Stochastic equivalent linearization in 3-D hysteretic frames

F.Casciati & L.Faravelli
Department of Structural Mechanics, University of Pavia, Italy

ABSTRACT

Stochastic equivalent linearization technique for hysteretic systems is extended to study the dynamic response of three dimensional frames with hysteretic constitutive laws in the potential plastic hinges. The constitutive law is idealized by an appropriate endochronic model. A general purpose finite element code is adopted in order to generate the matrices by which the equations of motion to be linearized are built.

1. INTRODUCTION

A recent state of art report (Lin et al., 1986) pointed out the features and the advantages of stochastic equivalent linearization techniques in the dynamic analysis of non-linear structural systems subjected to stochastic excitation. Except for the cumulant method (Wu and Lin, 1984), equivalent linearization is the only approach capable of handling both strong non linearities (i.e. hysteretic constitutive laws (Wen, 1980))and complex (i.e. multi-degree-of-freedom) structures (Baber and Wen, 1982; Casciati and Faravelli, 1985; Suzuki and Minai, 1985). It is widely used in research and some recent developments (Mochio, Samaras and Shinozuka, 1985; Casciati, Faravelli and Singh, 1987) provided this technique very efficient solution procedures.
Basically two approaches were pursued in past years:

- associate a constitutive law with each of the N degrees of freedom and solve an equivalent linear system whose first order differential equations are in number of 3N (Ang and Wen, 1983; Wen, 1986); multivariate constitutive laws were introduced in (Wen,Park and Ang, 1986)
- associate a constitutive law with each of the K regions of the structure where inelastic deformations concentrate (Casciati and Faravelli, 1985 a) and b); Baber, 1986) and solve an equivalent linear system of (2N + K) first-order differential equations.

The first approach offers the advantage that the dimension of the solving system is generally smaller. The second approach makes use of a constitutive law that can be derived from the geometrical and mechanical properties of the structural elements and, therefore, does not need (as the first approach does) the identification of force-displacement relationships from experimental data which often are not

available.
This paper shows the way the second approach can be generalized to include bi-axial bending moments and, hence, to study three dimensional frames.

2. EQUIVALENT LINEARIZATION GOVERNING RELATIONS

A multi-degree-of-freedom plane frame gives rise to the following equivalent linear system of equations(Casciati, Faravelli, 1985a)):

$$\dot{\underline{\theta}} = (\underline{Q}_2 - \underline{D})^{-1} (\underline{E}\,\underline{Q}_1\,\underline{u}_m - \underline{Q}_1\,\dot{\underline{u}}_m + \underline{E}\,\underline{Q}_2\,\underline{\theta}) \tag{01}$$

$$\ddot{\underline{u}} + \beta\,\underline{Q}_3\,\dot{\underline{u}}_m + \underline{Q}_3\,\underline{u} + \underline{Q}_4\,\underline{\theta} = -\underline{I}\psi(t) \tag{02}$$

Eq.(01) represents an endochronic form of the constitutive laws while Eq.(02) groups the equations of motion. In detail:

θ_j is a measure of the inelastic deformation concentrated in the j-th potential region $(j = 1,\dots,K)$

u_{mi} is the displacement associated with the i-th degree of freedom $(i = 1,\dots,N)$

D_{jj} is the first linearization coefficient relevant to the j-th potential region; $0 < D_{jj} < 1$; $D_{ji} = 0$ for $i \neq j$

E_{jj} is the second linearization coefficient relevant to the j-th potential region; $E_{jj} < 0$; $E_{ji} = 0$ for $i \neq j$

$\psi(t)$ is the stochastic excitation (f.i. the ground acceleration) assumed to be stationary in this paper

