ON JUSTIFYING THE INTUITIVE BASIS FOR THE METHOD OF WEIGHTED RANKINGS

by

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I. INTRODUCTION

Let X_{ij} be the observation on the j-th of m treatments in the i-th of n complete blocks, and consider testing the hypothesis of no treatment effects, specifically

$$H_0: \; X_{i1},\ldots,X_{im} \text{ are interchangeable for each } i \; .$$

(By definition random variables are *interchangeable* if their joint distribution function is invariant under permutations.) The alternatives under consideration are fairly general; however, two specific examples are:

(1) Additive treatment effects

H_1: There exist quantities $\tau_1,\ldots,\tau_m$ (treatment effects); not all equal to zero, such that for i=1,...,n, $X_{i1} - \tau_1,\ldots,X_{im} - \tau_m$ are interchangeable.

or

(2) Ordered treatment effects

H_2: The quantities $\tau_1,\ldots,\tau_m$ (as above) satisfy $\tau_1 \leq \ldots \leq \tau_m$, with at least one strict inequality.

Standard nonparametric procedures for attacking this problem are based on within-block rankings: for example, the tests of Friedman (1937) and Brown and Mood (1951) for H_1; and Lyerly (1952), Page (1963) and Jonckheere (1954) for H_2. The only assumption they require is that the blocks be independent, i.e.,

$$\text{(I)} \quad \begin{cases} \text{The random vectors } \underset{\sim}{X}_i = (X_{i1},\ldots,X_{im})', \text{ for } i=1,\ldots,n \\ \text{(the blocks), are mutually independent.} \end{cases}$$

However, to simplify the exposition it is convenient also to assume

$$\text{(II)} \quad P\{X_{ij} = X_{ij'}\} = 0 \quad \text{for} \quad j \neq j' \quad ,$$

so that with probability 1 there will be no ties within blocks. But suppose we make the additional assumption of additive block effects, as is common in the usual "parametric" approach to this problem:

$$
\text{(III)} \quad \begin{cases} \text{There exist quantities } \beta_1,\ldots,\beta_n \text{ (block effects)} \\ \text{such that the random vectors } (X_{i1} - \beta_i,\ldots,X_{im} - \beta_i)' \\ \text{are all identically distributed.} \end{cases}
$$

Then comparisons of observations are possible between blocks as well as within, and procedures which use only within-block comparisons waste information.

A method of *weighted* within-block rankings, which generalizes the standard method based on unweighted rankings, has been introduced by Quade (1972, 1979). The idea behind this method is that blocks in which the treatments are more distinct are more likely to reflect any underlying true ordering; hence these blocks, which may be referred to as more *credible*, should receive greater weight in the analysis. (in practice, credibility is measured by *apparent* variability; but note that by Assumption III the *true* variability is the same in all blocks.)

To determine the weight for the i-th block, use some location-free statistic $D_i = D(X_{i1},\ldots,X_{im})$ which measures the credibility of the block with respect to treatment ordering, and let Q_i be the rank of D_i among $D_1,\ldots,D_n$. Again for simplicity of exposition, make the (unessential) assumption

$$
\text{(IV)} \qquad P\{D_i = D_{i'}\} = 0 \quad \text{for} \quad i \neq i' \quad ,
$$

so that there will be no ties in the ranking of the blocks. Let $0 \leq b_1 \leq \ldots \leq b_n$ be a fixed set of block scores; and weight the i-th block proportionally to b_{Q_i}.

Then in testing against the ordered alternative one may use the *weighted average external rank correlation*

$$
W = \sum_{i=1}^{n} b_{Q_i} C_i \Bigg/ \sum_{i=1}^{n} b_i \quad ,
$$

where C_i is the correlation between the alternative ranking and the ranking observed within the i-th block. Against the general alternative one may use the *weighted average internal rank correlation*

$$C = \sum_{i \neq i'}\sum b_{Q_i} b_{Q_{i'}} C_{ii'} \Big/ ((\Sigma b_i)^2 - \Sigma b_i^2) \quad ,$$

where $C_{ii'}$ is the correlation between the rankings observed in blocks i and i'.

The purpose of this paper is to examine the notion on which these weighted rank correlation coefficients are based, which we may call the *credibility hypothesis*.

