Some Multiple Comparison Sign Tests

by

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ERRATA SHEET

Page vi - Table 6.2.3 is Page <u>64</u>.

Page 16 - Equation 2.3.3 - (1/4) should be (1/6).

Page 50 - k = . . . comparison   Should be k = . . . comparison<u>s</u>

Page 51 - Line 4 - γ = 0.0169   Should be $\underline{\alpha}$ = 0.0169

Page 53 - Reference 10 - "T" should be <u>The</u>

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1. INTRODUCTION

The assumptions that treatment and environmental effects are additive, and that the experimental errors are independently distributed in a normal distribution are made when the analysis of variance and customary tests of significance are used to analyze a set of data. In many situations the experimenter may not wish to make these assumptions but would still like to determine if significant differences exist among the various treatment means. More particularly, the experimenter may be interested in more than a test of whether any differences exist among the means of the set of populations sampled in the experiment; his interest may be in locating all differences whose presence is manifested by some test criterion or suspected on the basis of prior experience.

When the assumptions of the analysis of variance are met, the F-test which tests the over-all hypothesis of a common population mean has limited value in locating differences. This test is a joint test of as many independent comparisons as there are degrees of freedom in the numerator, say f_t. Using the familiar procedure of partitioning the treatment degrees of freedom and treatment sum of squares, one can obtain a set of f_t orthogonal or uncorrelated sums of squares to test corresponding comparisons among the population means. However, in practice, the most meaningful set of comparisons may not be an independent or uncorrelated set. Performance of correlated or non-orthogonal comparisons should not necessarily be discouraged, but it should be kept in mind that tabulated probabilities must be carefully related to a definition of error rate.

Considerable research has been conducted on multiple comparison procedures intended to indicate the locations of differences for non-orthogonal, as well as orthogonal, comparisons.

For proper appreciation of tabulated probability levels, these multiple comparison procedures may be grouped in terms of the definition appropriate to the Type I error rate. The two most common types of multiple comparison error rate are probably the per comparison and the experiment-wise error rate. The first is defined in terms of the expected proportion or long run average of erroneous inferences as:

$$\begin{array}{c}\text{Per comparison or}\\ \text{comparison-wise}\\ \text{error rate}\end{array} = \frac{\text{Number of erroneous inferences when the null hypothesis is true}}{\text{Total number of inferences attempted}} .$$

On the other hand, experiment-wise error rate is defined as the expected proportion of experiments with one or more erroneous statements when the null hypothesis is true. That is,

$$\begin{array}{c}\text{Experiment-wise}\\ \text{error rate}\end{array} = \frac{\text{Number of experiments with one or more erroneous statements when the null hypothesis is true}}{\text{Total number of experiments}} .$$

The least significant difference test, l.s.d., based upon the work of "Student" [13], a modification of this proposed by Fisher [7], and the studentized range test proposed by Tukey[1] are examples of multiple comparison range tests. Tests using as a test criterion a range dependent on the number of means involved are tests of the homogeneity of a set of sample means.

The l.s.d. test can be used, and the tabulated probability levels will be valid, when making orthogonal or non-orthogonal comparisons

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[1] Tukey, J. W., 1952, "Allowances for various types of error rates." Private circulation. Department of Mathematics, Princeton University, Princeton, New Jersey.

provided the treatments to be compared have been selected prior to observing the results of the experiment, that is, provided the tests are not suggested by the data. The calculated error rate is valid as a comparison-wise error rate. Tukey's test, sometimes called the honestly significant difference procedure, h.s.d., is an example of a test with an experiment-wise error rate, which allows the experimenter to make all possible pairwise comparisons among treatments simultaneously. The procedure may also be used to compute a joint set of confidence intervals. The above mentioned range test of Fisher, also allowing all possible pair-wise comparisons, has its Type I error rate in terms of the expected number of wrong statements per experiment (error rate per family), an error rate not formally defined here.

For the l.s.d. test, Fisher's test, and the h.s.d. test, only one value of the test criterion is calculated for the entire test. These tests are sometimes called fixed range tests. For other test procedures, the test statistic changes at each stage of the test. Thus we have the Newman-Keuls' test [10], Tukey's x-procedure[2], and Duncan's new multiple range test [4] as examples of multiple range tests.

The Newman-Keuls' test considers the number of treatments involved at any stage of the procedure, and the error rate is kept constant at each stage. That is, if the level of significance is set at α % then at each stage of the test procedure the probability of a Type I error is α %. Successive stages test the homogeneity of a decreasing number of treatment means with the same level of significance, on the assumption of the null hypothesis that no real differences exist among the

[2]Tukey, J. W., 1953. "The problem of multiple comparisons." Private circulation. Department of Mathematics, Princeton University, Princeton, New Jersey.

true means presently under test. If a set of means is declared significant, the next test of a smaller set is carried out on the assumption that the inference just drawn is a correct inference. Once a set of means is declared not significant no further tests are performed on this group. This permits a decision as to which differences are or are not significant. It is noted that the maximum value of the test criterion is the same as Tukey's h.s.d. procedure would require for a given experiment, while the minimum value is the same as the l.s.d. value.

Tukey's x-procedure is a compromise between his h.s.d. test procedure and the Newman-Keuls' test. The test statistic for each stage is the arithmetic mean of the h.s.d. value and the appropriate Newman-Keuls value. Obviously, this statistic, midway between the h.s.d. and the l.s.d., is smallest when only the homogeneity of two treatment means is tested and is largest, equal to the h.s.d., when the homogeneity of all treatment means is tested.

Duncan's new multiple comparison range test was developed to compare each treatment mean with every other treatment mean. The method calls for a set of significant differences of decreasing size. The size depends upon the closeness, that is the number in the set being tested, of the treatment sample means after ranking, with the largest value for the extreme means and the smallest for adjacent means. Once non-significance is declared between the extremes of a given set no further comparisons are made among the remaining means of the set. Instead of the usual idea of confidence limits, Duncan uses what he calls special protection levels against finding false significant differences; these levels, based upon the degrees of freedom, are given by $(1-\alpha)^{k-1}$ where k is the number of means in the set or subset under test. This test, although not as powerful as an earlier multiple comparisons test by the same author, has strong appeal because of its simplicity.

In many situations the experimenter may not wish to make all possible pair-wise comparisons among the treatments, but may wish only to compare the different treatments with a standard or a control. Dunnett [6] developed a test criterion for handling such a problem. Tables for one and two-tailed tests are available and require only a single difference for judging the significance of the observed comparisons. Dunnett's work also includes a method for computing a joint set of confidence intervals. Again, an experiment-wise rate applies.

The multiple comparison procedures mentioned thus far are all based upon the assumptions of normality and homogeneity of variance. Kramer [9] and Dunn [5] considered the multiple comparison problem when the data are normal but with unequal variances. When the experimenter finds he cannot make the normality assumption, he is likely to turn to nonparametric or distribution-free statistical tests in order to make the tests he desires.

The nonparametric tests considered in this paper are extensions of the sign test discussed by Dixon and Mood [2]. The sign test is one of the simplest nonparametric tests and is based on the signs of the differences between paired values. The null hypothesis is that the two treatment populations have the same medians rather than means. The only assumption necessary in applying this test is that the variable under consideration has a continuous distribution. The test does not make any assumptions about the form of the distribution, nor does it assume that all subjects are drawn from the same population. When the subjects are drawn from different populations, say with respect to age, sex, intelligence, etc., the only requirement is that within each pair the experimenter has achieved matching with respect to the relevant extraneous variables. The sign test when applied to a number of simultaneous comparisons is an analogue of the l.s.d. test in that the error rate is per comparison.

An extension of the sign test to a multiple comparison situation was proposed by Steel [11]. His proposal is for comparing all treatments in a randomized complete block design with a standard or control treatment. It is a nonparametric multiple comparison sign test. Analogous to Dunnett's procedure, and, like it, this test has an experiment-wise error rate.

In Chapter 2 of this study, the multiple comparison test for comparing all treatments against a single control is further developed. The general theory is presented. Exact distributions appropriate for making both one and two-tailed significance tests are extended beyond those originally obtained by Steel. Methods used for determining the exact distributions are discussed in detail. The approximate distributions, arising out of available tables of the distributions of certain statistics of equally correlated normal variates, are compared with the exact distributions. An approximation for determining the two-tailed critical values is suggested. Also, methods for constructing upper and lower confidence limits are given.

A second extension, considered in Chapter 3, of the sign test is a multiple comparison test for comparing all possible pairs of treatments. Douglas [3], working with Steel, introduced this procedure but unfortunately obtained only a severely limited number of usable results. This chapter presents the general theory for this sign test and shows the way in which critical values for the test criteria, analogous to the Newman-Keuls' and Tukey's tests, were obtained.

In the last chapter, the per-comparison and experiment-wise error rates for certain tabulated critical values are compared. A method for determining the corresponding per comparison error rate for a given experiment-wise error rate, and vice versa, is presented.

In summary, the purpose of this paper is to present general theory for two multiple comparison extensions of the sign test, to calculate exact and usable tables of critical values, to find suitable approximations where use of the exact distribution is impractical, and to give methods of constructing confidence intervals; also, to present a comparison of the tabulated critical values with an experiment-wise error rate and the corresponding critical values with a per comparison error rate.

2. A MULTIPLE COMPARISON SIGN TEST FOR COMPARING ALL TREATMENTS VERSUS A CONTROL

2.1. Introduction and Statement of the Problem

The text developed here was first considered by Steel [11] and is an extension of the sign test given by Dixon and Mood [2]. It is a nonparametric sign test, which has an experiment-wise error rate, applicable to comparing all treatments against a control in a randomized block design.

Let X_{oj} and X_{ij}, $i=1,\ldots,k$ and $j=1,\ldots,n$ be the measured responses on the control and i-th treatment in the j-th block. There are k comparisons with the control, and the proposed test considers only whether the differences, $d_{ij} = X_{ij} - X_{oj}$, are positive or negative. The test criterion is based on the number of plus signs (positive differences) and minus signs (negative differences) observed in each of the k set of n differences. The X_{oj} and X_{ij} need not have a common distribution over all replicates. However, it is required that the X_{oj} and X_{ij} be independent between replicates, that all X's within any replicate have a common distribution, and that the d_{ij}'s, $i=1,\ldots,k$, have the same median for all j.

The null and alternative hypotheses are stated in terms of the location of the medians of the multivariate distributions of the k-tuples of differences $(d_{1j}, d_{2j}, \ldots, d_{kj})$, where we define median $(d_{1j}, \ldots, d_{kj})$ equal to (median d_{1j}, ..., median d_{kj}).

Consider for the null hypothesis that the distributions of the $(d_{1j}, \ldots, d_{kj})$ have zero medians. Thus

H_o: Each k-tuple of differences $(d_{1j},\ldots,d_{kj}) = (X_{1j} - X_{oj},\ \ldots,\ X_{kj} - X_{oj})$ has a joint probability distribution with median zero.

For the one-sided test, the alternative hypothesis is:

H_1: The k-tuples of differences have joint probability distributions with common median in which at least one component is greater (or less) than zero.

(The expression "common median" implies that whatever the vector of medians is, and there is no need for the components to be identical, it remains constant from trial to trial.)

For the two-sided test, the alternative hypothesis is:

H_1: The k-tuples of differences have probability distribution with common median in which at least one component is not zero.

The usual modifications of the sign test may also be carried out for this multivariate sign test. In particular, testing for percentage or additive increases can be done by considering the null hypotheses, respectively, that the distributions of the k-tuples $(X_{oj} - a_1X_{1j},\ \ldots,\ X_{oj} - a_kX_{kj})$ or of the k-tuples $(X_{oj} - (a_1 + X_{1j}),\ \ldots,\ X_{oj} - (a_k + X_{kj}))$ have zero medians.

The test criterion is based on the distribution of $(r_1,\ldots,r_k)$ where r_i is the number of minus signs (or plus signs) observed for the differences $X_{ij} - X_{oj}$, for $j=1,2,\ldots,n$, a small number of minus signs indicating that the i-th treatment is superior to the control. Hence, record each of the k differences in $(X_{1j} - X_{oj},\ \ldots,\ X_{kj} - X_{oj})$ as one if the difference is negative and zero if positive. Do this for each replication and so obtain n vectors each of which is made up of k elements which are either zero or one. By adding these n-vectors one obtains $(r_1, r_2, \ldots, r_k)$. Obviously, the number of plus signs is $(n - r_1,\ n - r_2,\ \ldots,\ n - r_k)$.

When the null hypothesis is true, each difference, $X_{ij} - X_{oj}$, has probability .5 of being positive or negative in any given trial. Thus under H_o, there are $(k + 1)!$ equally likely arrangements, that is, orderings of X's in each block, giving rise to 2^k possible vectors. For example, if k=5, there are $6! = 720$ possible arrangements, but only $2^5 = 32$ possible vectors. The vector (0, 0, 0, 0, 0) appears in $5!$ possible arrangements, being the smallest observation; also, the vector (1, 1, 1, 0, 0) appears in $3!2! = 12$ possible arrangements; and so on. Hence, the ratio of the product of the number of possible arrangements of the observations on each side of the control to the total number of arrangements is the probability the vector appears in a given trial.

Steel [11] denotes the set of vectors in a single trial by v_1, v_2, ..., v_s where $s=2^k$ and their corresponding probabilities by p_1, p_2, ..., p_s. Then the probability of obtaining a given $(r_1, r_2, \ldots, r_k)$ as the sum of the vectors in n trials is the sum of the coefficients of the products of powers of the n_i's in one or more terms of the expansion of (2.1.1).

(2.1.1) $$(p_1 x_1 + p_2 x_2 + \ldots + p_s x_s)^n$$

To find an appropriate term, (2.1.2) must be solved for the n_i's subject to the condition $\sum_{i=1}^{s} n_i = n$.

$$\sum_{i=1}^{s} n_i v_i = (r_1, r_2, \ldots, r_k) \tag{2.1.2}$$

Each solution of equation (2.1.2) is a set of exponents of the x_i's in expression (2.1.1) and therefore determines a term.

Based on the distribution of the $(r_1, r_2, \ldots, r_k)$, the test criteria are developed. The test procedures proposed here are to have an experiment-wise error rate, defined as the proportion of experiments in which at least one wrong inference is made when the null hypothesis is true. An appropriate criterion for a test against the one-sided alternatives makes use of the minimum r_i, and for the two-sided alternative, the minimum r_i or minimum $(n-r_j)$, whichever is less.

2.2. The Probability of $(r_1, r_2, \ldots, r_k)$

A procedure for computing the probability associated with a particular value of $(r_1, r_2, \ldots, r_k)$ which satisfies equation (2.1.2) was given by Steel [11]. To illustrate this procedure, consider the following particular case.

For $k=3$, $n=5$ the method of obtaining $(r_1, r_2, r_3) = (1, 1, 2)$ is as follows. There are $s=2^k$ possible vectors, namely

$$\begin{array}{ll} v_1 = (111) & v_5 = (011) \\ v_2 = (110) & v_6 = (010) \\ v_3 = (101) & v_7 = (001) \\ v_4 = (100) & v_8 = (000) \end{array} \tag{2.2.1}$$

with their probabilities being $p_1 = p_8 = 1/4$ and $p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 1/12$. Now, using these vectors and the restriction $\sum_{i=1}^{s} n_i = n$, i.e., $\sum_{i=1}^{8} n_i = 5$, write equation (2.1.2) as equation (2.2.2).

$$(2.2.2) \qquad (n_1,\ldots,n_8)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (1,\ 1,\ 2,\ 5)$$

Lexicographically ordered, single trial vectors serve as the first k=3 elements of the row vectors in the coefficient matrix. The remaining elements, ones, provide for the restriction. Starting with the first entry in the r vector, i.e., $r_1=1$, and the restriction that n=5, observe that we need a single one and 4 zeros. In other words, we must choose one of the first 4 vectors and four from the remaining four vectors with repetition permitted. This is the first step towards a solution; see (2.2.3).

Step 1

(2.2.3)

First entry in v_i	Times v_i with specified first entry is chosen
1	1
0	4
Restriction	n=5

Note that $1 \times 1 + 0 \times 4 = 1 = r_1$. This part solution is carried into step 2 (2.2.4), where the second column of the coefficient matrix is considered. In step 2, there are seen to be two part solutions.

Step 2

(2.2.4)

Entry in v_i		Times v_i with two specified entries is chosen	
1st	2nd		
1	1	1	0
1	0	0	1
From (2.2.3)		1	1
0	1	0	1
0	0	4	3
From (2.2.3)		4	4

The two part solutions now available are read vertically. The entry r_1 has not altered; to check r_2, we now find: $1 \times 1 + 0 \times 0 + 1 \times 0 + 0 \times 4 = 1 = r_2$ for the first part solution and $1 \times 0 + 0 \times 1 + 1 \times 1 + 0 \times 3 = 1 = r_2$ for the second, as required.

Finally, we have step 3 (2.2.5), with the first part solution of step 2 providing two solutions, and the second providing four.

