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STUDIES OF SOME PROBLEMS IN NONPARAMETRIC INFERENCE

by

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The asymptotic behavior of some nonparametric test criteria used in analysis of variance (one-way and two-way classification) under the Pitman type of alternative is considered. Step-down procedure is suggested for bivariate location parameter problem. Wald's test is used for testing hypotheses in the categorical set-up.

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INTRODUCTION

This thesis is concerned with the asymptotic behavior of some nonparametric tests used in the analysis of variance (one-way and two-way classifications), when the alternative hypothesis is of the Pitman type; with a generalization of Mood's test to test the hypothesis of the equality of k bivariate distributions which are identical except for the location parameters; and with the use of Wald's test to test certain hypotheses in the categorical set-up.

In the analysis of variance with one-way classification, suppose we have $\{x_{ij}\}$ $(i = 1, 2, \ldots, c; \; j = 1, 2, \ldots, n_i)$,(real valued) $N = \Sigma_{i=1}^{c} n_i$ independent observations and further, suppose for each fixed i, $F_i(x)$ is the cumulative distribution function of X_{ij}. Then we are interested in testing the hypothesis

$$H_o: \; F_1(x) = F_2(x) = \ldots = F_c(x) \qquad .$$

Various nonparametric tests based on either the ranks of the observations or the quantiles of the combined sample have been proposed by Wallis and Kruskal [29][1], Mood [19], Massey [17], Terpstra [25] and Bhapkar [4].

Very little work has been done on the power of these tests. Andrews [2] has investigated the asymptotic power of the Wallis-Kruskal test and Mood's test when the alternative hypothesis is of

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[1] The numbers in square brackets refer to the bibliography listed at the end.

the Pitman type i.e. the alternative hypothesis H_N is

$$H_N: \quad F_i(x) = F(x + N^{-1/2}\theta_i) \quad (i = 1, 2, \ldots, c),$$

where not all θ_i's are equal.

In Chapter I of this thesis, we consider the asymptotic distribution of Bhapkar's test criterion and Massey's test criterion when the alternative hypothesis H_N holds. These two tests are compared with other nonparametric tests mentioned above and with the usual F-test. For some specified choices of $F(x)$ and c, their asymptotic relative efficiencies are tabulated.

Similarly, in a complete two-way design with t treatments and b blocks, we assume that the distribution of the random variable X_{ij} $(i = 1, 2, \ldots, t; \; j = 1, 2, \ldots, b)$, is $F(x + \nu + \alpha_i + \beta_j)$, where ν is the general effect, α_i is the i-th treatment effect and β_j is the j-th block effect. In this situation, the hypothesis of the equality of treatment effects is

$$H_o: \quad \alpha_1 = \alpha_2 = \ldots = \alpha_t \quad .$$

Mood [19] and Friedman [11] have proposed nonparametric tests for testing this hypothesis H_o and their tests have been generalized to the case of balanced incomplete block designs by Bhapkar [4] and Durbin [9] respectively.

Chapter II of the present work, deals with the asymptotic distribution (as $b \longrightarrow \infty$) of Mood's test criterion, its generalization and Friedman's test criterion, when the alternative hypothesis is

$$H_b: \ \alpha_i = b^{-1/2} \Theta_i \quad ,$$

where not all Θ_i's are equal.

As in Chapter I, asymptotic comparisons of these tests with each other and with the usual F-test are made and for specified choices of F(x) and t, the asymptotic relative efficiencies are calculated.

In the next chapter an exponential form i.e. $\text{Exp}(\beta x)$ is assumed for the median of the conditional distribution of y given x, and for each x, the conditional distribution of y is assumed to be symmetric about this median. Under this model, a nonparametric test is proposed for testing the hypothesis that β has a specified value β_o. Also the step-down procedure is used for testing the equality of k bivariate distributions, where the model is one in which they differ only in their locations.

In the last chapter, Wald's test [28] is used for testing the hypotheses of two-by-two independence in a 2 x 2 x 2 table and the equality of two marginal distributions in an r x r table. Problems of symmetry in two-way and three-way tables, when each classification has the same number of categories are considered. The corresponding chi-square test statistic and the non-centrality parameters are given.

CHAPTER I

ON THE ASYMPTOTIC BEHAVIOR OF SOME NONPARAMETRIC TEST CRITERIA USED IN ANALYSIS OF VARIANCE (ONE-WAY CLASSIFICATION)

1.1 Introduction.

In the usual analysis of variance when the assumption of normality is not realistic, one usually resorts to nonparametric procedures where no functional form of the distribution is assumed. Various nonparametric tests have been proposed in analysis of variance with one-way classification and c samples $(c \geq 2)$.

Let $\{X_{ij}\}$ $(i = 1, 2, \ldots, c;\ j = 1, 2, \ldots, n_i)$ be $N = \sum_{i=1}^{c} n_i$ independent observations (real valued) and suppose X_{ij} has a cumulative distribution function $F_i(x)$. Then the null hypothesis to be tested is given by

$$(1.1.1) \qquad H_o\colon\ F_1(x) = F_2(x) = \ldots\ldots = F_c(x)\ ,$$

for all real values of x.

Wallis and Kruskal [29] have considered a test based on the ranks of observations in N-fold over-all sample. They have shown that the asymptotic distribution of their test statistic, under H_o, is a chi-square distribution with $c-1$ degrees of freedom. Terpstra [25] has generalized Wilcoxon's two-sample test [30] to the case of c samples by considering the Wilcoxon test statistic for all possible pairs of samples. His test statistic, also, has a limit chi-square distribution with $\binom{c}{2}$ degrees of freedom when H_o is true. Another generalization of Wilcoxon's test is proposed by Bhapkar [4], who considers c-plets of observations, one from each sample. Under H_o, he shows that his test criterion has a limit chi-square distribution with c-1 degrees of freedom.

Instead of considering the ranks of the observations, Mood [19] considers the number of observations in each sample which are less than or equal to the median of the combined sample. This idea is extended by Massey [17] who considers more than one quantile, say $h \geq 2$, and bases his test statistic on the number of observations from each sample which lie in each of the $h+1$ intervals formed by the h quantiles. Their tests reduce to the conventional χ^2 for testing independence in $2 \times c$ and $(h+1) \times c$ contingency tables respectively, when the null hypothesis holds.

So far very little work has been done on the power of these tests. Andrews [2] has investigated the asymptotic power, in Pitman's sense, of the Kruskal-Wallis test and of Mood's test when the alternative hypothesis H_N, is specified by

$$(1.1.2) \qquad H_N: \; F_i(x) = F(x + N^{-\frac{1}{2}} \theta_i) \; (i = 1, 2, \ldots, c),$$

for all real values of x and where not all θ_i's are equal and $F(x)$ is a continuous cumulative distribution function. He also, has made a comparison of these tests with the usual F-test of the analysis of variance.

In this chapter, we investigate the asymptotic power, in Pitman's sense, of Bhapkar's test and Massey's test, under H_N defined in (1.1.2). An asymptotic comparison of these tests with other nonparametric tests and the usual F-test is made. It should be noted that some work on the asymptotic efficiency of Massey's test (or one form of it) has been considered in a paper by Chakravarti, Leone and Alanen to be published in the Bulletin of the International Statistical Institute. But as far as we know they have dealt with the two-sample case.

1.2 Some Definitions and Known Results.

Definition 1.2.1: Let $\{X_{ij}\}$ $(i = 1, 2, \ldots, c;\ j = 1, 2, \ldots, n_i)$ for a fixed i, be independent and identically distributed random variables with cumulative distribution function $F_i(x)$. Let $m_1 \leq n_1$, $m_2 \leq n_2 \ldots\ m_c \leq n_c$. Further, let

$$(1.2.1) \qquad \Phi(u_1, u_2, \ldots u_{m_1};\ v_1 v_2, \ldots, v_{m_2}; \ldots\ldots\ldots;\ w_1, w_2, \ldots w_{m_c})$$

be a function symmetric in each set of its arguments. Then the statistic

$$(1.2.2) \qquad U_N = \binom{n_1}{m_1}^{-1} \binom{n_2}{m_2}^{-1} \cdots \binom{n_c}{m_c}^{-1}$$

$$\times\ \Sigma\ \Phi(X_{1\alpha_1}, \ldots, X_{1\alpha_{m_1}}; \ldots\ldots; \ldots\ldots; X_{c\delta_1} \ldots; X_{c\delta_{m_c}})$$

where the summation runs over all subscripts $\alpha, \beta, \ldots, \delta$ such that

$$\begin{aligned} 1 &\leq \alpha_1 < \alpha_2 \cdots < \alpha_{m_1} \leq n_1 \\ 1 &\leq \beta_1 < \beta_2 \cdots < \beta_{m_2} \leq n_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 &\leq \delta_1 < \delta_2 \cdots < \delta_{m_c} \leq n_c \end{aligned}$$

is called a generalized U-statistic.

Theorem 1.2.1 (Bhapkar [4] and Sukhatme [24]):

Let $p_1, p_2, \ldots, p_c$ be c fixed positive numbers and let $n_i = Np_i$ where $\sum_{i=1}^{c} p_i = 1$. Suppose we consider the vectors

$$\underline{\Phi}' = (\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(g)})$$

and

$$\underline{U}_N' = (U_N^{(1)}, U_N^{(2)}, \ldots, U_N^{(g)})$$

where $U_N^{(i)}$ is a generalized U-statistic corresponding to $\Phi^{(i)}$. Let

$$\underline{\eta}_N' = (E(U_N^{(1)}), E(U_N^{(2)}), \ldots, E(U_N^{(g)})) .$$

Then under the assumption of the existence of second moments of Φ's, the limit distribution of

$$\sqrt{N}\ (\underline{U}_N - \underline{\eta}_N) \tag{1.2.3}$$

is a g-variate normal distribution with zero means and variance co-variance matrix, $\Sigma = (\sigma_{ij})$ $(i,j = 1, 2, \ldots, g)$, where

$$\sigma_{ij} = \sum_{k=1}^{c} \frac{m_k^{(i)} m_k^{(j)}}{p_k}\ \xi(i,j)_{\substack{0,0,\ldots,1,0,\ldots,0 \\ 1 \text{ at the k-th place.}}} \tag{1.2.4}$$

and

$$\xi(i,j)_{\substack{0,0,\ldots,1,0,\ldots,0 \\ 1 \text{ at the k-th place}}} = E(\Phi_1^{(i)}\ \Phi_{2,k}^{(j)}) - E(\Phi_1^{(i)})\ E(\Phi_1^{(j)}) , \tag{1.2.5}$$

$$\Phi_1^{(u)} = \Phi^{(u)}(X_{11}, X_{12}, \ldots, X_{1m_1}; X_{21}, X_{22}, \ldots, X_{2m_2}; \ldots; X_{c1}, \ldots, X_{cm_c}) , \tag{1.2.6}$$

and $\Phi_{2,k}^{(u)}$ is obtained from $\Phi_1^{(u)}$ by replacing all the X's by X''s excepting X_{k1}, the primes denoting a new set of independent random variables. This is a generalization of Hoeffding's theorem on U-statistic [13] extended by Lehmann [15] to the case of two samples to the case of c samples and random vectors of U-statistics.

1.3 Bhapkar's Test Statistic.

Under the assumptions of the preceding section, let

$$(1.3.1) \quad \phi^{(i)}(X_{1\alpha_1}, X_{2\alpha_2}, \ldots, X_{c\alpha_c}) = 1 \quad \text{if} \quad X_{i,\alpha_i} = \min(X_{1\alpha_1}, X_{2\alpha_2} \ldots X_{c\alpha_c})$$

$$= 0 \quad \text{otherwise.}$$

$$(i = 1, 2, \ldots c).$$

Then

$$(1.3.2) \qquad U_N^{(i)} = \prod_{j=1}^{c} n_j^{-1} \sum_{\alpha_1=1}^{n_1} \sum_{\alpha_2=1}^{n_2} \cdots \sum_{\alpha_c=1}^{n_c} \phi^{(i)}(X_{1\alpha_1}, X_{2\alpha_2}, \ldots, X_{c\alpha_c}).$$

Let

$$(1.3.3) \qquad \bar{U}_N = \sum_{i=1}^{c} p_i \, U_N^{(i)} \quad ,$$

and

$$(1.3.4) \qquad X_B^2 = N(2c-1) \sum_{i=1}^{c} p_i (U_N^{(i)} - \bar{U}_N)^2 .$$

Then, under the null hypothesis H_o, defined in (1.1.1), the limit distribution of X_B^2 as $N \longrightarrow \infty$ is a chi-square distribution with c-1 degrees of freedom.

1.4 Asymptotic Distribution of X_B^2 Under H_N:

Theorem 1.4.1 Suppose

(i) for each positive integer N, H_N is given by

$$H_N: \quad F_i(x) = F(x + N^{-\frac{1}{2}}\Theta_i)$$

for all real x and not all Θ_i's are equal and $F(x)$ is an absolutely continuous cumulative distribution function with $f(x)$ as derivative with respect to Lebesgue measure,

(ii) $f(x)$ is differentiable and $f(x)$ and $f'(x)$ are bounded almost everywhere.

Then, the limit distribution of χ^2_B is a non-central chi-square distribution with $c-1$ degrees of freedom and non-centrality parameter λ^2_B, given by

$$\lambda^2_B = c^2(2c-1)\, I^2\, \sigma^2_\Theta \ , \tag{1.4.1}$$

where

$$I = \int_{-\infty}^{\infty} [1 - F(y)]^{c-2} f^2(y)\, dy \ , \tag{1.4.2}$$

and

$$\sigma^2_\Theta = \sum_{i=1}^{c} p_i(\Theta_i - \bar{\Theta})^2 \ , \qquad \bar{\Theta} = \sum_{i=1}^{c} p_i\, \Theta_i \ . \tag{1.4.3}$$

Proof: The proof of theorem 1.4.1 will be a direct application of theorem 1.2.1. In order to apply theorem 1.2.1, we need to calculate η_N and Σ when H_N is true. We shall neglect terms of $O(N^{-1})$ and $O(N^{-\frac{1}{2}})$ in calculating η_N and Σ respectively.

$$\begin{aligned} \eta_N^{(i)} &= E(U_N^{(i)}) = E(\Phi^{(i)}) \\ &= \Pr.\,(X_{i\alpha_i} < X_{j\alpha_j},\ j = 1, 2, \ldots, c \text{ and } j \neq i) \\ &= \int_{-\infty}^{\infty} \prod_{j\neq i}^{c} (1 - F_j(x))\, f_i(x)\, dx \\ &= \int_{-\infty}^{\infty} \prod_{j\neq i}^{c} (1 - F(x + N^{-\frac{1}{2}}\Theta_j)) f(x + N^{-\frac{1}{2}}\Theta_i)\, dx \end{aligned} \tag{1.4.4}$$

$$(1.4.4)\text{(cont.)} \quad = \int_{-\infty}^{\infty} \prod_{j\neq i}^{c} (1 - F(y + N^{-\frac{1}{2}}(\theta_j - \theta_i)))\, f(y)\, dy \; .$$

Using Taylor's series expansion, we get

$$F(y + N^{-\frac{1}{2}}(\theta_j - \theta_i)) = F(y) + N^{-\frac{1}{2}}(\theta_j - \theta_i)\, f(y) + O(N^{-1}) \; .$$

Therefore,

$$(1.4.5) \quad \prod_{j\neq i}^{c} [1 - F(y + N^{-\frac{1}{2}}(\theta_j - \theta_i))]$$

$$= \prod_{j\neq i}^{c} [1 - F(y) - N^{-\frac{1}{2}}(\theta_j - \theta_i) f(y) - O(N^{-1})]$$

$$= [1 - F(y)]^{c-1} - N^{-\frac{1}{2}} \sum_{j\neq i}^{c} (\theta_j - \theta_i)[1 - F(y)]^{c-2}$$

$$\times \; f(y) + O(N^{-1}) \; .$$

Substituting (1.4.5) in (1.4.4) and integrating, we have

$$(1.4.6) \quad \eta_N^{(i)} = \frac{1}{c} - N^{-\frac{1}{2}} \sum_{j\neq i}^{c} (\theta_j - \theta_i) I + O(N^{-1}) \; .$$

$$= \frac{1}{c} - N^{-\frac{1}{2}} (\theta - c\theta_i) I + O(N^{-1}) \; ,$$

where

$$\theta = \sum_{i=1}^{c} \theta_i \; .$$

Next,

$$\underset{\substack{0,0,0,1\ldots,0 \\ \text{1 at the i-th place}}}{\xi(i,i)} = E(\phi_1^{(i)} \phi_{2,i}^{(i)}) - E^2(\phi_1^{(i)}) \; ,$$

and

$$E(\Phi_1^{(i)}\Phi_2^{(i)}) = E[\Phi^{(i)}(X_{11},X_{21},X_{i1},X_{c1})\Phi^{(i)}(X_{12},X_{22},\dots,X_{i1},X_{c2})]$$

$$= \Pr.(X_{i1}<X_{11},\dots X_{i1}<X_{c1};X_{i1}<X_{12},\dots X_{i1}<X_{c2})$$

$$= \int_{-\infty}^{\infty} \prod_{j\neq i}^{c} [1 - F_j(x)]^2 f_i(x)\, dx$$

$$= \int_{-\infty}^{\infty} \prod_{j\neq i}^{c} [1 - F(y + N^{-\frac{1}{2}}(\Theta_j - \Theta_i))]^2 f(y)\, dy .$$

Using (1.4.5), we get

$$\prod_{j\neq i}[1 - F(y + N^{-\frac{1}{2}}(\Theta_j - \Theta_i))]^2 = (1 - F(y))^{2c-2} + O(N^{-\frac{1}{2}}) .$$

Hence

$$(1.4.7) \qquad E(\Phi_1^{(i)}\ \Phi_{2,i}^{(i)}) = \int_{-\infty}^{\infty}(1 - F(y))^{2c-2}f(y)dy + O(N^{-\frac{1}{2}})$$

$$= \frac{1}{2c-1} + O(N^{-\frac{1}{2}}) .$$

Also, from (1.4.6)

$$(1.4.8) \qquad E^2(\Phi_1^{(i)}) = \frac{1}{c^2} + O(N^{-\frac{1}{2}}) .$$

Therefore,

$$\underset{\substack{0,0,\dots 1,0,\dots 0\\ \text{1 at the i-th place}}}{\xi(i,i)} = \frac{(c-1)^2}{c^2(2c-1)} + O(N^{-\frac{1}{2}}) .$$

Further,

$$\underset{\substack{0,0,\dots,1,0,\dots,0\\ \text{1 at the k-th place}\\ k\neq i}}{\xi(i,i)} = E(\Phi_1^{(i)}\ \Phi_{2,k}^{(i)}) - E^2(\Phi_1^{(i)}) ,$$

and

$$E(\Phi_1^{(i)}\ \Phi_{2,k}^{(i)}) = E(\Phi_1^{(i)}(X_{11},\ldots,X_{i1},\ldots X_{k1},\ldots X_{c1})$$

$$\times\ \Phi_1^{(i)}(X_{12},\ldots,X_{i2},\ \ldots X_{k1},\ldots X_{c2}))$$

$$= \Pr.\ (X_{i1}<X_{11},\ldots X_{i1}<X_{k1},\ldots X_{i1}<X_{c1};X_{i2}<X_{12},\ldots X_{i2},\ldots$$

$$X_{i2}<X_{k1},\ldots X_{i2}<X_{c2})$$

$$= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{y} \prod_{j\neq i,k} [\,1-F_j(x)]\ f_i(x)\Big]^2 f_k(y)dy\ .$$

Now

$$\int_{-\infty}^{y} \prod_{j\neq i,k} [1-F_j(x)]\ f_i(x)dx = \int_{-\infty}^{y+N^{-\frac{1}{2}}\Theta_i} \prod_{j\neq i,k} [1-F(z+N^{-\frac{1}{2}}(\Theta_j-\Theta_i))]\ f(z)dz$$

$$= \int_{-\infty}^{y+N^{-\frac{1}{2}}\Theta_i} [1-F(z)]^{c-2} f(z)dz + O(N^{-\frac{1}{2}})$$

(1.4.9)
$$= \frac{1}{c-1}\ [1-(1-F(y))^{c-1}\,] + O(N^{-\frac{1}{2}}).$$

Therefore,

$$E(\Phi_1^{(i)}\ \Phi_{2,k}^{(i)}) = \frac{1}{(c-1)^2} \int_{-\infty}^{\infty} [1-(1-F(y))^{c-1}\,]^2 f_K(y)dy + O(N^{-\frac{1}{2}})$$

(1.4.10)
$$= \frac{1}{(c-1)^2}\ [1 - \frac{2}{c} + \frac{1}{2c-1}\,] + O(N^{-\frac{1}{2}})$$

$$= \frac{2}{c(2c-1)} + O(N^{-\frac{1}{2}})\ .$$

Using (1.4.8) and (1.4.10), we get

$$(1.4.11)\qquad \underset{\substack{0,0,\ldots,1,0,\ldots,0 \\ 1 \text{ at the k-th place} \\ k \neq i}}{\xi(i,i)} = \frac{2}{c(2c-1)} - \frac{1}{c^2} + O(N^{-\frac{1}{2}})$$

$$= \frac{1}{c^2(2c-1)} + O(N^{-\frac{1}{2}}) .$$

This is true for all $k \neq i$, therefore by (1.2.4)

$$(1.4.12)\qquad \sigma_{ii} = \frac{1}{c^2(2c-1)} \left[\frac{(c-1)^2}{p_i} + \sum_{k \neq i}^{c} \frac{1}{p_k} \right] .$$

