator K such that $K \equiv 0$ on the null space of R_{ZW}, with $R_{XY} = R_{ZW}[I+K]R_{ZW}$, and $I+K$ is bounded away from zero on $\overline{\text{range}(R_{ZW})}$.

One now has

$$I + V = B(I+T)B^* + BAKA^*B^*$$

where $AA^* = I+T$, or $I-BB^*+V-BTB^* = BAKA^*B^*$. Since B has bounded inverse, and $I-BB^*$ is Hilbert-Schmidt, condition (b) implies that AKA^* is Hilbert-Schmidt. The null space of K contains the null space of R_{ZW}, so that $\text{range}(K) \subset \overline{\text{range}(R_{ZW})}$. But $R_{ZW}^{1/2} = A^*R_{Z\otimes W}^{1/2}$, so that $\text{range}(K) \subset \overline{\text{range}(A^*)}$. The assumption that $||\underset{\sim}{V}\underset{\sim}{u}||_2$ is bounded away from $||\underset{\sim}{u}||_1$ for $\underset{\sim}{u}$ in $\text{range}(\underset{\sim}{I}-\underset{\sim}{V}^*\underset{\sim}{V})$, and condition (c), imply that $|||A(\underset{\sim}{u},\underset{\sim}{v})|||^2 \geq \underset{\sim}{k}|||(\underset{\sim}{u},\underset{\sim}{v})|||$ for all $(\underset{\sim}{u},\underset{\sim}{v})$ in $\overline{\text{range}(A^*A)}$ and some $k > 0$. Moreover, $\overline{\text{range}(A^*A)} = \overline{\text{range}[(A^*A)^{1/2}]} = \overline{\text{range}(A^*)}$. Hence if $\{(\underset{\sim}{u}_n,\underset{\sim}{v}_n)\}$ is a c.o.n. set in $\text{range}(AA^*)$, then

$$\sum_n |||AKA^*(\underset{\sim}{u}_n,\underset{\sim}{v}_n)|||^2 \geq k \sum_n |||KA^*(\underset{\sim}{u}_n,\underset{\sim}{v}_n)|||^2 \geq k^2 \sum_n |||K(\underset{\sim}{u}_n,\underset{\sim}{v}_n)|||^2 ,$$

so that K is Hilbert-Schmidt and does not have -1 as an eigenvalue (the latter because $I+K$ is invertible).

Finally, it is easy to show that condition (d) implies $m_{XY}-m_{ZW}$ belongs to the range of $R_{XY}^{1/2}$. This concludes the proof.

REMARK: Note that if $R_{XY} = R_{ZW}$, and either $\underset{\sim}{m}_X = \underset{\sim}{m}_Z$ or $\underset{\sim}{m}_Y = \underset{\sim}{m}_W$, then condition (a) of Theorem 5 is necessary and sufficient for $\mu_{XY} \sim \mu_{ZW}$.

In the case where $\mu_{ZW} \equiv \mu_{X\otimes Y}$, simple and general condition for equivalence can be given.

THEOREM 6. $\mu_{XY} \sim \mu_X \otimes \mu_Y$ if and only if $||\underset{\sim}{V}|| < 1$ and $\underset{\sim}{V}$ is Hilbert-Schmidt.

PROOF: $R_{XY} = R_{X\otimes Y}^{\frac{1}{2}}[I+V]R_{X\otimes Y}^{\frac{1}{2}}$. From Theorem 3, V is Hilbert-Schmidt if and only if $\underset{\sim}{V}$ is Hilbert-Schmidt, and V has -1 as an eigenvalue if and only if $\underset{\sim}{V}\underset{\sim}{V}^*$ has $+1$ as an eigenvalue. If $\underset{\sim}{V}$ is Hilbert-Schmidt, $||\underset{\sim}{V}|| = 1$ if and only if $\underset{\sim}{V}\underset{\sim}{V}^*$ has 1 as an eigenvalue, since the set of limit points of the spectrum of a self-adjoint compact operator contains only zero. The result follows from Lemma 2.

COROLLARY: $\mu_{XY} \sim \mu_{ZW}$ if

(a) $\mu_X \sim \mu_Z$ and $\mu_Y \sim \mu_W$

(b) $||\underset{\sim}{V}|| < 1$ and $\underset{\sim}{V}$ is Hilbert-Schmidt.

(c) $||\underset{\sim}{T}|| < 1$ and $\underset{\sim}{T}$ is Hilbert-Schmidt.

PROOF: (b) implies $\mu_{XY} \sim \mu_X \otimes \mu_Y$, (c) implies $\mu_{ZW} \sim \mu_Z \otimes \mu_W$, and (a) implies $\mu_X \otimes \mu_Y \sim \mu_Z \otimes \mu_W$.