Step 3

Entry in v_i			Times fully specified v_i is chosen					
1st	2nd	3rd						
Solution number			1	2	3	4	5	6
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	0	0
From (2.2.4)			1	1	0	0	0	0
1	0	1	0	0	1	1	0	0
1	0	0	0	0	0	0	1	1
From (2.2.4)			0	0	1	1	1	1
0	1	1	0	0	1	0	1	0
0	1	0	0	0	0	1	0	1
From 2.2.4)			0	0	1	1	1	1
0	0	1	1	2	0	1	1	2
0	0	0	3	2	3	2	2	1
From (2.2.4)			4	4	3	3	3	3

(2.2.5)

We have now obtained six solutions to equation (2.2.2). Entries r_1 and r_2 have already been checked. We now note that

$1 \times 1 + 0 \times 0 + 1 \times 0 + 0 \times 0 + 1 \times 0 + 0 \times 0 + 1 \times 1 + 0 \times 3 = 2 = r_3$,

for the first solution. The remaining five solutions may be similarly checked.

The six solutions given in 2.2.5 determine six terms in the expansion of (2.1.1) for $k=3$, $n=5$, which give $(r_1, r_2, r_3) = (1, 1, 2)$. The required coefficients of these terms are shown in (2.2.6).

$$\frac{5!}{1!1!3!}(1/4)^4(1/12)^1;\ \frac{5!}{1!2!2!}(1/4)^2(1/12)^3;$$

(2.2.6) $$\frac{5!}{1!1!3!}(1/4)^3(1/12)^2;\ \frac{5!}{1!1!1!2!}(1/4)^2(1/12)^3;$$

$$\frac{5!}{1!1!1!2!}(1/4)^2(1/12)^3;\ \frac{5!}{1!1!2!1!}(1/4)\,(1/12)^4\ .$$

The sum of these probabilities is the probability that $(r_1, r_2, r_3) = (1, 1, 2)$ in 5 trials.

Due to symmetry, the probability that $(r_1, r_2, \ldots, r_k)$ will be observed is the same as that for $(n-r_1, n-r_2, \ldots, n-r_k)$, all k. That is, if r_i is the number of negative differences or the number of positive differences, the probability associated with a given $(r_1, \ldots, r_k)$ is the same. Also note, since the numbering of the treatments is random, the probabilities of any permutation of $(r_1, r_2, \ldots, r_k)$ are all equal. This is true for any number of treatments for a given number of replicates.

Unfortunately, the number of terms in the expansion of (2.1.1) increases rapidly as either k or n increases and thus other procedures are needed in order to construct the desired probability tables.

2.3. The Distribution of Minimum r_i and of Minimum$_i$ (min (r_i, n-r_i)); k=2

For k=2 the four vectors involved are given by (2.3.1).

$$
\begin{aligned}
v_1 &= (1, 1) \text{ with } p = 1/3 \\
v_2 &= (1, 0) \text{ with } p = 1/6 \\
v_3 &= (0, 1) \text{ with } p = 1/6 \\
v_4 &= (0, 0) \text{ with } p = 1/3
\end{aligned} \tag{2.3.1}
$$

The probability that v_1 occurs n_1 times, ..., v_4 occurs n_4 times in an experiment with n replications is

$$\frac{n!}{n_1!n_2!n_3!n_4!} (1/3)^{n_1 + n_4} (1/6)^{n_2 + n_3} \tag{2.3.2}$$

subject to $n_1 + n_2 + n_3 + n_4 = n$. It follows directly that the probability that $(r_1, r_2) = (a, b)$ is the sum of terms like (2.3.3),

$$\frac{n!}{\alpha!(a-\alpha)!\,(b-\alpha)!\,(n-a-b+\alpha)!} (1/3)^{n-a-b+2\alpha} (1/4)^{a+b-2\alpha} \tag{2.3.3}$$

where the summation is over all values of α which make the terms in the denominator non-negative.

The distributions of the min (r_i) and $\min_i$ (min (r_i, n-r_i)) are easily obtained by solving (2.3.3) for every possible value of a and b and classifying accordingly. For values of n=1(1)24, the distributions of the min (r_i) and $\min_i$ (min (r_i, n-r_i)) have been tabulated by means of a computer and the results given in Tables 6.2.1 and 6.2.3, respectively. Tables 6.2.2 and 6.2.4 give the corresponding cumulative probability distributions.

2.4. The Distribution of Minimum r_i and of $\underset{i}{\text{Minimum}}$ $(\min (r_i, n-r_i))$; $k > 2$

When k is greater than two, expressions similar to (2.3.3) can be obtained where summation is over more than two parameters. However, simplification does not seem possible. In this situation, it seems better to expand the multinomial (2.1.1), evaluating every term and recording its value as the partial sum of the $(r_1, r_2, \ldots, r_k)$ to which it belongs. For example, when k=3, the possible vectors are given in (2.2.1) and the required coefficients of the individual terms in the expansion of 2.1.1 are of the form (2.4.1).

$$\text{(2.4.1)} \qquad \frac{n!}{n_1!n_2!\ldots n_8!} (1/4)^{n_1 + n_8} (1/12)^{n-n_1-n_8} .$$

If $(n_1,\ldots,n_8) = (1, 1, 0, 0, 2, 2, 0, 0)$, the value of (2.4.1) is given by (2.4.2). Note that $\sum_{i=1}^{8} n_i = n$.

$$\text{(2.4.2)} \qquad \frac{6!}{1!1!2!2!} (1/4) (1/12)^5 = .0001808$$

This is one of the terms giving rise to $(r_1, r_2, r_3) = (2, 6, 3)$ as may be seen by evaluating the left hand of an equation like (2.2.2) using the n_i given above.

$$\text{(2.4.3)} \qquad \begin{aligned} r_1 &= n_1 + n_2 + n_3 + n_4 = 1 + 1 + 0 + 0 = 2 \\ r_2 &= n_1 + n_2 + n_5 + n_6 = 1 + 1 + 2 + 2 = 6 \\ r_3 &= n_1 + n_3 + n_5 + n_7 = 1 + 0 + 2 + 0 = 3 \end{aligned}$$

By this means, namely generating all probabilities, the distribution of $(r_1, \ldots, r_k)$ and, in turn, of min (r_i) and $\min_i (\min(r_i, n-r_i))$

have been obtained for k=3, n=1 (1) 15 and k=4, n=1 (1) 7. These results are given in Tables 6.2.5 and 6.2.7 respectively for k=3 and in Tables 6.2.9 and 6.2.11 for k=4. Corresponding cumulative distributions are given in Tables 6.2.6, 6.2.8, 6.2.10 and 6.2.12.

Appendix 6.1 gives the Fortran Program used in computing Tables 6.2.5 and 6.2.7 for the case of k=3. The program for k=4 is an extension of this case. As n and/or k increases, computer time soon becomes the major factor in restricting the tabulation of more extensive tables.

In view of the rapid increase in computations associated with an increase in either or both of k and n it is desirable to have an approximation for the calculation of the distribution of $(r_1,r_2,\ldots,r_k)$. Before contemplating an approximation, the correlations between the various r_i's of $(r_1,\ldots,r_k)$ must be obtained.

2.5. Correlation of the r_i's

Equal correlations exist between the various r_i's of $(r_1,\ldots,r_k)$. Consider first the case of k=2. Here, as stated before, the four possible vectors are

$$
\begin{aligned}
v_1 &= (1, 1)\\
v_2 &= (1, 0)\\
v_3 &= (0, 1)\\
v_4 &= (0, 0)
\end{aligned}
\tag{2.5.1}
$$

with respective probabilities 1/3, 1/6, 1/6, and 1/3. If $(r_1, r_2) = (r, s)$ then (2.5.2) follows.

$$\rho_{rs} = \frac{\sigma_{rs}}{(\sigma_{rr}\sigma_{ss})^{\frac{1}{2}}}$$

$$= \frac{E(rs) - E(r)E(s)}{[(E(r^2) - (E(r))^2)\ (E(s^2) - (E(s))^2)]^{\frac{1}{2}}}$$

$$(2.5.2) \qquad = \frac{\Sigma rs\ p(rs) - (\Sigma rp(r))\ (\Sigma sp(s))}{[\Sigma r^2 p(r) - (\Sigma rp(r))^2]^{\frac{1}{2}}\ [\Sigma s^2 p(s) - (\Sigma sp(s))^2]^{\frac{1}{2}}}$$

$$= \frac{0(2/3) + 1(1/3) - (0(1/2) + 1(1/2))\ (0)1/2) + 1(1/2))}{(0^2(1/2) + 1^2(1/2)) - (0(1/2) + 1(1/2))^2}$$

$$= 1/12 \div 1/4 = 1/3$$

where p(rs), p(r) and p(s) are probabilities derived from the joint distribution of r and s.

The addition of further treatments will not affect the probabilities, p(rs), p(r) or p(s). For example, if another treatment is added to the two used in the illustration, there will be eight possible vectors arising from the $(k + 1)! = 24$ possible arrangements. For a given ordering or permutation (relative to magnitude) of treatment one, treatment two and the control treatment, insert the new treatment. There are $k + 1 = 4$ possible different insertions, all of which leave the first two elements of the difference vector unchanged. Hence, the number of arrangements which give rise to the same first two elements in the difference vector for $k = 3$ is $k + 1 = 4$ times the number of ways in which it could arise in the case of $k = 2$. And since the total number of arrangements is $k + 1$ times the total arrangements of the previous case, the value of p(rs), p(r) and p(s) will be unchanged. By induction, this is true for any number of treatments.

2.6. A Multivariate Normal Approximation

The equal correlations, namely $\rho = 1/3$, between the various r_i's and the fact that the individual r_i's are distributed binomially with $p = 1/2$, suggest the multivariate normal distribution with all correlations equal to 1/3 as an appropriate approximation.

Gupta [8] has tabled, among other distributions, the probability that k=1(1)12 standard normal variables with common correlation equal to 1/3 are simultaneously less than or equal to H. To do this, he evaluates equation (2.6.1).

$$\text{(2.6.1)} \qquad \Pr(\text{all } x\text{'s} < H|\rho) = \int_{-\infty}^{\infty} F^k\left(\frac{H + \rho y^{\frac{1}{2}}}{(1-\rho)^{\frac{1}{2}}}\right) f(y)\, dy$$

where f(y) is the unit normal density function and

$$F(z) = \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}\, dt \quad .$$

From relation (2.6.2) for k=2

$$\text{(2.6.2)} \qquad \begin{aligned} \Pr(\min r_i < H|k=2) &= \Pr(\text{at least one } r_i < H|k=2) \\ &= \Pr(r_1 < H|k=1) + \Pr(r_2 < H|k=1) - \Pr(r_1, r_2 < H|k=2) \end{aligned}$$

and similar relations for larger k, along with Gupta's tables, the probability that the minimum r_i is equal to or less than some given value may be derived. Clearly, these results will be useful in tabulating critical values for one-tailed tests. The minimum r_i are standardized, as in equation (2.6.3), in order to tabulate the approximation probabilities.

(2.6.3) $$H = \frac{\text{minimum } r_i + 0.5 - np}{\sqrt{npq}} \quad \text{with } p=q=1/2$$

2.7. Exact Distribution Versus That Based on the Multivariate Normal Approximation

Comparison of the exact and approximate distributions follow, (2.7.1) through (2.7.7), for a number of values of k and n.

$k = 2,\ n = 4$

(2.7.1)

Minimum r_i	Exact	Approximate	Difference
	Cumulative Probabilities		
0	.1127	.1213	.0085
1	.4769	.4783	.0014
2	.8519	.8612	.0093
3	.9877	.9876	-.0001
4	1.0000	1.0000	--

$k = 3,\ n = 4$

(2.7.2)

Minimum r_i	Exact	Approximate	Difference
	Cumulative Probabilities		
0	.1544	.1671	.0127
1	.5786	.5851	.0065
2	.9145	.9241	.0096
3	.9961	.9963	.0002
4	1.0000	1.0000	--

$k = 4, n = 4$

(2.7.3)

		Cumulative Probabilities	
Minimum r_i	Exact	Approximate	Difference
0	.1900	.2065	.0165
1	.6478	.6582	.0104
2	.9446	.9532	.0086
3	.9984	.9985	.0001
4	1.0000	1.0000	--

Unfortunately, Gupta's tables give probabilities of the variate in increments of one tenth. Hence, in the following few situations, some errors may exist in the values of the approximation due to interpolation.

$k = 2, n = 9$

(2.7.4)

		Cumulative Probabilities	
Minimum r_i	Exact	Approximate	Difference
0	.0039	.0083	.0044
1	.0372	.0432	.0069
2	.1594	.1626	.0032
3	.4040	.4045	.0005
4	.6927	.6959	.0032
5	.8962	.8990	.0028
6	.9797	.9797	--
7	.9981	.9977	.0004
8	.9999	.9999	--
9	1.0000	1.0000	

k = 3, n = 9

(2.7.5)

Minimum r_i	Exact	Approximate	Difference
0	.0057	.0123	.0066
1	.0533	.0618	.0085
2	.2157	.2204	.0047
3	.5039	.5061	.0022
4	.7890	.7939	.0051
5	.9465	.9491	.0026
6	.9930	.9931	.0001
7	.9996	.9995	.0001
8	.9999	.9999	--
9	1.0000	1.0000	--

Cumulative Probabilities (Exact, Approximate, Difference)

k = 4, n = 7

(2.7.6)

Minimum r_i	Exact	Approximate	Difference
0	.0287	.0421	.0134
1	.1927	.2065	.0138
2	.5309	.5335	.0026
3	.8431	.8502	.0071
4	.9751	.9777	.0026
5	.9986	.9985	.0001
6	.9999	.9999	--
7	1.0000	1.0000	--

Cumulative Probabilities (Exact, Approximate, Difference) k = 2, n = 16

(2.7.7)

	Cumulative Probabilities		
Minimum r_i	Exact	Approximate	Difference
0	--	--	--
1	.0005	.0010	.0005
2	.0041	.0059	.0018
3	.0206	.0237	.0031
4	.0715	.0748	.0033
5	.1846	.1863	.0017
6	.3683	.3688	.0005
7	.5892	.5901	.0009
8	.7856	.7873	.0017

The true differences between the exact probability and its approximation are probably less than those above, since linear interpolation tends to over-estimate the cumulative probabilities in the left tail and, in all comparisons made, the approximate probability is somewhat higher in this region. As n increases, it is also to be noted that the extreme differences quickly diminish.

For probabilities of .05(.05).25, the critical values given by the exact distribution differ from those given by the approximation in only one case among those for k=2, n=4(1)24, for k=3, n=4(1)15, and for k=4, n=4(1)7; in the case of k=3, n=6, the true critical value of zero is not given by the approximation. The exact probability for the value zero is .043.

2.8. One-tailed Test

An experiment-wise error rate of say $\alpha = .05$ implies a high degree of caution in experiments with large numbers of treatments. This suggests that the significance level might be chosen according to the number of treatments, being larger as this number increases. For this reason, tables have been constructed with $\alpha = .05$, .10, .15, .20, and .25. Exact distributions (Tables 6.2.1 - 6.2.12) are supplemented with others based on the multivariate normal approximation. From the latter, it is possible to construct a one-tailed critical region table with an experiment-wise error rate.

Table 6.2.13 gives critical values of minimum r_i for making all comparisons of k treatments against a single control in an experiment with n replications of the set of observations. Tabulation is for $\alpha =$ .05(.05).25 and for n=4(1)50 and k=2(1)9.

The test procedure for the one-tailed alternative is:

(1) Compute the signed differences $X_{1j} - X_{oj}, \ldots, X_{kj} - X_{oj}$, for $j = 1, 2, 3, \ldots, n$.

(2) Observe the number of minus signs (or plus signs) for each of the k sets of n signs and record as r_i, $i = 1, \ldots, k$.

(3) Compare each r_i with the single tabulated critical value for the desired experiment-wise probability level. A significance statement is made for each of the k comparisons depending on whether each r_i is greater than the critical value or not. That is, reject H_o if $r_i \leq$ critical value, otherwise do not.

Steel [11] suggests Dunnett's t with infinite degrees of freedom as an approximate distribution for $(\min r - \mu_r)/\sigma_r$. However, Dunnett's t is calculated on the basis of 0.5 correlation, whereas for the distribution of $(r_1, \ldots, r_k)$, the correlation between r_i and r_j is 1/3. Comparison of the values from the approximation used in constructing Table 6.2.13 with those obtained by use of Dunnett's t show complete agreement for values of k=2 through 7 and n=4(1)20. Dunnett's t procedure gives the same or larger critical regions for $k > 7$. This is to be expected, for as the correlation increases, $\Pr(\min r_i < H)$ decreases. For example, from Gupta's table $\Pr(\min r_i < -2.00 | k = 2, \rho = 1/3) = .04319$ and $\Pr(\min r_i < -2.00 | k = 2, \rho = 1/2) = .04145$. Also when $\rho = 0$, $\Pr(\min r_i < -2.00 | k = 2)$ is .04508.