II. FORMULATION OF THE PROBLEM

Let $F(X_1,\ldots,X_m)$ be a distribution function defined on R^m. Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ be n independent observations from $F(\underset{\sim}{X})$. Let D be a measure of variability defined on R^m. Then D may be regarded as an "ordering function" on R^m. Note that, for simplicity of exposition, we assume

$$P\{D(\underset{\sim}{X}) = D(\underset{\sim}{Y})\} = 0 \quad \text{for all } \underset{\sim}{X} \text{ and } \underset{\sim}{Y} \in R^m \quad .$$

Using this notation, we may say

$$\underset{\sim}{X} < \underset{\sim}{Y} \text{ if and only if } D(\underset{\sim}{X}) < D(\underset{\sim}{Y}) \ , \quad \underset{\sim}{X} \text{ and } \underset{\sim}{Y} \in R^m \quad .$$

Let the random variables $D(\underset{\sim}{X}_1),\ldots,D(\underset{\sim}{X}_n)$ be arranged in ascending order of magnitude, as $D(\underset{\sim}{X}_{(1)})<\ldots<D(\underset{\sim}{X}_{(n)})$. Then we may write $\underset{\sim}{X}_{(1)}<\ldots<\underset{\sim}{X}_{(n)}$, and call $\underset{\sim}{X}_{(i)}$ the i-th ordered vector $(i=1,\ldots,n)$ with respect to D.

Let G be a "score" transformation defined on R^m. An example of G which we will keep in mind throughout the discussion is: $G_R: R^m \to S_m$, where S_m is the set of permutations of the integers $\{1,\ldots,m\}$ and $G_R(\underset{\sim}{X}) = G_R(X_1,\ldots,X_m) = (R_1,\ldots,R_m)$, where R_i is the rank of X_i among $\{X_1,\ldots,X_m\}$. We note that it is not necessary to restrict attention to the linear rank case; the discussion may be extended to $G(\underset{\sim}{X}) = (t_1,\ldots,t_m)$ where $\underset{\sim}{t}$ is a vector of scores. Now, the credibility hypothesis may be examined by studying some of the statistical properties of the random variables $G(\underset{\sim}{X}_{(i)})$, or in our case $G_R(\underset{\sim}{X}_{(i)})$, $i=1,\ldots,n$. Again we narrow discussion to the case of testing against ordered alternatives. Statements may be transferred to the general case with obvious modifications. One way of confirming the credibility hypothesis would be to show that

$$E[\mathrm{Corr}(G_R(\underset{\sim}{X}_{(i)}), \sigma_0)] < E[\mathrm{Corr}(G_R(\underset{\sim}{X}_{(j)}), \sigma_0)] \quad \text{for} \quad i < j \quad ,$$

where σ_0 is the true ordering of the m treatments (which we will assume is $\sigma_0 = \{1,\ldots,m\}$). This can be done by studying

$$E[G_R(\underset{\sim}{X}_{(i)})] = (ER_{(i),1},\ldots,ER_{(i),m}) \quad .$$

This is the criterion which will be used in this paper, although we mention another possible criterion:

$$P\{\sigma_0 = G_R(\underset{\sim}{X}_{(i)})\} < P\{\sigma_0 = G_R(\underset{\sim}{X}_{(j)})\} \quad , \quad i<j \quad .$$

In the next section we give an example illustrating these concepts in a special case.

III. THE CASE OF TWO TREATMENTS AND TWO BLOCKS

Consider the situation where we have two treatments X and Y, and two blocks. The raw data may be represented as shown on the left below; the ranks are as shown on the right.

	X	Y
I	X_1	Y_1
II	X_2	Y_2

	X	Y
I'	$R_{(1),1}$	$R_{(1),2}$
II'	$R_{(2),1}$	$R_{(2),2}$

Note by this notation that the range of the observations in the first block I' is less than the range of the second block II'.

Let

$$f_X(x) = \mu e^{-\mu x} \quad , \quad \mu > 0 \quad , \quad x \geq 0 \quad ,$$

and

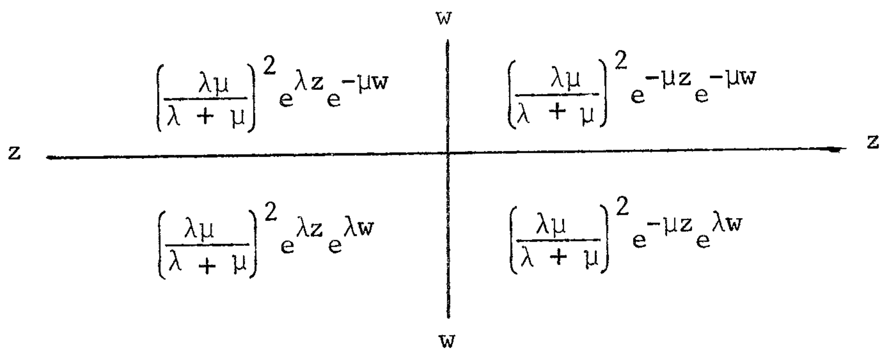
$$f_Y(y) = \lambda e^{-\lambda y} \quad , \quad \lambda > 0 \quad , \quad y \geq 0 \quad .$$