Next,

$$\underset{\substack{0,0,\ldots,1,0,\ldots,0 \\ 1 \text{ at the i-th place}}}{\xi(i,j)} = E(\Phi_1^{(i)} \Phi_{2,i}^{(j)}) - E(\Phi_1^{(i)}) E(\Phi_1^{(j)}) ,$$

and

$$E(\Phi_1^{(i)}\Phi_{2,i}^{(j)}) = E(\Phi^{(i)}(X_{11},\ldots,X_{i1},\ldots X_{c1})$$

$$\times \Phi^{(j)}(X_{12},X_{22},\ldots X_{j2},\ldots X_{i1},\ldots X_{c2}))$$

$$= \Pr(X_{i1} < X_{11},\ldots X_{i1} < X_{c1}; X_{j2} < X_{12},\ldots X_{j2} < X_{i1},\ldots X_{j2} < X_{c2})$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y} \prod_{k \neq i,j} \left[1-F_k(z)\right] f_j(z)dz \right] \prod_{\ell \neq i} \left[1-F_\ell(y)\right]$$

$$\times \ f_i(y)dy$$

$$= \int_{-\infty}^{\infty} \frac{\left[1-(1-F(y))^{c-1}\right]}{c-1} \prod_{\ell \neq i} \left[1-F_\ell(y)\right] f_i(y)dy + O(N^{-\frac{1}{2}})$$

Using (1.4.9)

$$= \int_{-\infty}^{\infty} \frac{1}{c-1} \left[1-(1-F(y))^{c-1}\right](1-F(y))^{c-1} f(y)dy + O(N^{-\frac{1}{2}})$$

$$= \frac{1}{c(2c-1)} + O(N^{-\frac{1}{2}}) .$$

Hence,

(1.4.13) $$\xi(i,j)_{\substack{0,0,\dots,1,0,\dots,0 \\ 1 \text{ at the i-th place}}} = \left(\frac{1}{c(2c-1)} - \frac{1}{c^2}\right) + O(N^{-\frac{1}{2}})$$

$$= \frac{-(c-1)}{c^2(2c-1)} + O(N^{-\frac{1}{2}}) .$$

Similarly,

$$\xi(i,j)_{\substack{0,0,\dots,1,0,\dots,0 \\ 1 \text{ at the j-th place}}} = \frac{-(c-1)}{c^2(2c-1)} + O(N^{-\frac{1}{2}}) ,$$

$$\xi(i,j)_{\substack{0,0,\dots,1,0,\dots,0 \\ 1 \text{ at the k-th place} \\ k \neq i,\, j}} = E(\Phi_1^{(i)} \Phi_{2,k}^{(j)}) - E(\Phi_1^{(i)}) E(\Phi_1^{(j)}) .$$

Now

$$E(\Phi_1^{(i)} \Phi_{2,k}^{(j)}) = E(\Phi^{(i)}(X_{11},\dots,X_{i1},\dots,X_{k1},\dots X_{c1})$$

$$\Phi^{(j)}(X_{12},\dots,X_{j2},\dots,X_{k1},\dots,X_{c2})).$$

$$= \Pr(X_{i1} < X_{11},\dots,X_{i1} < X_{k1},\dots,X_{i1} < X_{c1};$$

$$X_{j2} < X_{12},\dots,X_{j2} < X_{k1} \ \dots \ X_{j2} < X_{c2}).$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y} \prod_{\ell \neq i,k} [1-F_\ell(u)] f_i(u)\,du\right]$$

$$\left[\int_{-\infty}^{y} \prod_{m \neq i,k} [1-F_m(v)] f_j(v)\,dv\right] \times f_k(y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{(c-1)^2} \left[1-(1-F(y))^{c-1}\right]^2 f_k(y)\,dy + O(N^{-\frac{1}{2}})$$

$$= \frac{2}{c(2c-1)} + O(N^{-\frac{1}{2}}) .$$

Hence

$$(1.4.14) \qquad \underset{\substack{0,0,\ldots,1,\ldots 0 \\ 1 \text{ at the k-th place} \\ k \neq i,\, j}}{\xi(i,j)} = \left[\frac{2}{c(2c-1)} - \frac{1}{c^2} \right] + O(N^{-\frac{1}{2}})$$

$$= \frac{1}{c^2(2c-1)} + O(N^{-\frac{1}{2}}) .$$

Thus, from (1.2.4) we have

$$(1.4.15) \qquad \sigma_{ij} = \frac{1}{c^2(2c-1)} \left[\sum_{k=1}^{c} \frac{1}{p_k} - \frac{c}{p_i} - \frac{c}{p_j} \right] + O(N^{-\frac{1}{2}}) .$$

Therefore, by theorem 1.2.1, the asymptotic distribution of

$$\sqrt{N}\,(\underline{U}_N - \underline{\eta}_N)$$

is a multivariate normal distribution with zero means and covariance matrix $\Sigma = (\sigma_{ij})$, where

$$\sigma_{ii} = \frac{1}{c^2(2c-1)} \left[\frac{(c-1)^2}{p_i} + \sum_{k \neq i}^{c} \frac{1}{p_k} \right] ,$$

and

$$\sigma_{ij} = \frac{1}{c^2(2c-1)} \left[\sum_{k=1}^{c} \frac{1}{p_k} - \frac{c}{p_i} - \frac{c}{p_j} \right]$$

neglecting terms of order $O(N^{-\frac{1}{2}})$ in Σ .

Now

$$\sum_{i=1}^{c} U_N^{(i)} = \frac{\left[\sum_{i=1}^{c} \sum_{\alpha_1=1}^{n_1} \sum_{\alpha_2=1}^{n_2} \cdots \sum_{\alpha_c=1}^{n_c} \Phi^{(i)}(X_{1\alpha_1}, X_{2\alpha_2}, \ldots, X_{c\alpha_c}) \right]}{\prod_{i=1}^{c} n_i}$$

$$= 1.$$

Hence the joint distribution of U_N's is singular and therefore the asymptotic distribution of U_N's must also be singular. This can also be seen by noting that the matrix Σ, is singular, since

$$\sum_{j=1}^{c} \sigma_{ij} = 0 \qquad (i = 1, 2, \ldots, c).$$

We shall consider the first c-1 U_N's and denote by

$$\underline{U}'_{o,N} = (U_N^{(1)}, U_N^{(2)}, \ldots, U_N^{(c-1)}) ,$$

and

$$\underline{\eta}'_{o,N} = (E(U_N^{(1)}), E(U_N^{(2)}), \ldots E(U_N^{(c-1)})) .$$

The corresponding covariance matrix obtained from Σ, be deleting the last row and column of Σ is denoted by Σ_o. Further, let

$$\underset{1\times c-1}{\underline{J}'_o} = (1,1,\ldots, 1) ,$$

$\underset{c-1\times c-1}{I_{c-1}}$ = Diagonal matrix with unity as diagonal elements and off-diagonal elements zero,

and

$$\underset{1\times c-1}{\underline{\Theta}'_o} = (\Theta - c\Theta_1, \Theta - c\Theta_2, \ldots, \Theta - c\Theta_{c-1}) .$$

Then, from (1.4.6)

$$\underline{\eta}'_{o,N} = \frac{1}{c} \underline{J}'_o - (N^{-\frac{1}{2}} I) \underline{\Theta}'_o I_{c-1} ,$$

and the asymptotic distribution of

$$\sqrt{N} (\underline{U}_{O,N} - \frac{1}{c} J_o)$$

is

$$N((I)I_{c-1} \underline{\Theta}_o , \Sigma_o) .$$

Now in order to obtain the asymptotic distribution of X_B^2, we need the following lemma.

Lemma 1.4.1

Suppose the vector $\underset{r\times 1}{\underline{x}}$ has a multivariate normal distribution with mean vector $\underset{r\times 1}{\underline{\mu}}$ and nonsingular covariance matrix Σ_o, then the quadratic form $\underline{x}' \Sigma_o^{-1} \underline{x}$ has a non-central chi-square distribution with r degrees of freedom (where r is the rank of the matrix Σ_o) and non-centrality parameter λ^2 given by

$$\lambda^2 = \underline{\mu}' \Sigma_o^{-1} \underline{\mu} \quad .$$

A proof of this lemma is given by Roy [23].

We calculate Σ_o^{-1}. Let

$$\underline{p}_o' = (p_1, p_2, \ldots, p_{c-1}) \quad ,$$

$$\underline{q}_o' = (p_1^{-1}, p_2^{-1}, \ldots, p_{c-1}^{-1}) \quad ,$$

$$\underset{c-1\times c-1}{D_o} = (p_1^{-1}, p_2^{-1}, \ldots, p_{c-1}^{-1}) \quad ,$$

Diagonal matrix.

and

$$a = \sum_{k=1}^{c} p_k^{-1} \quad .$$

Then

$$c^2(2c-1)\,\Sigma_o = c^2 D_o + a\,\underline{J}_o\,\underline{J}_o' - c\,\underline{q}_o\,\underline{J}_o' - c\,\underline{J}_o\,\underline{q}_o' \;.$$

Let

$$\frac{1}{c^2(2c-1)}\;\Sigma_o^{-1} = c^{-2} D_o^{-1} + \alpha\,\underline{J}_o\underline{J}_o' + \beta\,\underline{p}_o\underline{J}_o' + \gamma\,\underline{J}_o\underline{p}_o' + \delta\,\underline{p}_o\underline{p}_o',$$

where α, β, γ and δ are some constants to be determined.

Now

$$\underset{c-1xc-1}{I} = \Sigma_o \Sigma_o^{-1} = \underset{c-1xc-1}{I} + [\alpha(\frac{c}{p_c} - a) + \beta(a - ap_c + c) - \frac{1}{c}] \underline{J}_o \underline{J}_o'$$

$$+ [c(\alpha - \beta(1-p_c))] \underline{q}_o \underline{J}_o'$$

$$+ [c\gamma - c\delta(1-p_c) - \frac{1}{c}] \underline{q}_o \underline{p}_o'$$

$$+ [\gamma(\frac{c}{p_c} - a) + \delta(a + c - ap_c) + \frac{a}{c^2}] \underline{J}_o \underline{p}_o' .$$

Equating the coefficients of $\underline{J}_o \underline{J}_o'$, $\underline{q}_o \underline{J}_o'$, $\underline{q}_o \underline{p}_o'$ and $\underline{J}_o \underline{p}_o'$ to zero and solving for α, β, γ and δ, we get

$$\alpha = p_c(1-p_c)/c^2, \quad \beta = p_c/c^2, \; \gamma = p_c/c^2 \text{ and } \delta = -1/c^2 .$$

Therefore,

$$\Sigma_o^{-1} = (2c-1) [D_o^{-1} + p_c(1-p_c)\underline{J}_o\underline{J}_o' + p_c(\underline{p}_o\underline{J}_o' + \underline{J}_o\underline{p}_o') - \underline{p}_o\underline{p}_o'] .$$

Hence

$$N(\underline{U}_{o,N} - \frac{1}{c}\underline{J}_o)' \Sigma_o^{-1}(\underline{U}_{o,N} - \frac{1}{c}\underline{J}_o) = N(2c-1)\sum_{i=1}^{c} p_i(U_N^{(i)} - \bar{U}_N)^2$$

$$= X_B^2 ,$$

and

$$\underline{\mu}' \Sigma_o^{-1} \underline{\mu} = I^2 \underline{\Theta}_o' \underline{\Sigma}_o^{-1} \underline{\Theta}_o$$

$$= c^2(2c-1) I^2 \sum_{i=1}^{c} p_i(\Theta_i - \bar{\Theta})^2$$

$$= \lambda_B^2 .$$

The proof of the theorem is now completed by using lemma 1.4.1 with a theorem of Mann and Wald [16].

The above result is also obtained independently by Bhapkar [4a].

1.5 Asymptotic Relative Efficiency.

The concept of asymptotic relative efficiency of one consistent test with respect to another, when both test statistics have a limit normal distribution under the null as well as alternative hypothesis, is due to Pitman [22]. An account of his method is given by Noether [21]. Briefly, the idea of asymptotic relative efficiency is to select a sequence of alternatives which depend on the sample sizes in such a manner that the powers of the two tests for this sequence of alternatives tend to a common limit less than one. The comparison is then made on a sample size basis. This method has been extended by Hannan [12] to consistent tests, whose test statistics have limit distributions of same analytical form which is not necessarily normal. In particular, in the case of chi-square distribution with the same number of degrees of freedom for both test statistics, the formula for asymptotic relative efficiency reduces to the ratio of their non-centrality parameters.

In this section, we shall compare Bhapkar's test with the Kruskal-Wallis test, Mood's test and the usual F-test for some specified cumulative distribution functions. The non-centrality parameters for the last three tests are given in Andrews [2].

By using Hannan's result, and denoting the asymptotic relative efficiencies of Bhapkar's test with respect to the Kruskal-Wallis test, Mood's test and the F-test by $e_{B,K}$, $e_{B,M}$ and $e_{B,F}$ respectively, we have

$$(1.5.1) \qquad e_{B,K} = c^2(2c-1)\, I^2 / 12 \left[\int_{-\infty}^{\infty} f^2(x)dx \right]^2 ,$$

(1.5.2) $$e_{B,M} = c^2(2c-1)\, I^2 / 4 [f(a)]^2$$

where a is the median of the distribution $F(x)$, and

(1.5.3) $$e_{B,F} = c^2(2c-1)\, I^2\, \sigma_F^2 ,$$

where $$\sigma_F^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx\right]^2 .$$

Example 1.5.1 Normal Distribution $N(0,1)$

In order to calculate the integral I defined in (1.4.2) for the normal distribution, we make use of the formulae given by Bose and Gupta [6] and Hojo [14]. Table 1.5.1 gives the values of a.r.e. of Bhapkar's test.

Table 1.5.1

c	2	3	4	5	6	7	8	10
$e_{B,K}$	1.00	0.94	0.86	0.80	0.74	0.69	0.65	0.58
$e_{B,M}$	1.50	1.41	1.29	1.20	1.11	1.03	0.97	0.87
$e_{B,F}$	0.95	0.90	0.82	0.76	0.71	0.66	0.62	0.55

Example 1.5.2 Exponential Distribution

$$F(x) = 1 - \text{Exp}[-x] \qquad (x \geq 0),$$
$$= 0 \qquad (x < 0).$$

Table 1.5.2

c	2	3	4	5	6	7	8	9	10	11
$e_{B,K}$	1.00	1.67	2.33	3.00	3.67	4.33	5.00	5.67	6.33	7.00
$e_{B,M}$	3	5	7	9	11	13	15	17	19	21
$e_{B,F}$	3	5	7	9	11	13	15	17	19	21

Example 1.5.3 Double Exponential Distribution:

$$F(x) = (1/2)\text{Exp}(x) \qquad (x \le 0),$$
$$= 1 - (1/2)\text{Exp}(-x) \qquad (x > 0).$$

Table 1.5.3.

c	2	3	4	5	6	7	8	9	10	11
$e_{B,K}$	1.00	0.94	0.79	0.66	0.55	0.47	0.40	0.35	0.31	0.28
$e_{B,M}$	0.75	0.70	0.59	0.49	0.41	0.35	0.30	0.26	0.23	0.21
$e_{B,F}$	1.50	1.40	1.18	0.98	0.82	0.70	0.60	0.52	0.46	0.42

Example 1.5.4 Rectangular Distribution

$$F(x) = 0 \qquad (x < 0),$$
$$= x \qquad (0 \le x \le 1),$$
$$= 1 \qquad (x > 1).$$

Table 1.5.4

c	2	3	4	5	6	7	8	9	10	11
$e_{B,K}$	1.00	0.94	1.04	1.17	1.32	1.47	1.63	1.79	1.95	2.12
$e_{B,M}$	3.00	2.81	3.11	3.52	3.96	4.42	4.90	5.38	5.86	6.35
$e_{B,F}$	1.00	0.94	1.04	1.17	1.32	1.47	1.63	1.79	1.95	2.12

Remarks

For non-normal alternatives it would be of interest to compare these tests with a test which is optimal or "good" when these particular distributions are assumed.

For exponential alternatives the F-test is surely very inefficient.

1.6 Massey's Test

Let $\{X_{ij}\}$ for $i = 1, 2, \ldots, c_j$ $j = 1, 2, \ldots, n_i$ be a set of independent (real valued) random variables. The probability distribution function of X_{ij} is denoted by $F_i(x)$. Let $N = \sum_{i=1}^{c} n_i$ and further, let $Z_1, Z_2, \ldots, Z_h$ be the ρ_1-th, $(\rho_1+\rho_2)$-th, ... $(\rho_1 + \rho_2 + \ldots + \rho_h)$-th - quantiles of the combined sample, where $\rho_j > 0$ $j = 1, 2, \ldots, h$ and $\sum_{j=1}^{h} \rho_j < 1$ and further $\rho_{h+1} = 1 - \sum_{j=1}^{h} \rho_j$. For convenience, let $Z_0 = -\infty$ and $Z_{h+1} = \infty$ and further, let

$$(1.6.1) \quad M_{i,t} = \{\text{Number of } j\text{'s such that } Z_{t-1} < X_{ij} \leq Z_t\} \quad (i = 1, 2, \ldots, c;\ t = 1, 2, \ldots, h+1).$$

Then

$$(1.6.2) \quad \sum_{i=1}^{c} M_{i,t} = \rho_t N \quad (t = 1, 2, \ldots, h+1),$$

and

$$(1.6.3) \quad \sum_{t=1}^{h+1} M_{i,t} = n_i \quad .$$

Define

$$(1.6.4) \quad X^2_{Ma.} = \sum_{i=1}^{c} \sum_{t=1}^{h+1} (M_{i,t} - \rho_t n_i)^2 / \rho_t n_i \quad .$$

Then, under the null hypothesis, H_0 defined in (1.1.1) the asymptotic distribution of X^2_{Ma} as $n_i \rightarrow \infty$ such that $n_i/N \rightarrow p_i > 0$ $i = 1, 2, \ldots, c$ is a chi-square distribution with $(c-1)h$ degrees of freedom. We shall consider the distribution of $X^2_{Ma.}$ under H_N.

1.7 Notation and Definitions.

Corresponding to the random vector $\underline{Z}$ where $\underline{Z}' = (Z_1, Z_2, \ldots, Z_h)$ define

$$(1.7.1) \quad G_{i,t}(\underline{Z}) = F_i(Z_t) - F_i(Z_{t-1}) \quad i = 1, 2, \ldots, c$$

$$G_t(\underline{Z}) = F(Z_t) - F(Z_{t-1}) , \quad t = 1, 2, \ldots, h+1,$$

where

$$F_i(Z_o) = 0, \quad F_i(Z_{h+1}) = 1 \qquad (i = 1, 2, \ldots, c).$$

$$F(Z_o) = 0, \quad F(Z_{h+1}) = 1$$

Let

(1.7.2) $\delta_{i,t} = 1$ if Z_t belongs to the i-th sample,

$= 0$ otherwise $\quad i = 1, 2, \ldots, c$, $t = 1, 2, \ldots, h$,

and

$$\delta_{i,h+1} = 0 \qquad \text{for all } i .$$

Further, let

$$\underline{\delta}' = (\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1,h}; \delta_{2,1}, \ldots, \delta_{2,h}; \ldots; \delta_{c,1}, \delta_{c,2}, \ldots \delta_{c,h})$$

1.8 Asymptotic Distribution of $\chi^2_{Ma.}$ Under H_N.

Theorem 1.8.1

Suppose

(i) $F(x)$ is an absolutely continuous distribution function having a continuous, bounded derivative, $f(x)$ with respect to Lebesgue measure,

(ii) $f'(x)$ exists and is bounded

(iii) for each positive integer N,

$$H_N: \quad F_i(x) = F(x + N^{-1/2}\, \theta_i)$$

for all real x and not all θ_i's are equal, and

(iv) for each i, as $n_i \longrightarrow \infty$, $n_i/N \longrightarrow p_i > 0$ where $\sum_{i=1}^{c} p_i = 1$.

Then the limit distribution of χ^2_{Ma} is a non-central chi-square distribution with $(c-1)h$ degrees of freedom and non-centrality parameter $\lambda^2_{Ma.}$ given by

(1.8.1) $$\lambda^2_{Ma.} = \left(\sum_{j=1}^{h+1} a_j^2/\rho_j\right)\left(\sum_{i=1}^{c} p_i\,(\theta_i - \bar{\theta})^2\right),$$

where

$$\bar{\Theta} = \sum_{i=1}^{c} p_i \, \Theta_i \quad ,$$

$$a_j = f(\lambda_j) - f(\lambda_{j-1}) \qquad (j = 1, 2, \ldots, h+1),$$

$$f(\lambda_o) = f(\lambda_{h+1}) = 0$$

and $\lambda_1, \lambda_2, \ldots, \lambda_h$ are the respective h quantiles of the distribution function $F(x)$.

Proof: The joint probability density function of $\{M_{i,t}\}$ and $\{Z_t\}$ is given by

(1.8.2)
$$f(m_{11}, m_{12}, \ldots m_{1,h+1}; \ldots\ldots; m_{c1}, m_{c2}, \ldots, m_{c,h+1}; z_1, z_2, z_1, z_2, \ldots, z_h)$$
$$= \sum_{\underline{\delta}} \frac{\prod_{i=1}^{c} n_i!}{\prod_{i=1}^{c} \prod_{t=1}^{h+1} m_{i,t}!} \prod_{i=1}^{c} \prod_{t=1}^{h+1} m_{i,t}^{\delta_{i,t}} \prod_{i=1}^{c} \prod_{t=1}^{h+1} [G_{i,t}(\underline{z})]^{m_{i,t} - \delta_{i,t}}$$
$$\times \prod_{i=1}^{c} \prod_{t=1}^{h} [f_i(z_t)]^{\delta_{i,t}}$$

where the summation Σ is over all c^h possible values of the vector $\underline{\delta}$.