Finally, for values of n or k outside the ranges given in the exact tables, namely Tables 6.2.2, 6.2.6 and 6.2.10, and in Table 6.2.13, the approximate probability of observing a minimum $r_i \leq r$ is obtained as follows:

1. Standardize r, i.e., find $H = \dfrac{r + .5 - n/2}{\sqrt{n/4}}$.

2. Then calculate the probability, a generalization to k-variables of (2.6.2),

$$\Pr(\min r_i < H) = \sum_{i=1}^{k} (-1)^{i-1} \frac{k!}{i!(k-i)!} \Pr(\text{all } r_i \leq H | i\text{-variates}, \rho=1/3)$$

where

$$\Pr\,(\text{all } r_i \leq H | i\text{-variates}, \rho = 1/3)$$

is the probability that all i variates from an i variate multivariate normal distribution with $\rho = 1/3$ are simultaneously less than or equal to H (Gupta's tables).

For example, the probability of having a minimum r_i of 39 or less in 100 replications when three treatments are compared with a control is .04910. The computations follow.

Here

1. $H = \dfrac{r + .5 - n/4}{\sqrt{n/4}} = \dfrac{39 + .5 - 50}{5} = \dfrac{-10.5}{5} = -2.1$

and

2. $\Pr(\min r_i \leq 39|k=3) = 3 \Pr(\text{all } r_i \leq 39|k=1) - 3 \Pr$ $(\text{all } r_i \leq 39|k=2) + \Pr(\text{all } r_i \leq 39|k=3) = 3(.01786) - 3(.00159) + (.00029) = .04910.$

2.9. Two-tailed Test

The two-tailed test is used to test the null hypothesis that the k-tuples of differences have probability distributions with zero medians against an alternative in which at least one component is not zero. For this situation, the test procedure is as follows:

(1) Compute the signed difference $X_{1j} - X_{oj}$, $X_{2j} - X_{oj}$, ..., $X_{kj} - X_{oj}$, for j=1, 2, ..., n.

(2) Observe the number of times the less frequent sign occurs for each of the k sets of n signs and record as r_i or $n - r_i$, $i = 1, \ldots, k$, whichever is less.

(3) To judge significance, compare each r_i or $n - r_i$ with the critical value read from the appropriate table. If any r_i or $n - r_i$ is less than or equal to the critical value, then the null hypothesis is rejected in favor of the alternative. As in the one-tailed case, a significance statement is made for each of the k comparisons.

Exact probabilities for a few values of $\min_i (\min(r_i, n - r_i))$ are presented in Tables 6.2.4, 6.2.8 and 6.2.12. Table 6.2.4 gives the cumulative probabilities of $\min_i (\min(r_i, n - r_i))$ for k=2, n=1(1)24. Tables 6.2.8 and 6.2.12 list the same for k=3, n=1(1)15 and k=4, n=1(1)7 respectively.

As in the one-tailed situation, a multivariate normal distribution with equal correlations, $\rho = 1/3$, would probably produce a reasonable approximation to the exact distribution of $\min_i (\min(r_i, n - r_i))$. Unfortunately, no one seems to have tabulated the distribution of $\Pr(\max |x_i| \geq H)$ for the equally correlated, $\rho = 1/3$, multivariate normal which is needed for the approximation. Stuart [13], David and Thigpen [1], among others, have considered this problem. David and Thigpen [1] considered the distribution of the extremes in a normal sample when the variables are equally correlated with common, known mean and common variance. Their report gives the upper 100 α (α = .10, .05, .01) percentage points of the distribution of the maximum of the modulus of k = 2(1)9 standard normal variables with equicorrelations ρ, (ρ = 0, .2, .4, .8, 1).

Steel [11] computed a table of critical values by taking the integral part of the number computed by means of equation (2.9.1) with t from Dunnett's [6] tables. It was assumed that no value should be declared significant when the computation gave negative values.

$$r = \frac{n-1}{2} - \frac{5\sqrt{n}}{2} \tag{2.9.1}$$

where t is Dunnett's t with an infinite number of degrees of freedom and based on ρ = .5. Replacing the values of t, k = 2(1)9, in equation (2.9.1) by the corresponding values obtained by David [1] for $\rho = .2$ and again for $\rho = .4$, the resulting two tables of critical values differ only slightly from that obtained by Steel [11].

The critical value for each n, k, and α obtained using $\rho = .2$ was compared with the corresponding critical value found when $\rho = .4$. When these two values differed, interpolation to $\rho = 1/3$ was used to obtain the critical values. By supplementing the critical values obtained from the exact distributions with those from the approximation, Table 6.2.14 is formed. This table gives critical values for $\alpha = .10$, .05 and .01 for the test criterion $\min_i (\min(r_i, n - r_i))$, used for comparing k treatments against a control in n sets of observations. Tabulation is for $n = 6(1)50$ and $k = 2(1)9$.

A comparison shows that critical regions for $\alpha = .10$, .05 and .01 corresponding to tabulated exact distributions differ only slightly from those obtained by using the approximation. These discrepancies occurred for $k = 2$, $\alpha = .10$ at $n = 17$ and 22, for $k = 2$, $\alpha = .05$ at $n = 10$, 13, for $k = 2$, $\alpha = .01$ at $n = 9$, 13, 16, 19, for $k = 3$, $\alpha = .10$ and $n = 15$, for $k = 3$, $\alpha = .05$ and $n = 7$, and for $k = 3$, $\alpha = .01$ and $n = 10$.

In each of the above discrepancies, the critical region of the approximation was smaller than that of exact distribution by one integer. Although this approximation underestimates the critical region in some cases, it is felt that for practical purposes it is usable for all values of k.

2.10. Confidence Limits

The multiple comparison sign test for comparing all treatments versus a control and the multiple comparison sign test for comparing all pairs of treatments (See Chapter 3) may readily be converted into joint confidence statement procedures concerning the medians of the distributions of differences. Let $(c_1, c_2, \ldots, c_k)$ be any vector such that the null hypothesis that each k-tuple of differences $(d_{1j}, d_{2j}, \ldots, d_{kj}) = (X_{1j} - X_{oj} - c_1, X_{2j} - X_{oj} - c_2, \ldots, X_{kj} - X_{oj} - c_k)$ has a probability distribution with median zero will not be rejected in favor of the alternative hypothesis. Let c_i' and c_i'' be, respectively, the smallest and largest possible values of c_i not leading to rejection of the null hypothesis. Obviously, the possible values of the c_i's are dependent upon the alternative hypothesis.

To construct joint, one-sided, $1 - \alpha$ level, lower (upper), confidence limits, determine the set of c_i' (c_i'') for which the null hypothesis is not rejected in favor of the alternative that the k-tuples of differences have joint probability distribution in which at least one component is greater (less) than zero. The test for this hypothesis against the alternative is based upon the α % critical value of minimum r_i, where r_i is the number of negative (positive) differences in $X_{ij} - X_{oj}$, $i = 1,\ldots,k$; $j = 1,\ldots,n$. That is, to find joint one-sided, $1 - \alpha$ level, lower (upper) confidence limits first find from Table 6.2.13 the critical value of r_i, say r^*, for the given values of k, n and α. Now the smaller (larger) the value of c_i the less the number of negative (positive) differences. Thus, c_i' (c_i'') is the smallest (largest)

c_i such that the resulting number of negative (positive) differences r_i is greater than $r*$ and such that $c* = c_i' - \delta$ $(c* = c_i'' + \delta)$, for any $\delta > 0$, produces $r_i \leq r*$. The procedure is illustrated in Section 2.11.

For joint two-sided, $1 - \alpha$ level confidence intervals, the two sets of c's, i.e., $(c_1', c_2', \ldots, c_k')$ and $(c_1'', c_2'', \ldots, c_k'')$, are determined using the alternative hypothesis that the k-tuples of differences have probability distributions with common median in which at least one component is not zero. The test statistic for this alternative is the α percent critical value of $\underset{i}{\text{minimum}}\ (\min(r_i, n - r_i))$. Thus,

$$\Pr\ (c_i' \leq m_i - m_o \leq c_i'', \text{ for all } i) = 1 - \alpha$$

where m_i and m_o are the medians of the i-th treatment and control respectively.

2.11. Example

The accompanying data are a small part of the results of an experiment designed to compare three methods of measuring serum cholesterol with the method that was being used at the time of the experiment.

Table 2.1. Total cholesterol data

		Method		
Patient No.	Control	1	2	3
1	260	240 (-20)[a]	270 (+10)	200 (-60)
2	300	290 (-10)	290 (-10)	240 (-60)
3	290	320 (+30)	320 (+30)	240 (-50)
4	250	240 (-10)	270 (+20)	210 (-40)
5	270	250 (-20)	260 (-10)	190 (-80)
6	180	220 (+40)	230 (+50)	160 (-20)
7	200	190 (-10)	210 (+10)	140 (-60)
8	220	230 (+10)	250 (+30)	180 (-40)
9	410	420 (+10)	430 (+20)	270 (-140)
10	310	300 (-10)	320 (+10)	200 (-110)
Number of Minuses, r_i		6	2	10
Number of Pluses, $(n-r_i)$		4	8	0

[a]Differences $X_{ij} - X_{oj}$ are given in parentheses.

For example, to perform a 5% one-sided test of the null hypothesis that each three-tuple of differences (d_{1j}, d_{2j}, d_{3j}) has a probability distribution with median zero against the one-sided alternative that the three-tuples of differences have probability distributions with common median in which at least one component is greater than zero, find from Table 6.2.6 or from Table 6.2.13 the critical value of minimum r_i (r_i = number of negative signs). For k = 3, n = 10 the .05 critical value is 1. Since all comparisons have 2 or more negative signs the null hypothesis is not rejected. Inequality 2.11.2 gives the corresponding 95% lower confidence limits.

$$(2.11.2) \qquad \begin{bmatrix} -20 \\ -10 \\ -110 \end{bmatrix} \leq \begin{bmatrix} m_1 - m_o \\ m_2 - m_o \\ m_3 - m_o \end{bmatrix}$$

For the 5% two-sided test of the same null hypothesis against the null hypothesis that the three-tuples of differences (d_{1j}, d_{2j}, d_{3j}) have probability distributions with common median in which at least one component is different from zero, Table 6.2.8 and Table 6.2.14 give the critical value of $\underset{i}{\text{minimum}}$ (min $(r_i, n - r_i)$) to be 0. In this situation, Method 3 is declared to be significantly different from the control. In inequality 2.11.3, the corresponding 95% confidence limit band of the differences is given.

$$(2.11.3) \qquad \begin{bmatrix} -20 \\ -10 \\ -140 \end{bmatrix} \leq \begin{bmatrix} m_1 - m_o \\ m_2 - m_o \\ m_3 - m_o \end{bmatrix} \leq \begin{bmatrix} 40 \\ 50 \\ -20 \end{bmatrix}$$

3. A MULTIPLE COMPARISON SIGN TEST FOR COMPARING ALL PAIRS OF TREATMENTS

3.1. Introduction and Statement of Problem

The proposed multiple comparison sign test for comparing all pairs of treatments is similar to that for comparing all treatments with a control. For data arising from a randomized complete block design, the test is applicable to significance testing for all possible paired comparisons. This multivariate sign test is an extension of the Dixon-Mood sign test [2] and was first considered by Douglas [3].

As in the previous chapter, the observation X_{ij}, $i = 1,2,\ldots,k$ and $j = 1,\ldots,n$ is the measured response on the i-th treatment in the j-th block. Since all pairs of treatments are to be compared, there are $\binom{k}{2} = k(k-1)/2$ paired comparisons among the k treatments. The test is based on the number of plus signs (positive differences) or minus signs (negative differences) observed in each of the $k(k-1)/2$ comparisons for n replicates. The null and alternative hypotheses are stated in terms of the locations of the medians of the differences instead of the means of the $k(k-1)/2$ = c-tuples of differences $(d_{1j},\ldots,d_{cj})$. Both one-sided and two-sided tests can be performed, depending upon the interest as stated in the alternative hypothesis. The null and alternative hypotheses for the one-sided and two-sided tests are identical, respectively, to those for comparing all treatments against a control. That is, for the situation where a one-sided test is used, the null and alternative hypotheses are as follows:

H_o: Each c-tuple of differences $(d_{1j},\ldots,d_{cj})$ has a probability distribution with median zero.

H_1: The c-tuples of differences have probability distributions with common median in which at least one component is greater than zero.

For a two-sided test, the alternative hypothesis becomes

H_1: The c-tuples of differences have probability distributions with common median in which at least one component is different from zero.

It should be noted that for the one-sided test, the experimenter must make a statement about the relative worth of the two treatments in every paired comparison when H_o is false, prior to conducting the experiment. This implies a complete ranking of treatments. For example, we might write, for a particular situation, H_1: At least one inequality in $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_c$ holds.

The test criterion is based on the distribution of $(r_1,\ldots,r_c)$, where r_i, for example, is the number of negative signs (or positive signs) observed for the i-th comparison in the n replicates.

There are $s=k!$ permutations of the k observations in a block and each permutation gives rise to a possible vector of zeros and ones. Since under the null hypotheses each permutation is equally likely, the probability that a given vector will occur in a single trial is $1/k!$ It should be noted that all possible arrangements of zero and one in the difference vectors are not possible. For example, for $k = 3$, the vectors are based on the differences $(X_{1j} - X_{2j}, X_{1j} - X_{3j}, X_{2j} - X_{3j})$.

The vector (+, -, +) or (0, 1, 0) implies $X_{1j} > X_{2j}$ and $X_{1j} < X_{3j}$, and at the same time $X_{2j} > X_{3j}$, which is impossible. Thus, for k = 3, there are $k! = 6$ vectors, namely 111, 110, 100, 011, 001, 000, instead of 2^3 which might be thought to be true. Unfortunately, $k! > 2^k$ for $k > 3$.

As in Chapter 2, let the possible single trial vectors be denoted by $v_1, v_2, \ldots, v_s$ where $s = k!$, and their respective probabilities by $p_1, \ldots, p_s$. Since all permutations of the observations are assumed to be equally likely and each permutation yields one and only one vector, we have $p_1 = p_2 = \cdots = p_s = 1/k!$. By summing the coefficients of the one or more appropriate terms in the expansion of (3.1.1) the probability of $(r_1,\ldots,r_c)$ occurring can be obtained.

$$\begin{aligned} & (p_1x_1 + p_2x_2 + \cdots + p_sx_s)^n \\ (3.1.1) \qquad & = p^n(x_1 + x_2 + \cdots + x_s)^n \\ & = (1/k!)^n \, (x_1 + \cdots + x_s)^n \end{aligned}$$

Equation (3.1.2) is solved for a set of n_i subject to the restriction $\sum_{i=1}^{s} n_i = n$ to find a particular term. Usually, there will be more than one solution.

$$(3.1.2) \qquad \sum_{i=1}^{s} n_i v_i = (r_1,\ldots,r_c)$$

Each solution in (3.1.2) is a set of exponents of products of powers of the x_i's in expression (3.1.1) and thus determines a term. The sum of these coefficients is the probability of $(r_1,\ldots,r_c)$.

3.2. Distribution of $(r_1, r_2, \ldots, r_c)$

The distribution of $(r_1, r_2, \ldots, r_c)$ for n = 2(1)9 was obtained by Douglas [3] using the procedure suggested by Steel [11]. This is the procedure described in Section 2.2 of this paper. Douglas points out that as k or n increases, the calculation of the exact distribution of $(r_1, \ldots, r_c)$ by direct computation becomes prohibitive.

When finding the distribution of $(r_1, \ldots, r_c)$ by hand calculation, a second procedure has probably more appeal. This method uses the distribution of $(r_1, \ldots, r_c)$ for n - 1 replications to determine the distribution for n replications. For example, for k = 3, the possible vectors are given in the matrix 3.2.1.

$$(3.2.1) \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now let N_n (x, y, z) be the number of ways $(r_1, r_2, r_3) = (x, y, z)$ can occur in n replications. If in a particular experiment for which $(r_1, r_2, r_3) = (x, y, z)$, the n-th replication produced the vector v_i, then the number of ways in which (x, y, z) could occur is N_{n-1} $((x, y, z) - v_i)$. Hence, for k = 3, this fact and the set of vectors given in (3.2.1) lead to equation (3.2.2).

$$N_n(x, y, z) = N_{n-1}(x-1, y-1, z-1) + N_{n-1}(x-1, y-1, z)$$

$$(3.2.2) \qquad + N_{n-1}(x-1, y, z) + N_{n-1}(x, y-1, z-1)$$

$$+ N_{n-1}(x, y, z-1) + N_{n-1}(x, y, z)$$

$$= \sum_{i=1}^{s} N_{n-1}((x, y, z) - v_i) .$$

In general, we have equation(3.2.3).

$$(3.2.3) \qquad N_n(r_1, \ldots, r_c) = \sum_{i=1}^{s} N_{n-1}((r_1, \ldots, r_c) - v_i).$$

Unfortunately, the number of terms increases so rapidly as either k or n increases that computer core storage soon limits this approach when a computer is used to generate the distribution of $(r_1, \ldots, r_c)$.