Assume that X and Y are independent. Let $D(X,Y) = |X-Y|$ and $G_R(X,Y) = (R_1,R_2)$.

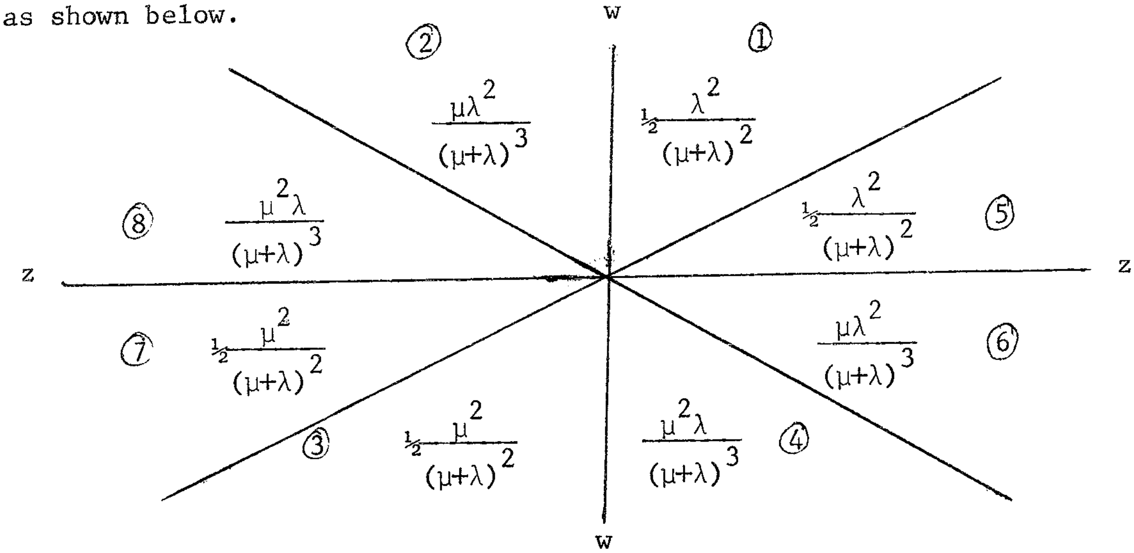
Our object is to find an expression for $E[R_{(i),j}]$ and hence the expected correlation with the true ordering for each block. To do that, let $Z = X_1 - Y_1$, so that we have

$$f_Z(z) = \begin{cases} \dfrac{\lambda\mu}{\mu+\lambda}\, e^{\lambda z}, & z \le 0 \\[2ex] \dfrac{\lambda\mu}{\mu+\lambda}\, e^{-\mu z}, & z > 0 \end{cases}$$

Similarly, by letting $W = X_2 - Y_2$, we have $f_W(w) = f_Z(w)$, and Z and W are independent. Thus the joint density function of Z and W is given by $g(z,w)$, as shown on the following graph.



To finish the calculations, we need to evaluate the probabilities of the 8 regions $A_1,\ldots,A_8$ determined by the lines $w=0$, $z=0$, $w=z$, and $w=-z$, as shown below.



The respective probabilities $P_1,\ldots,P_8$ have been written on the graph, for simplicity of presentation. That $P_1=P_5$, $P_2=P_6$, $P_3=P_7$, and $P_4=P_8$ is not a special feature of the example, but follows from the fact that Z and W are independent and identically distributed. In particular, $P_1+P_2+P_3+P_4=P\{|Z|\leq|W|\}=1/2$. Now,

$$E[R_{(1),1}] = 1 \cdot P\{R(X)=1 \mid |Z|\leq|W|\} + 2 \cdot P\{R(X)=2 \mid |Z|\leq|W|\}$$

$$= \frac{P\{R(X)=1,\ |Z|\leq|W|\} + 2P\{R(X)=2,\ |Z|\leq|W|\}}{P\{|Z|\leq|W|\}}$$

$$= \frac{(P_2+P_3) + 2(P_1+P_4)}{1/2}$$

since R(X)=1 (or 2) if and only if Z, or equivalently W, is negative (or positive); or in general

$$E[R_{(1),1}] = 1 + 2(P_1+P_4) \quad .$$

In the example,

$$E[R_{(1),1}] = 1 + \frac{\lambda^3 + \lambda^2\mu + 2\lambda\mu^2}{(\mu+\lambda)^3} \quad .$$