Introduce the new random variables $\{U_{i,t}\}$ and $\{W_t\}$ by the following transformations:

(1.8.3)
$$M_{i,t} = n_i \, G_{i,t} (\underline{\lambda}) + n_i^{1/2} \, U_{i,t} \quad ,$$
$$Z_t = \lambda_t + N^{-1/2} \, W_t \qquad (i=1, 2, \ldots, c; \; t=1, 2, \ldots, h+1),$$

where the λ's satisfy

$$(1.8.4)\qquad \sum_{i=1}^{c} n_i\, G_{i,t}(\underline{\lambda}) = \rho_t\, N \qquad (t=1,2,\ldots,h+1)\ .$$

Obviously,

$$(1.8.5)\qquad \sum_{t=1}^{h+1} U_{i,t} = 0 \qquad (i=1,2,\ldots,c)\ ,$$

and

$$(1.8.6)\qquad \sum_{i=1}^{c} n_i^{1/2}\, U_{i,t} = 0 \qquad (t=1,2,\ldots,h+1)\ .$$

Then we obtain from (1.8.2), the probability density function of $U_{i,t}$ and W_t as

$$(1.8.7)\quad f(u_{1,1},\ldots,u_{1,h+1};\ldots;u_{c,1},\ldots,u_{c,h+1};\ w_1,w_2,\ldots,w_h)$$

$$= N^{-h/2} \prod_{i=1}^{c} n_i^{h/2} \sum_{\underline{\delta}} \frac{\prod_{i=1}^{c} n_i!\ \prod_{i=1}^{c}\prod_{t=1}^{h+1}\left(n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t}\right)^{\delta_{i,t}}}{\prod_{i=1}^{c}\prod_{t=1}^{h+1}\left(n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2}\, u_{i,t}\right)!} \times$$

$$\prod_{i=1}^{c}\ \prod_{t=1}^{h+1} \left(G_{i,t}[\underline{\lambda} + N^{-1/2}\,\underline{w}]\right)^{n_i G_{i,t}(\underline{\lambda})+n_i^{1/2}\, u_{i,t} - \delta_{i,t}}$$

$$\times \prod_{i=1}^{c}\ \prod_{t=1}^{h} \left[f_i(\lambda_t + N^{-1/2}\, w_t)\right]^{\delta_{i,t}}\ .$$

Consider a particular term, say s_g, in Σ corresponding to a value $\underline{\delta}_g$ of the random vector $\underline{\delta}$ and let

$$(1.8.8)\qquad \delta_{i,g} = \sum_{t=1}^{h} \delta_{i,t;g}\ .$$

Then

$$\sum_{i=1}^{c} \delta_{i,g} = h\ .$$

For large values of n_i's, using Stirling's approximation to factorials, we get

$$(1.8.9)\quad \log s_g = -(ch/2)\log(2\pi) - \sum_{i=1}^{c} n_i + \sum_{i=1}^{c} (n_i + \tfrac{1}{2}) \log n_i$$

$$+ \sum_{i=1}^{c}\sum_{t=1}^{h+1} (n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t})$$

$$- \sum_{i=1}^{c}\sum_{t=1}^{h+1} (n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t} + \tfrac{1}{2}) \log(n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t})$$

$$+ \sum_{i=1}^{c}\sum_{t=1}^{h+1} \delta_{i,t;g} \log(n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t})$$

$$+ \sum_{i=1}^{c}\sum_{t=1}^{h+1} (n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t} - \delta_{i,t;g}) \times$$

$$\log G_{i,t}(\underline{\lambda} + N^{-1/2} \underline{w})$$

$$+ \sum_{i=1}^{c}\sum_{t=1}^{h} \delta_{i,t;g} \log f_i(\lambda_t + N^{-1/2} w_t) \quad .$$

Now

$$(1.8.10)\quad G_{i,t}(\underline{\lambda} + N^{-1/2}\underline{w}) = F_i(\lambda_t + N^{-1/2} w_t) - F_i(\lambda_{t-1} + N^{-1/2} w_{t-1})$$

$$= G_{i,t}(\underline{\lambda}) + N^{-1/2}(w_t f_{i,\ell}(\lambda_t) - w_{t-1} f_i(\lambda_{t-1}))$$

$$+ O(N^{-1}) ,$$

and

$$f_i(\lambda_t + N^{-1/2} w_t) = f_i(\lambda_t) + O(N^{-1/2}) \quad .$$

Substituting these values in (1.8.9), and after some simplification, we get

$$\log s_g = -(ch/2)\log(2\pi) - (h/2)\sum_{i=1}^{c} \log n_i - \sum_{i=1}^{c}\sum_{t=1}^{h+1} (n_i G_{i,t}(\lambda) + n_i^{1/2} u_{i,t} + \tfrac{1}{2}) \log G_{i,t}(\underline{\lambda})$$

$$- \sum_{i=1}^{c}\sum_{t=1}^{h+1} (n_i G_{i,t}(\underline{\lambda}) + n_i^{1/2} u_{i,t} + \tfrac{1}{2}) \times \left(\frac{n_i^{-1/2} u_{i,t}}{G_{i,t}(\underline{\lambda})} - \frac{n_i^{-1} u_{i,t}^2}{2G_{i,t}^2(\underline{\lambda})} \right).$$

Therefore

$$(1.8.11) \qquad s_g = c_g \operatorname{Exp}[-\tfrac{1}{2}\psi] + O(N^{-1/2}),$$

where

$$\psi = \sum_{i=1}^{c}\sum_{t=1}^{h+1} u_{i,t}^2 / G_{i,t}(\underline{\lambda}) + \sum_{i=1}^{c}\sum_{t=1}^{h+1} \frac{p_i [w_t f_i(\lambda_t) - w_{t-1} f_i(\lambda_{t-1})]^2}{G_{i,t}(\underline{\lambda})}$$

$$-2\sum_{i=1}^{c}\sum_{t=1}^{h+1} p_i^{1/2} u_{i,t} [w_t f_i(\lambda_t) - w_{t-1} f_i(\lambda_{t-1})] / G_{i,t}(\underline{\lambda}),$$

and

$$c_g = (2\pi)^{-ch/2} \prod_{i=1}^{c} n_i^{-(h/2-\delta_{i,g})} \prod_{i=1}^{c}\prod_{t=1}^{h+1} [G_{i,t}(\underline{\lambda})]^{-1/2} \times \prod_{i=1}^{c}\prod_{t=1}^{h} [f_i(\lambda_t)]^{\delta_{i,t;g}}.$$

Substituting the value of s_g, given above, in (1.8.7) and after some simplification, we obtain

$$(1.8.12) \quad f(u_{11}, u_{12}, \ldots u_{1,h+1}; u_{21}, \ldots u_{2,h+1}; \ldots u_{c1}, u_{c2}, \ldots u_{c,h+1}; w_1\, w_2, \ldots, w_h)$$

$$= C[\operatorname{Exp}[-1/2\,\psi] + O(N^{-1/2})]$$

where

$$C = (2\pi)^{-ch/2} \prod_{i=1}^{c} n_i^{1/2} \prod_{i=1}^{c} \prod_{t=1}^{h+1} [G_{i,t}(\underline{\lambda})]^{-1/2}$$

$$\sum_{g} \prod_{i=1}^{c} \prod_{t=1}^{h} [f_i(\lambda_t)]^{\delta_{i,t;g}} \prod_{i=1}^{c} p_i^{\delta_{i,g}} .$$

Thus, the asymptotic distribution of $\{U_{i,t}\}$ and $\{W_t\}$ is a multivariate normal distribution. But from (1.8.5) and (1.8.6) we observe that not all $U_{i,t}$'s are independent. Therefore, we shall consider the joint distribution of $h(c-1)$ $U_{i,t}$'s say $U_{11},\ldots,U_{1,h};$ $U_{21}, U_{22},\ldots,U_{2h};\ldots U_{c-1,1},U_{c-1,2}\cdots U_{c-1,h}$ and $W_1, W_2,\ldots, W_h$. Then from (1.8.12) we have

(1.8.13) $$f(u_{11},u_{12},\ldots u_{1,h};u_{21},u_{22},\ldots u_{2,h};\ldots u_{c-1,1},u_{c-1,2},\ldots u_{c-1,h}; w_1,\ldots,w_h)$$

$$= C \operatorname{Exp}[-1/2\ \Phi] ,$$

where

$$\Phi = \sum_{i=1}^{c-1} \sum_{i'=1}^{c-1} \sum_{t=1}^{h} \sum_{t'=1}^{h} a_{i,t;i',t'}\, u_{i,t}\, u_{i',t'} + \sum_{t=1}^{h} \sum_{t'=1}^{h} b_{t,t'}\, w_t\, w_{t'}$$

$$- 2 \sum_{i=1}^{c-1} \sum_{t=1}^{h} \sum_{t'=1}^{h} d_{i,t;t'}\, u_{i,t}\, w_{t'} ,$$

and

(1.8.14) $$a_{i,t;i',t'} = p_c^{-1} [p_i/G_{c,t}(\underline{\lambda}) + p_c/G_{i,t}(\underline{\lambda}) + p_i/G_{c,h+1}(\underline{\lambda}) + p_c/G_{i,h+1}(\underline{\lambda})] \quad \text{if } i = i',\ t = t' ,$$

$$= p_c^{-1}(p_i p_{i'})^{1/2} [1/G_{c,t}(\underline{\lambda}) + 1/G_{c,h+1}(\underline{\lambda})] \text{ if } i \neq i', t=t' ,$$

$$= p_c^{-1} [p_i/G_{c,h+1}(\underline{\lambda}) + p_c/G_{i,h+1}(\underline{\lambda})] \quad \text{if } i = i', t \neq t'$$

$$= (p_i p_{i'})^{1/2}/p_c G_{c,h+1}(\underline{\lambda}) \qquad \text{if } i \neq i', t \neq t' ,$$

(1.8.15) $$b_{t,t'} = \sum_{i=1}^{c} p_i f_i(\lambda_t) [1/G_{i,t}(\underline{\lambda}) + 1/G_{i,t+1}(\underline{\lambda})] \text{ if } t=t',$$

$$= - \sum_{i=1}^{c} p_i f_i(\lambda_t) f_i(\lambda_{t+1})/G_{i,t+1}(\underline{\lambda}) \quad \text{if } t'=t+1 ,$$

$$= 0 \quad \text{otherwise.}$$

Further,

(1.8.16) $$d_{i,t;t'} = p_i^{1/2} [f_i(\lambda_t)/G_{i,t}(\underline{\lambda}) - f_c(\lambda_t)/G_{c,t}(\underline{\lambda})] \text{ if } t=t' \; t=1,2,\ldots,h,$$

$$= p_i^{1/2} [f_c(\lambda_{t-1})/G_{c,t}(\underline{\lambda}) - f_i(\lambda_{t-1})/G_{i,t}(\underline{\lambda})] \text{ if } t'=t-1,$$

$$= p_i^{1/2} [f_i(\lambda_h)/G_{i,h}(\underline{\lambda}) - f_c(\lambda_h)/G_{c,h}(\underline{\lambda})] \text{ if } t'=h.$$

Now under H_N,

$$F_i(x) = F(x + N^{-1/2}\,\theta_i) ,$$

and

$$f_i(x) = f(x + N^{-1/2}\,\theta_i) \quad \text{for all real } x .$$

Therefore,

(1.8.17) $$G_{i,t}(\underline{\lambda}) = F(\lambda_t + N^{-1/2}\,\theta_i) - F(\lambda_{t-1} + N^{-1/2}\,\theta_i)$$

$$= F(\lambda_t) - F(\lambda_{t-1}) + N^{-1/2} a_t\,\theta_i + O(N^{-1})$$

$$= G_t(\lambda) + N^{-1/2} a_t\,\theta_i + O(N^{-1})$$

where the a_t's are defined in (1.8.1) .

Using the above relation in (1.8.4), we get

$$(1.8.18)\quad N\,\rho_t = \sum_{i=1}^{c} n_i\, G_{i,t}(\underline{\lambda})$$

$$= N\,G_t(\underline{\lambda}) + N^{-1/2} a_t \sum_{i=1}^{c} n_i\,\theta_i + O(1)\ .$$

Hence as $n_i \rightarrow \infty$,

$$(1.8.19)\quad G_t(\underline{\lambda}) = \rho_t - N^{-1/2} a_t\,\overline{\theta} + O(N^{-1})\ ,$$

and

$$(1.8.20)\quad G_{i,t}(\underline{\lambda}) = \rho_t + N^{-1/2} a_t(\theta_i - \overline{\theta}) + O(N^{-1})\ .$$

Substituting the values of $G_{i,t}(\underline{\lambda})$ given by (1.8.20) in (1.8.14) and (1.8.16), we get

$$(1.8.21)\quad a_{i,t;i',t'} = p_c^{-1}(p_i+p_c)[1/\rho_t + 1/\rho_{h+1}], \text{if } i=i',\ t=t'$$

$$= p_c^{-1}(p_i p_{i'})^{1/2}[1/\rho_t \pm 1/\rho_{h+1}]$$

$$\text{if } i\neq i',\ t=t'$$

$$= (p_i + p_c)/p_c\rho_{h+1}\ , \quad \text{if } i=i',\ t\neq t',$$

$$= (p_i p_{i'})^{1/2}/p_c\rho_{h+1} \quad \text{if } i\neq i',\ t\neq t',$$

and

$$d_{i,t;t'} = O(N^{-1/2}) \qquad \text{for } i = 1, 2, \ldots, c-1$$

$$t, t' = 1, 2, \ldots, h\ .$$

Therefore $\{U_{i,t}\}$ $(i = 1, 2, \ldots, c-1;\ t = 1, 2, \ldots, h)$ and $\{W_t\}$ $(t = 1, 2, \ldots, h)$ are asymptotically independent and the limit distribution of $\{U_{i,t}\}$ $(i = 1, 2, \ldots, c-1;\ t = 1, 2, \ldots, h)$ is a multivariate normal distribution with zero means and covariance matrix, Σ, where

$$\Sigma^{-1} = ((a_{i,t;i',t'}))$$

and $a_{i,t;i',t'}$ are defined in (1.8.21) .

Substituting the values of $G_{i,t}(\underline{\lambda})$ given by (1.8.20) in (1.8.3), we get

$$M_{i,t} = n_i \left[\rho_t + N^{-1/2} a_t(\Theta_i - \bar{\Theta}) \right] + n_i^{1/2} U_{i,t} + O(1) .$$

Therefore, as $n_i \longrightarrow \infty$ and $n_i/N \longrightarrow p_i$,

$$U_{i,t} = n_i^{-1/2} (M_{i,t} - n_i \rho_t) - p_i^{1/2} a_t(\Theta_i - \bar{\Theta}) .$$

Let

$$\underset{1\times h(c-1)}{\underline{M}'} = (n_1^{-1/2}(M_{11} - \rho_1 n_1), \ldots, n_1^{-1/2}(M_{1,h} - \rho_h n_1); \ldots . \\ \ldots . \; n_{c-1}^{-1/2}(M_{c-1,1} - \rho_1 n_{c-1}) \ldots n_{c-1}^{-1/2} \\ (M_{c-1,h} - \rho_h n_{c-1}))$$

and

$$\underline{\Theta}' = (p_1^{1/2} a_1(\Theta_1 - \bar{\Theta}), \ldots p_1^{1/2} a_h(\Theta_1 - \bar{\Theta}), \ldots ; p_{c-1}^{1/2} a_1(\Theta_{c-1} - \bar{\Theta}), \ldots \ldots ; \\ p_{c-1}^{1/2} a_h(\Theta_{c-1} - \bar{\Theta}))$$

Then the asymptotic distribution of the random vector $\underline{M}$ is multivariate normal with mean vector $\underline{\Theta}$ and covariance matrix Σ, where

$$\Sigma^{-1} = ((a_{i,t;i',t'})) .$$

Therefore, by Lemma 1.4.1, the asymptotic distribution of

$$\underline{M}' \Sigma^{-1} \underline{M} = \chi^2_{Ma.}$$

is a non-central chi-square with (c-1)h degrees of freedom and non-centrality parameter $\lambda^2_{Ma.}$ given by

$$\lambda^2_{Ma.} = \underline{\Theta}' \Sigma^{-1} \underline{\Theta} = (\sum_{j=1}^{h+1} a_j^2/\rho_j)(\sum_{i=1}^{c} p_i(\Theta_i - \bar{\Theta})^2) .$$

This completes the proof of theorem 1.8.1.

Remark: In case $h = 1$, $\rho_1 = \rho_2 = 1/2$,

$$\lambda^2_{Ma.} = 4[f(\lambda_1)]^2 (\sum_{i=1}^{c} p_i(\Theta_i - \bar{\Theta})^2)$$

where λ_1 is the median of the distribution $F(x)$. The above expression for $\lambda^2_{Ma.}$ agrees with the one given in Andrews [2] for Mood's test.

1.9 Asymptotic Relative Efficiency of Massey's Test.

In this section, we shall compare asymptotically Massey's test with other nonparametric tests and the F-test. First, we observe that for a specified choice of $F(x)$, the non-centrality parameter depends only upon $\sum_{i=1}^{c} p_i(\Theta_i - \bar{\Theta})^2$. Secondly, for $h \geq 2$, Massey's test statistic has asymptotic non-central chi-square distribution with $(c-1)h$ 'degrees of freedom, while the other test statistics under consideration have asymptotic non-central chi-square distribution with $(c-1)$ degrees of freedom. Thus Hannan's simple formula mentioned in section 1.5 is not applicable here. Instead we shall use the following consideration.

For each positive integer n, denote

$$\underline{N}_n^{(i)'} = (n_1^{(i)}(n), n_2^{(i)}(n), \ldots, n_c^{(i)}(n))$$

$i = 1, 2$ where each $n_j^{(i)}(n)$ is a sequence of increasing positive integers.

Let

$$N_n^{(i)} = \sum_{j=1}^{c} n_j^{(i)}(n) \quad ,$$

and

$$\underline{\Theta}_n' = (N^{-1/2}(n)\, \Theta_1 \, , \; N^{-1/2}(n)\, \Theta_2, \ldots N^{-1/2}(n)\, \Theta_c) \, ,$$

where for some pair (j,k) $\Theta_j \neq \Theta_k$ and suppose

(1.9.1) $$\lim_{n \to \infty} n_j^{(i)}(n)/N_{(n)}^{(i)} = p_j > 0 \quad \text{where} \quad \sum_{j=1}^{c} p_j = 1 \, .$$

Further, let $\phi_n^{(1)}$ and $\phi_n^{(2)}$ be sequences of Massey's test and some other competitive test under consideration based on $\underline{N}_n^{(1)'}$ and $\underline{N}_n^{(2)'}$ respectively and having the same fixed significance level α_o for testing the null hypothesis defined in (1.1.1). We shall consider alternatives of type H_n: $F_j(x) = F(x+N^{-1/2}(n)\, \Theta_j)$, $j = 1, 2, \ldots, c$, where $F(x)$ is a specified distribution satisfying the conditions of theorem 1.8.1.

Denote by $p^{(i)}\,(\phi_n^{(i)}, F, \underline{\Theta}_n')$ the power of the i-th test. Now if there exist two sequences $\underline{N}_n^{(i)'}$ $i = 1, 2$ which satisfy (1.9.1) and are such that

(i) for each positive integer n, $\phi_n^{(1)}$ and $\phi_n^{(2)}$ have same significance level α_o ,

(ii) $$\lim_{n \to \infty} p^{(1)}\,(\phi_n^{(1)}, F, \underline{\Theta}_n^{(1)'}) = \lim_{n \to \infty} p^{(2)}(\phi_n^{(2)}, F, \Theta_n^{(2)'})$$

$$= \beta_o \, ,$$

where $0 < \beta_o < 1$; and

$$(1.9.2) \qquad \Theta_j^{(1)}/\sqrt{N_n^{(1)}} = \Theta_j^{(2)}/\sqrt{N_n^{(2)}} \qquad \text{for all } j.$$

Then the asymptotic relative efficiency of test $\phi^{(1)}$ with respect to test $\phi^{(2)}$ is defined as

$$e_{1,2}(\alpha_o, \beta_o) = \lim_{n \to \infty} N^{(2)}(n)/N^{(1)}(n) .$$

We know that when H_n holds, Massey's test statistic and the test statistic corresponding to any other competitive test under consideration have limit non-central chi-square distribution with $(c-1)h$ and $(c-1)$ degrees of freedom. Therefore to satisfy the requirement (ii) we must have

$$(1.9.3) \quad \lambda^2_{\alpha_o,\beta_o,h(c-1)} = c_{1,F} \sum_{j=1}^{c} p_j \left(\Theta_j^{(1)} - \bar{\Theta}^{(1)}\right)^2 ,$$

and

$$(1.9.4) \quad \lambda^2_{\alpha_o,\beta_o,(c-1)} = c_{2,F} \sum_{j=1}^{c} p_j \left(\Theta_j^{(2)} - \bar{\Theta}^{(2)}\right)^2$$

where $\lambda^2_{\alpha_o,\beta_o,h(c-1)}$ and $\lambda^2_{\alpha_o,\beta_o,(c-1)}$ are the tabulated values of non-centrality parameters given by Fix [10] and $c_{1,F}$, $c_{2,F}$ are constants depending on $F(x)$.