3.3. The Distribution of (r_1, r_2, r_3); $k = 3$

To determine the number of ways a particular value of $(r_1, r_2, r_3) = (x, y, z)$ can occur, first consider the six possible difference vectors of equation(3.3.1). From these, n are chosen with replacement. Let n_i be the number of times v_i is chosen.

$$(3.3.1) \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

To have $r_1 = x$ in n replications or trials is to imply that of those vectors chosen, there are x with one and $n - x$ with zero as the first entry. This can occur in $\binom{n}{x}$ ways. Now suppose that α of these x vectors with one's as first element also have a one for the second element, i.e., are vectors v_1 or v_2 so that $n_1 + n_2 = \alpha$. Since $r_2 = y$, there must be $y - \alpha$ vectors with a zero first element and a one for the second element, i.e., $y - \alpha$ must be vector v_4 so that $n_4 = y - \alpha$ and $n_5 + n_6 = n - x - y + \alpha$. The total number of ways in which this may happen is $\binom{n}{x}\binom{x}{\alpha}\binom{n-x}{y-\alpha}$. Vectors v_1, v_4 and v_5 must occur a total of $r_3 = z$ times. But from above, $(n_1 + n_2) + (n_5 + n_6) = n - x - y + 2\alpha$. Hence, $z - y + \alpha$ one's must be chosen from $n - x - y + 2\alpha$ possibilities. It follows that equation (3.3.2),

$$(3.3.2)\ N_n = \Sigma\binom{n}{x}\binom{x}{\alpha}\binom{n-x}{y-\alpha}\binom{n-x-y+2\alpha}{z-y+\alpha} = \Sigma\binom{n}{\alpha,\ x-\alpha,\ n-x-y+\alpha}\binom{n-x-y+2\alpha}{z-y+\alpha}$$

where the summation is over all values of α which make the terms in the denominator non-negative, gives the number of ways $(r_1, r_2, r_3) = (x, y, z)$ can occur in n replications. Note that if (3.3.2) is summed over z, the result is the same as that given in (2.3.3).

Due to symmetry, the probability that (r_1, r_2, r_3) will be observed is the same as that for $(n - r_1, n - r_2, n - r_3)$ for any number of replications. Also the probability of (r_1, r_2, r_3) = probability (r_3, r_2, r_1).

Equation (3.3.2) was solved for all values of x, y and z for $n = 1(1)24$ by use of an IBM 704 computer. The distributions of the $\min_i (r_i)$ and $\underset{i}{\text{minimum}} (\min(r_i, n - r_i))$ were also tabulated in Tables

6.3.1 and 6.3.3, respectively. Tables 6.3.2 and 6.3.4 give the corresponding cumulative probability distributions.

3.4. Calculating the Distribution of Minimum r_i and Minimum $(\min(r_i, n - r_i))$; $k = 4$

For values of k greater than three, equations similar to (3.3.2) are difficult to obtain. In order to obtain the distribution of minimum r_i and $\underset{i}{\text{minimum}}$ $(\min(r_i, n - r_i))$, a procedure much like that described in Section 2.4 is used. That is, expand equation (3.1.1), evaluate every term and record each probability value, i.e., the probability that this combination of vectors will occur. Next, add those probabilities for combinations resulting in the same $(r_1, \ldots, r_c)$. The two required distributions are then obtained by classifying each term in the expansion of (3.1.1) according to its minimum r_i and its $\underset{i}{\text{minimum}}$ $(\min(r_i, n - r_i))$, respectively.

For example, when $k = 4$ the twenty-four possible and equally likely sign vectors are those given in Table 3.1, and the probabilities arising from the individual terms of equation (3.1.1) are of the form (3.4.1).

$$\text{(3.4.1)} \qquad \frac{n!}{(n_1)!\,(n_2)!\,\ldots\,(n_{23})!\,(n_{24})!} \cdot \frac{1}{24^n}$$

where n_i is the number of times v_i occurs. The coefficient of $1/24^n$ in (3.4.1) for a given set of n_i's is part of the number of ways $(r_1, r_2, \ldots, r_6)$ can be obtained, where r_i is given by equation (3.4.2).

Table 3.1. The twenty-four possible and equally likely sign vectors when comparing all pairs among k = 4 treatments

Permutations from largest to smallest	Vector No.	A - B	A - C	A - D	B - C	B - D	C - D
A B C D	v_1	1	1	1	1	1	1
A B D C	v_2	1	1	1	1	1	0
A D B C	v_3	1	1	1	1	0	0
A C B D	v_4	1	1	1	0	1	1
A C D B	v_5	1	1	1	0	0	1
A D C B	v_6	1	1	1	0	0	0
D A B C	v_7	1	1	0	1	0	0
D A C B	v_8	1	1	0	0	0	0
C A B D	v_9	1	0	1	0	1	1
C A D B	v_{10}	1	0	1	0	0	1
C D A B	v_{11}	1	0	0	0	0	1
D C A B	v_{12}	1	0	0	0	0	0
B A C D	v_{13}	0	1	1	1	1	1
B A D C	v_{14}	0	1	1	1	1	0
B D A C	v_{15}	0	1	0	1	1	0
D B A C	v_{16}	0	1	0	1	0	0
B C A D	v_{17}	0	0	1	1	1	1
C B A D	v_{18}	0	0	1	0	1	1
B C D A	v_{19}	0	0	0	1	1	1
B D C A	v_{20}	0	0	0	1	1	0
D B C A	v_{21}	0	0	0	1	0	0
C B D A	v_{22}	0	0	0	0	1	1
C D B A	v_{23}	0	0	0	0	0	1
D C B A	v_{24}	0	0	0	0	0	0

$$r_1 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12}$$

$$r_2 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_{13} + n_{14} + n_{15} + n_{16}$$

$$r_3 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_9 + n_{10} + n_{13} + n_{14} + n_{17} + n_{18}$$

$$r_4 = n_1 + n_2 + n_3 + n_7 + n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{19} + n_{20} + n_{21}$$

$$r_5 = n_1 + n_2 + n_4 + n_9 + n_{13} + n_{14} + n_{15} + n_{17} + n_{18} + n_{19} + n_{20} + n_{22}$$

$$r_6 = n_1 + n_4 + n_5 + n_9 + n_{10} + n_{11} + n_{13} + n_{17} + n_{18} + n_{19} + n_{22} + n_{23}$$

(3.4.2)

The value of (3.4.1) is then stored or classified according to minimum r_i and $\underset{i}{\text{minimum}}\,(\min\,(r_i,\ n - r_i))$. By evaluating each term in the expansion of (3.1.1), we obtain the desired two distributions of minima.

The number of vectors $s = k!$ greatly increases as k increases. Even when a high speed computer is used, the number of terms which must be evaluated in order to determine the above two distributions is so large that it is impractical to consider the method for values of k greater than four.

A limited number $n = 1(1)6$ of exact distributions of minimum r_i and $\underset{i}{\text{minimum}}\,(\min\,(r_i,\ n - r_i))$ are given in Tables 6.3.5 and 6.3.7 respectively for $k = 4$, while their corresponding cumulative distributions are given in Tables 6.3.6 and 6.3.8.

3.5. Correlations and Possible Approximate Distribution of the r_i's

The correlation between r_i and r_j depends upon the treatments involved in the i-th and j-th comparisons. Four possible situations can occur involving two differences from the vector $(d_{1j}, d_{2j}, \ldots, d_{cj})$. For the differences $x_i - x_j$ and $x_h - x_k$, the two differences are clearly independent and hence the correlation is zero. For $x_i - x_j$ and $x_i - x_h$, where $i < j, h$ and for $x_j - x_i$ and $x_h - x_i$ where $j, h < i$ the correlation is 1/3. The remaining situation $x_i - x_j$ and $x_j - x_h$ where $i < j < h$ has a correlation of $- 1/3$.

In Section 2.5, the correlation between the various r_i's was shown to be 1/3. There, the differences were formed by taking the i-th treatment value from the control or vice versa. Thus, by considering the i-th treatment, when comparing all possible pairs, as the control it follows that the correlation between $x_i - x_j$ and $x_i - x_k$ where $i < j$, h and also between $x_j - x_i$ and $x_h - x_i$ where $j, h < i$ is equal to 1/3. Also the correlation between the two differences $x_i - x_j$ and $x_j - x_h$ for $i < j < h$ is the negative of the correlation obtained if the differences had been formed as $x_i - x_j$ and $x_h - x_j$ for $i < j < h$. From Chapter 2, taking the j-th treatment as the control, the correlation between the latter differences is seen to be 1/3. Hence, the correlation between $x_i - x_j$ and $x_j - x_h$ for $i < j < h$ is $- 1/3$.

Examination of the exact distributions of the minimum r_i for $k = 3$, $n = 1(2)23$ reveals that for all values of min r_i such that expression

$$\frac{\text{Min } r_i + .5 - n/2}{\sqrt{n/4}} \tag{3.5.1}$$

(3.5.1) is zero, the cumulative probabilities are almost constant. See Table 3.2.

Table 3.2. Cumulative probabilities for some values of min r_i which when standardized equal zero

Value of n	Min r_i	Cumulative Probabilities
1	0	.8333
3	1	.8426
5	2	.8449
7	3	.8458
9	4	.8463
11	5	.8466
13	6	.8468
15	7	.8470
17	8	.8471
19	9	.8472
21	10	.8473
23	11	.8473

This is a good indication that the standardized minimum r_i have approximately the same distribution for all values of n and a given value of k.

A multivariate normal approximation for the joint distribution of $(r_1, \ldots, r_c)$ would probably give the best approximation. However, at present no one has tabulated the multivariate normal distribution needed.

3.6. One and Two-tailed Test of H_o

For the test procedure described here for all paired differences, as for that in the previous chapter, the error rate is an experimentwise error rate, i.e., is applicable to the joint set of significance statements. It is assumed that a common median exists from trial to trial and that the trials are independent.

The one-sided test is used to test the null hypothesis that each c-tuple of differences has a probability distribution with median zero, against the alternative that the c-tuples of differences have probability distributions (with common medians) in which at least one component is greater than zero. In this situation, the experimenter, prior to conducting the experiment, must make a statement about the relative worth of the two treatments in every paired comparison if H_o is false. For example, we might write, for a particular situation, H_1: At least one inequality in $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_c$ holds. The one-sided test procedure is:

(1) Compute the signed differences $(x_{1j} - x_{2j}, \ldots, x_{k-1,j} - x_{kj})$ for $j = 1,\ldots,n$.

(2) Observe the number of times a negative sign occurs for each of the c sets of n signs and record as r_i, $i = 1,\ldots,c$.

(3) To judge significance, compare each r_i with the critical value read from the appropriate table. If any r_i is less than, or equal to, the critical value, the null hypothesis is rejected in favor of the alternative.

If the experimenter is interested in the alternative hypothesis that the c-tuples of differences have probability distributions with common medians in which at least one component is less than zero, then the test procedure is the same as above after (2) is replaced by (2').

(2') Observe the number of times a positive sign occurs for each of the c-sets of n signs and record as r_i, $i = 1, \ldots, c$.

When the alternative hypothesis considered is a two-sided one, i.e. when

H_1: The c-tuples of differences have probability distributions with common medians in which at least one component is different from zero,

the two-tailed test procedure is as follows:

(1) Compute the signed differences

$X_{1j} - X_{2j}$, $X_{1j} - X_{3j}$, $\ldots$, $X_{k-1,j} - X_{kj}$, for $j = 1, \ldots, n$.

(2) Observe the number of times the less frequent sign occurs for each of the $c = \binom{k}{2}$ sets of n signs and record as r_i or $n - r_i$, $i = 1, \ldots, c$, whichever is less.

(3) To judge significance, compare each $\min(r_i, n - r_i)$ with the critical value read from the appropriate table. If any r_i or $n - r_i$ is less than or equal to the critical value, then the null hypothesis is rejected in favor of the alternative.

In all of the above considerations, a significance statement is made for each of the c comparisons. The significance level applies to the joint set of significance statements.

Tables 6.3.1 through 6.3.8 give exact and cumulative probabilities associated with a minimum r_i and $\min_i$imum $(\min_i (r_i, n - r_i))$ for k = 3, 4 and a limited number of values of n and are appropriate for both one and two-tailed tests. Tables 6.3.9 and 6.3.10 give, respectively, the α level (α = .01, .05, .10, .15, .20, .25) critical values for a few values of n and k appropriate for performing one and two-tailed tests.

3.7. Example

The following data are a small part of the results of the Cooperative Uniform Soybean Tests, 1956, for the North Central States. The data, yields in bushels per acre, were obtained from locations with widely differing conditions. Two locations in Ontario were used, three in Ohio, one in Michigan, two in Wisconsin, two in Minnesota, two in North Dakota, and one in South Dakota.

Table 3.3. Mean yield of soybeans in bushels per acre obtained from thirteen locations

Location	Strain			Difference		
	1	2	3	1-2	1-3	2-3
A	29.2	33.8	31.3	-4.6	-2.1	2.5
B	21.4	29.3	29.5	-7.9	-8.1	- .2
C	36.3	23.9	24.4	12.4	1.9	- .5
D	40.7	33.3	30.8	7.4	9.9	2.5
E	39.2	37.4	37.4	1.8	1.8	0
F	45.6	46.4	43.5	- .8	2.1	2.9
G	20.5	28.4	28.4	-7.9	-7.9	0
H	26.2	30.3	29.8	-4.1	-3.6	.5

Table 3.3 (continued

Location	Strain			Difference		
	1	2	3	1-2	1-3	2-3
I	34.4	32.5	33.5	1.9	.9	-1.0
J	46.1	47.1	44.5	-1.0	1.6	2.6
K	6.0	10.0	9.0	-4.0	-3.0	1.0
L	19.8	25.7	29.1	-5.9	-9.3	-3.4
M	24.0	20.2	24.5	3.8	- .5	-4.3
Number of Minuses r_i				8	7	5
Number of $n - r_i$				5	6	6

A 10% two-sided test of the null hypothesis that each three-tuple of differences (d_{1j}, d_{2j}, d_{3j}) has a probability distribution with median zero against the alternative that the three-tuples of differences have probability distributions with common median in which at least one component is different from zero follows. From Table 6.3.4 look up the critical value of $\underset{i}{\text{minimum}}\ (\min\ (r_i, n - r_i))$ for $k = 3$ and $n = 13$. This critical value is 2. Hence, do not reject the null hypothesis.

A 90% confidence interval for each difference (3.7.2) can be constructed using the procedure outlined in Section 2.10. In general, if $r*$ is the critical value for an α level two-sided test of the null hypothesis, then $1 - \alpha$ % joint confidence limits may be constructed as follows. First, rank the d_{ij}, $j = 1,\ldots,n$ for each i. Let $d_{i(j)}$ be the j-th ordered d_{ij}. If less than $d_{i(r* + 1)}$ or more than $d_{i(n - r*)}$ is subtracted from each d_{ij}, then $\min\ (\underset{i}{\min}(r_i, n - r_i)) \leq r*$. Hence, the $1 - \alpha$ % joint confidence limits are $d_{i(r^* + 1)}$ to $d_{i(n - r^*)}$ for all i. For the example $d_{i(r^* + 1)} = d_{i(3)}$ and $d_{i(n - r^*)} = d_{i(11)}$.

$$\begin{bmatrix} -5.9 \\ -7.9 \\ -1.0 \end{bmatrix} \leq \begin{bmatrix} m_1 - m_2 \\ m_1 - m_3 \\ m_2 - m_3 \end{bmatrix} \leq \begin{bmatrix} 3.8 \\ 1.9 \\ 2.5 \end{bmatrix} \tag{3.7.2}$$

4. COMPARISON OF ERROR RATES

4.1. Comparisonwise Versus Experimentwise Error Rate

Tests having comparisonwise error rates at the customary levels of α should be chosen in situations where the risk of failing to detect real treatment differences is more serious than the risk of stating that a difference exists when it does not. When the converse situation is true a test should be chosen with a specified experimentwise error rate. However, the same aims can be accomplished with either test by adjusting the levels of α.

In some situations an experimenter has a specified experimentwise error rate (specified comparisonwise error rate) and would like to determine the corresponding comparisonwise error rate (experimentwise error rate). If the comparisons are orthogonal the corresponding comparisonwise error rate (experimentwise error rate) for a given experimentwise error rate (comparisonwise error rate) can be obtained by use of formula (4.1.1).