We give an example of two equivalent joint Gaussian measures which satisfy neither condition (b) or condition (c) of the corollary. Suppose that $\mu_X \sim \mu_Z$, and let $\underset{\sim}{W}$ be any bounded linear operator mapping $H_1 \to H_2$. Define μ_Y by $\mu_Y[B] = \mu_X\{\underset{\sim}{x}: \underset{\sim}{W}\underset{\sim}{x} \in B\}$ for $B \in \Gamma_2$, and define μ_W by $\mu_W[B] = \mu_Z\{x: \underset{\sim}{W}z \in B\}$. Also define μ_{XY} and μ_{ZW} by

$$\mu_{XY}[C] = \mu_X\{\underset{\sim}{x}: (\underset{\sim}{x}, \underset{\sim}{W}\underset{\sim}{x}) \in C\}, \qquad C \in \Gamma_1 \times \Gamma_2$$

$$\mu_{ZW}[C] = \mu_Z\{\underset{\sim}{x}: (\underset{\sim}{x}, \underset{\sim}{W}\underset{\sim}{x}) \in C\}.$$

Then $\mu_{XY} \sim \mu_{ZW}$, and both μ_{XY} and μ_{ZW} are Gaussian if μ_X and μ_Z are Gaussian. However, $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X^{\frac{1}{2}}\underset{\sim}{U}\underset{\sim}{R}_Y^{\frac{1}{2}}$ ($\underset{\sim}{R}_Y = \underset{\sim}{W}\underset{\sim}{R}_X\underset{\sim}{W}^*$, and $\underset{\sim}{R}_{XY} = \underset{\sim}{R}_X\underset{\sim}{W}^*$), where $\underset{\sim}{U}$ is a partially isometric map of H_2 into H_1, similarly, $\underset{\sim}{R}_{ZW} = \underset{\sim}{R}_Z^{\frac{1}{2}}\underset{\sim}{S}\underset{\sim}{R}_W^{\frac{1}{2}}$, where $\underset{\sim}{S}$ is a partial isometry of H_2 into H_1. Thus $||\underset{\sim}{U}||=||\underset{\sim}{S}||= 1$, and neither $\underset{\sim}{U}$ nor $\underset{\sim}{S}$ is Hilbert-Schmidt if range($\underset{\sim}{R}_Y$) is infinite-dimensional.

From the proof of the corollary, it is clear that $\mu_{XY} \perp \mu_{ZW}$ if (b) or (c) (but not both (b) and (c)) of the corollary is satisfied.

Applications to Information Theory

The average mutual information (AMI) of a joint measure μ_{XY} is defined as

$$AMI(\mu_{XY}) \equiv \int_{H_1 \times H_2} \frac{d\mu_{XY}}{d\mu_X \otimes \mu_Y}(\underset{\sim}{u},\underset{\sim}{v}) \, \ell og \left[\frac{d\mu_{XY}}{d\mu_X \otimes \mu_Y}(\underset{\sim}{u},\underset{\sim}{v}) \right] d\mu_X \otimes \mu_Y (\underset{\sim}{u},\underset{\sim}{v})$$

if $\mu_{XY} << \mu_X \otimes \mu_Y$, and equal to $+\infty$ otherwise. For Gaussian μ_{XY}, $AMI(\mu_{XY})$ is clearly finite if and only if $\mu_{XY} \sim \mu_X \otimes \mu_Y$. The most comprehensive results on AMI are due to Gel'fand and Yaglom [12]. In [3], necessary and sufficient conditions for $AMI(\mu_{XY})$ to be finite, stated in terms of covariance operators, were obtained for μ_{XY} Gaussian, and $H_1 = H_2$. That result was an extension of the work of Gel'fand and Yaglom. Theorem 6 gives necessary and sufficient conditions for $AMI(\mu_{XY})$ to be finite without requiring $H_1 = H_2$, provided μ_{XY} is Gaussian, and is independent of the results of Gel'fand and Yaglom.

Our final result is of interest in information theory applications. Let $H_1 = H_2 = H$, $\Gamma_1 = \Gamma_2 = \Gamma$, and consider the measure μ_{Y-X} defined by $\mu_{Y-X}(A) = \mu_{XY}\{(\underset{\sim}{x},\underset{\sim}{y}): \underset{\sim}{y}-\underset{\sim}{x} \in A\}$ (this is a well-defined Gaussian measure, since $f(\underset{\sim}{u},\underset{\sim}{v}) \equiv \underset{\sim}{v}-\underset{\sim}{u}$ is a continuous linear map from $(H \times H, \Gamma \times \Gamma)$ to (H, Γ)). In information theory applications, μ_{Y-X} represents "noise", μ_X "signal" and μ_Y "signal-plus-noise". Typically, these measures are induced by measurable mean-square continuous stochastic processes, with $H_1 = H_2 = H = L_2[0,T]$ for some finite T. Of interest is the $AMI(\mu_{XY})$ (the "average information about the signal obtained by observing signal-plus-noise") and the equivalence or orthogonality of μ_Y and μ_{Y-X}. For, if $AMI(\mu_{XY}) < \infty$, one might intuitively expect that $\mu_{Y-X} \sim \mu_Y$, and conversely, since $\mu_Y \perp \mu_{Y-X}$ allows one to discriminate perfectly between noise and signal-plus-noise. It is known that $\mu_{Y-X} \sim \mu_Y$ does not imply $AMI(\mu_{XY}) < \infty$, [13], [3]. In the case where