Substituting the values of $\Theta_j^{(2)}$ given by (1.9.2) in (1.9.4), we get

$$(1.9.5) \qquad \lambda^2_{\alpha_o,\beta_o,c-1} = [N^{(2)}(n)/N^{(2)}(n)]c_{2,F} \sum_{j=1}^{c} p_j \left(\Theta_j^{(1)} - \bar{\Theta}^{(1)}\right)^2$$

Therefore, from (1.9.3) and (1.9.4) we have

$$e_{1,2}(\alpha_o, \beta_o) = \lim_{n \to \infty} N^{(2)}_{(n)}/N^{(1)}_{(n)}$$

$$= [\lambda^2_{\alpha_o,\beta_o,(c-1)}/\lambda^2_{\alpha_o,\beta_o,h(c-1)}][c_{1,F}/c_{2,F}].$$

1.10 Examples.

Example 1.10.1 Normal Distribution:

$$F(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \operatorname{Exp}[-1/2\, t^2]dt \qquad (-\infty \leq x \leq \infty)$$

(i) $h = 3, \quad \rho_j = 1/4 \ (j = 1, 2, 3, 4)$,

$$\sum_{j=1}^{4} a_j^2/\rho_j = 0.8629 .$$

(ii) $h = 4, \quad \rho_j = 1/5, \ (j = 1, 2, 3, 4, 5)$,

$$\sum_{j=1}^{5} a_j^2/\rho_j = 0.8969 .$$

(iii) $h = 7, \quad \rho_j = 1/8, \ (j = 1, 2, \ldots, 8)$,

$$\sum_{j=1}^{8} a_j^2/\rho_j = 0.9146 .$$

(iv) $h = 9, \quad \rho_j = 1/10, \ (j = 1, 2, \ldots, 10)$,

$$\sum_{j=1}^{10} a_j^2/\rho_j = 0.9590 .$$

Table 1.10.1

Asymptotic Relative Efficiency of Massey's Test

for Normal Alternatives

$\alpha_o = 0.05$ $\beta_o = 0.90$

No. of Samples	No. of quantiles	A. R. E. w.r.t.		
		Mood's Test	Kruskal Test	F-test
c	h	$e_{Ma.M}$	$e_{Ma.K}$	$e_{Ma.F}$
2	3	1.00	0.67	0.64
3	3	0.98	0.66	0.63
4	3	0.97	0.65	0.62
5	3	0.96	0.64	0.61
2	4	0.96	0.64	0.61
3	4	0.93	0.62	0.59
4	4	0.91	0.61	0.58
5	4	0.90	0.60	0.57
2	7	0.87	0.58	0.55
3	7	0.83	0.56	0.53
4	7	0.81	0.54	0.51
5	7	0.79	0.53	0.50
2	9	0.80	0.53	0.51
3	9	0.76	0.51	0.48
4	9	0.73	0.49	0.47
5	9	0.71	0.47	0.45

Example 1.10.2 Exponential Distribution

$$F(x) = 1 - \text{Exp}[-x] \qquad (x \geq 0) ,$$
$$= 0 \qquad (x < 0) .$$

For any h,

$$e_{Ma.M}(\alpha_o,\beta_o) = (\rho_1^{-1}-1)\, \lambda^2_{\alpha_o,\beta_o,(c-1)}/\lambda^2_{\alpha_o,\beta_o,h(c-1)}$$

$$= e_{Ma.F}(\alpha_o,\beta_o) ,$$

$$e_{Ma.,K} = (\rho_1^{-1} - 1)\lambda^2_{\alpha_o,\beta_o,(c-1)}/3\lambda^2_{\alpha_o,\beta_o,h(c-1)} .$$

In this case, the asymptotic relative efficiency of Massey's test depends only on ρ_1, h and c and for given h and c, it can be made arbitrarily large by choosing ρ_1 sufficiently small.

Example 1.10.3 Rectangular Distribution

$$F(x) = 0 \qquad (x < 0) ,$$
$$= x \qquad (0 \leq x \leq 1) ,$$
$$= 1 \qquad (x > 1) .$$

For any h,

$$e_{Ma,M}(\alpha_o,\beta_o) = (\rho_1^{-1} + \rho_{h+1}^{-1})\, \lambda^2_{\alpha_o,\beta_o,(c-1)}/4\lambda^2_{\alpha_o,\beta_o,h(c-1)},$$

$$e_{Ma,K}(\alpha_o,\beta_o) = (\rho_1^{-1} + \rho_{h+1}^{-1})\, \lambda^2_{\alpha_o\beta_o,(c-1)}/12\lambda^2_{\alpha_o,\beta_o,h(c-1)}$$

$$= e_{Ma,F}\,(\alpha_o,\beta_o) .$$

In this case, also, the asymptotic relative efficiency for given h and c, can be made arbitrarily large by proper choices of ρ_1 and ρ_{h+1} .

Example 1.10.4 Double Exponential Distribution

$$F(x) = (1/2)\ \mathrm{Exp}\ (x) \qquad (-\infty \leq x \leq 0),$$
$$= (1/2)[2-\mathrm{Exp}(-x)] \qquad (x > 0)\ .$$

For given h, let k denote the least integer less than or equal to h such that

$$\sum_{j=1}^{k} \rho_j \leq 1/2 \quad ,$$

and let

$$\sum_{j=1}^{k} \rho_j = A_k \leq 1/2$$

Then

$$e_{Ma,M}(\alpha_o,\beta_o) = \frac{[(1-2A_k)^2 - \rho_{k+1}(1-4A_k)]}{2\ \rho_{k+1}} \times \frac{\lambda^2_{\alpha_o,\beta_o,(c-1)}}{\lambda^2_{\alpha_o,\beta_o,h(c-1)}} \quad ,$$

$$e_{Ma,K}(\alpha_o,\beta_o) = \frac{4[(1-2A_k)^2 - \rho_{k+1}(1-4A_k)]}{3\rho_{k+1}} \quad \frac{\lambda^2_{\alpha_o,\beta_o,(c-1)}}{\lambda^2_{\alpha_o,\beta_o,h(c-1)}} \quad ,$$

$$e_{Ma,F}(\alpha_o,\beta_o) = \frac{2[(1-2A_k)^2 - \rho_{k+1}(1-4A_k)]}{\rho_{k+1}} \times \frac{\lambda^2_{\alpha_o,\beta_o,(c-1)}}{\lambda^2_{\alpha_o,\beta_o,h(c-1)}} \quad .$$

1.11 Asymptotic Power.

The power of a test is defined as the probability of rejecting the null hypothesis when the alternative hypothesis is true. We have already seen that when the alternative hypothesis of type H_N: $F_i(x) = F(x + N^{-1/2}\ \theta_i)$ is true, both Bhapkar's test criterion and Massey's test criterion have limit non-central chi-square distributions. Moreover, for a specified choice of $F(x)$, the non-centrality parameters depend on $\theta_1, \theta_2, \ldots, \theta_c$ only through

$\sum_{i=1}^{c} p_i\ (\theta_i - \bar{\theta})^2$. Therefore, for a given choice of $F(x)$ and given sample sizes $n_1, n_2, \ldots, n_c$ and a given alternative $F_i(x) = F(x + \delta_i)$ where δ_i's are fixed numbers, we compute

$$\theta_i = N^{1/2}\ \delta_i\ ,$$

and

$$\sum_{i=1}^{c} p_i\ (\theta_i - \bar{\theta})^2 = N \sum_{i=1}^{c} p_i(\delta_i - \bar{\delta})^2\ .$$

Corresponding to the calculated value of non-centrality parameter and number of degrees of freedom, the power is obtained by referring to the tables of non-central chi-square distribution given by Fix [10] .

1.12 Remarks.

We have made an asymptotic comparison of Massey's test with other non-parametric tests when the alternative distributions differ only in locations. It would be interesting to make a comparison when the alternative distributions differ in more than one parameter. However, the difficulty in this case, is not only in finding the asymptotic distributions of the test criteria, but also except in the case of location and scale alternatives, in specifying the alternative distributions in convenient forms so that they differ in more than one parameter.

In the case of location and scale alternatives, if we assume

$$F_i(x) = F\,[(1 + N^{-1/2}\ \delta_i)(x + N^{-1/2}\ \epsilon_i)] \qquad (1.12.1)$$

and some further conditions on $F(x)$, then it can be shown that the asymptotic distribution of $\chi^2_{Ma.}$ is a non-central chi-square distribution with $(c-1)h$ degrees of freedom and non-centrality parameter λ^2_{Ma} given by

$$\lambda^2_{Ma.} = \sum_{t=1}^{h+1} \frac{1}{\rho_t} \sum_{i=1}^{c} p_i(a_t(\epsilon_i-\bar{\epsilon}) + b_t(\delta_i-\bar{\delta}))^2 ,$$

where

$$a_t = f(\lambda_t) - f(\lambda_{t-1}) ,$$

$$b_t = \lambda_t f(\lambda_t) - \lambda_{t-1} f(\lambda_{t-1}) ,$$

$$f(\lambda_o) = f(\lambda_{h+1}) = 0 ,$$

$$\bar{\epsilon} = \sum_{i=1}^{c} p_i \epsilon_i , \qquad \bar{\delta} = \sum_{i=1}^{c} p_i \delta_i$$

and the λ's are defined in section 1.8. .

In particular, for Mood's test

$$\lambda^2_{M.} = 4f^2(\lambda_1) \left[\sum_{i=1}^{c} ((\epsilon_i-\bar{\epsilon}) + \lambda_1(\delta_i-\bar{\delta}))^2 \right],$$

where λ_1 is the median of the distribution $F(x)$.

Unfortunately, in this situation, the asymptotic relative efficiency depends on ϵ_i's and δ_i's. It would be interesting to compare these two tests for different choices of values of ϵ_i's and δ_i's and also to study the behavior of other nonparametric tests based on ranks when the alternative defined in (1.12.1) is true.

CHAPTER II

ON THE ASYMPTOTIC BEHAVIOR OF SOME NON-PARAMETRIC TESTS USED IN ANALYSIS OF VARIANCE (TWO-WAY CLASSIFICATION)

2.1 Introduction

In this chapter we shall consider the asymptotic distribution of some non-parametric tests used in the analysis of variance set-up with t treatments and b blocks.

Let $\{X_{ij}\}$ $(i = 1, 2, \ldots, t;\ j = 1, 2, \ldots, b)$ be bt independent (real valued) random variables. Let $F_{ij}(x)$ denote the cumulative distribution function of X_{ij} and suppose for each (i,j)

$$F_{ij}(x) = F(x + \nu + \alpha_i + \beta_j)$$

where α_i denotes the i-th treatment effect, β_j denotes the j-th block effect and $F(x)$ is a continuous cumulative distribution function. Then the null hypothesis to be tested is

(1.2.1) $\quad H_o\colon\ \alpha_1 = \alpha_2 = \ldots = \alpha_t$.

Mood and Brown [20] have proposed a non-parametric test for testing the hypothesis H_o in a complete two-way classification. Let $\tilde{x}_j$ $(j = 1, 2, \ldots, b)$ be the median of observations in the j-th block and in two-way table, let the observation x_{ij} be replaced by $+1$ if it exceeds $\tilde{x}_j$ or by 0 if it does not. Let m_i be the number of $+1$'s in the i-th row. Then the test statistic used by Mood and Brown is

(1.2.2) $\chi^2_{M.} = [t(t-1)/d(t-d)] \sum_{i=1}^{t} (m_i - \frac{bd}{t})^2$, where $d=t/2$ if t is even and $d = (t-1)/2$ if t is odd. Unless b is small, Mood and Brown have shown that the limit distribution of χ^2_M is a chi-square distribution with t-1 degrees of freedom when H_o is true.

Another test criterion suggested by Friedman [11] makes use of the ranks of the observations. Let r_{ij} denote the rank of x_{ij} when the observations in the j-th block are arranged in ascending order. Then calculate the usual F-test statistic based on r_{ij}'s. An equivalent test statistic is

$$(2.1.3) \qquad \chi^2_{Fr.} = [12b/t(t+1)] \sum_{i=1}^{t} (r_{i.} - \frac{t+1}{2})^2$$

where

$$r_{i.} = \sum_{j=1}^{b} r_{ij}/b \quad .$$

Under the null hypothesis H_o, the limit distribution of $\chi^2_{Fr.}$ is a chi-square with t-1 degrees of freedom.

These test statistics have been generalized to the case of incomplete block designs by Bhapkar [4] and Durbin [9].

In this chapter, we shall consider the asymptotic distribution as $b \to \infty$, of the Mood-Brown test statistic, of its generalization to incomplete block designs and of Friedman's test statistic when the alternative hypothesis is

$$H_b: \quad \alpha_i = b^{-1/2} \theta_i \qquad \text{where } \theta_i \neq \theta_{i'}$$

for some pair (i,i'). The asymptotic distribution of Durbin's test statistics under H_b, has already been considered by Van Elteren and Noether [27].

Finally an asymptotic comparison, in Pitman's sense, of the Mood Brown test is made with Friedman's test and the F-test and also the Mood-Brown test in case of balanced incomplete block design is compared with Durbin's test.

2.2 Asymptotic Distribution of χ^2_M Under H_b.

Theorem 2.2.1:

Let $\{X_{ij}\}$ $(i = 1, 2, \ldots, t;\ j = 1, 2, \ldots, b)$ be bt independent random variables and let $F_{ij}(x)$ denote the cumulative distribution function of X_{ij}. Further, suppose

(i) $F_{ij}(x) = F(x + \nu + \alpha_i + \beta_j)$ where $F(x)$ is a continuous cumulative distribution function with bounded first and second derivatives $f(x)$ and $f'(x)$ respectively, and

(ii) H_b: $\alpha_i = b^{-1/2}\,\theta_i$,

where $\theta_i \neq \theta_{i'}$ for some pair (i, i').

Let

$$\underline{M}' = (m_1 - \frac{bd}{t},\ m_2 - \frac{bd}{t},\ \ldots,\ m_t - \frac{bd}{t}),$$

where m_i's are defined in section 2.1.

Then under the hypothesis H_b, as $b \longrightarrow \infty$

(a) the asymptotic distribution of the random vector

$$b^{-1/2}\,\underline{M}$$

is a multivariate singular normal distribution with mean vector $\underline{\mu}$ and covariance matrix $\Sigma = (\sigma_{i,i'})$, where

(2.2.1a) $$\mu_i = -t\binom{t-2}{d-1}(\Theta_i-\bar{\Theta})\int_{-\infty}^{\infty}[1-F(x)]^{d-1}F^{d}(x)f^2(x)dx$$
if t is odd,

(2.2.1b) $$= -t\binom{t-2}{d-1}(\Theta_i-\bar{\Theta})\int_{-\infty}^{\infty}[1-F(x)]^{d-1}F^{d-1}(x)f^2(x)dx$$
if t is even ,

and

(2.2.2) $$\sigma_{i,i'} = d(t-d)/t^2 \quad \text{if } i = i' ,$$
$$= -d(t-d)/t^2(t-1) \quad \text{if } i \neq i' ,$$

and where $t\bar{\Theta} = \sum_{i=1}^{t} \Theta_i$, $d = t/2$ if t is even and $d = (t-1)/2$ if t is odd,

and

(b) the asymptotic distribution of χ_M^2 is a non-central chi-square distribution with t-1 degrees of freedom and non-centrality parameter λ_M^2, given by

(2.2.3a) $$\lambda_M^2 = \frac{t^3(t-1)}{d(t-d)}\binom{t-2}{d-1}^2\left[\int_{-\infty}^{\infty}[1-F(x)]^{d-1}F^{d}(x)f^2(x)dx\right]^2 \sum_{i=1}^{c}(\Theta_i-\bar{\Theta})^2 \quad \text{if } t \text{ is odd,}$$

(2.2.3b) $$= \frac{t^3(t-1)}{d(t-d)}\binom{t-2}{d-1}^2\left[\int_{-\infty}^{\infty}[1-F(x)]^{d-1}F^{d-1}(x)f^2(x)dx\right]^2 \sum_{i=1}^{c}(\Theta_i-\bar{\Theta})^2 \quad \text{if } t \text{ is even.}$$

In order to prove this theorem we need the following multivariate form of the central limit theorem due to Bernstein.

<u>Theorem 2.2.2</u> (Bernstein [3])

Given a sequence $(X_{1,nk}, X_{2,nk},\ldots,X_{t,nk})$ $(n = 1, 2, \ldots;$ $k = 1, 2, \ldots \nu(n);$ $\lim_{n\to\infty} \nu = \infty)$ of sets of random vectors in R_t, independent for each fixed n, with $E(X_{i,nk}) = 0$ $(i = 1, 2, \ldots, t)$, let

$$E(X^p_{r,nk} X^q_{s,nk}) = \mu^{(nk)}_{p,q}(r,s) \qquad (p+q=2;\ r,s=1,2,\ldots,t),$$

$$\rho^3_{nk} = \max\left(E\,|X_{1,nk}|^3,\ E|X_{2,nk}|^3, \ldots, E|X_{t,nk}|^3 \right),$$

$$\rho^3_n = \sum_{k=1}^{\nu} \rho^3_{nk} .$$

Assume that

$$\lim_{n \to \infty} \nu^{-1} \sum_{k=1}^{\nu} \mu^{(nk)}_{p,q}(r,s) = \mu_{p,q}(r,s) \qquad (p+q=2).$$

Then as $n \to \infty$, the random vector

$$\nu^{-1/2} \left(\sum_{k=1}^{\nu} X_{1,nk},\ \sum_{k=1}^{\nu} X_{2,nk}, \ldots, \sum_{k=1}^{\nu} X_{t,nk} \right)$$

has a limiting normal distribution with mean $(0, 0, \ldots, 0)$ and covariances $\mu_{p,q}(r,s)$.

Proof of Theorem 2.2.1:

Let $\tilde{X}_j$ denote the median of observations in the j-th block and define new random variables $\{U_{i,j}\}$ as follows:

$$(2.2.4) \qquad U_{i,j} = 1 \quad \text{if } X_{ij} > \tilde{X}_j, \qquad = 0 \quad \text{otherwise}$$

and let

$$\underline{U}'_j = (U_{1,j}, U_{2,j}, \ldots, U_{t,j}) .$$

Then

$$(2.2.5) \qquad m_i = \sum_{j=1}^{b} U_{i,j} .$$

Let

$$(2.2.6) \qquad \eta_{i.j} = E(U_{i,j}) \quad ,$$

and

$$(2.2.7) \qquad \eta_i = E(m_i) = \sum_{j=1}^{b} \eta_{i,j} \quad .$$

For a fixed t, let $T(t)$ denote the set of integers $1, 2, \ldots, t$. We shall consider first the case when t is odd.

Case 1: $(t = 2d+1)$

$$(2.2.8) \qquad \eta_{i,j} = \Pr.\left(X_{ij} > \tilde{X}_j\right) \quad .$$

For a fixed $u \neq i$, where $i, u \in T(t)$, let

$$(2.2.9) \quad {}_u p^{(j)}_{i;(i_1,i_2,\ldots i_d)} = \Pr.(X_{i_1,j} \leq X_{u,j}, X_{i_2,j} \leq X_{u,j}, \ldots\ldots X_{i_d,j} \leq X_{u,j}; X_{i,j} > X_{u,j} \ldots\ldots X_{i_{d+1},j} > X_{u,j}, \ldots X_{i_{2d-1},j} > X_{u,j}),$$

where $i_k \neq i$ or u and $(i_1,i_2,\ldots,i_d)$ is a specified combination of d integers from the subset $T(t)-\{i,u\}$. Then

$$(2.2.10) \quad \eta_{i,j} = \sum_{u \neq i} \sum_{(i_1,i_2,\ldots i_d)} {}_u p^{(j)}_{i;(i_1,i_2,\ldots,i_d)} = \sum_{u \neq i} {}_u p^{(j)}_i \quad ,$$

where the inner summation is over all combinations of d integers from the subset $T(t)-\{i,u\}$ and the outer summation is over all treatments other than the i-th treatment.

Now

$$(2.2.11)\quad {}_u p^{(j)}_{i;(i_1,i_2,\dots i_d)} = \int_{-\infty}^{\infty} [1-F_{i,j}(x)] \prod_{k=1}^{d} F_{i_k,j}(x) \prod_{k=d+1}^{2d-1} [1-F_{i_k,j}(x)] \times f_{u,j}(x)dx.$$

Substituting the values of $F_{i,j}(x)$ from (i) and (ii) (2.2.11), we get

$$(2.2.12)\quad {}_u p^{(j)}_{i;(i_1,i_2,\dots i_d)} = \int_{-\infty}^{\infty} [1-F(x+\nu+\beta_j+b^{-1/2}\theta_i)] \prod_{k=1}^{d} F(x+\nu+\beta_j+b^{-1/2}\theta_{i_k}) \times \prod_{k=d+1}^{2d-1} [1-F(x+\nu+\beta_j+b^{-1/2}\theta_{i_k})] f(x+\nu+\beta_j+b^{-1/2}\theta_u) \times dx .$$

Put

$$y = x + \nu + \beta_j + b^{-1/2}\theta_u .$$

Then using Taylor's series expansion, we get

$$F(x+\nu+\beta_j+b^{-1/2}\theta_i) = F(y+b^{-1/2}(\theta_i-\theta_u))$$

$$(2.2.13)\quad = F(y)+b^{-1/2}(\theta_i-\theta_u)f(y)+0(b^{-1}),$$

and similar expressions for other F's in (2.2.12).