(4.1.1) $$\gamma = 1 - (1 - \alpha)^k$$

where:

k = number of independent comparison

α = comparisonwise error rate

γ = experimentwise error rate

or

$$1 - \gamma = (1 - \alpha)^k .$$

For example, if $k = 3$, $\gamma = .05$, then (4.1.1) becomes

$$.05 = 1 - (1 - \alpha)^3$$

$$\alpha = 1 - (0.95)^{1/3}$$

$$\gamma = 0.0169 \quad .$$

Also if $k = 3$, $\alpha = .05$, then (3.1.1) becomes

$$\gamma = 1 - (1 - .05)^3$$

$$\gamma = 1 - (.95)^3$$

$$\gamma = .1426 \; .$$

For dependent comparisons this formula is not appropriate. Thus for the multiple comparison sign test for comparing all treatments versus a control and for the multiple comparison sign test for comparing all pairs of treatments, other methods must be devised which give the two corresponding error rates.

For a given experimentwise error rate for both of the multiple comparison sign tests, the corresponding per comparison error rate can be obtained as follows: Note the experimentwise critical value of min r_i. Then using tables of the cumulative binomial with $p = q = 1/2$ and the same number of replications determine α so that the per comparison critical value will be the same as the experimentwise. The probability associated with this value is the required comparisonwise error rate. By reversing the process the corresponding experimentwise error rate for a given comparisonwise error rate can also be determined.

For example, in the comparison of $k = 2$ treatments with a control and $n = 24$ replications, if the Type I experimentwise error rate (one-sided test) is to be no more than 10%, the critical value of min r_i is 7. Table 6.2.2 gives the exact probability of observing a min r_i of 7 or less to be .05996 or .060. From tables of the binomial distribution with $p = q = 1/2$, the probability of obtaining a 7 or less in 24 replications is .032. Hence, the corresponding comparisonwise error rate is 3.2%.

4.2. Experimentwise Versus Comparisonwise Error Rate

Going from a comparisonwise error rate to the corresponding experimentwise error rate is now illustrated. From the tables of the binomial distribution when $n = 7$, $p = q = 1/2$, the probability of observing a 1 or less is given to be .0625. For this comparisonwise error rate, the corresponding experimentwise error rate is .113 for $k = 2$ treatments versus a control (Table 6.2.2); is .156 for $k = 3$ treatments versus a control (Table 6.2.3); and is .193 for $k = 4$ treatments versus a control (Table 6.2.4).

5. LIST OF REFERENCES

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6. APPENDICES

6.1. Fortran Program

This appendix gives the Fortran Program used to generate the distributions of minimum r_i and $\underset{i}{\text{minimum}}$ (min $(r_i, n - r_i)$) when comparing k = 3 treatments against a control treatment. The program was written in Fortran applicable to an IBM 1620 Computer. With a few modifications it can be run on other IBM systems. In order to reduce computer running time for large values of n, this program was also translated into GAT so that the much faster Univac 1105A could be utilized.

Fortran Input

```
C       A MULTIPLE COMPARISON SIGN TEST, TREATMENT VS. CONTROL K = 3
        DIMENSION F(25)
        DIMENSION H(24), G(24), IR(3), AK(8), M(8)
        DO23I = 1, 8
 23     AK(I) = 0
        DO24I = 1, 3
 24     IR(I) = 0
        F(1) = 1
        DO25I = 2, 25
        AI = I - 1
        J = I - 1
 25     F(I) = F(J)*AI
        DO252I = 1, 24
        G(I) = 0
252     H(I) = 0
        AN = 1.0
        N = 2
135     DO123MA1 = 1, N
        NMA = N-MA1+1
        DO123MA2 = 1,NMA
        NLA = N-MA1-MA2+2
        B = (6.0)**(MA1+MA2-2)*(2.0)**(NLA-1)
        DO123MA3 = 1, NLA
        NLB = N-MA1-MA2-MA3+3
        DO123MA4 = 1, NLB
        NAS = MA1+MA2+MA3+MA4
        NLE = N-NAS+4
        DO123MB1 = 1, NLE
```

```
      NLF = N-MB1-NAS+5
      DO123MB2 = 1, NLF
      NLG = N-MB1-MB2-NAS+6
      DO123MB3 = 1, NLG
      MB4 = NLG-MB3+1
      IR(1) = MB1+MB2+MB3+MA2-4
      IR(2) = MA3+MA4+MB2+MA2-4
      IR(3) = MA2+MA4+MB3+MB4-4
      C = (AN*B)/(F(MA1)*F(MA2)*F(MA3)*F(MA4)*F(MB1)*F(MB2)*F(MB3)*F(MB4))
      IA = IR(1)
      DO7I = 2. 3
      IF(IA-IR(I))7, 7, 6
    6 IA = IR(I)
    7 CONTINUE
      IB = IR(1)
      DO9I = 2, 3
      IF(IB-IR(I))8, 9, 9
    8 IB = IR(I)
    9 CONTINUE
      IF(IA-N+IB+1)11, 10, 10
   10 IC = N-1-IB
      GO TO 12
   11 IC = IA
   12 IA = IA + 1
      G(IA) = G(IA)+C
      IC = IC+1
  123 H(IC) = H(IC)+C
      ANN = N-1
      PRINT200, ANN
      DO141 - 1, N
      PRINT 100, G(I)
   14 CONTINUE
      PRINT 300
      NA = (N+2)/(2)
      DO15I = 1, NA
      PRINT 100, H(I)
   15 CONTINUE
      PRINT 400
      PRINT 200, ANN
      DO66I:1,N
      ZZZ:G(*)/((24.0)**(N-1))
      PRINT 100, ZZZ
   66 G(I) = 0
      PRINT 300
      DO67I = 1, NA
      ZZZ = H(I)/((24.0)**(N-1))
      PRINT 100, ZZZ
   67 H(I) = 0
      D = AN*(ANN+1.0)
      AN = D
      N = N+1
      IF(26-N)17, 17, 135
```

```
 17     STOP
100     FORMAT(E14.8)
200     FORMAT(18H MINIMUM VALUES N = ,E14.8)
300     FORMAT(23H MINIMUM MAXIMUM VALUES)
400     FORMAT(14H PROBABILITIES)
        END
```

6.2. Tables of the Distributions of Minimum r_i and Minimum (min(r_i, n - r_i)) When Comparing k Treatments against a Control

Table 6.2.1. The exact probability distributions of minimum r_i when comparing k = 2 treatments against a control

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.666667	.388889	.212963	.112654	.058385	.029878
1	.333333	.500000	.472222	.364198	.250772	.160751
2		.111111	.277778	.375000	.380658	.327289
3			.037037	.135802	.244342	.312500
4				.012346	.061728	.141461
5					.004115	.026749
6						.001372

Table 6.2.1. (continued)

Minimum r_i equal to	Number of replications (n) equal to 7	8	9	10	11	12
0	.015168	.007660	.003855	.001936	.000971	.000468
1	.098172	.057928	.033327	.018812	.010463	.005752
2	.252508	.180608	.122221	.079317	.049830	.030511
3	.325824	.296401	.244639	.187793	.136333	.094718
4	.221051	.273438	.288616	.271583	.234311	.188935
5	.075617	.141099	.203572	.246094	.261363	.251445
6	.011203	.038142	.083486	.138573	.189809	.225586
7	.000457	.004572	.018404	.046582	.087955	.135274
8		.000152	.001829	.008573	.024799	.052764
9			.000051	.000720	.003881	.012704
10				.000017	.000279	.001715
11					.000006	.000107
12						.000002

Table 6.2.1 (continued)

Minimum r_i equal to	Number of replications (n) equal to 13	14	15	16	17	18
0	.000244	.000122	.000061	.000030	.000015	.000008
1	.003133	.001694	.000910	.000486	.000259	.000137
2	.018300	.010792	.006276	.003607	.002053	.001158
3	.063523	.041393	.026338	.016428	.010076	.006092
4	.144349	.105559	.074458	.050963	.034007	.022208
5	.223792	.187087	.148581	.113076	.083026	.059136
6	.240349	.234814	.213834	.183754	.150427	.118192
7	.178596	.209473	.223536	.220842	.204656	.179714
8	.090417	.131763	.169225	.196381	.209705	.208921
9	.030212	.057297	.091647	.128279	.161237	.185471
10	.006301	.016633	.034707	.060631	.092097	.124926
11	.000743	.003041	.008855	.020226	.038416	.063083
12	.000041	.000316	.001433	.004579	.011395	.023440
13	.000001	.000015	.000133	.000661	.002309	.006233
14		[a]	.000006	.000055	.000300	.001138
15			[a]	.000002	.000022	.000134
16				[a]	.000001	.000009
17					[a]	[a]
18						[a]

[a]Less than .000001.

Table 6.2.1 (continued)

Minimum r_i equal to	Number of replications (n) equal to 19	20	21	22	23	24
0	.000004	.000002	.000001	a	a	a
1	.000072	.000038	.000020	.000010	.000005	.000003
2	.000649	.000361	.000200	.000109	.000060	.000033
3	.003638	.002149	.001257	.000730	.000420	.000240
4	.014236	.008980	.005586	.003432	.002086	.001255
5	.041036	.027843	.018525	.012117	.007806	.004962
6	.089665	.066002	.047325	.033167	.022781	.015372
7	.150747	.121572	.094763	.071707	.052868	.038094
8	.196281	.175387	.150083	.123690	.098640	.076418
9	.198077	.198613	.188660	.171001	.148783	.124885
10	.154317	.176197	.188130	.189598	.181721	.166681
11	.092041	.121745	.148246	.168188	.179500	.181633
12	.041468	.064882	.091653	.118747	.142860	.161180
13	.013835	.026285	.043980	.066188	.091046	.115931
14	.003321	.007929	.016130	.028788	.046050	.067119
15	.000550	.001729	.004425	.009615	.018260	.030982
16	.000059	.000261	.000881	.002412	.005582	.011257
17	.000004	.000025	.000122	.000440	.001286	.003165
18	a	.000001	.000011	.000056	.000216	.000673
19	a	a	.000001	.000005	.000025	.000105
20		a	a	a	.000002	.000011
21			a	a	a	.000001
22				a	a	a
23					a	a
24						a

[a]Less than .000001.

Table 6.2.2. The exact cumulative probability distributions of minimum r_i when comparing k = 2 treatments against a control

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.666667	.388889	.212963	.112654	.058385	.029878
1	1.000000	.888889	.685185	.476852	.309157	.190629
2		1.000000	.962963	.851852	.689815	.517918
3			1.000000	.987654	.934157	.830418
4				1.000000	.995885	.971879
5					1.000000	.998628
6						1.000000

Table 6.2.2 (continued)

Minimum r_i equal to	Number of replications (n) equal to 7	8	9	10	11	12
0	.015168	.007660	.003855	.001936	.000971	.000486
1	.113340	.065588	.037182	.020748	.011434	.006238
2	.365848	.246196	.159403	.100065	.061264	.036749
3	.691672	.542597	.404042	.287858	.197597	.131467
4	.912723	.816035	.692658	.559441	.431908	.320402
5	.988340	.957134	.896230	.805535	.693271	.571847
6	.999543	.995276	.979716	.944108	.883080	.797433
7	1.000000	.999848	.998120	.990690	.971035	.932707
8		1.000000	.999949	.999263	.995834	.985471
9			1.000000	.999983	.999715	.998175
10				1.000000	.999994	.999890
11					1.000000	.999997
12						1.000000

Table 6.2.2 (continued)

Minimum r_i equal to	Number of replications (n) equal to 13	14	15	16	17	18
0	.000244	.000122	.000061	.000030	.000015	.000008
1	.003377	.001816	.000971	.000516	.000274	.000145
2	.021677	.012608	.007247	.004123	.002327	.001303
3	.085200	.054001	.033585	.020551	.012403	.007395
4	.229549	.159560	.108043	.071514	.046410	.029603
5	.453341	.346647	.256624	.184590	.129436	.088739
6	.693690	.581461	.470458	.368344	.279863	.206931
7	.872286	.790934	.693994	.589186	.484519	.386645
8	.962703	.922697	.863219	.785567	.694224	.595566
9	.992915	.979994	.954866	.913846	.855461	.781037
10	.999216	.996627	.989573	.974477	.947558	.905963
11	.999959	.999668	.998428	.994703	.985974	.969046
12	[a]	.999984	.999861	.999282	.997369	.992486
13	1.000000	.999999	.999994	.999943	.999678	.998719
14		1.000000	[a]	.999998	.999978	.999857
15			1.000000	[a]	[a]	.999991
16				1.000000	[a]	[a]
17					1.000000	[a]
18						1.000000

[a] Greater than .999999.

Table 6.2.2 (continued)

Minimum r_i equal to	Number of replications (n) equal to 19	20	21	22	23	24
0	.000004	.000002	.000001	[a]	[a]	[a]
1	.000076	.000040	.000021	.000010	.000005	.000003
2	.000725	.000401	.000221	.000119	.000065	.000036
3	.004363	.002550	.001478	.000849	.000485	.000276
4	.018599	.011530	.007064	.004281	.002571	.001531
5	.059635	.039373	.025589	.016398	.010377	.006493
6	.149300	.105375	.072914	.049565	.033158	.021865
7	.300047	.226947	.167677	.121272	.086026	.059959
8	.496328	.402334	.317760	.244962	.184666	.136377
9	.694405	.600947	.506420	.415963	.333449	.261262
10	.848722	.777144	.694550	.605561	.515170	.427943
11	.940763	.898889	.842796	.773749	.694670	.609576
12	.982231	.963771	.934449	.892496	.837530	.770756
13	.996066	.990056	.978429	.958684	.928576	.886687
14	.999387	.997985	.994559	.987472	.974626	.953806
15	.999937	.999714	.998984	.997087	.992886	.984788
16	.999996	.999975	.999865	.999499	.998468	.996045
17	[b]	[b]	.999987	.999939	.999754	.999210
18	[b]	[b]	.999998	.999995	.999970	.999883
19	1.000000	[b]	.999999	[b]	.999995	.999988
20		1.000000	[b]	[b]	.999997	.999999
21			1.000000	[b]	[b]	[b]
22				1.000000	[b]	[b]
23					1.000000	[b]
24						1.000000

[a] Less than .000001.
[b] Greater than .999999.

Table 6.2.3. The exact probability distributions of $\underset{i}{\text{minimum}}$ $(\min(r_i, n-r_i))$ when comparing k = 2 treatments against a control

$\underset{i}{\text{Minimum}}$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.722222	.416667	.223765	.116512	.059714	.030328
1	.277778	.583333	.623457	.474537	.315072	.194895
2			.152778	.408951	.520190	.460455
3					.105024	.314322

$\underset{i}{\text{Minimum}}$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 8	9	10	11	12	13
0	.015319	.007711	.003872	.001942	.000973	.000487
1	.115541	.066588	.037609	.020923	.011504	.006266
2	.347846	.240719	.157657	.099415	.060963	.036586
3	.441234	.429848	.354593	.265837	.187357	.126447
4	.080060	.255133	.381566	.397220	.349389	.278247
5			.064702	.214664	.335522	.366707
6					.054293	.185259

Table 6.2.3 (continued)

Minimum i $(\min(r_i, n-r_i)$ equal to	Number of replications (n) equal to 14	15	16	17	18	19
0	.000244	.000122	.000061	.000031	.000015	.000008
1	.003387	.001820	.000972	.000517	.000274	.000145
2	.021581	.012551	.007214	.004105	.002316	.001297
3	.082621	.052632	.032846	.020150	.012183	.007275
4	.207541	.147761	.101571	.067910	.044386	.028464
5	.338731	.282855	.220767	.164143	.117629	.081866
6	.299124	.339326	.325777	.282649	.228971	.176488
7	.046770	.162932	.269713	.315090	.312110	.279477
8			.041079	.145404	.245491	.293701
9					.036623	.131279

Table 6.2.3 (continued)

Minimum i $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 20	21	22	23	24
0	.000004	.000002	.000001	a	a
1	.000076	.000040	.000021	.000011	.000006
2	.000721	.000399	.000220	.000120	.000066
3	.004298	.002515	.001459	.000841	.000481
4	.017958	.011171	.006864	.004171	.002511
5	.055622	.037032	.024228	.015611	.009924
6	.130967	.094291	.066214	.045524	.030733
7	.233567	.185600	.141888	.105169	.075987
8	.298530	.274500	.235567	.192150	.150722
9	.225217	.274795	.285436	.268457	.235701
10	.033039	.119654	.208008	.258025	.273013
11			.030094	.109919	.193225
12					.027631

[a]Less than .000001.