$\mu_{X-Y,X} = \mu_{Y-X} \otimes \mu_X$, Hajek [9] has shown that $\mathrm{AMI}(\mu_{XY}) < \infty$ if and only if μ_Y and μ_{Y-X} are strongly equivalent; i.e., $R_{Y-X} = R_Y^{\frac{1}{2}}[I+W]R_Y^{\frac{1}{2}}$, where W is *trace-class* and does not have -1 as an eigenvalue. The significance of strong equivalence is partly because one can then explicitly express the Radon-Nikodym derivative in series form [13], [9]. In the following, we show that not only is strong equivalence of μ_{Y-X} and μ_Y *not* equivalent to $\mu_{XY} \sim \mu_X \otimes \mu_Y$ when $\mu_{Y-X,X} \neq \mu_{Y-X} \otimes \mu_X$, but also that $\mu_{XY} \sim \mu_X \otimes \mu_Y$ does not even imply that $\mu_{Y-X} \sim \mu_Y$.

THEOREM 7. If $\mu_{XY} \sim \mu_X \otimes \mu_Y$, and $||m_X|| = ||m_Y|| = 0$, then $\mu_{Y-X} \sim \mu_Y$ if and only if $R_X = R_Y^{\frac{1}{2}}QR_Y^{\frac{1}{2}}$ for Q Hilbert-Schmidt.

PROOF: Define $\mu_{Y\ominus X}$ by $\mu_{Y\ominus X}(A) = \mu_X \otimes \mu_Y\{(x,y)\colon y-x \in A\}$, $A \in \Gamma$. Note that $\mu_{XY} \sim \mu_X \otimes \mu_Y \Rightarrow \mu_{Y-X} \sim \mu_{Y\ominus X}$.

Suppose first that $\mu_{Y-X} \sim \mu_Y$; then $\mu_Y \sim \mu_{Y\ominus X}$, and $\mu_Y \sim \mu_{Y\ominus X} \Rightarrow R_{Y\ominus X} = R_Y + R_Y^{\frac{1}{2}}QR_Y^{\frac{1}{2}}$, Q Hilbert-Schmidt, -1 not an eigenvalue of Q. But $R_{Y\ominus X} = R_Y + R_X$; hence $R_X = R_Y^{\frac{1}{2}}QR_Y^{\frac{1}{2}}$, where Q is Hilbert-Schmidt.

Conversely, suppose $R_X = R_Y^{\frac{1}{2}}QR_Y^{\frac{1}{2}}$, Q Hilbert-Schmidt. Then $\mu_{Y\ominus X} \sim \mu_Y$, since Q is non-negative. Hence $\mu_Y \sim \mu_{Y-X}$.

As an example where $\mu_{XY} \sim \mu_X \otimes \mu_Y$ but $\mu_{Y-X} \perp \mu_Y$, one can take any covariance operator R_Y such that $||R_Y|| < 1$, and define $R_X = R_Y$, $R_{XY} = R_Y^{3/2}$. Then the Gaussian joint measure μ_{XY} having μ_X and μ_Y as projections and $R_Y^{3/2}$ as cross-covariance operator is equivalent to $\mu_X \otimes \mu_Y$. However, $\mu_{Y-X} \perp \mu_Y$ by Theorem 7, since $R_X = R_Y^{\frac{1}{2}}IR_Y^{\frac{1}{2}}$.

Although the result of Hajek quoted above does not hold when $\mu_{Y-X,X} \neq \mu_{Y-X} \otimes \mu_X$, it is true that if $R_Y \geq R_{Y-X}$, then $\mu_{XY} \sim \mu_X \otimes \mu_Y \Rightarrow \mu_{Y-X}$ is strongly equivalent to μ_Y [3].

These results indicate that the concept of mutual information is not completely satisfactory, since one may be able to discriminate perfectly between

μ_X and μ_{Y-X} while having only a finite value of $AMI(\mu_{XY})$, or, $AMI(\mu_{XY})$ can be infinite while it is impossible to discriminate perfectly between μ_X and μ_{Y-X}.

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