Therefore,

$$(2.2.14)\quad [1-F(y+b^{-1/2}(\theta_i-\theta_u))] \prod_{i=1}^{d} F(y+b^{-1/2}(\theta_{i_k}-\theta_u)) \prod_{k=d+1}^{2d-1} [1-F(y+b^{-1/2}(\theta_{i_k}-\theta_u))]$$

$$= [(1-F(y))^d - b^{-1/2}(\Theta_i + \sum_{k=d+1}^{2d-1} \Theta_{i_k} - d\Theta_u)(1-F(y))^{d-1} f(y) + O(b^{-1})]$$

$$\times [F^d(y) + b^{-1/2}(\sum_{k=1}^{d} \Theta_{i_k} - d\,\Theta_u)(F^{d-1}(y)f(y) + O(b^{-1})]$$

$$= (1-F(y))^d F^d(y) - b^{-1/2}(\Theta_i + \sum_{k=d+1}^{2d-1} \Theta_{i_k} - d\,\Theta_u)(1-F(y))^{d-1} F^d(y) f(y)$$

$$+ b^{-1/2}(\sum_{k=1}^{d} \Theta_{i_k} - d\,\Theta_u)(1-F(y))^d F^{d-1}(y) f(y) + O(b^{-1}).$$

Hence, substituting the above value in the integrand of (2.2.12) and integrating out the first term, we obtain

$$(2.2.15)\quad {}_u p^{(j)}_{i,(i_1,i_2,\ldots,i_d)} = \frac{\sqrt{d+1}\,\sqrt{d+1}}{\sqrt{2d+2}} - b^{-1/2}(\Theta_i + \sum_{k=d+1}^{2d-1} \Theta_{i_k} - d\Theta_u) \times \int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^d(y) f^2(y)\,dy$$

$$+ b^{-1/2}(\sum_{k=1}^{d} \Theta_{i_k} - d\,\Theta_u) \int_{-\infty}^{\infty} [1-F(y)]^d F^{d-1}(y) d^2(y)\,dy$$

$$+ O(b^{-1}).$$

Now out of $\binom{2d-1}{d}$ combinations of type $(i_1, i_2, \ldots, i_d)$, a particular integer $s \neq i$ or u will occur $\binom{t-3}{d-1}$ times in $(i_1, i_2, \ldots, i_d)$ and $\binom{t-3}{d-2}$ times in $(i_{d+1}, \ldots, i_{2d-1})$. Therefore

$$(2.2.16)\quad \sum_{(i_1,i_2,\ldots i_d)} {}_u p^{(j)}_{i;(i_1,i_2,\ldots i_d)} = \binom{2d-1}{d} \Big[\frac{\sqrt{d+1}\,\sqrt{d+1}}{\sqrt{2d+2}}$$

$$- b^{-1/2}(\Theta_i - d\,\Theta_u) \times$$

$$\int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^{d}(y) f^{2}(y)\,dy$$

$$-b^{-1/2} d\, \Theta_u \int_{-\infty}^{\infty} [1-F(y)]^{d} F^{d-1}(y) f^{2}(y)\,dy\,]$$

$$-b^{-1/2} \binom{t-3}{d-2} \Big(\sum_{s \neq i,u} \Theta_s \Big) \int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^{d}(y) f^{2}(y)\,dy$$

$$+\, b^{-1/2} \binom{t-3}{d-1} \Big(\sum_{s \neq i,u} \Theta_s \Big) \int_{-\infty}^{\infty} [1-F(y)]^{d} F^{d-1}(y) f^{2}(y)\,dy$$

$$+\, O(b^{-1}) \,.$$

Now

$$(2.2.17) \qquad \sum_{s \neq i,u}^{t} \Theta_s = t\,\bar{\Theta} - \Theta_i - \Theta_u \,.$$

Substituting from (2.2.17) in (2.2.16), we get

$$(2.2.18) \quad {}_u p_i^{(j)} = \sum \; {}_u p^{(j)}_{i,(i_1, i_2, \ldots, i_d)}$$

$$= d/t(t-1) - \binom{2d-1}{d} b^{-1/2} [(\Theta_i - d\Theta_u) \int_{-\infty}^{\infty} [1-F(y)]^{d-1}$$

$$\times\, F^{d}(y) f^{2}(y)\,dy$$

$$-\, d\, \Theta_u \int_{-\infty}^{\infty} [1-F(y)]^{d} F^{d-1}(y) f^{2}(y)\,dy\,]$$

$$-b^{-1/2} \binom{t-3}{d-2} (t\,\bar{\Theta} - \Theta_i - \Theta_u) \int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^{d}(y) f^{2}(y)\,dy$$

$$+\, b^{-1/2} \binom{t-3}{d-1} (t\,\bar{\Theta} - \Theta_i - \Theta_u) \int_{-\infty}^{\infty} [1-F(y)]^{d} F^{d-1}(y) f^{2}(y)\,dy$$

$$+\, O(b^{-1}) \,.$$

Hence

$$(2.2.19) \qquad \eta_{ij} = \sum_{u \neq i} {}_u p_i^{(j)}$$

$$= \frac{d}{t} - \binom{2d-1}{d} b^{-1/2} \Big\{ \Big[\int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^d(y) f^2(y) dy \Big] \times \sum_{u \neq i}^{t} (\Theta_i - d\,\Theta_u) - d \sum_{u \neq i}^{t} \Theta_u$$

$$\int_{-\infty}^{\infty} [1-F(y)]^d F^{d-1}(y) f^2(y) dy \Big\}$$

$$- b^{-1/2} \binom{t-3}{d-2} \Big(\sum_{u \neq i} (t\,\overline{\Theta} - \Theta_i - \Theta_u) \Big(\int_{-\infty}^{\infty} (1-F(y))^{d-1}$$

$$\times F^d(y) f^2(y) dy \Big)$$

$$+ b^{-1/2} \binom{t-3}{d-1} \Big(\sum_{u \neq i} t\,\overline{\Theta} - \Theta_i - \Theta_u \Big) \Big(\int_{-\infty}^{\infty} (1-F(y))^d \times$$

$$\times F^{d-1}(y) f^2(y) dy \Big)$$

$$+ O(b^{-1}) .$$

Simplifying the right hand side of the above expression by using

$$\sum_{u \neq i} \Theta_u = t\,\overline{\Theta} - \Theta_i ,$$

we obtain

$$(2.2.20) \qquad \eta_{ij} = d/t - t\, b^{-1/2} (\Theta_i - \overline{\Theta}) \int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^d(y) f^2(y) dy$$

$$+ O(b^{-1}) ,$$

and thus

(2.2.21) $\eta_i = bd/t - tb^{1/2}(\Theta_i - \bar{\Theta})\int_{-\infty}^{\infty}[1-F(y)]^{d-1}F^d(y)f^2(y)dy+O(1).$

Next, let

$$(2.2.22) \quad \mu_{1,1}^{(j)}(i,i) = E(U_{ij}^2) - \eta_{ij}^2$$

$$= E(U_{ij}) - \eta_{ij}^2$$

$$= d(t-d)/t^2 + O(b^{-1/2}) ,$$

and

$$\mu_{1,1}^{(j)}(i,i') = E(U_{ij}U_{i'j}) - \eta_{ij}\ \eta_{i'j} \quad (i \neq i') \ .$$

As before, for a fixed $u \neq i$ or i', let

$$_{u}p^{(j)}_{(i,i');(i_1,i_2,\ldots,i_d)} = \Pr.(X_{i_1j} \leq X_{uj},\ldots X_{i_d,j} \leq X_{uj}; X_{ij} > X_{uj} ,$$

$$X_{i'j} > X_{uj}, X_{i_{d+1}j} > X_{uj}\ldots X_{i_{2d-2},j} > X_{u,j}) ,$$

where $(i_1, i_2,\ldots i_d)$ is a specified combination of d integers from the subset $T(t)-\{i,i',u\}$. Then

$$E(U_{ij}U_{i'j}) = \sum_{u\neq i,i'}\ \sum_{(i_1,i_2,\ldots i_d)}\ {}_{u}p^{(j)}_{(i,i');(i_1,i_2,\ldots i_d)} \ .$$

Following the same procedure as above and after considerable simplification, we get

$$(2.2.23) \quad E(U_{ij}U_{i'j}) = d(d-1)/t(t-1) + O(b^{-1/2}) ,$$

and hence

$$\mu_{1,1}^{(j)}(i,i') = -d(t-d)/t^2(t-1) + O(b^{-1/2}) \,. \tag{2.2.24}$$

Further, let

$$\rho_j^3 = \underset{1 \leq i \leq t}{\text{Max.}} \quad (E|U_{ij} - \eta_{ij}|^3).$$

Then

$$\rho_j^3 \leq 1 \,.$$

Case II $t = 2d$

As in case I,

$$E(m_i) = \sum_{j=1}^{b} E(U_{ij})$$

and

$$E(U_{ij}) = Pr.(X_{ij} > \tilde{X}_j)$$

Let

$$_u p^{(j)}_{i;(i_1,i_2,\ldots i_d)} = Pr[X_{i,j} \leq X_{uj},\ldots,X_{i_d j} \leq X_{uj}; X_{i,j} > X_{u,j}$$

$$X_{i_{d+1}j} > X_{u,j},\ldots\ X_{i_{2d-2}j} > X_{u,j}]$$

where $u \neq i$, $i_k \neq i$ or u and $(i_1,i_2,\ldots i_d)$ is a combination of d integers from the set $T(t) - \{i,u\}$. Further, let

$$_i p^{(j)}_{i;(i_1,i_2,\ldots i_d)} = Pr[X_{i,j} \leq X_{i,j},\ldots X_{i_d j} \leq X_{ij};$$

$$X_{i_{d+1}j} > X_{ij}\ \ldots,\ X_{i_{2d-1}j} > X_{ij}]$$

where $(i_1,i_2,\ldots i_d)$ is a combination of d integers from the subset $T(t) - \{i\}$.

Then

$$E(u_{ij}) = \sum_{(i_1 \ldots i_d) \in T(t) - \{i\}} i \, {}_i p^{(j)}_{i;(i_1, i_2, \ldots, i_d)}$$

$$+ \sum_{u \neq i} \sum_{(i_1, \ldots i_d) \in T(t) - \{i,u\}} {}_u p^{(j)}_{i;(i_1, i_2, \ldots i_d)}$$

Following the same procedure as in case I, we obtain

$$(2.2.25) \quad E(u_{ij}) = d/t - tb^{-1/2}\binom{t-2}{d-1}(\Theta_i - \bar{\Theta}) \int_{-\infty}^{\infty} (1-F(y))^{d-1} F^{d-1}(y) f^2(y)dy + O(b^{-1}) .$$

Similarly,

$$(2.2.26) \quad \mu^{(j)}_{1,1}(i,i) = d(t-d)/t^2 + O(b^{-1/2}) ,$$

$$(2.2.27) \quad \mu^{(j)}_{1,1}(i,i') = -d(t-d)/t^2(t-1) + O(b^{-1/2}) ,$$

and

$$\rho^3_j \leq 1 .$$

The proof of part (a) of theorem 2.2.1 is completed by verifying that the conditions of theorem 2.2.2 hold.

We shall take $\nu(n) = n$ and

$$X_{ij} = U_{ij} - \eta_{ij}$$

Then from (2.2.23), (2.2.24), (2.2.26), and (2.2.27), we have

$$\lim_{b \to \infty} b^{-1} \sum_{j=1}^{b} \mu^{(j)}_{1,1}(i,i') = -d(t-d)/t^2(t-1) \quad \text{if } i \neq i' ,$$

$$= d(t-d)/t^2 \quad \text{if } i \neq i' ,$$

and

$$\lim_{b \to \infty} b^{-1/2} \rho_b = 0$$

Hence as $b \to \infty$, the random vector

$$b^{-1/2}(m_1 - \eta_1, m_2 - \eta_2, \ldots, m_t - \eta_t)$$

has a limiting normal distribution with mean vector (0, 0, ..., 0) and covariance matrix $\Sigma = ((\sigma_{i,i'}))$ where

$$\begin{aligned} \sigma_{i,i'} &= d(t-d)/t^2 && \text{if } i = i' , \\ &= -d(t-d)/t^2(t-1) && \text{if } i \neq i' . \end{aligned}$$

Moreover $\sum_{i'=1}^{t} \sigma_{i,i'} = 0$ for $i = 1, 2, \ldots, t$. Therefore, Σ is a singular matrix. Thus, as $b \to \infty$ the random vector

$$b^{-1/2} \underline{M}$$

has a limiting multivariate singular normal distribution with mean vector $\underline{\mu}$ and covariance matrix $\Sigma = ((\sigma_{i,i'}))$ where μ_i and $\sigma_{i,i'}$ are defined in (2.2.1) and (2.2.2) respectively.

(b) We note first that

$$\sum_{i=1}^{t} m_i = bd$$

and therefore m_i's are linearly dependent. Consider the first t-1 m_i's and denote by

$$\underline{M}_o' = (m_1 - \frac{bd}{t} , m_2 - \frac{bd}{t} , \ldots m_{t-1} - \frac{bd}{t}) .$$

Then by part (a)

$$b^{-1/2} \underline{M}_o$$

has a limit normal distribution with mean vector $\underline{\mu}_o$ and covariance matrix $\Sigma_o = ((\sigma_{0;i,i'}))$ where

$$\mu_{o,i} = \mu_i \quad (i = 1, 2, \ldots, t-1) ,$$

$$\sigma_{0,i,i'} = \sigma_{i,i'} \; (i, i' = 1, 2, \ldots, t-1) .$$

Let

$$\Sigma_o^{-1} = ((\sigma_o^{i,i'})) .$$

Then

$$\sigma_o^{i,i'} = 2t(t-1)/d(t-d) \qquad \text{if } i = i' ,$$

$$= t(t-1)/d(t-d) \qquad \text{if } i \neq i' .$$

Therefore by lemma 1.1.4, the asymptotic distribution of

$$\text{(2.2.28)} \qquad X_M^2 = [t(t-1)/bd(t-d)] \sum_{i=1}^{t} (m_i - \frac{bd}{t})^2$$

$$= \frac{1}{b} \underline{M}_o' \Sigma_o^{-1} \underline{M}_o$$

is a non-central chi-square distribution with t-1 degrees of freedom and non-centrality parameter λ_M^2 given by

$$\lambda_M^2 = \underline{\mu}_o' \Sigma_o^{-1} \underline{\mu}_o$$

$$\text{(2.2.29a)} \qquad = \frac{t^3(t-1)}{d(t-d)} \binom{t-2}{d-1}^2 \left[\int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^d(y) f^2(y) dy\right]^2 \times$$

$$\sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 \qquad \text{if } t \text{ is odd,}$$

$$(2.2.29b) \qquad = \frac{t^3(t-1)}{d(t-d)} \binom{t-2}{d-1}^2 \left[\int_{-\infty}^{\infty} [1-F(y)]^{d-1} F^{d-1}(y) f^2(y)\,dy\right]^2 \times \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 \quad \text{if } t \text{ is even} .$$

2.3 Asymptotic Distribution of $X^2_{Fr.}$ Under H_b .

Theorem 2.3.1:

Let $\{X_{ij}\}$ $(i = 1, 2, \ldots, t;\ j = 1, 2, \ldots, b)$ be bt independent random variables and let

$$F_{ij}(x) = F(x + \nu + \alpha_i + \beta_j)$$

be the cumulative distribution function of X_{ij} where α_i is the effect of the i-th treatment and β_j is the effect of j-th block. Suppose

(i) $F(x)$ is a continuous cumulative distribution function with bounded first and second derivatives $f(x)$ and $f'(x)$ respectively,

(ii) for each b, H_b: $\alpha_i = b^{-1/2}\,\theta_i$, where $\theta_i \neq \theta_{i'}$ for some pair (i, i') .

Let r_{ij} denote the rank of the observation x_{ij} when the t observations in the j-th block are arranged in ascending order of magnitude and further, let

$$r_{i.} = b^{-1} \sum_{j=1}^{b} r_{ij}$$

The the asymptotic distribution as $b \rightarrow \infty$, of

$$(2.3.1) \qquad X^2_{Fr.} = \frac{12b}{t(t+1)} \sum_{i=1}^{t} (r_{i.} - \frac{t+1}{2})^2$$

is a non-central chi-square with t-1 degrees of freedom and non-centrality parameter $\lambda^2_{Fr.}$ given by

$$(2.3.2) \quad \lambda^2_{Fr.} = (12t/(t+1)) \left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2 \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 ,$$

where

$$t\,\bar{\theta} = \sum_{i=1}^{t} \theta_i .$$

Proof: Define new random variables $U^{(j)}_{ii'}$ by

$$(2.3.3) \quad U^{(j)}_{ii'} = 1 \qquad \text{if } X_{ij} > X_{i'j} ,$$
$$= 0 \qquad \text{otherwise.}$$

Since we have assumed that $F_{ij}(x)$ is a continuous cumulative distribution function, therefore

$$\Pr.\,\{X_{ij} = X_{i'j};\ i \neq i'\} = 0 \qquad \text{for each } j.$$

Hence

$$(2.3.4) \qquad r_{ij} = \sum_{i' \neq i}^{t} U^{(j)}_{ii'} + 1.$$

Let

$$\underline{R}' = (r_{11}, r_{21}, \ldots r_{t1}; r_{12}, r_{22}, \ldots r_{t2}; \ldots . r_{1b}, r_{2b}, \ldots, r_{tb}),$$

and

$$(2.3.5) \quad \eta_{ij} = E(r_{ij})$$
$$= \sum_{i' \neq i}^{t} E(U^{(j)}_{ii'}) + 1 .$$

Now under H_b ,

$$E(U^{(j)}_{ii'}) = \Pr.\,(X_{ij} > X_{i'j}) \qquad i \neq i'$$
$$= \int_{-\infty}^{\infty} F(x + \nu + \beta_j + b^{-1/2}\theta_{i'})f(x+\nu+\beta_j+b^{-1/2}\theta_i)\,dx$$

$$= \int_{-\infty}^{\infty} F(y+b^{-1/2}\ (\Theta_{i'} - \Theta_i))f(y)dy\ .$$

Using Taylor's series expansion, we have

$$F(y+b^{-1/2}(\Theta_{i'}-\Theta_i)) = F(y) + b^{-1/2}(\Theta_{i'} - \Theta_i)f(y)+O(b^{-1}).$$

Therefore,

(2.3.6) $$E(U_{ii'}^{(j)}) = \int_{-\infty}^{\infty} F(y)f(y)dy + b^{-1/2}(\Theta_{i'}-\Theta_i) \int_{-\infty}^{\infty} f^2(y)dy+O(b^{-1})$$
$$= 1/2 + b^{-1/2}(\Theta_{i'}-\Theta_i)\int_{-\infty}^{\infty} f^2(y)dy + O(b^{-1})\ .$$

Thus

(2.3.7) $$\eta_{ij} = E(r_{ij})$$
$$= \sum_{i'\neq i}^{t} [1/2+b^{-1/2}(\Theta_{i'}-\Theta_i)\int_{-\infty}^{\infty} f^2(y)dy+O(b^{-1})]+ 1$$
$$= (t+1)/2 + tb^{-1/2}(\bar{\Theta} - \Theta_i)\int_{-\infty}^{\infty} f^2(y)dy + O(b^{-1})\ .$$