A similar argument shows that

$$E[R_{(2),1}] = 1 + \frac{\lambda^3 + 3\lambda^2\mu}{(\mu+\lambda)^3}$$

Since $E[R_{(i),1}+R_{(i),2}] = 3$, for i=1,2 we have the following expectations:

	X	Y	
I'	$1 + \frac{\lambda^3 + \lambda^2\mu + 2\lambda\mu^2}{(\mu+\lambda)^3}$	$2 - \frac{\lambda^3 + \lambda^2\mu + 2\lambda\mu^2}{(\mu+\lambda)^3}$	3
II'	$1 + \frac{\lambda^3 + 3\lambda^2\mu}{(\mu+\lambda)^3}$	$2 - \frac{\lambda^3 + 3\lambda^2\mu}{(\mu+\lambda)^3}$	3
	$3 + \frac{\mu-\lambda}{\mu+\lambda}$	$3 - \frac{\mu-\lambda}{\mu+\lambda}$	6

To evaluate $E[\mathrm{corr}(R_{(i)},\sigma_0)]$ we take $\sigma_0 = [1,2]$, which is true in the example if $\lambda<\mu$. Then $\mathrm{corr}(R_{(1)},\sigma_0) = 1$ if $R_{(1),1} = 1$ and -1 if $R_{(1),1} = 2$. Thus

$$E[\mathrm{corr}(R_{(1)},\sigma_0)] = P\{R(X)=1 \big| |Z|\le|W|\} - P\{R(X)=2 \big| |Z|\le|W|\}$$

$$= \frac{P\{R(X)=1,\ |Z|\le|W|\} - P\{R(X)=2 \big| |Z|\le|W|\}}{P\{|Z|\le|W|\}}$$

$$= \frac{P_2 + P_3 - P_1 - P_4}{P_1 + P_2 + P_3 + P_4}$$

and in the example this is

$$E[\mathrm{corr}(R_{(1)},\sigma_0)] = \frac{(\mu-\lambda)(\mu^2+\lambda^2)}{(\mu+\lambda)^3} \quad .$$

In a similar manner we find that

$$E[\mathrm{corr}(R_{(2)},\sigma_0)] = \frac{P_3 + P_4 - P_1 - P_2}{P_1 + P_2 + P_3 + P_4}$$

$$= \frac{(\mu-\lambda)(\mu^2+4\mu\lambda+\lambda^2)}{(\mu+\lambda)^3}$$

and hence clearly $E[\mathrm{Corr}(R_{(2)},\sigma_0)] > E[\mathrm{Corr}(R_{(1)},\sigma_0)]$ if $\lambda<\mu$.

IV. A CONJECTURE

Based on these results, we state the following conjecture for the ordered alternative case:

There exists a treatment j_0, $1 \le j_0 \le m$, such that

$$E[R_{(i),j}] \le E[R_{(k),j}] \quad \text{for all} \quad i > k \quad \text{and} \quad j \le j_0$$

and

$$E[R_{(i),j}] \ge E[R_{(k),j}] \quad \text{for all} \quad i > k \quad \text{and} \quad j \ge j_0 \quad .$$

Hence

$$E[\mathrm{Corr}(R_{(i)},\sigma_0)] \ge E[\mathrm{Corr}(R_{(k)},\sigma_0)] \quad \text{for all} \quad i \ge k \quad .$$

This conjecture is examined in the next section, using a simulation experiment.