Table 6.2.4. The exact cumulative probability distributions of minimum$_i$ ($\min(r_i, n-r_i)$) when comparing k = 2 treatments against a control

Minimum$_i$ ($\min(r_i,n-r_i)$) equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.722222	.416667	.223765	.116512	.059714	.030328
1	1.000000	1.000000	.847222	.591049	.374786	.225223
2			1.000000	1.000000	.894976	.685678
3					1.000000	1.000000

Minimum$_i$ ($\min(r_i,n-r_i)$) equal to	Number of replications (n) equal to 8	9	10	11	12	13
0	.015319	.007711	.003872	.001942	.000973	.000487
1	.130860	.074299	.041481	.022865	.012477	.006753
2	.478706	.315018	.199138	.122280	.073440	.043339
3	.919940	.744866	.553731	.388117	.260797	.169786
4	1.000000	1.000000	.935297	.785337	.610186	.448033
5			1.000000	1.000000	.945708	.814740
6					1.000000	1.000000

Table 6.2.4 (continued)

Minimum $\underset{i}{\text{}}$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 14	15	16	17	18	19
0	.000244	.000122	.000061	.000031	.000015	.000008
1	.003631	.001942	.001033	.000548	.000289	.000153
2	.025212	.014493	.008247	.004653	.002605	.001450
3	.107833	.067125	.041093	.024803	.014788	.008725
4	.315374	.214886	.142664	.092713	.059174	.037189
5	.654105	.497741	.363431	.256856	.176803	.119055
6	.953229	.837067	.689208	.539505	.405774	.295543
7	1.000000	1.000000	.158921	.854595	.717884	.575020
8			1.000000	1.000000	.963375	.868721
9					1.000000	1.000000

Table 6.2.4 (continued)

Minimum over i $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 20	21	22	23	24
0	.000004	.000002	.000001	[a]	[a]
1	.000080	.000042	.000022	.000011	.000006
2	.000801	.000441	.000242	.000131	.000072
3	.005099	.002956	.001701	.000972	.000553
4	.023057	.014127	.008565	.005143	.003064
5	.078679	.051159	.032793	.020754	.012988
6	.209646	.145450	.099007	.066278	.043721
7	.443212	.331050	.240895	.171447	.119708
8	.741743	.605550	.476462	.363597	.270430
9	.966960	.880345	.761898	.632054	.506131
10	1.000000	1.000000	.969906	.890079	.779144
11			1.000000	1.000000	.972369
12					1.000000

[a]Less than .000001.

Table 6.2.5. The exact probability distributions of minimum r_i when comparing k = 3 treatments against a control

Minimum r_i equal to	Number of replicates (n) equal to 2	3	4	5	6	7	8
0	.479167	.279514	.154369	.082381	.043004	.022127	.011277
1	.458333	.498264	.424190	.313404	.211365	.133920	.081158
2	.062500	.206597	.335937	.388937	.368155	.305122	.230318
3		.015625	.081597	.184462	.277456	.326339	.325391
4			.003906	.029839	.089413	.169178	.241649
5				.000977	.010362	.039787	.092440
6					.000244	.003465	.016620
7						.000061	.001126
8							.000015

Table 6.2.5 (continued)

Minimum r_i equal to	Number of replications (n) equal to 9	10	11	12	13	14	15
0	.005711	.002880	.001448	.000727	.000364	.000182	.000091
1	.047605	.027249	.015309	.008478	.004642	.002518	.001356
2	.162336	.108644	.069859	.043531	.026454	.015755	.009229
3	.288264	.233786	.177149	.127250	.087584	.058231	.037632
4	.285072	.292740	.270672	.230624	.184117	.139441	.101136
5	.157506	.216867	.255469	.267146	.254368	.224625	.186711
6	.046534	.093225	.148137	.198411	.233004	.246554	.239774
7	.006611	.021970	.051138	.092890	.140375	.183981	.215260
8	.000357	.002528	.009848	.026368	.054306	.091976	.133794
9	.000004	.000111	.000936	.004228	.012912	.029940	.056495
10		[a]	.000034	.000337	.001751	.006052	.015701
11			[a]	.000010	.000119	.000703	.002732
12				[a]	.000003	.000041	.000274
13					[a]	.000001	.000014
14						[a]	[a]
15							[a]

[a] Less than .000001.

Table 6.2.6. The exact cumulative probability distributions of minimum r_i when comparing k = 3 treatments against a control

Minimum r_i equal to	Number of replications (n) equal to 2	3	4	5	6	7	8
0	.479167	.279514	.154369	.082381	.043004	.022127	.011277
1	.937500	.777778	.578559	.395785	.254369	.156047	.092435
2	1.000000	.984375	.914496	.784722	.622524	.461169	.322753
3		1.000000	.996093	.969184	.899980	.787508	.648144
4			1.000000	.999023	.989393	.956686	.889793
5				1.000000	.999755	.996473	.982233
6					1.000000	.999938	.998853
7						1.000000	.999979
8							1.000000

Table 6.2.6 (continued)

Minimum r_i equal to	Number of replications (n) equal to 9	10	11	12	13	14	15
0	.005711	.002880	.001448	.000727	.000364	.000182	.000091
1	.053316	.030129	.016757	.009205	.005006	.002700	.001447
2	.215652	.138773	.086616	.052736	.031460	.018455	.010676
3	.503916	.372559	.263765	.179986	.119044	.076686	.048308
4	.788988	.665299	.534437	.410610	.303161	.216127	.149444
5	.946494	.882166	.789906	.677756	.557529	.440752	.335955
6	.993028	.975391	.938043	.876167	.790533	.687306	.575729
7	.999639	.997361	.989181	.969057	.930908	.871287	.790989
8	.999996	.999889	.999029	.995425	.985214	.963263	.924783
9	1.000000	.999999	.999965	.999654	.998126	.993203	.981278
10		1.000000	.999999	.999990	.999877	.999255	.996979
11			1.000000	.999999	.999996	.999958	.999711
12				1.000000	.999999	.999999	.999985
13					1.000000	.999999	.999999
14						1.000000	.999999
15							1.000000

Table 6.2.7. The exact probability distributions of $\min_i (\min(r_i, n-r_i))$ when comparing k = 3 treatments against a control

$\min_i (\min (r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7	8
0	.833333	.534722	.304398	.164014	.085881	.044232	.022550
1	.166667	.465278	.629630	.560137	.405382	.263734	.161401
2			.065972	.275849	.471764	.505554	.425863
3					.036973	.186476	.365745
4							.024433

$\min_i (\min (r_i, n-r_i))$ equal to	Number of replications (n) equal to 9	10	11	12	13	14	15
0	.011421	.005760	.002896	.001454	.000729	.000364	.000183
1	.095015	.054458	.030610	.016954	.009283	.005037	.002713
2	.314398	.214493	.139003	.086887	.052867	.031499	.018456
3	.442601	.414745	.335898	.248657	.173439	.115979	.075135
4	.136565	.292856	.386110	.391677	.340722	.269001	.198981
5		.017689	.105482	.240803	.338347	.364985	.336164
6				.013569	.084613	.202301	.298556
7						.010834	.069812

Table 6.2.8. The exact cumulative probability distributions of $\min_i$ ($\min(r_i, n-r_i)$) when comparing k = 3 treatments against a control

$\min_i(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7	8
0	.833333	.534722	.304398	.164014	.085881	.044232	.022550
1	1.000000	1.000000	.934028	.724151	.491263	.307966	.183951
2			1.000000	1.000000	.963027	.813520	.609814
3					1.000000	1.000000	.975559
4							1.000000

$\min_i(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 9	10	11	12	13	14	15
0	.011421	.005760	.002896	.001454	.000729	.000364	.000183
1	.106436	.060218	.033506	.018404	.010012	.005401	.002896
2	.420834	.274711	.172509	.105295	.062879	.036900	.021352
3	.863435	.689456	.508407	.353952	.236318	.152879	.096487
4	1.000000	.982312	.894517	.745629	.577040	.421880	.295468
5		1.000000	1.000000	.986432	.915387	.786865	.631632
6				1.000000	1.000000	.989166	.930188
7						1.000000	1.000000

Table 6.2.9. The exact probability distributions of minimum r_i when comparing $k = 4$ treatments against a control

Minimum r_i equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.543333	.332278	.189951	.103895	.055182	.028738
1	.416667	.499722	.457824	.357304	.251050	.164011
2	.040000	.160000	.296875	.378706	.385724	.338165
3		.008000	.053750	.143092	.242413	.312153
4			.001600	.016683	.060662	.132007
5				.000320	.004906	.023527
6					.000064	.001386
7						.000013

Table 6.2.10. The exact cumulative probability distributions of minimum r_i when comparing $k = 4$ treatments against a control

Minimum r_i equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.543333	.332278	.189951	.103895	.055182	.027738
1	.960000	.832000	.647775	.461199	.306232	.192749
2	1.000000	.992000	.944650	.839905	.691956	.530914
3		1.000000	.998400	.982997	.934369	.843067
4			1.000000	.999680	.995031	.975074
5				1.000000	.999937	.998601
6					1.000000	.999987
7						1.000000

Table 6.2.11. The exact probability distributions of $\min_i$ $(\min(r_i, n-r_i))$, when comparing k = 4 treatments against a control

$\min_i (\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.893333	.621667	.371743	.206340	.110115	.057434
1	.106667	.378333	.598296	.603283	.470721	.320254
2			.029961	.190377	.405668	.509094
3					.013496	.113218

Table 6.2.12. The exact cumulative probability distributions of $\min_i$ $(\min(r_i, n-r_i))$ when comparing k = 4 treatments against a control

$\min_i (\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.893333	.621667	.371743	.206340	.110115	.057434
1	1.000000	1.000000	.970039	.809623	.580836	.377688
2			1.000000	1.000000	.986504	.886782
3					1.000000	1.000000

Table 6.2.13. Critical values of minimum r_i for comparison of k treatments against one control in n sets of observations: a one-tailed critical region with an experiment-wise error rate

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
4	.25	0 (.113)[a]	0 (.154)	0 (.190)	0	-	-	-	-
	.20	0 (.113)	0 (.154)	0 (.190)	-	-	-	-	-
	.15	0 (.113)	-	-	-	-	-	-	-
	.10	-	-	-	-	-	-	-	-
	.05	-	-	-	-	-	-	-	-
5	.25	0 (.058)	0 (.082)	0 (.104)	0	0	0	0	-
	.20	0 (.058)	0 (.082)	0 (.104)	0	0	-	-	-
	.15	0 (.058)	0 (.082)	0 (.104)	0	-	-	-	-
	.10	0 (.058)	0 (.082)	-	-	-	-	-	-
	.05	-	-	-	-	-	-	-	-
6	.25	1 (.191)	0 (.043)	0 (.055)	0	0	0	0	0
	.20	1 (.191)	0 (.043)	0 (.055)	0	0	0	0	0
	.15	0 (.030)	0 (.043)	0 (.055)	0	0	0	0	0
	.10	0 (.030)	0 (.043)	0 (.055)	0	0	-	-	-
	.05	0 (.030)	0 (.043)	-	-	-	-	-	-
7	.25	1 (.113)	1 (.156)	1 (.193)	1	0	0	0	0
	.20	1 (.113)	1 (.156)	1 (.193)	0	0	0	0	0
	.15	1 (.113)	0 (.022)	0 (.029)	0	0	0	0	0
	.10	0 (.015)	0 (.022)	0 (.029)	0	0	0	0	0
	.05	0 (.015)	0 (.022)	0 (.029)	0	-	-	-	-
8	.25	2 (.246)	1 (.092)	1	1	1	1	1	1
	.20	1 (.066)	1 (.092)	1	1	1	1	0	0
	.15	1 (.066)	1 (.092)	1	0	0	0	0	0
	.10	1 (.066)	1 (.092)	0	0	0	0	0	0
	.05	0 (.008)	0 (.011)	0	0	0	0	0	0
9	.25	2 (.159)	2 (.216)	1	1	1	1	1	1
	.20	2 (.159)	1 (.053)	1	1	1	1	1	1
	.15	1 (.037)	1 (.053)	1	1	1	1	1	1
	.10	1 (.037)	1 (.053)	1	1	0	0	0	0
	.05	1 (.037)	0 (.006)	0	0	0	0	0	0
10	.25	2 (.100)	2 (.139)	2	2	2	1	1	1
	.20	2 (.100)	2 (.139)	2	1	1	1	1	1
	.15	2 (.100)	2 (.139)	1	1	1	1	1	1
	.10	1 (.021)	1 (.030)	1	1	1	1	1	1
	.05	1 (.021)	1 (.030)	1	0	0	0	0	0

[a]( ) Exact cumulative probability

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
11	.25	3 (.198)[a]	2 (.087)	2	2	2	2	2	2
	.20	3 (.198)	2 (.087)	2	2	2	2	2	1
	.15	2 (.061)	2 (.087)	2	2	1	1	1	1
	.10	2 (.061)	2 (.087)	1	1	1	1	1	1
	.05	1 (.011)	1 (.017)	1	1	1	1	0	0
12	.25	3 (.131)	3 (.180)	3	2	2	2	2	2
	.20	3 (.131)	3 (.180)	2	2	2	2	2	2
	.15	3 (.131)	2 (.053)	2	2	2	2	2	2
	.10	2 (.037)	2 (.053)	2	2	1	1	1	1
	.05	2 (.037)	1 (.009)	1	1	1	1	1	1
13	.25	4 (.230)	3 (.119)	3	3	3	3	2	2
	.20	3 (.085)	3 (.119)	3	3	2	2	2	2
	.15	3 (.085)	3 (.119)	2	2	2	2	2	2
	.10	3 (.085)	2 (.031)	2	2	2	2	2	2
	.05	2 (.022)	2 (.031)	2	1	1	1	1	1
14	.25	4 (.160)	4 (.216)	3	3	3	3	3	3
	.20	4 (.160)	3 (.077)	3	3	3	3	3	3
	.15	3 (.054)	3 (.077)	3	3	3	2	2	2
	.10	3 (.054)	3 (.077)	2	2	2	2	2	2
	.05	2 (.013)	2 (.018)	2	2	2	2	1	1
15	.25	4 (.108)	4 (.149)	4	4	3	3	3	3
	.20	4 (.108)	4 (.149)	4	3	3	3	3	3
	.15	4 (.108)	4 (.149)	3	3	3	3	3	3
	.10	3 (.034)	3 (.048)	3	3	3	2	2	2
	.05	3 (.034)	3 (.034)	2	2	2	2	2	2
16	.25	4 (.185)	5	4	4	4	4	4	4
	.20	5 (.185)	4	4	4	4	4	3	3
	.15	4 (.072)	4	4	3	3	3	3	3
	.10	4 (.072)	3	3	3	3	3	3	3
	.05	3 (.021)	3	3	3	2	2	2	2
17	.25	5 (.129)	5	5	4	4	4	4	4
	.20	5 (.129)	5	4	4	4	4	4	4
	.15	5 (.129)	4	4	4	4	4	3	3
	.10	4 (.046)	4	4	3	3	3	3	3
	.05	4 (.046)	3	3	3	3	3	2	2

[a]( ) Exact cumulative probability.

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
18	.25	6 (.206)[a]	5	5	5	5	5	4	4
	.20	5 (.089)	5	5	5	4	4	4	4
	.15	5 (.089)	5	4	4	4	4	4	4
	.10	5 (.089)	4	4	4	4	4	3	3
	.05	4 (.029)	4	3	3	3	3	3	3
19	.25	6 (.149)	6	5	5	5	5	5	5
	.20	6 (.149)	5	5	5	5	5	5	4
	.15	6 (.149)	5	5	5	4	4	4	4
	.10	5 (.060)	5	4	4	4	4	4	4
	.05	4 (.019)	4	4	4	3	3	3	3
20	.25	7 (.227)	6	6	6	6	5	5	5
	.20	6 (.105)	6	6	5	5	5	5	5
	.15	6 (.105)	6	5	5	5	5	5	5
	.10	5 (.039)	5	5	5	4	4	4	4
	.05	5 (.039)	4	4	4	4	4	3	3
21	.25	7 (.168)	7	6	6	6	6	6	6
	.20	7 (.168)	6	6	6	6	5	5	5
	.15	6 (.073)	6	6	5	5	5	5	5
	.10	6 (.073)	5	5	5	5	5	5	5
	.05	5 (.026)	5	5	4	4	4	4	4
22	.25	8 (.245)	7	7	7	6	6	6	6
	.20	7 (.121)	7	6	6	6	6	6	6
	.15	7 (.121)	6	6	6	6	6	5	5
	.10	6 (.050)	6	6	5	5	5	5	5
	.05	6 (.050)	5	5	5	4	4	4	4
23	.25	8 (.185)	8	7	7	7	7	6	6
	.20	8 (.185)	7	7	7	6	6	6	6
	.15	7 (.086)	7	6	6	6	6	6	6
	.10	7 (.086)	6	6	6	6	5	5	5
	.05	6 (.033)	6	5	5	5	5	5	5
24	.25	8 (.136)	8	8	7	7	7	7	7
	.20	8 (.136)	8	7	7	7	7	7	6
	.15	8 (.136)	7	7	7	6	6	6	6
	.10	7 (.060)	7	6	6	6	6	6	6
	.05	6 (.022)	6	6	5	5	5	5	5

[a]( )Exact cumulative probability.