Let

$$\mu_{1,1}^{(j)}(i,i) = E(r_{ij}^2) - \eta_{ij}^2\ .$$

Now

$$E(r_{ij}^2) = E[\sum_{i'\neq i}^{t} U_{ii'}^{(j)}]^2 + 2E[\sum_{i'\neq i}^{t} U_{ii'}^{(j)}] + 1$$
$$= \sum_{i'\neq i}^{t} E(U_{ii'}^{(j)})^2 + \sum_{i'\neq i} \sum_{i''\neq i,i'} E(U_{ii'}^{(j)}\ U_{ii''}^{(j)}) + 2\sum_{i'\neq i} E(U_{ii'}^{(j)})+1\ ,$$
$$E(U_{ii'}^{(j)})^2 = E(U_{ii'}^{(j)})$$
$$= 1/2 + b^{-1/2}(\Theta_{i'}-\Theta_i)\int_{-\infty}^{\infty} f^2(y)dy + O(b^{-1}),$$

and

$$E(U_{ii'}^{(j)}\ U_{ii''}^{(j)}) = Pr.\ (X_{ij} > X_{i'j};\ X_{ij} > X_{i''j})$$
$$= \int_{-\infty}^{\infty} F(x+\nu+\beta_j+b^{-1/2}\Theta_{i'})F(x+\nu+\beta_j+b^{-1/2}\Theta_{i''})\ \times$$
$$f(x+\nu+\beta_j+b^{-1/2}\Theta_i)\ dx$$

$$= \int_{-\infty}^{\infty} F(y+b^{-1/2}(\Theta_{i'}-\Theta_i))F(y+b^{-1/2}(\Theta_{i''}-\Theta_i))f(y)dy$$

$$= \int_{-\infty}^{\infty} F^2(y)f(y)dy + O(b^{-1/2})$$

$$= \quad 1/3 + O(b^{-1/2}) \quad .$$

Therefore,

$$E(r_{ij}^2) = (t-1)/2 + (t-1)(t-2)/3+(t-1) + 1 + O(b^{-1/2})$$

$$= \ (t-1)(2t+1)/6 + O(b^{-1/2}) \ ,$$

and hence

$$(2.3.8) \quad \mu_{1,1}^{(j)}(i,i) = (t+1)(2t+1)/6 - (t+1)^2/4 + O(b^{-1/2})$$

$$= (t^2-1)/12 + O(b^{-1/2}) \ .$$

Next, let

$$\mu_{1,1}^{(j)}(i,i') = E(r_{ij}r_{i'j}) - \eta_{ij}\ \eta_{i'j} \qquad (i \neq i')$$

$$= E\left[\sum_{k\neq i}^{t} U_{ik}^{(j)} + 1\right]\left[\sum_{k'\neq i'}^{t} U_{i'k'}^{(j)} + 1\right] - \eta_{ij}\ \eta_{i'j}$$

$$= E\left[\sum_{k\neq i}^{t} U_{ik}\right]\left[\sum_{k'\neq i'}^{t} U_{i'k'}^{(j)}\right] + E\left[\sum_{k\neq i}^{t} U_{ik}^{(j)}\right]$$

$$+ E\left[\sum_{k'\neq i'}^{t} U_{i'k'}^{(j)}\right] + 1 - \eta_{ij}\ \eta_{i,j}$$

$$= E\left[\sum_{k\neq i}^{t} U_{ik}^{(j)}\right]\left[\sum_{k'\neq i'} U_{i'k'}^{(j)}\right] - (t-1)^2/4 + O(b^{-1/2})$$

$$E\left(\sum_{k\neq i}^{t} U_{ik}^{(j)}\right)\left(\sum_{k'\neq i'}^{t} U_{i'k'}^{(j)}\right) = E\left(U_{ii'}^{(j)} \sum_{k'\neq i,i'}^{t} U_{i'k'}^{(j)}\right)$$

$$+ \ E\left(U_{i'i}^{(j)} \sum_{k\neq i,i'}^{t} U_{ik}^{(j)}\right)$$

$$+ \ E\left(\sum_{k\neq i,i'}^{t} U_{ik}^{(j)}\ U_{i'k'}^{(j)}\right)$$

$$+ \ E\left(\sum_{k\neq i,i'}^{t} \sum_{k'\neq i,i',k} U_{ik}^{(j)}\ U_{i'k'}^{(j)}\right) ,$$

$$E(U_{ii'}^{(j)} U_{i'k'}^{(j)}) = \Pr.\ (X_{ij} > X_{i'j};\ X_{i'j} > X_{k'j}) \qquad (i \neq i';k' \neq i,i')$$

$$= \int_{-\infty}^{\infty} F(x+\nu+\Theta_{k'}b^{-1/2}+\beta_j)[1-F(x+\nu+\beta_j+b^{-1/2}\Theta_i)]$$

$$\times f(x+\nu+\beta_j+b^{-1/2}\Theta_{i'})dx$$

$$= \int_{-\infty}^{\infty} F(y)\ [1-F(y)]f(y)dy + O(b^{-1/2})$$

$$= \quad 1/6 + O(b^{-1/2})\ .$$

Similarly,

$$E(U_{i'i}^{(j)} U_{ik}^{(j)}) = 1/6 + O(b^{-1/2}) \qquad \text{if } k \neq i,\ i'\ ,$$

$$E(U_{ik}^{(j)} U_{i'k}^{(j)}) = \Pr.\ (X_{ij} > X_{kj};\ X_{i'j} > X_{kj})$$

$$= \int_{-\infty}^{\infty} [1-F(x+\nu+\beta_j+b^{-1/2}\ \Theta_i)][1-F(x+\nu+\beta_j+b^{-1/2}\Theta_{i'})]$$

$$\times f(x+\nu+\beta_j+b^{-1/2}\ \Theta_k)\ dx$$

$$= \int_{-\infty}^{\infty} (1-F(y))^2 f(y)dy + O(b^{-1/2})$$

$$= \quad 1/3 + O(b^{-1/2})\ ,$$

and

$$E(U_{ik}^{(j)}U_{i'k'}^{(j)}) = \Pr\ (X_{ij} > X_{kj};\ X_{i'j} > X_{k'j}) \qquad \begin{matrix} k \neq i,\ i' \\ k' \neq i,i',k \end{matrix}$$

$$= \left[\int_{-\infty}^{\infty} [1-F(x+\nu+\beta_j+b^{-1/2}\Theta_i)]f(x+\nu+\beta_j+b^{-1/2}\Theta_k)dx\right]$$

$$\times \left[\int_{-\infty}^{\infty} [1-F(x+\nu+\beta_j+b^{-1/2}\Theta_{i'})]f(x+\nu+\beta_j+b^{-1/2}\Theta_{k'})\right.$$

$$\left.\times\ dx\right]$$

$$= \left[\int_{-\infty}^{\infty} [1-F(y)]f(y)dy\right]^2 + O(b^{-1/2})$$

$$= \quad 1/4 + O(b^{-1/2}) \ .$$

Therefore,

(2.3.9) $$\mu_{1,1}^{(j)}(i,i') = \frac{t-2}{6} + \frac{t-2}{6} + \frac{t-2}{3} + \frac{(t-2)(t-3)}{4} - \frac{(t-1)^2}{4} + O\,(b^{-1/2})$$

$$= -(t+1)/12 + O(b^{-1/2}) \qquad i \neq i' \ .$$

Similarly, it may be shown that

$$\rho_j^3 = \max(E|r_{1j} - \eta_{1j}|^3, E(|r_{2j}-\eta_{2j}|^3), \ldots, E|r_{tj} - \eta_{tj}|^3)$$

$$\leq \max\,(E^{3/4}(r_{1j}-\eta_{1j})^4, E^{3/4}(r_{2j}-\eta_{2j})^4, \ldots E^{3/4}(r_{tj}-n_{tj})^4)$$

$$\leq [(3t^2-1)(t^2-1)/240 + O(b^{-1/2})]^{3/4} \ .$$

Therefore,

$$\lim_{b \to \infty} b^{-3/2} \sum_{j=1}^{b} \rho_j^3 = 0 \ .$$

Hence by theorem 2.2.2, as $b \to \infty$, the asymptotic distribution of the random vector

$$b^{-1/2}\left(\sum_{j=1}^{b}(r_{1j}-\eta_{1j}),\ \sum_{j=1}^{b}(r_{2j}-\eta_{2j}),\ \ldots\ \sum_{j=1}^{b}(r_{tj}-\eta_{tj})\right)$$

is a multivariate normal distribution with mean vector (0, 0,...,0) and covariance matrix $\Sigma = ((\sigma_{i,i'}))$, where

(2.3.10) $$\sigma_{i,i'} = (t^2-1)/12 \qquad \text{if } i = i' \ ,$$

$$= -(t+1)/12 \qquad \text{if } i \neq i' \ ,$$

and

$$(2.3.11) \qquad \eta_{ij} = (t+1)/2 + b^{-1/2} t(\bar{\Theta} - \Theta_i) \int_{-\infty}^{\infty} f^2(x)dx \ .$$

Now

$$\sum_{i=1}^{t} r_{ij} = t(t+1)/2 \qquad ,$$

and

$$\sum_{i'=1}^{t} \sigma_{i,i'} = 0 \ . \qquad (i-1,2,\ldots,t)$$

Therefore, the asymptotic distribution is singular, considering the first t-1 of r_{ij}'s and applying lemma 1.4.1, it may be shown that as $b \longrightarrow \infty$, the asymptotic distribution of $X^2_{Fr.}$ is a non-central chi-square with t-1 degrees of freedom and non-centrality parameter $\lambda^2_{Fr.}$ given by

$$(2.3.12) \qquad \lambda^2_{Fr.} = (12t/t+1) \left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2 \sum_{i=1}^{t} (\Theta_i - \bar{\Theta})^2 \ .$$

2.4 Asymptotic Relative Efficiency of Mood's Test With Respect To Friedman's Test and The F-Test.

In this section, we compare Mood's test with Friedman's test and the F-test. As already noted in section 1.4, the asymptotic relative efficiency of one test with respect to another is the ratio of their non-centrality parameters. Thus

$$(2.4.1a) \quad e_{M.,Fr.} = \frac{t^2(t^2-1)}{12d(t-d)} \binom{t-2}{d-1}^2 \left[\int_{-\infty}^{\infty} \left[1-F(x)\right]^{d-1} F^d(x) f^2(x)dx\right]^2 \Big/ \left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2 \text{ if } t=2d+1,$$

$$(2.4.1b) \qquad = \frac{t^2(t^2-1)}{12d(t-d)} \binom{t-2}{d-1}^2 \left[\int_{-\infty}^{\infty} \left[1-F(x)\right]^{d-1} F^{d-1}(x) f^2(x)dx\right]^2 \Big/ \left[\int_{-\infty}^{\infty} f^2(x)dx\right]^2 \text{ if } t=2d,$$

and compared with F-test,

$$(2.4.2a)\quad e_{M.F} = \frac{t^3(t-1)}{d(t-d)} \binom{t-2}{d-1}^2 \Big[\sigma\int_{-\infty}^{\infty} [1-F(x)]^{d-1}F^{d}(x)f^2(x)dx\Big]^2$$

if $t=2d+1$,

$$(2.4.2b)\quad = \frac{t^3(t-1)}{d(t-d)} \binom{t-2}{d-1}^2 \Big[\sigma\int_{-\infty}^{\infty} [1-F(x)]^{d-1}F^{d-1}(x)f^2(x)dx\Big]^2$$

if $t=2d$,

where

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \Big[\int_{-\infty}^{\infty} x\, f(x)dx\Big]^2 .$$

Example 2.4.1: Normal Distribution

$$F(x) = \int_{-\infty}^{x} (2\pi)^{-1/2}\, \mathrm{Exp}[-1/2\ t^2]dt \quad (-\infty < x < \infty)$$

Table 2.4.1

t	2	3	4	5	6	7	8	9	10
$e_{M.Fr.}$	1.00	0.75	0.77	0.72	0.73	0.70	0.71	0.70	0.70
$e_{M.F.}$	0.77	0.54	0.59	0.57	0.59	0.60	0.60	0.60	0.61

t	11	12	13
$e_{M.Fr.}$	0.69	0.69	0.69
$e_{M.F}$	0.61	0.61	0.61 .

Example 2.4.2: Exponential Distribution.

$$F(x) = 1 - \mathrm{Exp}[-x] \qquad (x \geq 0),$$
$$\quad = 0 \qquad (x < 0).$$

(2.4.3a) $\quad e_{M.Fr.} = 1/3 \qquad$ t is odd ,

(2.4.3b) $\quad = (t+1)/3(t-1) \qquad$ t is even ,

(2.4.4a) $\quad e_{M.F} = t/(t+1) \qquad$ t is odd,

(2.4.4b) $\quad = t/(t-1) \qquad$ t is even.

Table 2.4.2

t	2	3	4	5	6	7	8	9	10
$e_{M.Fr.}$	1.00	0.33	0.56	0.33	0.47	0.33	0.43	0.33	0.41
$e_{M.F.}$	2.00	0.75	1.33	0.83	1.20	0.87	1.14	0.90	1.11

t	11	12	13
$e_{M.Fr.}$	0.33	0.39	0.33
$e_{M.F.}$	0.92	1.09	0.93 .

Example 2.4.3: Rectangular Distribution.

$$F(x) = 0 \qquad (x < 0),$$
$$\quad = x \qquad (0 \leq x \leq 1),$$
$$\quad = 1 \qquad (x > 1).$$

(2.4.5a) $\quad e_{M.Fr.} = t^3/3(t-1)^2 \qquad$ if t is odd,

(2.4.5b) $\quad = (t+1)/3(t-1) \qquad$ if t is even,

(2.4.6a) $\quad e_{M.F} = t^3/3(t+1)(t-1)^2 \qquad$ if t is odd,

(2.4.6b) $\quad = t/3(t-1) \qquad$ if t is even .

Table 2.4.3

t	2	3	4	5	6	7	8	9	10
$e_{M.Fr.}$	1.00	0.75	0.56	0.52	0.47	0.45	0.43	0.42	.41
$e_{M.F}$	0.67	0.56	0.44	0.43	0.40	0.40	0.38	0.38	.37

t	11	12	13
$e_{M.Fr.}$	0.40	0.39	0.39
$e_{M.F}$	0.37	0.36	0.36

Example 2.4.4: Double Exponential Distribution.

$$F(x) = 1/2\ \mathrm{Exp}[x] \qquad (x < 0)\ ,$$
$$= 1 - 1/2\ \mathrm{Exp}[-x] \qquad (x \geq 0)\ .$$

(2.4.7a) $$e_{M.Fr.} = \frac{4t^2(t^2-1)}{3d(t-d)} \binom{t-2}{d-1}^2 [2^{-t}(\beta(d,1/2) - \beta(d,1))]^2$$

for all t,

(2.4.7b) $$e_{M.F} = \frac{4t^3(t-1)}{d(t-d)} \binom{t-2}{d-1}^2 [2^{-t}(\beta(d,1/2)-\beta(d,1))]^2 \text{ for all } t,$$

where

$$\beta(m,n) = \Gamma(m)\ \Gamma(n)\ /\ \Gamma(m+n)\ .$$

Table 2.4.4

t	2	3	4	5	6	7	8	9
$e_{M.Fr.}$	1.00	0.75	0.87	0.81	0.88	0.86	0.91	0.89
$e_{M.F.}$	1.00	0.84	1.04	1.02	1.13	1.13	1.21	1.20

t	10	11	12	13
$e_{M.Fr.}$	0.93	0.92	0.94	0.94
$e_{M.F.}$	1.26	1.26	1.31	1.31

2.5 Asymptotic Distribution of Mood's Test Criterion Generalized To Balanced Incomplete Block Designs.

Bhapkar [4] has generalized Mood's test for two-way classification with one or equal number of observations in each cell to the case of incomplete block designs. In particular, for a balanced incomplete block design with parameters t, b, r, k and λ, the test statistics is

$$(2.5.1)\quad \chi^2_{M;B.I.B.} = \frac{k^2(k-1)}{d(k-d)\lambda t} \sum_{i=1}^{t} \left(m_i - \frac{rd}{k}\right)^2 ,$$

where $d = k/2$ if k is even and $(k-1)/2$ if k is odd, and m_i is as defined before. Unless b is small, Bhapkar has shown that the asymptotic distribution of $\chi^2_{M.;B.I.B.}$ is chi-square with $t-1$ degrees of freedom when the null hypothesis of equality of treatment effects holds.

In this section, we shall consider the asymptotic distribution of $\chi^2_{M.;B.I.B.}$ as $r \to \infty$, under H_r where H_r is defined in (2.5.3).

Theorem 2.5.1.

Consider a balanced incomplete block design with parameters t, b, r, k and λ. Let X_{ij} (real valued) denote the chance variable when the i-th treatment occurs in j-th block and let $F_{ij}(x)$ denote the cumulative distribution function of X_{ij}. Suppose

$$(2.5.2)\qquad F_{ij}(x) = F(x + \nu + \beta_j + \alpha_i) ,$$

where $F(x)$ is a continuous cumulative distribution function with bounded first and second derivatives $f(x)$ and $f'(x)$ respectively, α_i is the i-th treatment effect and β_j is the j-th block effect. Consider the alternatives H_r given by

(2.5.3) H_r: $\alpha_i = r^{-1/2}\,\theta_i$, where $\theta_i \neq \theta_{i'}$ for some pair (i,i').

Then as $r \to \infty$, the asymptotic distribution of $\chi^2_{M;B.I.B.}$ is non central chi-square with t-1 degrees of freedom and non-centrality parameter λ^2 , given by

$$(2.5.4a)\quad \lambda^2_{M.B.I.B.} = \frac{vk^2(k-1)^2}{(v-1)d(k-d)} \left[\int_{-\infty}^{\infty} [1-F(x)]^{d-1} F^{d}(x) f^2(x)\,dx\right]^2 \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 \quad \text{if } k = 2d+1 ,$$

$$(2.5.4b)\quad = \frac{vk^2(k-1)^2}{(v-1)(k-d)} \left[\int_{-\infty}^{\infty} [1-F(x)]^{d-1} F^{d-1}(x) f^2(x)\,dx\right]^2 \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 \quad \text{if } k = 2d ,$$

where $t\,\bar{\theta} = \sum_{i=1}^{t} \theta_i$.

Proof: Suppose the i-th treatment occurs in $j_1^{(i)}, j_2^{(i)}, \ldots j_r^{(i)}$ blocks. Further, let $\tilde{X}_j$ denote the median of the observations in the j-th block. Defining new random variables U_{ij} as in (2.2.4), we have

$$m_i = \sum_{k=1}^{r} U_{i,j_k^{(i)}} ,$$

and

$$\eta_i = E(m_i) = \sum_{k=1}^{r} E(U_{i,j_k^{(i)}}) .$$

Then it may be shown as before, that when H_r holds,

$$(2.5.5a) \qquad \eta_i = rd/k - \frac{r^{1/2}(t)(k-1)}{(t-1)} \binom{k-2}{d-1} (\theta_i - \bar{\theta}) \times \int_{-\infty}^{\infty} [1-F(x)]^{d-1} F^d(x) f^2(x) dx + O(r^{-1})$$

if $k = 2d+1$,

$$(2.5.5b) \qquad = rd/k - \frac{r^{1/2}\ t(k-1)}{(t-1)} \binom{k-2}{d-1} (\theta_i - \bar{\theta}) \times \int_{-\infty}^{\infty} [1-F(x)]^{d-1} F^{d-1}(x) f^2(x) dx + O(r^{-1})$$

if $k = 2d$,

and

$$(2.5.6) \qquad \text{var}\ m_i = \sigma_{i,i} = rd(k-d)/k^2 + O(r^{-1/2}),$$

$$(2.5.7) \qquad \text{cov}\ (m_i, m_{i'}) = \sigma_{i,i'} = -d\ \lambda(k-d)/k^2(k-1) + O(r^{-1/2}).$$

The asymptotic normality of m_i's as $r \to \infty$ can be proved by using Bernstein's theorem. Hence using lemma 1.4.1, the asymptotic distribution of $\chi^2_{M.B.I.B.}$ is non-central chi-square with t-1 degrees of freedom and non-centrality parameter $\lambda^2_{M.B.I.B.}$ given by (2.5.4a) and (2.5.4b). We observe that when $k = t$, (2.5.4a) and (2.5.4b) reduce to the formulae given by (2.2.3a) and (2.2.3b) respectively.

2.6 Asymptotic Relative Efficiency Of Mood's Test For Balanced Incomplete Block Designs Compared With Durbin's Test and The F-Test.

Durbin [9] has generalized Friedman's rank test for randomized blocks to the case of balanced incomplete block designs. His test statistics $\chi^2_{D;\, B.I.B.}$, is as follows:

$$(2.6.1)\quad \chi^2_{D.;B.I.B.} = [12/t\lambda(k+1)] \sum_{i=1}^{t} [r_i - \frac{r(k+1)}{2}]^2,$$

where

r_i = sum of the ranks of observations on the i-th treatment.

It is shown by Van Elteren and Noether [27], that as $r \rightarrow \infty$, the asymptotic distribution of $\chi^2_{D.;B.I.B.}$ is a non-central chi-square with t-1 degrees of freedom and non-centrality parameter, $\lambda^2_{D.;B.I.B.}$ given by

$$(2.6.2)\quad \lambda^2_{D.;B.I.B.} = [12t(k-1)/(t-1)(k+1)][\int_{-\infty}^{\infty} f^2(x)dx]^2 \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2 .$$

Also, for the F-statistic, the corresponding non-centrality parameter given in Anderson and Bancroft [1] is

$$(2.6.3)\quad \lambda^2_{F.B.I.B.} = \frac{t(k-1)}{(t-1)k} \sum_{i=1}^{t} (\theta_i - \bar{\theta})^2/\sigma^2,$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - [\int_{-\infty}^{\infty} x\, f(x)dx]^2 .$$

Thus, by using Hannan's result [12], we have

$$(2.6.4a)\quad e_{M.D.;(B.I.B)} = \frac{\binom{k-2}{d-1}^2 \frac{k^2(k^2-1)}{12d(k-d)} [\int_{-\infty}^{\infty} [1-F(x)]^{d-1} F^d(x) f^2(x)dx]^2}{[\int_{-\infty}^{\infty} f^2(x)dx]^2}$$

if $k = 2d+1$,

(2.6.4b)
$$= \frac{\binom{k-2}{d-1}^2 k^2(k^2-1) \left[\int_{\infty} \left[1-F(x)\right]^{d-1} F^{d-1}(x) f^2(x) dx\right]^2}{12d(k-d)\left[\int_{-\infty}^{\infty} f^2(x) dx\right]^2}$$

if $k = 2d$,

and

(2.6.5a)
$$e_{M.F.(B.I.B.)} = \frac{k^3(k-1)}{d(k-d)} \binom{k-2}{d-1}^2 \left[\sigma \int_{-\infty}^{\infty} \left[1-F(x)\right]^{d-1} F^d(x) f^2(x) dx\right]^2$$

if $k = 2d + 1$,

(2.6.5b)
$$= \frac{k^3(k-1)}{d(k-d)} \binom{k-2}{d-1}^2 \left[\sigma \int_{-\infty}^{\infty} \left[1-F(x)\right]^{d-1} F^{d-1}(x) f^2(x) dx\right]^2$$

if $k = 2d$.

We note that both $e_{M.D.(B.I.B)}$ and $e_{M.F.(B.I.B)}$ depend only on $F(x)$ and k i.e. the block size.

CHAPTER III

TESTS FOR HYPOTHESES UNDER THE EXPONENTIAL REGRESSION MODEL AND FOR HYPOTHESES ON BIVARIATE LOCATION PARAMETERS

3.1. Introduction

Suppose we have a random vector, $z = (x, y)$ whose cumulative distribution function, $F(x, y)$ is continuous. Let $z_1, z_2, \ldots, z_n$ be a random sample of n paired observations each having the same cumulative distribution function $F(x, y)$. Furthermore, we assume that the variate y given x, is continuous and has median of the form $m(x)$, which is an unknown function of x. Mood and Brown [20] have considered $m(x) = \alpha + \beta x$ and suggested methods for estimating α and β and testing hypotheses about the values of α and β. Later, Bhapkar [4] has considered some additional regression problems along the lines of Mood and Brown. An entirely different approach for estimating α and β, by confidence intervals has been presented in Theil [26]. He assumes only a symmetric distribution about the origin for the deviations from the regressions. A generalization to other forms of $m(x)$ has also been indicated in Theil [26].

In section 2, we assume in the model, that the conditional distribution of y given x, is symmetric about the median $m(x)$ which is of the form $\mathrm{Exp}(\beta x)$ and then test for $\beta = \beta_0$ by using Theil's procedure. Another test based on the quantiles of the conditional distribution of y given x, has been proposed recently by Bhattacharya [5] for testing under a perfectly general model, the hypothesis that the median is a completely specified function of x.