β is the ratio between the stiffness and the damping matrices of the elastic frame

and $\underline{Q}_1$, $\underline{Q}_2$, $\underline{Q}_3$ and $\underline{Q}_4$ are appropriate structural matrices.
Introducing the vector $\underline{y}^T = \{\underline{u}_m^T, \dot{\underline{u}}_m^T, \underline{\theta}^T\}$, Eq.(02) can be split in two first order differential equations and the system (01)-(02) becomes

$$\dot{\underline{y}} + \underline{G}\,\underline{y} = -\,\underline{b}\,\psi(t) \tag{03}$$

where $\underline{b} = \{\underline{0}^T, \underline{1}^T, \underline{0}^T\}$. Matrix $\underline{G}$ contains the linearization coefficients which are unknown but given functions of the covariance matrix $\underline{\Sigma}_y$ of $\underline{y}$. This matrix is the solution of the classical equation

$$\underline{G}\,\underline{\Sigma}_y + \underline{\Sigma}_y\,\underline{G}^T = \underline{B} \tag{04}$$

with

$$\underline{B} = \langle \underline{b}\,\underline{b}^T \rangle \tag{05}$$

Guessing initial values for D_{jj} and E_{jj}, an iterative procedure can be started. At convergence, one finds the covariance matrix $\underline{\Sigma}_y$ whose use in damage analysis is discussed in (Casciati and Faravelli, 1985b) and Wen, 1986).

3. MULTIVARIATE CONSTITUTIVE LAW

Previous Section assumes that the inelastic deformation can be expressed by a single scalar associated with a static quantity allowed to vary in the range between its yielding values. This is true for

plane frames in which bending moments predominate or in spatial systems of pin-ended elements, but, generally, the static aspects of the problem require a multivariate description (stress tensor in general, or two or more internal actions in frame schemes). In the space of this static components, yielding is a surface and inelastic deformation is normal to this surface for materials with associated flow rule.
Without loss of generality, one focuses attention on beam elements of tridimensional frames for which both the bending moments M_1 and M_2 have to be considered.In this case Eq.(02) becomes

$$\underline{\ddot{u}}_m + \beta \underline{Q}_3 \underline{\dot{u}}_m + \underline{Q}_3 \underline{u}_m + \underline{Q}_{41} \underline{\theta}_1 + \underline{Q}_{42} \underline{\theta}_2 = -\underline{I} \psi(t) \qquad (06)$$

where θ_1 and θ_2 denote the inelastic rotations associated with the bending moments M_1 and M_2 respectively. The analogous of Eq. (01), however, cannot be written, the constitutive laws (M_1,θ_1) and (M_2,θ_2) being dependent on the values of M_2 and M_1 respectively.
Nevertheless, the matrices Q_1 and Q_2 of Eq(02) can still be computed for each rotation: they will be denoted as Q_{11}, Q_{12} and Q_{21}, Q_{22}.
Their mechanical meaning (Casciati and Faravelli, 1985 b) is to relate the bending moment vector with u_m and θ:

$$\begin{aligned} \underline{M}_1 &= \underline{Q}_{11} \underline{u}_m + (\underline{Q}_{21} + \underline{B}_1)\underline{\theta}_1 \\ \underline{M}_2 &= \underline{Q}_{12} \underline{u}_m + (\underline{Q}_{22} + \underline{B}_2)\underline{\theta}_2 \end{aligned} \qquad (07)$$

$\underline{B}_1$ and $\underline{B}_2$ being diagonal matrices with the post-yelding stiffness as entries. In order to simplify the notation, it is convenient to build the matrices in such a way that two contiguous rows are the bending moments M_{1j} and M_{2j} in the j-th potential inelastic region and two contiguous columns are the contributions of the corresponding inelastic rotations θ_{1j} and θ_{2j}. Put $\underline{M}_j^T = \{\underline{M}_{1j}^T, \underline{M}_{2j}^T\}$ and $\underline{\theta}_j^T = \{\underline{\theta}_{1j}, \underline{\theta}_{2j}\}$, then Eq.(06) becomes

$$\underline{M} = \underline{Q}_{1A} \underline{u}_m + (\underline{Q}_{2A} + \underline{B}_A)\, \underline{\theta} \qquad (08)$$

where $\underline{Q}_{1A}$, $\underline{Q}_{2A}$ and $\underline{B}_A$ are assembled forms of the matrices in Eq. (07)
In the plane M_1, M_2 the yielding condition can be piece-wise linearized according to classical theory of plasticity. For reinforced concrete, in particular,(CEB 1982) the yielding surface can be linearized by eight straigth lines, i.e. by four lines if one considers the half-plane $M_1 > 0$, the other half-plane being described by simmetry. Let α_{jk} (k=1,...,4) be the angle the k-th line of the yielding condition in the j-th potential inelastic region forms with the axis M_1 . The associated plastic multiplier λ_{jk} is such that