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
25	.25	9	8	8	8	8	7	7	7
	.20	8	8	8	7	7	7	7	7
	.15	8	8	7	7	7	7	7	7
	.10	7	7	7	7	6	6	6	6
	.05	7	6	6	6	6	6	5	5
26	.25	9	9	8	8	8	8	8	8
	.20	9	8	8	8	8	8	7	7
	.15	8	8	8	7	7	7	7	7
	.10	8	7	7	7	7	7	7	6
	.05	7	7	6	6	6	6	6	6
27	.25	10	9	9	9	8	8	8	8
	.20	9	9	9	8	8	8	8	8
	.15	9	8	8	8	8	8	7	7
	.10	8	8	8	7	7	7	7	7
	.05	7	7	7	7	6	6	6	6
28	.25	10	10	9	9	9	9	9	9
	.20	10	9	9	9	9	8	8	8
	.15	9	9	9	8	8	8	8	8
	.10	9	8	8	8	8	7	7	7
	.05	8	8	7	7	7	7	7	6
29	.25	11	10	10	10	9	9	9	9
	.20	10	10	9	9	9	9	9	9
	.15	10	9	9	9	9	8	8	8
	.10	9	9	8	8	8	8	8	8
	.05	8	8	8	7	7	7	7	7
30	.25	11	11	10	10	10	10	9	9
	.20	11	10	10	10	9	9	9	9
	.15	10	10	9	9	9	9	9	9
	.10	10	9	9	9	8	8	8	8
	.05	9	8	8	8	8	8	7	7
31	.25	12	11	11	10	10	10	10	10
	.20	11	11	10	10	10	10	9	9
	.15	11	10	10	10	9	9	9	9
	.10	10	10	9	9	9	9	9	8
	.05	9	9	8	8	8	8	8	8

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
32	.25	12	11	11	11	11	10	10	10
	.20	12	11	11	10	10	10	10	10
	.15	11	11	10	10	10	10	9	9
	.10	10	10	10	9	9	9	9	9
	.05	9	9	9	9	8	8	8	8
33	.25	12	12	12	11	11	11	11	11
	.20	12	12	11	11	11	10	10	10
	.15	12	11	11	10	10	10	10	10
	.10	11	10	10	10	10	9	9	9
	.05	10	10	9	9	9	9	8	8
34	.25	13	12	12	12	11	11	11	11
	.20	12	12	12	11	11	11	11	11
	.15	12	11	11	11	11	10	10	10
	.10	11	11	10	10	10	10	10	10
	.05	10	10	10	9	9	9	9	9
35	.25	13	13	12	12	12	12	12	11
	.20	13	12	12	12	12	11	11	11
	.15	12	12	12	11	11	11	11	11
	.10	12	11	11	11	10	10	10	10
	.05	11	10	10	10	10	9	9	9
36	.25	14	13	13	13	12	12	12	12
	.20	13	13	12	12	12	12	12	11
	.15	13	12	12	12	11	11	11	11
	.10	12	12	11	11	11	11	11	10
	.05	11	11	10	10	10	10	10	10
37	.25	14	14	13	13	13	13	12	12
	.20	14	13	13	13	12	12	12	12
	.15	13	13	12	12	12	12	12	11
	.10	13	12	12	11	11	11	11	11
	.05	12	11	11	11	10	10	10	10
38	.25	15	14	14	13	13	13	13	13
	.20	14	14	13	13	13	13	12	12
	.15	14	13	13	13	12	12	12	12
	.10	13	13	12	12	12	12	11	11
	.05	12	12	11	11	11	11	10	10

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
39	.25	15	15	14	14	14	14	13	13
	.20	15	14	14	13	13	13	13	13
	.15	14	14	13	13	13	13	12	12
	.10	14	13	13	12	12	12	12	12
	.05	12	12	12	11	11	11	11	11
40	.25	16	15	15	14	14	14	14	14
	.20	15	15	14	14	14	13	13	13
	.15	15	14	14	13	13	13	13	13
	.10	14	13	13	13	13	12	12	12
	.05	13	12	12	12	12	11	11	11
41	.25	16	15	15	15	15	14	14	14
	.20	16	15	15	14	14	14	14	14
	.15	15	15	14	14	14	13	13	13
	.10	14	14	13	13	13	13	13	13
	.05	13	13	13	12	12	12	12	12
42	.25	17	16	16	15	15	15	15	14
	.20	16	16	15	15	15	14	14	14
	.15	16	15	15	14	14	14	14	14
	.10	15	14	14	14	13	13	13	13
	.05	14	13	13	13	12	12	12	12
43	.25	17	16	16	16	15	15	15	15
	.20	17	16	16	15	15	15	15	14
	.15	16	15	15	15	14	14	14	14
	.10	15	15	14	14	14	14	13	13
	.05	14	14	13	13	13	13	12	12
44	.25	18	17	16	16	16	16	15	15
	.20	17	16	16	16	15	15	15	15
	.15	16	16	15	15	15	15	15	14
	.10	16	15	15	14	14	14	14	14
	.05	15	14	14	13	13	13	13	13
45	.25	18	17	17	17	16	16	16	16
	.20	17	17	16	16	16	16	15	15
	.15	17	16	16	16	15	15	15	15
	.10	16	16	15	15	15	14	14	14
	.05	15	14	14	14	14	13	13	13

Table 6.2.13 (continued)

n	Level of significance for min r_i	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
46	.25	18	18	17	17	17	17	16	16
	.20	18	17	17	17	16	16	16	16
	.15	17	17	16	16	16	16	15	15
	.10	17	16	16	15	15	15	15	15
	.05	15	15	15	14	14	14	14	14
47	.25	19	18	18	17	17	17	17	17
	.20	18	18	17	17	17	17	16	16
	.15	18	17	17	16	16	16	16	16
	.10	17	16	16	16	15	15	15	15
	.05	16	15	15	15	14	14	14	14
48	.25	19	19	18	18	18	17	17	17
	.20	19	18	18	17	17	17	17	17
	.15	18	18	17	17	17	16	16	16
	.10	17	17	16	16	16	16	16	15
	.05	16	16	15	15	15	15	14	14
49	.25	20	19	19	18	18	18	18	18
	.20	19	19	18	18	18	17	17	17
	.15	19	18	18	17	17	17	17	17
	.10	18	17	17	17	16	16	16	16
	.05	17	16	16	16	15	15	15	15
50	.25	20	19	19	19	19	18	18	18
	.20	20	19	19	18	18	18	18	17
	.15	19	18	18	18	17	17	17	17
	.10	18	18	17	17	17	17	16	16
	.05	17	17	16	16	16	16	14	15

Table 6.2.14. Critical values of $\min_i(\min(r_i, n-r_i))$ for comparison of k treatments against one control in n sets of observations: a two-tailed critical region with an experiment-wise error rate

n	Level of significance for $\min_i(\min(r_i,n-r_i))$	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
6	.10	0 (.060)[a]	0 (.085)	-	-	-	-	-	-
	.05	-	-	-	-	-	-	-	-
	.01	-	-	-	-	-	-	-	-
7	.10	0 (.030)	0 (.044)	0 (.057)	-	-	-	-	-
	.05	0 (.030)	-	-	-	-	-	-	-
	.01	-	-	-	-	-	-	-	-
8	.10	0 (.015)	0 (.023)	0	0	0	0	0	0
	.05	0 (.015)	0 (.023)	0	-	-	-	-	-
	.01	-	-	-	-	-	-	-	-
9	.10	1 (.074)	0 (.011)	0	0	0	0	0	0
	.05	0 (.008)	0 (.011)	0	0	0	0	-	-
	.01	-	-	-	-	-	-	-	-
10	.10	1 (.041)	1 (.060)	1	0	0	0	0	0
	.05	1 (.041)	0 (.006)	0	0	0	0	0	0
	.01	0 (.004)	0 (.006)	-	-	-	-	-	-
11	.10	1 (.023)	1 (.034)	1	1	1	1	0	0
	.05	1 (.023)	1 (.034)	0	0	0	0	0	0
	.01	0 (.002)	0 (.003)	0	-	-	-	-	-
12	.10	2 (.073)	1 (.018)	1	1	1	1	1	1
	.05	1 (.012)	1 (.018)	1	1	0	0	0	0
	.01	0 (.001)	0 (.001)	0	0	0	0	-	-
13	.10	2 (.043)	2 (.063)	2	1	1	1	1	1
	.05	2 (.043)	1 (.011)	1	1	1	1	1	1
	.01	1 (.007)	0 (.001)	0	0	0	0	0	0
14	.10	2 (.025)	2 (.037)	2	2	2	2	1	1
	.05	2 (.025)	2 (.037)	1	1	1	1	1	1
	.01	1 (.004)	1 (.005)	0	0	0	0	0	0

[a]( ) Exact cumulative probability

Table 6.2.14 (continued)

n	Level of significance for $\min_i(\min(r_i, n-r_i))$	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
15	.10	3 (.067)	3 (.096)	2	2	2	2	2	2
	.05	2 (.014)	2 (.021)	2	2	1	1	1	1
	.01	1 (.002)	1 (.003)	1	1	0	0	0	0
16	.10	3 (.041)	3	3	2	2	2	2	2
	.05	3 (.041)	2	2	2	2	2	2	2
	.01	2 (.008)	1	1	1	1	1	1	0
17	.10	4 (.093)	3	3	3	3	3	2	2
	.05	3 (.024)	3	2	2	2	2	2	2
	.01	2 (.005)	1	1	1	1	1	1	1
18	.10	4 (.059)	4	3	3	3	3	3	3
	.05	3 (.015)	3	3	3	2	2	2	2
	.01	2 (.003)	2	2	1	1	1	1	1
19	.10	4 (.037)	4	4	4	3	3	3	3
	.05	4 (.037)	3	3	3	3	3	3	3
	.01	3 (.009)	2	2	2	2	2	2	1
20	.10	5 (.079)	4	4	4	4	4	4	3
	.05	4 (.023)	4	3	3	3	3	3	3
	.01	3 (.005)	2	2	2	2	2	2	2
21	.10	5 (.051)	5	4	4	4	4	4	4
	.05	4 (.014)	4	4	4	4	3	3	3
	.01	3 (.003)	3	3	2	2	2	2	2
22	.10	6 (.099)	5	5	5	5	4	4	4
	.05	5 (.033)	4	4	4	4	4	4	4
	.01	4 (.009)	3	3	3	3	3	2	2
23	.10	6 (.066)	5	5	5	5	5	5	5
	.05	5 (.021)	5	5	4	4	4	4	4
	.01	4 (.005)	3	3	3	3	3	3	3
24	.10	6 (.043)	6	6	5	5	5	5	5
	.05	6 (.043)	5	5	5	5	5	4	4
	.01	4 (.003)	4	4	3	3	3	3	3
25	.10	7	6	6	6	6	6	5	5
	.05	6	6	5	5	5	5	5	5
	.01	4	4	4	4	4	4	3	3

Table 6.2.14 (continued)

n	Level of significance for $\min_i(\min(r_i, n-r_i))$	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
26	.10	7	7	6	6	6	6	6	6
	.05	6	6	6	6	5	5	5	5
	.01	5	5	4	4	4	4	4	4
27	.10	7	7	7	7	6	6	6	6
	.05	7	6	6	6	6	6	6	5
	.01	5	5	5	5	4	4	4	4
28	.10	8	7	7	7	7	7	7	6
	.05	7	7	6	6	6	6	6	6
	.01	6	5	5	5	5	5	5	4
29	.10	8	8	8	7	7	7	7	7
	.05	8	7	7	7	6	6	6	6
	.01	6	6	5	5	5	5	5	5
30	.10	9	8	8	8	8	7	7	7
	.05	8	8	7	7	7	7	7	7
	.01	6	6	6	6	5	5	5	5
31	.10	9	9	8	8	8	8	8	8
	.05	8	8	8	7	7	7	7	7
	.01	7	6	6	6	6	6	6	5
32	.10	10	9	9	9	8	8	8	8
	.05	9	8	8	8	8	7	7	7
	.01	7	7	6	6	6	6	6	6
33	.10	10	10	9	9	9	9	9	8
	.05	9	9	8	8	8	8	8	8
	.01	7	7	7	7	7	6	6	6
34	.10	10	10	10	9	9	9	9	9
	.05	10	9	9	9	8	8	8	8
	.01	8	7	7	7	7	7	7	7
35	.10	11	10	10	10	10	9	9	9
	.05	10	9	9	9	9	9	9	8
	.01	8	8	8	7	7	7	7	7
36	.10	11	11	10	10	10	10	10	10
	.05	10	10	10	9	9	9	9	9
	.01	9	8	8	8	8	7	7	7

Table 6.2.14 (continued)

n	Level of significance for $\min_i(\min(r_i, n-r_i))$	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
37	.10	12	11	11	11	10	10	10	10
	.05	11	10	10	10	10	9	9	9
	.01	9	9	8	8	8	8	8	8
38	.10	12	12	11	11	11	11	10	10
	.05	11	11	10	10	10	10	10	10
	.01	9	9	9	9	8	8	8	8
39	.10	12	12	12	11	11	11	11	11
	.05	12	11	11	11	10	10	10	10
	.01	10	9	9	9	9	9	8	8
40	.10	13	12	12	12	12	11	11	11
	.05	12	12	11	11	11	11	11	10
	.01	10	10	9	9	9	9	9	9
41	.10	13	13	12	12	12	12	12	12
	.05	12	12	12	11	11	11	11	11
	.01	11	10	10	10	9	9	9	9
42	.10	14	13	13	13	12	12	12	12
	.05	13	12	12	12	12	11	11	11
	.01	11	11	10	10	10	10	10	9
43	.10	14	14	13	13	13	13	13	12
	.05	13	13	12	12	12	12	12	12
	.01	11	11	11	10	10	10	10	10
44	.10	15	14	14	13	13	13	13	13
	.05	14	13	13	13	12	12	12	12
	.01	12	11	11	11	11	10	10	10
45	.10	15	14	14	14	14	13	13	13
	.05	14	14	13	13	13	13	12	12
	.01	12	12	11	11	11	11	11	11
46	.10	15	15	15	14	14	14	14	14
	.05	14	14	14	13	13	13	13	13
	.01	13	12	12	12	11	11	11	11
47	.10	16	15	15	15	14	14	14	14
	.05	15	14	14	14	14	13	13	13
	.01	13	12	12	12	12	12	11	11

Table 6.2.14 (continued)

n	Level of significance for $\min_i(\min(r_i, n-r_i))$	k = number of treatments (excluding control) 2	3	4	5	6	7	8	9
48	.10	16	16	15	15	15	15	15	14
	.05	15	15	14	14	14	14	14	14
	.01	13	13	13	12	12	12	12	12
49	.10	17	16	16	16	15	15	15	15
	.05	16	15	15	15	14	14	14	14
	.01	14	13	13	13	13	12	12	12
50	.10	17	17	16	16	16	16	15	15
	.05	16	16	15	15	15	15	14	14
	.01	14	14	13	13	13	13	13	12

6.3. Tables of the Distributions of Minimum r_i and Minimum$_i$ $(\min(r_i, n-r_i))$ When Comparing All Pairs among k Treatments

Table 6.3.1. The exact probability distributions of minimum r_i when comparing all pairs among k = 3 treatments

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.83333	.52778	.30093	.16281	.08552	.04413
1	.16667	.44444	.54167	.46913	.34272	.22711
2		.02778	.15278	.32408	.41667	.40606
3			.00463	.04321	.14403	.26706
4				.00077	.01093	.05305
5					.00013	.00257
6						.00002

Table 6.3.1 (continued)

Minimum r_i equal to	Number of replications (n) equal to 7	8	9	10	11	12
0	.02252	.01141	.00576	.00290	.00145	.00073
1	.14151	.08458	.04907	.02785	.01556	.00857
2	.33585	.25022	.17366	.11456	.07277	.04490
3	.34596	.35728	.31802	.25570	.19118	.13544
4	.13640	.23230	.29982	.31990	.29867	.25327
5	.01718	.05902	.12973	.20831	.26701	.29060
6	.00058	.00507	.02252	.06261	.12389	.19048
7	[a]	.00012	.00139	.00780	.02688	.06475
8		[a]	.00003	.00036	.00250	.01049
9			[a]	.00001	.00009	.00075
10				[a]	[a]	.00002
11					[a]	[a]
12						[a]

[a]Probability less than .00001.

Table 6.3.1 (continued)

Minimum r_i equal to	Number of replications (n) equal to 13	14	15	16	17	18
0	.000365	.000183	.000091	.000046	.000023	.000011
1	.004679	.002533	.001362	.000728	.000388	.000205
2	.027076	.016030	.009348	.005383	.003068	.001733
3	.092049	.060545	.038784	.024310	.014964	.009071
4	.199974	.149510	.107089	.074101	.049841	.032738
5	.280413	.247264	.203516	.158719	.118556	.085488
6	.242288	.267066	.263923	.239744	.203873	.164483
7	.118782	.176552	.222883	.247742	.249227	.231718
8	.030372	.065978	.114262	.165291	.207172	.231576
9	.003782	.013000	.033167	.066604	.110236	.155941
10	.000215	.001276	.005152	.015294	.035403	.066833
11	.000005	.000059	.000408	.001910	.006543	.017347
12	[a]	.000001	.000015	.000124	.000669	.002622
13	[a]	[a]	[a]	.000004	.000036	.000222
14		[a]	[a]	[a]	[a]	.000010
15			[a]	[a]	[a]	[a]
16				[a]	[a]	[a]
17					[a]	[a]
18						[a]

[a]Probability less than .000001.