In the last section, a step-down procedure is suggested for testing the equality of k bivariate distributions which are identical in form except for the location parameters. It should be remembered that while considering this same hypothesis, Bhapkar [4] has assumed in his model that the conditional distributions of y's given x's are identical except for a location parameter which is a linear function of x. However, in our case, we have relaxed that assumption. This is possible because we have reduced the problem to testing the equality of marginal distributions of x's and then testing the equality of conditional distributions of y's given that the corresponding x is less than the sample median of x's.

A further extension to two-way classification is under consideration.

3.2 A Nonparametric Test For Testing the Hypothesis $\beta = \beta_0$, When the Conditional Median of y given x, is assumed to be $Exp[\beta x]$ Under the Model.

Let $(x_1,y_1), (x_2,y_2),\ldots,(x_n,y_n)$ be an independent random sample from a population with continuous cumulative distribution function $F(x,y)$ and continuous marginal distribution functions $F_1(x)$ and $F_2(y)$. Suppose for given x, the distribution of y is symmetric about $Exp[\beta x]$, where β is an unknown parameter. We want to test the hypothesis

$$H_o: \ \beta = \beta_o \quad ,$$

where β_o is a given constant.

Suppose $x_{[1]} \leq x_{[2]} \cdots \leq x_{[n]}$ denote the x's $x_1, x_2, \ldots, x_n$ arranged in an ascending order of magnitude. Because of the continuity of $F_1(x)$, the probability of the event $[x_i = x_j \quad i \neq j]$ is zero so we may consider $x_{[i]}$'s to be in a strictly ascending order. Let $n_1 = (n-1)/2$ if n is odd and equal to $[\frac{n}{2}]$, i.e., the largest integer contained in $n/2$ if n is even. We shall assume $x_{[i]}$'s to be fixed and define new random variables $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ as follows:

$$y_i = \mathrm{Exp}[\beta \ x_i] + \epsilon_i \quad , \tag{3.2.1}$$

Then the distribution of ϵ_i is symmetric about the origin. Let $y_{[i]}$ and $\epsilon_{[i]}$ denote the y and ϵ corresponding to $x_{[i]}$.

Consider

$$y_{[i]} = e^{\beta x_{[i]}} + \epsilon_{[i]} \quad ,$$

and

$$y_{[n_1+i]} = \mathrm{Exp}[\beta x_{[n_1+i]} + \epsilon_{[n_1+i]}]$$

$$(i = 1, 2, \ldots, n_1) \ .$$

Then

$$(y_{[n_1+i]} - \epsilon_{[n_1+i]})/(y_{[i]} - \epsilon_{[i]}) = \mathrm{Exp}[\beta(x_{[n_1+i]} - x_{[i]})].$$

Therefore,

$$y_{[n_1+i]} - y_{[i]}\ \text{Exp}[\beta(x_{[n_1+i]} - x_{[i]})]$$

$$= \epsilon_{[n_1+i]} - \epsilon_{[i]}\ \text{Exp}[\beta(x_{[n_1+i]} - x_{[i]})].$$

Since $\epsilon_{[n_1+i]}$ and $\epsilon_{[i]}$ have distributions which are symmetric about the origin, therefore $\epsilon_{[n_1+i]} - \epsilon_{[i]}$ $\text{Exp}[\beta(x_{[n_1+i]} - x_{[i]})]$ has a symmetric distribution about the origin.

Thus

$$(3.2.2)\qquad \Pr.[y_{[n_1+i]} \leq y_{[i]}\ \text{Exp}\ [\beta(x_{[n_1+i]} - x_{[i]})]]$$

$$= \Pr.[y_{[n_1+i]} > y_{[i]}\ \text{Exp}\ \beta(x_{[n_1+i]} - x_{[i]})]$$

$$= 1/2\ ,$$

and therefore,

$$(3.2.3)\quad \Pr.\left\{ \frac{\log y_{[n_1+i]} - \log y_{[i]}}{x_{[n_1+i]} - x_{[i]}} \leq \beta \right\} =$$

$$= \Pr.\left\{ \frac{\log y_{[n_1+i]} - \log y_{[i]}}{x_{[n_1+i]} - x_{[i]}} \geq \beta \right\}$$

$$= 1/2\ .$$

If any one of the y_i's is negative, we replace it by its absolute value .

Define new random variables $z_{[i]}$ by

$$z_{[i]} = \log|y_{[n_1+i]}| - \log|y_{[i]}|/x_{[n_1+i]} - x_{[i]} \tag{3.2.4}$$

Then, under the null hypothesis $z_{[1]}, z_{[2]}, \ldots, z_{[n_1]}$ have medians β_o and hence we can use the sign test.

Instead of considering the y's corresponding to $x_{[i]}$ and $x_{[n_1+i]}$, we could as well have considered the y's corresponding to $x_{[i]}$ and $x_{[j]}$ $(i < j)$. Then the test would have been based on $n(n-1)/2$ observations on z's and for large n, this would involve more computations.

K-Samples:

Let (x_{ij}, y_{ij}) $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$, be $N = \sum_{i=1}^{k} n_i$ independent observations and suppose for each i, the conditional distribution of y given x, is symmetric about $\mathrm{Exp}\,[\beta_i\, x]$. We want to test the hypothesis

$$H_k\colon\ \beta_1 = \beta_2 = \ldots = \beta_k = \beta_o \ (\beta_o \text{ a given constant})$$

Then following the above procedure, we introduce the random variables z_{ij} $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_{1,i})$ where $n_{1,i} = (n_i - 1)/2$ if n_i is odd or $n_i/2$ if n_i is even. Then under H_k, each z_{ij} has median β_o and hence we can use the sign test.

3.3 Test for the Equality of Location Parameters for Bivariate Distributions

Let (x_{ij}, y_{ij}) $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ be $N = \sum_{i=1}^{k} n_i$ independent observations. For each i, let $F_i(x,y)$

denote cumulative distribution function of (x_{ij}, y_{ij}). Suppose

$$F_i(x,y) = F(x - \xi_i, y - \eta_i),$$

where $F(x, y)$ is a continuous cumulative distribution function with density function, $f(x,y)$ and marginal densities $f_1(x)$ and $f_2(y)$. The null hypothesis to be tested is given by

$$(3.3.1) \qquad H_o: \begin{array}{l} \xi_1 = \xi_2 = \cdots = \xi_k, \\ \eta_1 = \eta_2 = \cdots = \eta_k. \end{array}$$

We resolve the hypothesis H_o into two hypotheses H_{01} and H_{02} given by

$$(3.3.2) \qquad H_{01}: \xi_1 = \xi_2 = \cdots = \xi_k,$$

$$(3.3.3) \qquad H_{02}: \eta_1 = \eta_2 = \cdots = \eta_k,$$

and first test H_{01} and if H_{01} is accepted, then test H_{02}.

(a) Test for H_{01}:

Suppose $N = 4r + 1$ so that the sample medians determined later are unique. We shall denote by $x'_{(1)} \leq x'_{(2)} \cdots \leq x'_{(N)}$ the N observations x_{ij} $(i = 1,2,\ldots,k;\ j = 1,2,\ldots,n_i)$ arranged in ascending order of magnitude. Let a_i denote the number of observations from the i-th sample which are less than or equal to $x'_{(2r+1)}$ i.e. the sample median of x_{ij}'s. Further, let

$$(3.3.4) \qquad P_i(x'_{(2r+1)}) = \int_{-\infty}^{x'(2r+1)} f_1(x - \xi_i)\, dx.$$

Then, under the hypothesis, H_{01}

$$(3.3.5)\quad P_1(x'_{(2r+1)}) = P_2(x'_{(2r+1)}) = \dots = P_k(x'_{2r+1)}) = P_o(x'_{(2r+1)}) \text{ say}$$

and the joint probability density function of $\{a_i\}$ and $P_o(x'_{(2r+1)})$ is given by

$$(3.3.6)\quad f(a_1,a_2,\dots a_k;\ P_o(x'_{(2r+1)}) = \prod_{i=1}^{k}\binom{n_i}{a_i} \times \sum_{i=1}^{k} a_i$$

$$P_o^{2r}(x'_{(2r+1)})(1-P_o(x'_{(2r+1)}))^{2r} f_1(x'_{(2r+1)}) ,$$

where the i-th term in the summation signifies that $x'_{(2r+1)}$ belongs to the i-th sample.

Since $\sum_{i=1}^{k} a_i = (2r+1)$, therefore we have

$$(3.3.7)\quad f(a_1,a_2,\dots,a_k;\ P_o(x'_{(2r+1)}) = (2r+1) \prod_{i=1}^{k} \binom{n_i}{a_i}$$

$$P_o^{2r}(x'_{(2r+1)})(1 - P_o(x'_{(2r+1)}))^{2r} x' f_1(x'_{(2r+1)}) .$$

Thus $\{a_i\}$ and $x'_{(2r+1)}$ are independently distributed under the hypothesis H_{01} and moreover, the marginal distribution of a_i is a hypergeometric distribution. Hence for large N, we can use Mood's test which is given by the test statistic

$$(3.3.8)\quad X^2_{1,M} = \frac{N(N-1)}{2r(2r+1)} \sum_{i=1}^{k} \frac{1}{n_i}\left(a_i - \frac{2n_i r}{N}\right)^2 .$$

(b) <u>Test for H_{02} assuming H_{01} holds:</u>

If H_{01} is not rejected, then we proceed to test H_{02}, consider the conditional density of y_{ij}, given $x_{ij} \leq x'_{(2r+1)}$ and denote by

$$(3.3.9) \quad \psi_i(y|x'_{(2r+1)}) = \int_{-\infty}^{x'_{(2r+1)}} f(x-\xi_i, y-\eta_i)dx \Big/ \int_{-\infty}^{\infty}\int_{-\infty}^{x'_{(2r+1)}} f(x-\xi_i, y-\eta_i)dx\, dy .$$

Then if H_{02} is true, when H_{01} is not rejected, we obtain

$$(3.3.10) \quad \psi_1(y|x'_{(2r+1)}) = \psi_2(y|x'_{(2r+1)}) = \ldots = \psi_k(y|x'_{(2r+1)}) = \psi_o(y|x'_{(2r+1)}) \text{ say.}$$

Let us denote by y^* the y for which $x \leq x'_{(2r+1)}$ and let y^*_{ij} $(i=1,2,\ldots,k;\ j=1,2,\ldots,a_i)$ denote the y_{ij}'s from the i-th sample for which the corresponding $x_{ij} \leq x'_{(2r+1)}$. Further, denote by $y^*_{(1)} \leq y^*_{(2)} \ldots \leq y^*_{(2r+1)}$ the y^*_{ij}'s $(i=1,2,\ldots,k;$ $j=1,2,\ldots,a_i)$, arranged in an ascending order of magnitude and let b_i denote the number observations from the i-th sample such that $y^*_{ij} \leq y^*_{(r+1)}$, that is, less than the median of the observations y^*_{ij}.

We shall write

$$(3.3.11) \quad \overline{P}_i = \int_{-\infty}^{y^*_{(r+1)}} \psi_i(y|x'_{(2r+1)})\, dy ,$$

and if (3.3.10) holds, we have

$$(3.3.12) \quad \overline{P}_1 = \overline{P}_2 = \ldots = \overline{P}_k = \overline{P}_o$$

and the joint probability density function of $\{b_i\}$ and $\bar{P}_o$ given $\{a_i\}$ and $x'_{(2r+1)}$ is given by

$$(3.3.13)\quad f(b_1,b_2,\ldots,b_k;\ \bar{P}_o \mid a_1,a_2,\ldots,a_k; x'_{(2r+1)}) = \prod_{i=1}^{k} \binom{a_i}{b_i} \bar{P}_o^{r} (1-\bar{P}_o)^{r} \sum_{i=1}^{k} b_i .$$

But $\sum_{i=1}^{k} b_i = r+1$,

therefore,

$$(3.3.14)\quad f(b_1,b_2,\ldots,b_k;\ P_o \mid a_1,a_2,\ldots,a_k;\ x'_{(2r+1)}) = \left[\prod_{i=1}^{k} \binom{a_i}{b_i} \bar{P}_o^{r} (1-\bar{P}_o)^{r}\right](r+1) .$$

Thus $\{b_i\}$ and $\bar{P}_o$ given $\{a_i\}$ and $x'_{(2r+1)}$ are independently distributed and $\{b_i\}$ have a hypergeometric distribution. Hence we can again use Mood's test given by the test statistic

$$(3.3.15)\quad X^2_{2,M} = \frac{(2r+1)2r}{r(r+1)} \sum_{i=1}^{k} \frac{1}{a_i} \left(b_i - \frac{a_i r}{2r+1}\right)^2$$

For large values of N, the asymptotic distribution of $X^2_{1,M}$ can be approximated by a chi-square distribution with k-1 degrees of freedom and the distribution of $X^2_{2,M}$ can also be approximated by a chi-square distribution with k-1 degrees of freedom even if 2r is only of the order of twenty provided all the a_i's are at least five Mood (19). Thus, in this sense, $X^2_{1,M}$ and $X^2_{2,M}$ are asymptotically independent. Therefore, if the two tests are performed at the significance levels α_1 and α_2 respectively, then the over-all significance for H_0 is equal to $1-(1-\alpha_1)(1-\alpha_2)$.

Finally, we observe two things about the above procedure. Firstly, we could have used y's first and test for H_{02} and then test for H_{01}. In many problems, there might be some natural way to effect the resolution, for example, the more important hypothesis is tested first, since the test stops if the first hypothesis is rejected. Secondly, instead of considering the y's such that the corresponding x's $\leq x'_{(2r+1)}$, we may have considered the y's such that the corresponding x's $> x'_{(2r+1)}$. Then we would have obtained another test statistic $X^2_{3,M}$ and since both $X^2_{2,M}$ and $X^2_{3,M}$ are asymptotically independent, therefore, $X^2_{2,M} + X^2_{3,M}$ will also be asymptotically distributed as chi-square with (2k-2) degrees of freedom. Thus at the second stage, we have three test statistics $X^2_{2,M}$, $X^2_{3,M}$ and $X^2_{2,M} + X^2_{3,M}$.

CHAPTER IV

SOME PROBLEMS IN THE CATEGORICAL SET-UP

4.1 Introduction.

In this chapter, we shall be concerned with some problems which arise when the data are presented in the form of observed frequencies. We shall call a variable a factor or a response according as the corresponding marginal frequencies are held fixed or not. The appropriate probability model will be a single multinomial or a product multinomial according as whether all the variables are responses or only some of them are responses and others factors. It is assumed that the cell probabilities are functions of k unknown parameters, $\theta_1, \theta_2, \ldots, \theta_k$ which can be represented as a point θ in k-dimensional Euclidean space. The null hypothesis usually specifies that θ lies in a subset which can be represented by r functional relationships $h_i(\theta) = 0$ $i = 1, 2, \ldots, r \leq k$. Various hypotheses both in the spirit of dependence and analysis of variance have been posed by Roy [23], Mitra [18] and Bhapkar [4]. The most common tests used in this field are the Wilks' likelihood ratio test, Karl Pearson's chi-square test, Neyman's modified chi-square test and his other test obtained by using linearization technique. The corresponding test statistics are all asymptotically equivalent and have asymptotic chi-square distribution when the null hypothesis is true. The major difficulty which is common to all the above methods lies in the fact that it is extremely difficult to solve the minimizing or maximizing equations and to obtain the estimates of the parameters explicitly, when they are subject to constraints. One way of avoiding this difficulty is to use Wald's method

[28] wherein the test statistic involves only the estimates of the parameters without any restrictions. The calculation of the corresponding test statistic involves only an inversion of a matrix which can be done by the use of an electronic computer.

In Section 2, we use this approach to test two-by-two independence in a 2 x 2 x 2 table and also to test the equality of two marginal distributions in an r x r table.

In the last section, we consider some hypotheses which can be posed in the categorical set-up where each response has the same number of categories. The hypotheses of symmetry and independence are considered, and the non-centrality parameters of the corresponding test statistics under alternatives of the Pitman type are obtained.

4.2 Wald's Test and Its Use for Testing Some Hypotheses in the Categorical Set-Up.

We first state Wald's test. Let $f(x_1, x_2, \ldots, x_m; \ \theta_1, \theta_2, \ldots, \theta_k)$ be the joint probability density function of the variates $x_1, x_2, \ldots, x_m$ involving k unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$ which can be represented by a point θ of a subset Ω of k-dimensional Euclidean space. Ω may or may not be the entire k-dimensional space. Let ω be the subset of Ω defined by the equations

$$(4.2.1) \qquad h_1(\theta) = h_2(\theta) = \ldots = h_r(\theta) = 0 \qquad (r \leq k) \quad .$$

Denote by H_ω the hypothesis that the true parameter point θ is in ω.

Let $\hat{\theta}_n$ denote the point with coordinates $\hat{\theta}_{1,n}, \hat{\theta}_{2,n}, \ldots, \hat{\theta}_{k,n}$ where $\hat{\theta}_{i,n}$ is the unrestricted maximum likelihood estimate of θ_i based on n independent observations on $x_1, x_2, \ldots, x_m$. The expected value of

$$-\partial^2 \log f(x_1, x_2, \ldots, x_m; \Theta)/\partial\Theta_i\partial\Theta_j$$

is denoted by $c_{ij}(\Theta)$ and $C(\Theta)$ denotes the matrix $((c_{ij}(\Theta)))$. Let

$$\Sigma = ((\sigma_{ij}(\Theta))) = ((c_{ij}(\Theta)))^{-1} \quad ,$$

and denote

(4.2.2) $$\sum_{\ell=1}^{k} \sum_{m=1}^{k} (\partial h_p(\Theta)/\partial\Theta_\ell)(\partial h_q(\Theta)/\partial\Theta_m)\sigma_{\ell,m}(\Theta)$$

by $\sigma^*_{p,q}$ and

(4.2.3) $$B(\Theta) = ((b_{p,q})) = ((\sigma^*_{p,q}(\Theta)))^{-1} \quad .$$

Then Wald [28] has shown that when the number of observations n is large, the statistic

(4.2.4) $$X_W^2 = n \sum_{p=1}^{r} \sum_{q=1}^{r} h_p(\hat{\Theta}_n)h_q(\hat{\Theta}_n)b_{p,q}(\hat{\Theta}_n)$$

has a chi-square distribution with r degrees of freedom when the hypothesis H_ω is true.

The matrix B and the test statistic, X_W^2 can be written in the following form:

Let

(4.2.5) $$\underset{1 \times r}{\underline{h}'(\Theta)} = (h_1(\Theta), h_2(\Theta), \ldots, h_r(\Theta)) \quad ,$$

and let H_Θ denote the k x r matrix

$$((\partial h_j(\Theta)/\partial\Theta_i)) \qquad (i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, r).$$

Then

(4.2.6) $$B_\Theta = (H_\Theta' C_\Theta^{-1} H_\Theta)^{-1} \quad ,$$

and

$$(4.2.7) \qquad X_w^2 = nh'(\hat{\Theta}_n)B(\hat{\Theta}_n)h(\hat{\Theta}_n) \quad .$$

Example 4.2.1 Two by two independence in a 2 x 2 x 2 table.

Consider three responses "i", "j" and "k" each with two categories. Suppose we have a sample of n observations and let n_{ijk} denote the number of observations in (i, j, k) cell. The corresponding -rpbabo'otu density model is

$$(4.2.8) \qquad \phi = \frac{n!}{\prod_{i,j,k}^{2} n_{ijk}!} \prod_{i,j,k}^{2} p_{ijk}^{n_{ijk}} \quad ,$$

where

$$\sum_{i,j,k}^{2} p_{ijk} = 1 \quad , \quad \text{and} \quad p_{ijk} > 0 \quad .$$

Let

$$p_{ijo} = \sum_{k=1}^{2} p_{ijk} \quad , \quad p_{iok} = \sum_{j=1}^{2} p_{ijk} \quad ,$$

$$p_{ojk} = \sum_{i=1}^{2} p_{ijk} \quad , \quad p_{ioo} = \sum_{k=1}^{2} \sum_{j=1}^{2} p_{ijk} \quad ,$$

$$p_{ojo} = \sum_{i=1}^{2} \sum_{k=1}^{2} p_{ijk} \text{ and } \quad p_{ook} = \sum_{i=1}^{2} \sum_{j=1}^{2} p_{ijk} \quad .$$

Similarly, let

$$n_{ijo} = \sum_{k=1}^{2} n_{ijk} \quad , \quad n_{iok} = \sum_{j=1}^{2} n_{ijk} \quad ,$$

$$n_{ojk} = \sum_{i=1}^{2} n_{ijk} \quad , \quad n_{ioo} = \sum_{k=1}^{2} \sum_{j=1}^{2} n_{ijk} \quad ,$$

$$n_{ojo} = \sum_{i=1}^{2} \sum_{k=1}^{2} n_{ijk} \quad , \qquad n_{ook} = \sum_{i=1}^{2} \sum_{j=1}^{2} n_{ijk} \quad .$$

The null hypothesis H_o is given by

$$(4.2.9) \qquad p_{ijo} = p_{ioo}p_{ojo},\ p_{iok} = p_{ioo}p_{ook} \text{ and } p_{ojk} = p_{ojo}p_{ook} \ .$$

Alternatively, we can write H_o as H_o' where

$$(4.2.10) \qquad H_o': \ h_1(p) = p_{11o}p_{22o} - p_{12o}p_{21o} = 0 \quad ,$$

$$h_2(p) = p_{1o1}p_{2o2} - p_{1o2}p_{2o1} = 0 \quad ,$$

$$h_3(p) = p_{o11}p_{o22} - p_{o12}p_{o21} = 0 \quad .$$

It is easy to see that $H_o \Longleftrightarrow H_o'$, so we shall consider the hypothesis H_o'.