$$\underline{Z}_{jk} \lambda_{jk} = \begin{Bmatrix} \sin \alpha_{jk} \\ \cos \alpha_{jk} \end{Bmatrix} \lambda_{jk} = \begin{Bmatrix} \theta_{1j} \\ \theta_{2j} \end{Bmatrix} = \underline{\theta}_j \qquad (09)$$

where θ_{1j} is the inelastic rotation for $M_{2j} = 0$ and θ_{2j} is the one for $M_{1j} = 0$.
The static counterpart of the parameter λ_{jk} is the quantity

$$\phi_{jk} = \underline{Z}_{jk}^T \underline{M}_j \qquad (10)$$

This quantity ϕ_{jk} varies in the range $(-\phi_{jkL}, \phi_{jkL})$, ϕ_{jkL} being

the distance from the origin of the k-th linear part of the yielding line. Therefore, an endochronic constitutive law between ϕ_{jk} and λ_{jk} with the same mathematical form of the one between M_j and θ_j in Eq.(01) can be introduced.
From Eqs. (08) and (10) one finds

$$\underline{\phi}_k = \underline{Z}_k^T \underline{M} = \underline{Z}_k^T \underline{Q}_{1A} \underline{u}_m + \underline{Z}_k^T(\underline{Q}_{2A} + \underline{B}_A) \underline{Z}_k \underline{\lambda}_k \tag{11}$$

with $\underline{\phi}_k^T = \{\phi_{1k}, \dots, \phi_{Kk}\}$, $\underline{\lambda}_k^T = \{\lambda_{1k}, \dots, \lambda_{kK}\}$ and $\underline{Z}_k^T$ assembled of consequence. The endocronic relationship between ϕ_{jk} and λ_{jk} is easily written and linearized in the form:

$$\dot{\underline{\lambda}}_k = (\underline{Z}_k^T\underline{Q}_{2A}\underline{Z}_k - \underline{D}_k)^{-1}(\underline{E}_k\underline{Z}_k^T\underline{Q}_{1A}\underline{u}_m - \underline{Z}_k^T\underline{Q}_{1A}\dot{\underline{u}}_m + \underline{E}_k\underline{Z}_k^T\underline{Q}_{2A}\underline{Z}_k\underline{\lambda}_k) \tag{12}$$

Eqs.(09) to (12) hold when only one plastic multipler is activated in the single inelastic region. However, during the dynamic excitation more than one straight line of the yielding surface can be achieved. In this case Eq.(09) must take into account the plastic rotations may arise from different plastic multipliers, i.e.

$$\underline{\theta}_j = \sum_k \underline{Z}_{jk} \lambda_{jk} \tag{13}$$

and

$$\phi_{jk} = \underline{Z}_k^T \underline{Q}_{1A} \underline{u}_m + \sum_h \underline{Z}_k^T (\underline{Q}_{2A} + \underline{B}_A) \underline{Z}_h \underline{\lambda}_h \tag{14}$$

As a consequence Eq.(12), assembled over the index k, provides

$$\dot{\underline{\Lambda}} = (\underline{W}_2 - \underline{D})^{-1} (\underline{E}\,\underline{W}_1 \underline{u}_m - \underline{W}_1 \dot{\underline{u}}_m + \underline{E}\,\underline{W}_2 \underline{\Lambda}) \tag{15}$$

where: $\underline{\Lambda} = \{\underline{\lambda}_1, \dots, \underline{\lambda}_K^T\}$; $\underline{D} = \begin{bmatrix} \underline{D}_1 & 0 & 0 \\ 0 & \underline{D}_2 & 0 \\ 0 & 0 & \underline{D}_K \end{bmatrix}$; $\underline{E} = \begin{bmatrix} \underline{E}_1 & 0 & 0 \\ 0 & \underline{E}_2 & 0 \\ 0 & 0 & \underline{E}_K \end{bmatrix}$;