V. AN EXPERIMENT TO EVALUATE THE CREDIBILITY HYPOTHESIS

It was decided to evaluate $E[R_{(i),j}]$ for i=1,...,n and j=1,...,m for experiments with up to n=10 blocks and m=5 treatments. Each experiment was simulated 500 times. The measure of credibility used was the block range, and the distributions used for X_{ij} were as follows:

(Appendix I) Normal with mean j (and unit variance)

(Appendix II) Uniform on the interval (j, j+5)

(Appendix III) Exponential with mean j

(Appendix IV) Cauchy with median j (and unit scale parameter)

Each Appendix contains 36 tables (j=2,3,4,5; n=2,...,10) in the same format, showing $E[R_{(i),j}]$ in the i-th row and j-th cloumn. For example, the first table is as follows:

1	1.332	1.668
2	1.155	1.845

This shows the expected ranks in an experiment with two blocks and two treatments, for the case where in each block (row) the observation on the first treatment (column) is normally distributed with mean 1 and variance 1, while the observation on the second treatment is normal with mean 2 and variance 1. In particular, (say), the expected rank of the second treatment in the block with larger range is estimated to be 1.845. Estimates of the expected ranks in a block chosen at random can be obtained as the column averages.

The results of this Monte Carlo experiment clearly confirm the conjecture of Section IV, and hence the credibility hypothesis, for the normal, uniform, and exponential data: within each table the expectations decrease monotonically in the left-hand columns and increase monotonically in the right-hand columns, except for very slight discrepancies ascribable to sampling error. (The magnitude of these discrepancies can be appreciated on inspecting the middle columns for the 3-treatment and 5-treatment experiments, where by symmetry the true values must be 2 and 3, respectively; or by noting also that within each row the true expectations must be symmetrically placed.)

For the Cauchy data, however, the results are clearly contrary to the conjecture and hypothesis. The within-column trends are not clear for this case, although there is some suggestion that the expectations are most spread out in the middle rows: i.e., that the blocks with middling range are more credible than those with extremely large or small ranges.

These results are consistent with Monte Carlo results reported by Silva and Quade (1980) for the general alternative, and by Salama and Quade (1981) for the ordered alternative, in which simulations indicated weighted ranking procedures to be preferred over unweighted for normal and uniform data. These previous papers did not consider exponential or Cauchy errors; instead, they looked at Laplace errors, but with rather inconclusive results.

VI. THE CASE OF LINEAR WEIGHTS

In defining the weighted average internal correlation, suppose *linear weights* are used: that is, suppose

$$b_{Q_i} = Q_i \quad \text{for} \quad i=1,\dots,n \quad .$$

Then

$$C = \sum_{i \neq i'}\sum Q_i Q_{i'} C_{ii'} / \frac{(n^3-n)(3n+2)}{12} \quad ,$$

and a little algebra shows that

$$E[C] = \frac{12}{(n+1)(3n+2)} E[Q_1 Q_2 C_{12}] \quad .$$

Now,

$$Q_1 = 1 + I\{D_2 < D_1\} + \sum_{i=3}^{n} I\{D_i < D_1\}$$

and

$$Q_2 = 1 + I\{D_1 < D_2\} + \sum_{j=3}^{n} I\{D_j < D_2\} \ ;$$

multiplying,

$$Q_1Q_2 = 1 + I\{D_1<D_2\} + \sum_{j=3}^{n} I\{D_j<D_2\}$$

$$+ I\{D_2<D_1\} + I\{D_2<D_1\}I\{D_1<D_2\} + I\{D_2<D_1\} \sum_{j=3}^{n} I\{D_j<D_2\}$$

$$+ \sum_{i=3}^{n} I\{D_i<D_1\} + I\{D_1<D_2\} \sum_{i=3}^{n} I\{D_i<D_1\} + \sum_{i=3}^{n} I\{D_i<D_1\} \sum_{j=3}^{n} I\{D_j<D_2\} \quad .$$

Since $I\{D_1<D_2\} + I\{D_2<D_1\} = 1$ and $I\{D_2<D_1\}I\{D_1<D_2\} = 0$, we have

$$Q_1Q_2 = 2 + \sum_{i=3}^{n} I\{D_i<D_1\} + \sum_{j=3}^{n} I\{D_j<D_2\}$$

$$+ \sum_{j=3}^{n} I\{D_j<D_2<D_1\} + \sum_{i=3}^{n} I\{D_i<D_1<D_2\} + \sum_{i=3}^{n} \sum_{j=3}^{n} I\{D_i<D_1\}I\{D_j<D_2\} \quad .$$