Let

$$\underline{p}' = (p_{111}, p_{112}, \ldots, p_{221}) \quad ,$$

where $p_{111}, p_{112}, \ldots, p_{221}$ are seven independent parameters such that $0 < p_{ijk} < 1$. Then the maximum likelihood estimates of p_{ijk}'s are given by

$$\hat{p}_{ijk} = n_{ijk}/n$$

(4.2.11) and

$$\hat{p}_{ijo} = n_{ijo}/n \quad , \quad \hat{p}_{iok} = n_{iok}/n, \quad \hat{p}_{ojk} = n_{ojk}/n$$

$$(4.2.12)\quad C^{-1}(\hat{\underline{p}}) = \begin{bmatrix} \hat{p}_{111}(1-\hat{p}_{111}) & -\hat{p}_{111}\hat{p}_{112} & \cdots & -\hat{p}_{111}\hat{p}_{221} \\ -\hat{p}_{111}\hat{p}_{112} & \hat{p}_{112}(1-\hat{p}_{112}) & \cdots & -\hat{p}_{112}\hat{p}_{221} \\ \vdots & \vdots & & \vdots \\ -\hat{p}_{111}\hat{p}_{221} & -\hat{p}_{112}\hat{p}_{221} & \cdots & \hat{p}_{221}(1-\hat{p}_{221}) \end{bmatrix}$$

$$(4.2.13)\quad \underset{7\times 3}{H(\hat{\underline{p}})} = \begin{bmatrix} h_1 & h_2 & h_3 \\ \hat{p}_{220}-\hat{p}_{110} & \hat{p}_{202}-\hat{p}_{101} & \hat{p}_{022}-\hat{p}_{011} \\ \hat{p}_{220}-\hat{p}_{110} & -\hat{p}_{101}-\hat{p}_{201} & -\hat{p}_{021}-\hat{p}_{011} \\ -\hat{p}_{110}-\hat{p}_{210} & \hat{p}_{202}-\hat{p}_{101} & -\hat{p}_{012}-\hat{p}_{011} \\ -\hat{p}_{110}-\hat{p}_{210} & -\hat{p}_{201}-\hat{p}_{101} & 0 \\ -\hat{p}_{110}-\hat{p}_{120} & -\hat{p}_{102}-\hat{p}_{101} & \hat{p}_{022}-\hat{p}_{011} \\ -\hat{p}_{110}-\hat{p}_{120} & 0 & -\hat{p}_{021}-\hat{p}_{011} \\ 0 & -\hat{p}_{102}-\hat{p}_{101} & -\hat{p}_{012}-\hat{p}_{011} \end{bmatrix}$$

Let

$$(4.2.14)\quad \underset{3\times 1}{A(\hat{\underline{p}})} = \begin{bmatrix} \hat{p}_{110}+\hat{p}_{220} \\ \hat{p}_{101}+\hat{p}_{202} \\ \hat{p}_{011}+\hat{p}_{022} \end{bmatrix},$$

and

$$(4.2.15)\quad \underset{3\times 3}{D(\hat{\underline{p}})} = \begin{bmatrix} -\hat{p}_{120}\hat{p}_{210} & \begin{matrix} +\hat{p}_{100}\hat{p}_{101}\hat{p}_{212} \\ -\hat{p}_{100}\hat{p}_{102}\hat{p}_{211} \\ +\hat{p}_{200}\hat{p}_{202}\hat{p}_{121} \\ -\hat{p}_{200}\hat{p}_{201}\hat{p}_{112} \end{matrix} & \begin{matrix} +\hat{p}_{010}\hat{p}_{011}\hat{p}_{122} \\ -\hat{p}_{010}\hat{p}_{012}\hat{p}_{121} \\ +\hat{p}_{020}\hat{p}_{022}\hat{p}_{211} \\ -\hat{p}_{020}\hat{p}_{021}\hat{p}_{212} \end{matrix}, \\ \begin{matrix} +\hat{p}_{100}\hat{p}_{110}\hat{p}_{221} \\ -\hat{p}_{100}\hat{p}_{120}\hat{p}_{211} \\ +\hat{p}_{200}\hat{p}_{220}\hat{p}_{112} \\ -\hat{p}_{200}\hat{p}_{210}\hat{p}_{122} \end{matrix}, & -\hat{p}_{102}\hat{p}_{201}, & \begin{matrix} +\hat{p}_{001}\hat{p}_{011}\hat{p}_{122} \\ -\hat{p}_{001}\hat{p}_{021}\hat{p}_{112} \\ +\hat{p}_{002}\hat{p}_{022}\hat{p}_{211} \\ -\hat{p}_{002}\hat{p}_{012}\hat{p}_{221} \end{matrix}, \\ \begin{matrix} +\hat{p}_{010}\hat{p}_{110}\hat{p}_{221} \\ -\hat{p}_{010}\hat{p}_{210}\hat{p}_{121} \\ +\hat{p}_{020}\hat{p}_{220}\hat{p}_{112} \\ -\hat{p}_{020}\hat{p}_{120}\hat{p}_{212} \end{matrix}, & \begin{matrix} +\hat{p}_{001}\hat{p}_{101}\hat{p}_{212} \\ -\hat{p}_{001}\hat{p}_{201}\hat{p}_{112} \\ +\hat{p}_{002}\hat{p}_{202}\hat{p}_{121} \\ -\hat{p}_{002}\hat{p}_{102}\hat{p}_{221} \end{matrix}, & -\hat{p}_{012}\hat{p}_{021} \end{bmatrix}$$

Then

$$(4.2.16)\quad B^{-1}(\hat{\underline{p}}) = H'(\hat{\underline{p}})\, C^{-1}(\hat{\underline{p}})\, H(\hat{\underline{p}})$$

$$= 4h(\hat{\underline{p}})\, h'(\hat{\underline{p}}) - A(\hat{\underline{p}})\, h'(\hat{\underline{p}}) + D(\hat{\underline{p}}) ,$$

and

$$(4.2.17)\quad X^2_W = n \left[h'(\hat{\underline{p}}) \left[4h(\hat{\underline{p}})h'(\hat{\underline{p}}) - A(\hat{\underline{p}})h'(\hat{\underline{p}}) + D(\hat{\underline{p}}) \right]^{-1} h(\hat{\underline{p}}) \right]$$

an asymptotic chi-square distribution with three degrees of freedom.

<u>Example 4.2.2</u> <u>Equality of two marginal distributions in an rxr table:</u>

Let "i" and "j" denote two responses each with r categories.

Suppose we have a sample of n observations and n_{ij} observations lie in the (i, j)-th cell. The corresponding probability model is

$$(4.2.18) \quad \phi = \frac{n!}{\prod_{i,j}^{r} n_{ij}!} \prod_{i,j}^{r} p_{ij}^{n_{ij}}, \quad \sum_{i=1}^{r} \sum_{j=1}^{r} p_{ij} = 1, \; p_{ij} > 0 .$$

Let

$$p_{io} = \sum_{j=1}^{r} p_{ij}, \quad p_{oj} = \sum_{i=1}^{r} p_{ij}, \quad n_{io} = \sum_{j=1}^{r} n_{ij}, \; n_{oj} = \sum_{i=1}^{r} n_{ij}$$

The null hypothesis to be tested is given by

$$(4.2.19) \quad H_1: \; p_{io} = p_{oi} \qquad (i = 1, 2, \ldots, r).$$

This can be written equivalently as H_1', where

$$(4.2.20) \quad H_1' : \; h_i(\underline{p}) = p_{io} - p_{oi} = 0 \quad (i = 1, 2, \ldots, r-1).$$

Consider

$$\underline{p}' = (p_{11}, p_{12}, \ldots, p_{1r}; p_{21}, p_{22}, \ldots, p_{2r}; \ldots p_{r1}, p_{r2}, \ldots, p_{r,r-1})$$

as a vector of r^2-1 independent parameters. Then the maximum likelihood estimates of p_{ij} are

$$\hat{p}_{ij} = n_{ij}/n, \quad \hat{p}_{io} = n_{io}/n, \quad \hat{p}_{oi} = n_{oi}/n \quad ,$$

and

$$(4.2.21) \quad \underset{r^2-1 \times r^2-1}{C^{-1}(\hat{\underline{p}})} = \begin{bmatrix} \hat{p}_{11}(1-\hat{p}_{11}) & -\hat{p}_{11}\hat{p}_{12} & \cdots & -\hat{p}_{11}\hat{p}_{r,r-1} \\ -\hat{p}_{11}\hat{p}_{12} & \hat{p}_{12}(1-\hat{p}_{12}) & \cdots & -\hat{p}_{12}\hat{p}_{r,r-1} \\ \vdots & & & \\ -\hat{p}_{11}\hat{p}_{r,r-1} & -\hat{p}_{12}\hat{p}_{r,r-1} & \cdots & \hat{p}_{r,(r-1)}(1-\hat{p}_{r,r-1}) \end{bmatrix}$$

Let us denote the element in the i-th row and j-th column of $H'(\hat{\underline{p}})$ by

$$h_{ij} = \left[\partial hi(\underline{p}) / \partial p_{k\ell} \right]_{\underline{p} = \hat{\underline{p}}} ,$$

where $j = r(k-1) + \ell$ and for $k=1,2,\ldots,(r-1)$, ℓ takes the values $1, 2, \ldots, r$ and for $k=r$, ℓ takes the values $1, 2, \ldots, (r-1)$. Then

$$\begin{aligned} h_{ij} &= 1 && \text{if } k=i \text{ and } \ell = 1,2,\ldots,(r-1) \\ &=-1 && \text{if } \ell=i \text{ and } k = 1,2,\ldots,(i-1),(i+1),\ldots,r, \\ &= 0 && \text{otherwise,} \end{aligned}$$

(4.2.22)

and

$$\begin{aligned} h_{ij} &= -1 && \text{if } k=r \text{ and } \ell = i, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Further, let $D(\hat{\underline{p}})$ denote the $r-1 \times r-1$ matrix whose element in the i-th row and j-th column is

(4.2.23) $$d_{ij} = \delta_{ij}(\hat{p}_{io} + \hat{p}_{oj}) - \hat{p}_{ij} - \hat{p}_{ji} ,$$

where

$$\begin{aligned} \delta_{ij} &= 1 && \text{if } i = j, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then

(4.2.24) $$\begin{aligned} B^{-1}(\hat{\underline{p}}) &= H'(\hat{\underline{p}})\, C^{-1}(\hat{\underline{p}})\, H(\hat{\underline{p}}) \\ &= D(\hat{\underline{p}}) - h(\hat{\underline{p}})\, h'(\hat{\underline{p}}) , \end{aligned}$$

and

$$(4.2.25) \qquad \chi^2_W = n \, [\, h'(\underline{p}) \, [D(\hat{\underline{p}}) - h(\hat{\underline{p}}) h'(\hat{\underline{p}})]^{-1} \, h(\hat{\underline{p}}) \,]$$

has an asymptotic chi-square distribution with $r-1$ degrees of freedom.

In case $r=3$, let

$$u_{ij}(\hat{\underline{p}}) = \hat{p}_{ij} + \hat{p}_{ji} \qquad i \neq j \,,$$

$$h_3(\hat{\underline{p}}) = -(h_1(\hat{\underline{p}}) + h_2(\hat{\underline{p}})) \,,$$

$$K(\hat{\underline{p}}) = h_3^2(\hat{\underline{p}}) \, u_{12}(\hat{\underline{p}}) + h_2^2(\hat{\underline{p}}) \, u_{13}(\hat{\underline{p}}) + h_1^2(\hat{\underline{p}}) \, u_{23}(\hat{\underline{p}}) \,,$$

$$L(\hat{\underline{p}}) = u_{12}(\hat{\underline{p}}) u_{13}(\hat{\underline{p}}) + u_{12}(\hat{\underline{p}}) u_{23}(\hat{\underline{p}}) + u_{13}(\hat{\underline{p}}) u_{23}(\hat{\underline{p}}) \,.$$

Then

$$(4.2.26) \qquad \chi^2_W = n \, k(\hat{\underline{p}})/L(\hat{\underline{p}}) - k(\hat{\underline{p}}) \quad .$$

4.3 Hypotheses of Symmetry

4.3.1 Hypothesis of Symmetry in a two-way table:

Consider a two-way table with two responses "i" and "j" each having r categories. The corresponding probability model is given by

$$(4.3.1) \qquad \phi = \frac{n!}{\prod_{i,j=1}^{r} n_{ij}!} \prod_{i,j=1}^{r} p_{ij}^{n_{ij}} \,, \quad \sum_{i=1}^{r} \sum_{j=1}^{r} p_{ij} = 1 \quad p_{ij} > 0 \,.$$

We shall consider the hypothesis of symmetry which is

$$(4.3.2) \qquad H_2: \; p_{ij} = p_{ji} \qquad (i, j = 1,2,\ldots, r).$$

The corresponding test statistic can be easily obtained by minimizing

$$X^2 = \sum_{i=1}^{r} \sum_{j=1}^{r} (n_{ij} - n\, p_{ij})^2 / n\, p_{ij}$$

with respect to p_{ij}'s subject to the conditions

$$p_{ij} = p_{ji} , \qquad \sum_{i=1}^{r} \sum_{j=1}^{r} p_{ij} = 1$$

The minimizing estimates of p_{ij}'s are

$$\hat{p}_{ij} = \hat{p}_{ji} = (n_{ij} + n_{ji})/2n ,$$

and

(4.3.3) $$X^2 = \sum_{i=1}^{r} \sum_{i<j} (n_{ij} - n_{ji})^2 / (n_{ij} + n_{ji})$$

has an asymptotic chi-square distribution with $r(r-1)/2$ degrees of freedom when H_2 is true. The same result has been obtained previously by Bowkar [7]. In case $r = 2$, the hypothesis of symmetry $\Longrightarrow$ the hypothesis of the equality of two marginal distributions, and for $r > 2$, the hypothesis of symmetry does imply the hypothesis of the equality of two marginal distributions but the latter does not imply the former. Hence for $r > 2$, we cannot replace H_1 considered in section 4.2 by H_2.

4.3.2 Hypothesis of Symmetry and Independence:

In a two-way table, consider the hypothesis of independence

(4.3.4) H_3: $p_{ij} = p_{io} p_{oj}$.

Suppose

(4.3.5) $H_4 = H_3 \cap H_2$.

Then for testing H_4 i.e. symmetry and independence, the corresponding test statistic is given by

$$(4.3.6)\quad X^2 = 4n \sum_{i=1}^{r} \sum_{j=1}^{r} \left[n_{ij} - \frac{(n_{io}+n_{oi})(n_{jo}+n_{oj})}{4n} \right]^2 / \left[(n_{io}+n_{oi})(n_{jo}+n_{oj})\right]$$

with $r(r-1)$ degrees of freedom.

4.3.3 Hypothesis of Independence Under Symmetry.

We might consider H_3 under the model $\cap H_2$. The corresponding test statistic is given by

$$(4.3.7)\quad X^2 = 4n \sum_{i=1}^{r} \sum_{j=1}^{r} \left[n_{ij} - \frac{(n_{io}+n_{oi})(n_{jo}+n_{oj})}{4n} \right]^2 / \left[(n_{io}+n_{oi})(n_{jo}+n_{oj})\right] - \sum_{i=1}^{r} \sum_{i<j} \frac{(n_{ij} - n_{ji})^2}{n_{ij}+n_{ji}}$$

with $r(r-1)/2$ degrees of freedom.

4.3.4 Hypothesis of Symmetry In A Three-way Table:

Consider three responses "i", "j" and "k" each with r categories. The corresponding probability model is given by

$$(4.3.8)\qquad \phi = \frac{n!}{\prod_{i,j,k} n_{ijk}!} \prod_{i,j,k} p_{ijk}^{n_{ijk}} .$$

We shall consider hypotheses of two-by-two symmetry which are given by

$$(4.3.9)\qquad H_5:\ p_{ijk} = p_{jik} \qquad (i \neq j,\ k = 1, 2, \ldots, r),$$

$$(4.3.10)\qquad H_6:\ p_{ijk} = p_{kji} \qquad (i \neq k,\ j = 1, 2, \ldots, r),$$

$$(4.3.11)\qquad H_7:\ p_{ijk} = p_{ikj} \qquad (j \neq k;\ i = 1, 2, \ldots, r).$$

The corresponding chi-square test statistics are given by

$$(4.3.12)\quad \chi^2_{(5)} = \sum_{k=1}^{r} \sum_{i<j} (n_{ijk}-n_{jik})^2/(n_{ijk}+n_{jik}) \quad ,$$

$$(4.3.13)\quad \chi^2_{(6)} = \sum_{j=1}^{r} \sum_{i<k} (n_{ijk}-n_{kji})^2/(n_{ijk}+n_{kji}) \quad ,$$

$$(4.3.14)\quad \chi^2_{(7)} = \sum_{i=1}^{r} \sum_{j<k} (n_{ijk}-n_{ikj})^2/(n_{ijk}+n_{ikj}) \quad ,$$

each with $r^2(r-1)/2$ degrees of freedom.

We might as well consider the hypothesis of complete symmetry which is given by

$$(4.3.15)\quad \begin{aligned} H_8:\ & p_{ijk} = p_{ikj} = p_{jik} = p_{jki} = p_{kij} = p_{kji} \quad i\neq j\neq k, \\ & p_{iik} = p_{iki} = p_{kii} \quad (i\neq k). \end{aligned}$$

The corresponding chi-square test statistic is

$$(4.3.16)\quad \chi^2 = 6n \sum_{i\neq j\neq k} \frac{[n_{ijk}-(n_{ijk}+n_{ikj}+n_{jik}+n_{jki}+n_{kij}+n_{kji})/6n]^2}{n_{ijk}+n_{ikj}+n_{jik}+n_{jki}+n_{kij}+n_{kji}}$$

$$+ 3n \sum_{i\neq j} \left[n_{iji} - \frac{(n_{iij}+n_{iji}+n_{jii})}{3n}\right]^2 / (n_{iij}+n_{iji}+n_{jii})$$

$$+ 3n \sum_{i\neq k} \left[n_{iik} - \frac{(n_{iik}+n_{iki}+n_{kii})}{3n}\right]^2 / (n_{iik}+n_{iki}+n_{kii})$$

$$+ 3n \sum_{j\neq k} \left[n_{jkk} - \frac{(n_{jkk}+n_{kjk}+n_{kkj})}{3n}\right]^2 / (n_{jkk}+n_{kjk}+n_{kkj})$$

with $2r(r-1)$ degrees of freedom.

It can be easily seen that

$$H_5 \cap H_6 \cap H_7 \Longrightarrow H_8$$

but

$$H_8 \not\Longrightarrow H_5 \cap H_6 \cap H_7 .$$

4.4 <u>The Non-centrality Parameters For The Hypothesis of Symmetry.</u>

The non-centrality parameters for the test statistics used in testing the hypothesis of symmetry can be easily obtained by using the theorems given by Diamond, Mitra and Roy [8]. We list below in each case the hypothesis, the test-statistics and the corresponding non-centrality parameter.

(1) Two way table "i" and "j" response:

$$H_2: \; p_{ij} = p_{ji} \qquad\qquad i,j=1,2,\ldots,r \;,$$

$$H_{2,n}: \; p_{ij} = p^o_{ji} + n^{-1/2}\,\delta_{ij} \;,$$

where

$$p^o_{ij} = p^o_{ji} \quad .$$

$$\chi^2_{(2)} = \sum_{i<j} (n_{ij}-n_{ji})^2/(n_{ij}+n_{ji}) \;.$$

(4.4.1) $$\Delta^2_{(2)} = 1/2n \sum_{i<j} \delta^2_{ij}/p^o_{ij} \;.$$

(2) Three way table "i", "j" and "k" responses

$$H_5 \;: \; p_{ijk} = p_{jik} \qquad i \neq j \;,$$

$$H_{5,n}: \; p_{ijk} = p^o_{jik} + n^{-1/2}\,\delta_{ijk}, \quad p^0_{ijk} = p^o_{jik}$$

$$\chi^2_{(5)} = \sum_{k=1}^{r} \sum_{i<j} (n_{ijk}-n_{jik})^2/n_{ijk}+n_{jik} \;.$$

(4.4.2) $$\Delta^2_{(5)} = 1/2n \sum_{k=1}^{r} \sum_{i<j} \delta^2_{ijk}/p^o_{ijk} .$$

$H_6 \; : \; p_{ijk} = p_{kji} \qquad (i \neq k) ,$

$H_{6,n} : \; p_{ijk} = p^o_{kji} + n^{-1/2}\delta_{ijk} , \quad p^o_{ijk} = p^o_{kji} .$

$$\chi^2_{(6)} = \sum_{j=1}^{r} \sum_{i<k} (n_{ijk}-n_{kji})^2/(n_{ijk}+n_{kji}) .$$

(4.4.3) $$\Delta^2_{(6)} = 1/2n \sum_{j=1}^{r} \sum_{i<k} \delta^2_{ijk}/p^o_{ijk} .$$

$H_7 \; : \; p_{ijk} = p_{ikj} \qquad (j \neq k) ,$

$H_{7,n} : \; p_{ijk} = p^o_{ikj} + \delta_{ijk}/\sqrt{n} , \quad p^o_{ijk} = p^o_{ikj} .$

$$\chi^2_{(7)} = \sum_{i=1}^{r} \sum_{j<k} (n_{ijk}-n_{ikj})^2/n_{ijk}+n_{ikj} .$$

(4.4.4) $$\Delta^2_{(7)} = 1/2n \sum_{i=1}^{r} \sum_{j<k} \delta^2_{ijk}/p^o_{ijk} .$$

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