$$\underline{W}_1 = \begin{Bmatrix} \underline{Z}_1^T \underline{Q}_{1A} \\ \underline{Z}_2^T \underline{Q}_{1A} \\ \cdots\cdots \\ \underline{Z}_k \underline{Q}_{1A} \end{Bmatrix} \quad \text{and} \quad \underline{W}_2 = \begin{bmatrix} \underline{Z}_1^T\underline{Q}_{2A}\underline{Z}_1 & \underline{Z}_1^T\underline{Q}_{2A}\underline{Z}_2 & \underline{Z}_1^T\underline{Q}_{2A}\underline{Z}_K \\ \cdots & \cdots & \cdots \\ \underline{Z}_K^T\underline{Q}_{2A}\underline{Z}_1 & \cdots\cdots & \underline{Z}_K^T\underline{Q}_{2A}\underline{Z}_K \end{bmatrix}$$

The kinematic uncknown variables are now $\underline{u}_m$ and $\underline{\lambda}_k$ and hence also Eq.(16) has to be updated:

$$\ddot{\underline{u}}_m + \beta\, \underline{Q}_3 \dot{\underline{u}}_m + \underline{Q}_3 \underline{u}_m + \sum_k \underline{Q}_{4A} \underline{Z}_k \underline{\lambda}_k = -\underline{I}\psi(t) \tag{16}$$

where Q_{4A} assembles in the appropriate way $\underline{Q}_{41}$ and $\underline{Q}_{42}$. Eq. (16) is fully correct for rigid plastic materials: when the constitutive law is elastic plastic, the sum of the terms $\underline{\lambda}_k$ introduces an error which is more significant more the structural response is close to the elastic solution. With this approximation, Eqs.(15) and (16) extend Eqs(01) and (02) to the analysis of tridimensional frames.

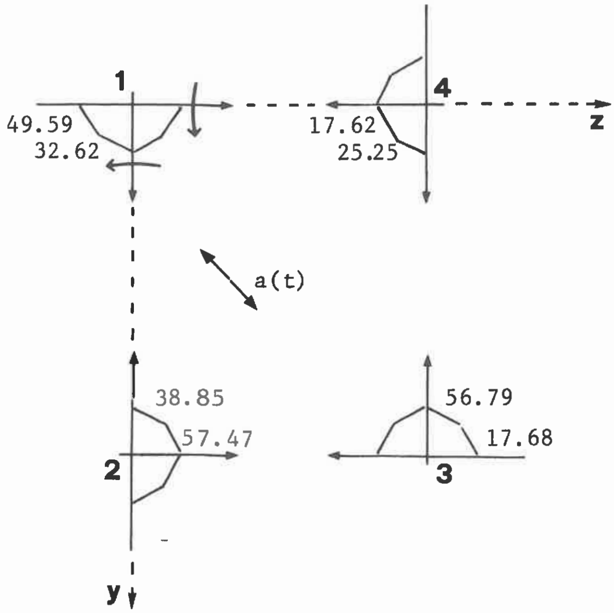
4. A NUMERICAL EXAMPLE

Consider four columns of length 300 cm connected by four beams of length 550 cm along axis y and 450 cm along z. The dimension of the reinforced concrete structural elements are 30x30, 30x50 and 30x50 respectively. The yielding surfaces are obtained from (CEB, 1982) considering the reinforcement ratio to be 1. Each of the columns 1 and 2 (along axis y) supports a mass of 200 kg s /cm ; each of the other two columns 100 kg s /cm. The structure is excited by a seismic action along the bisector of the plane (y,z). The seismic action is modeled as a filtered white noise of power spectral density function of Kanai Tajimi type with a =15-6 rad/s and =.69. The intensity is such that the expected peak acceleration is $.2\sqrt{2}$g. The results of the anlysis are in term of displacements along direcions y and z. They are summarized in Table I. Table II provides the variances of the plastic multipliers which correspond to the larger spread of inelastic deformation (all in the columns).

Table I - Variances of the displacements (cm) at the top of the columns

Direction	Column number			
	1	2	3	4
axis y	10.09	10.09	5.19	5.19
axis z	6.83	8.89	8.88	6.82

Table II - Variances of the plastic multipliers ($\lambda\ 10^6$) at column bottom.



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