Table 6.3.1 (continued)

Minimum r_i equal to	Number of replications (n) equal to 19	20	21	22	23	24
0	.000006	.000003	.000001	.000001	a	a
1	.000109	.000057	.000030	.000016	.000008	.000004
2	.000971	.000541	.000299	.000165	.000090	.000049
3	.005427	.003211	.001881	.001092	.000629	.000360
4	.021078	.013340	.008318	.005120	.003116	.001877
5	.059861	.040890	.027344	.017954	.011601	.007391
6	.127151	.094876	.068718	.048525	.033523	.022720
7	.202288	.167811	.133484	.102515	.076420	.055526
8	.236166	.223691	.199258	.169409	.138051	.108668
9	.194144	.217835	.224538	.215916	.196072	.169777
10	.106625	.148015	.183132	.205998	.214146	.208508
11	.037196	.066792	.103363	.141187	.173677	.195681
12	.007910	.019174	.038637	.066569	.100397	.135223
13	.000992	.003384	.009227	.020791	.039796	.066218
14	.000071	.000357	.001368	.004174	.010478	.022221
15	.000003	.000022	.000123	.000525	.001786	.004975
16	a	a	.000006	.000040	.000193	.000727
17	a	a	a	.000002	.000013	.000068
18	a	a	a	a	.000001	.000004
19	a	a	a	a	a	a
20		a	a	a	a	a
21			a	a	a	a
22				a	a	a
23					a	a
24						a

[a]Probability less than .000001.

Table 6.3.2. The exact cumulative probability distributions of minimum r_i when comparing all pairs among k = 3 treatments

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.83333	.52778	.30093	.16281	.08552	.04413
1	1.00000	.97222	.84260	.63194	.42824	.27124
2		1.00000	.99538	.95602	.84491	.67730
3			1.00000	.99923	.98894	.94436
4				1.00000	.99987	.99741
5					1.00000	.99998
6						1.00000

Minimum r_i equal to	Number of replications (n) equal to 7	8	9	10	11	12
0	.02252	.01141	.00576	.00290	.00145	.00073
1	.16403	.09599	.05483	.03075	.01701	.00930
2	.49988	.34621	.22849	.14531	.08978	.05420
3	.84584	.70349	.54651	.40101	.28096	.18964
4	.98224	.93579	.84633	.72091	.57963	.44291
5	.99942	.99481	.97606	.92922	.84664	.73351
6	[a]	.99988	.99858	.99183	.97053	.92399
7	1.00000	[a]	.99997	.99963	.99741	.98874
8		1.00000	[a]	.99999	.99991	.99923
9			1.00000	[a]	[a]	.99998
10				1.00000	[a]	[a]
11					1.00000	[a]
12						1.00000

[a] Greater than .99999.

Table 6.3.2 (continued)

Minimum r_i equal to	Number of replications (n) equal to 13	14	15	16	17	18
0	.000365	.000183	.000091	.000046	.000023	.000011
1	.005044	.002716	.001453	.000774	.000411	.000216
2	.032120	.018746	.010801	.006157	.003479	.001949
3	.124169	.079291	.049585	.030467	.018443	.011020
4	.324143	.228801	.156674	.104568	.068284	.043758
5	.604556	.476065	.360190	.263287	.186840	.129246
6	.846844	.743131	.624113	.503031	.390713	.293729
7	.965626	.919683	.846996	.750773	.639940	.525447
8	.995998	.985661	.961258	.916064	.847112	.757023
9	.999780	.998661	.994425	.982668	.957348	.912964
10	.999995	.999937	.999577	.997962	.992751	.979797
11	[a]	.999996	.999985	.999872	.999294	.997144
12	[a]	[a]	[a]	.999996	.999963	.999766
13	1.000000	[a]	[a]	[a]	.999999	.999988
14		1.000000	[a]	[a]	[a]	.999998
15			1.000000	[a]	[a]	[a]
16				1.000000	[a]	[a]
17					1.000000	[a]
18						1.000000

[a]Greater than .999999.

Table 6.3.2 (continued)

Minimum r_i equal to	Number of replications (n) equal to 19	20	21	22	23	24
0	.000006	.000003	.000001	.000001	[a]	[a]
1	.000115	.000060	.000031	.000017	.000008	.000004
2	.001086	.000601	.000330	.000182	.000098	.000053
3	.006513	.003812	.002211	.001274	.000727	.000413
4	.027591	.017152	.010529	.006394	.003843	.002290
5	.087452	.058042	.037873	.024348	.015444	.009681
6	.214603	.152918	.106591	.072873	.048967	.032401
7	.416891	.320729	.240075	.175388	.125387	.087927
8	.653057	.544420	.439603	.344797	.263438	.196595
9	.847201	.762255	.664141	.560713	.459510	.366372
10	.953826	.910270	.847273	.766711	.673656	.574880
11	.991022	.977063	.950636	.907898	.847333	.770561
12	.998932	.996236	.989273	.974467	.947730	.905784
13	.999927	.999620	.998500	.995258	.987526	.972002
14	.999995	.999977	.999868	.999432	.998004	.994223
15	.999998	.999999	.999991	.999957	.999790	.999198
16	[b]	[b]	.999997	.999997	.999983	.999925
17	[b]	[b]	[b]	.999999	.999996	.999993
18	[b]	[b]	[b]	[b]	.999997	.999997
19	1.000000	[b]	[b]	[b]	[b]	[b]
20		1.000000	[b]	[b]	[b]	[b]
21			1.000000	[b]	[b]	[b]
22				1.000000	[b]	[b]
23					1.000000	[b]
24						1.000000

[a] Less than 8.000001.

[b] Greater than .999999.

Table 6.3.3. The exact probability distributions of $\min_i$ $(\min(r_i, n-r_i))$ when comparing all pairs among k = 3 treatments

$\min_i$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.83333	.52778	.30093	.16281	.08552	.04413
1	.16667	.47222	.69213	.55170	.39815	.26001
2			.00694	.28549	.47646	.50007
3					.03987	.19579

$\min_i$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 8	9	10	11	12	13
0	.02252	.01141	.00576	.00290	.00145	.00073
1	.15987	.09446	.05426	.03055	.01693	.00928
2	.41769	.30864	.21148	.13765	.08634	.05266
3	.37329	.44063	.40743	.32910	.24431	.17116
4	.02663	.14486	.30168	.38708	.38602	.33375
5			.01939	.11272	.25001	.34150
6					.01494	.09093

Table 6.3.3 (continued)

Minimum over i $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 14	15	16	17	18	19
0	.000365	.000183	.000091	.000046	.000023	.000011
1	.005035	.002712	.001452	.000773	.000410	.000217
2	.031425	.018430	.010657	.006091	.003447	.001935
3	.114914	.074677	.047291	.029326	.017874	.010736
4	.263708	.195789	.138830	.095001	.063183	.041051
5	.361175	.329614	.273858	.213179	.158117	.112951
6	.211431	.303240	.336199	.320690	.277748	.224915
7	.011949	.075355	.181777	.271124	.312480	.309285
8			.009844	.063769	.158426	.244022
9					.008292	.054876

Table 6.3.3 (continued)

Minimum ($\min(r_i, n-r_i)$) equal to	Number of replications (n) equal to 20	21	22	23	24
0	.000006	.000003	.000001	[a]	[a]
1	.000114	.000060	.000031	.000016	.000009
2	.001079	.000597	.000329	.000180	.000099
3	.006370	.003740	.002175	.001255	.000719
4	.026156	.016392	.010128	.006181	.003732
5	.078282	.052915	.035023	.022767	.014572
6	.173254	.128359	.092149	.064448	.044087
7	.277433	.232327	.184851	.141315	.104592
8	.290538	.296747	.274317	.236469	.193502
9	.139660	.220987	.270497	.283866	.269365
10	.007108	.047873	.124315	.201258	.252300
11			.006182	.042243	.111582
12					.005441

[a]Less than .000001.

Table 6.3.4. The exact cumulative probability distributions of $\underset{i}{\text{minimum}}$ $(\min(r_i, n-r_i))$ when comparing all pairs among $k = 3$ treatments

$\underset{i}{\text{Minimum}}$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 2	3	4	5	6	7
0	.83333	.52778	.30093	.16281	.08552	.04413
1	1.00000	1.00000	.99306	.71451	.48367	.30414
2			1.00000	1.00000	.96013	.80421
3					1.00000	1.00000

$\underset{i}{\text{Minimum}}$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 8	9	10	11	12	13
0	.02252	.01141	.00576	.00290	.00145	.00073
1	.18239	.10587	.06002	.03345	.01838	.01001
2	.60008	.41451	.27150	.17110	.10472	.06267
3	.97337	.85514	.67893	.50020	.34903	.23383
4	1.00000	1.00000	.98061	.88728	.73505	.56758
5			1.00000	1.00000	.98506	.90908
6					1.00000	1.00000

Table 6.3.4 (continued)

Minimum (min(r_i, n-r_i)) equal to	Number of replications (n) equal to 14	15	16	17	18	19
0	.000365	.000183	.000091	.000046	.000023	.000011
1	.005400	.002895	.001543	.000819	.000433	.000228
2	.036825	.021325	.012200	.006910	.003880	.002163
3	.151739	.096002	.059491	.036236	.021754	.012899
4	.415447	.291791	.198321	.131237	.084937	.053950
5	.776622	.621405	.472179	.344416	.243054	.166901
6	.988053	.924645	.808378	.665106	.520802	.391816
7	1.000000	1.000000	.990155	.936230	.833282	.701101
8			1.000000	1.000000	.991708	.945123
9					1.000000	1.000000

Table 6.3.4 (continued)

Minimum $(\min_i(r_i, n-r_i)$ equal to	Number of replications (n) equal to 20	21	22	23	24
0	.000006	.000003	.000001	[a]	[a]
1	.000120	.000063	.000032	.000016	.000009
2	.001199	.000660	.000361	.000196	.000108
3	.007569	.004400	.002536	.001451	.000827
4	.033725	.020792	.012664	.007632	.004559
5	.112007	.073707	.047687	.030399	.019131
6	.285261	.202066	.139836	.094847	.063218
7	.562694	.434393	.324687	.236162	.167810
8	.853232	.731140	.599004	.472631	.361312
9	.992892	.952127	.869501	.756497	.630677
10	1.000000	1.000000	.993816	.957755	.882977
11			1.000000	1.000000	.994559
12					1.000000

[a]Less than .00001.

Table 6.3.5. The exact probability distributions of minimum r_i when comparing all pairs among k = 4 treatments

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.95833	.73785	.47866	.28098	.15544	.08292
1	.04167	.26042	.48676	.55647	.48457	.35951
2		.00174	.03451	.15923	.32709	.43103
3			.00007	.00332	.03264	.12162
4				[a]	.00027	.00490
5					[a]	.00002
6						[a]

[a]Less than .00001.

Table 6.3.6. The exact cumulative probability distributions of minimum r_i when comparing all pairs among k = 4 treatments

Minimum r_i equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	.95833	.73785	.47866	.28098	.15544	.08292
1	1.00000	.99827	.96542	.83745	.64001	.44243
2		1.00000	.99993	.99668	.96710	.87346
3			1.00000	.99999	.99974	.99508
4				1.00000	.99999	.99998
5					1.00000	.99999
6						1.00000

Table 6.3.7. The exact probability distributions of $\min_i$ $(\min(r_i, n-r_i))$ when comparing all pairs among $k = 4$ treatments

$\min_i$ $(\min(r_i, n-r_i))$ equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	1.0000	.95833	.73785	.47866	.28097	.15544
1		.04167	.26215	.51418	.61709	.54462
2				.00716	.10194	.29763
3						.00231

Table 6.3.8. The exact cumulative probability distributions of $\min_i$ $(\min(r_i, n-r_i))$ when comparing all pairs among $k = 4$ treatments

$\min_i$ $(\min(r_i, n-r_i)$ equal to	Number of replications (n) equal to 1	2	3	4	5	6
0	1.00000	.95833	.73785	.47866	.28097	.15544
1		1.00000	1.00000	.99284	.89806	.70006
2				1.00000	1.00000	.99769
3						1.00000

Table 6.3.9. Critical values of minimum r_i for comparison of all pairs among $k = 3$ treatments in n sets of observations

n	.25	.20	.15	.10	.05	.01
	Level of significance for minimum r_i					
4	0 (.163)[a]	0 (.163)	-	-	-	-
5	0 (.086)	0 (.086)	0 (.086)	0 (.086)	-	-
6	0 (.044)	0 (.044)	0 (.044)	0 (.044)	0 (.044)	-
7	1 (.164)	1 (.164)	0 (.023)	0 (.023)	0 (.023)	-
8	1 (.096)	1 (.096)	1 (.096)	1 (.096)	0 (.011)	-
9	2 (.228)	1 (.055)	1 (.055)	1 (.055)	0 (.006)	0 (.006)
10	2 (.145)	2 (.145)	2 (.145)	1 (.031)	1 (.031)	0 (.003)
11	2 (.090)	2 (.090)	2 (.090)	2 (.090)	1 (.017)	0 (.001)
12	3 (.190)	3 (.190)	2 (.054)	2 (.054)	1 (.009)	1 (.009)
13	3 (.124)	3 (.124)	3 (.124)	2 (.032)	2 (.032)	1 (.005)
14	4 (.229)	3 (.079)	3 (.079)	3 (.079)	2 (.019)	1 (.003)
15	4 (.157)	4 (.157)	3 (.050)	3 (.050)	3 (.050)	1 (.001)
16	4 (.105)	4 (.105)	4 (.105)	3 (.030)	3 (.030)	2 (.006)
17	5 (.187)	5 (.187)	4 (.068)	4 (.068)	3 (.018)	2 (.003)
18	5 (.129)	5 (.129)	5 (.129)	4 (.044)	4 (.044)	2 (.002)
19	6 (.215)	5 (.087)	5 (.087)	5 (.087)	4 (.028)	3 (.006)
20	6 (.153)	6 (.153)	5 (.058)	5 (.058)	4 (.017)	3 (.004)
21	7 (.240)	6 (.106)	6 (.106)	5 (.038)	5 (.038)	3 (.002)
22	7 (.175)	7 (.175)	6 (.073)	6 (.073)	5 (.024)	4 (.006)
23	7 (.125)	7 (.125)	7 (.125)	6 (.049)	6 (.049)	4 (.004)
24	8 (.197)	8 (.197)	7 (.088)	7 (.088)	6 (.032)	5 (.010)

[a]( ) Exact cumulative probability.

Table 6.3.10. Critical values of $\min_i$ (min(r_i, n-r_i)) for comparison of all pairs among k=3 treatments in n sets of observations

n	Level of significance for $\min_i$ (min(r_i, n-r_i)) .25	.20	.15	.10	.05	.01
4	-	-	-	-	-	-
5	0 (.163)*	0 (.163)	-	-	-	-
6	0 (.086)	0 (.086)	0 (.086)	0 (.086)	-	-
7	0 (.044)	0 (.044)	0 (.044)	0 (.044)	0 (.044)	-
8	1 (.182)	1 (.182)	0 (.023)	0 (.023)	0 (.023)	-
9	1 (.106)	1 (.106)	1 (.106)	0 (.011)	0 (.011)	-
10	1 (.060)	1 (.060)	1 (.060)	1 (.060)	0 (.006)	0 (.006)
11	2 (.171)	2 (.171)	1 (.033)	1 (.033)	1 (.033)	0 (.003)
12	2 (.105)	2 (.105)	2 (.105)	1 (.018)	1 (.018)	0 (.001)
13	3 (.234)	2 (.063)	2 (.063)	2 (.063)	1 (.010)	1 (.010)
14	3 (.152)	3 (.152)	2 (.037)	2 (.037)	2 (.037)	1 (.005)
15	3 (.096)	3 (.096)	3 (.096)	3 (.096)	2 (.021)	1 (.003)
16	4 (.198)	4 (.198)	3 (.059)	3 (.059)	2 (.012)	1 (.002)
17	4 (.131)	4 (.131)	4 (.131)	3 (.036)	3 (.036)	2 (.007)
18	5 (.243)	4 (.085)	4 (.085)	4 (.085)	3 (.022)	2 (.004)
19	5 (.167)	5 (.167)	5 (.167)	4 (.054)	4 (.054)	2 (.002)
20	5 (.112)	5 (.112)	5 (.112)	4 (.034)	4 (.034)	3 (.008)
21	6 (.202)	5 (.074)	5 (.074)	5 (.074)	4 (.021)	3 (.004)
22	6 (.140)	6 (.140)	6 (.140)	5 (.048)	5 (.048)	3 (.003)
23	7 (.236)	6 (.095)	6 (.095)	6 (.095)	5 (.030)	4 (.007)
24	7 (.168)	7 (.168)	6 (.063)	6 (.063)	5 (.019)	4 (.005)

[a]( ) Exact cumulative probability.

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