Hence

$$E[Q_1Q_2C_{12}] = 2E[C_{12}] + \sum_{i=3}^{n} E[C_{12}I\{D_i<D_1\}] + \sum_{j=3}^{n} E[C_{12}I\{D_j<D_2\}]$$

$$+ \sum_{j=3}^{n} E[C_{12}I\{D_j<D_2<D_1\}] + \sum_{i=3}^{n} E[C_{12}I\{D_i<D_1<D_2\}]$$

$$+ \sum_{i=3}^{n} \sum_{j=3}^{n} E[C_{12}I\{D_i<D_1\}I\{D_j<D_2\}]$$

$$= 2E[C_{12}] + 2(n-2)E[C_{12}I\{D_3<D_1\}]$$

$$+ 2(n-2)E[C_{12}I\{D_3<D_2<D_1\}]$$

$$+ \Big((n-2)E[C_{12}I\{D_3<D_1\}I\{D_3<D_2\}]$$

$$+ (n-2)(n-3)E[C_{12}I\{D_3<D_1\}I\{D_4<D_2\}]\Big)$$

Consider the situation where there are n=4 blocks. Let P_i for i=1,...,24 denote the 24 possible permutations of (1,2,3,4) and let Q be the observed ranking of the 4 blocks with respect to credibility; then $Pr\{Q=P_i\} = 1/24$ for i=1,...,n (and this is true regardless of whether the hypothesis of interchangeability of treatments within blocks holds or not). Then

$$E[C_{12}I\{D_3<D_1\}] = \sum_{i=1}^{24} E[C_{12}I\{D_3<D_1\}|Q=P_i]P\{Q=P_i\}$$

$$= \frac{1}{24} \sum_{i=1}^{24} E[C_{12}I\{D_3<D_1\}|Q=P_i] \quad .$$

Similarly, we can find $E[C_{12}I\{D_3<D_2<D_1\}]$, etc.

Let E_{ij} = expected correlation between the i-th and j-th least credible blocks among 4, $1\le i<j\le 4$. We then find

$$E[C_{12}I\{D_3<D_1\}] = \frac{1}{24}\{E_{13} + 2E_{14} + 2E_{23} + 3E_{24} + 4E_{34}\} ,$$

$$E[C_{12}I\{D_3<D_2<D_1\}] = \frac{1}{24}\{E_{23} + E_{24} + 2E_{34}\} ,$$

$$E[C_{12}I\{D_3<D_1\}I\{D_3<D_2\}] = \frac{1}{24}\{2E_{23} + 2E_{24} + 4E_{34}\} ,$$

$$E[C_{12}I\{D_3<D_1\}I\{D_4<D_2\}] = \frac{1}{24}\{2E_{24} + 4E_{34}\} \quad .$$

Letting

$$E[C_{12}] = \frac{1}{6}\{E_{12} + E_{13} + E_{23} + E_{14} + E_{24} + E_{34}\} = \Psi \quad \text{(say)},$$

We have

$$E[Q_1Q_2C_{12}] = 24 + \frac{2(n-2)}{24}\{E_{13} + 2E_{14} + 2E_{23} + 3E_{24} + 4E_{34}\}$$

$$+ \frac{2(n-2)}{24}\{E_{23} + E_{24} + 2E_{34}\}$$

$$+ \frac{(n-2)}{24}\{2E_{23} + 2E_{24} + 4E_{34}\}$$

$$+ \frac{(n-2)(n-3)}{24}\{2E_{24} + 4E_{34}\}$$

$$= 2\psi + \frac{(n-2)}{12}E_{13} + \frac{(n-2)}{6}E_{14} + \frac{(n-2)}{3}E_{23}$$

$$+ \frac{(n+2)(n-2)}{12}E_{24} + \frac{(n+1)(n-2)}{6}E_{34} \quad .$$

i	P_i	$I\{D_3<D_1\}$	$I\{D_3<D_2<D_1\}$	$I\{D_3<D_1\}I\{D_3<D_2\}$	$I\{D_3<D_1\}I\{D_4<D_2\}$	$E[C_{12}]$
1	1234	0	0	0	0	E_{12}
2	1243	0	0	0	0	E_{12}
3	1324	0	0	0	0	E_{13}
4	1342	0	0	0	0	E_{13}
5	1423	0	0	0	0	E_{14}
6	1432	0	0	0	0	E_{14}
7	2134	0	0	0	0	E_{12}
8	2143	0	0	0	0	E_{12}
9	2314	1	0	1	0	E_{23}
10	2341	0	0	0	0	E_{23}
11	2413	1	0	1	1	E_{24}
12	2431	0	0	0	0	E_{24}
13	3124	1	0	0	0	E_{13}
14	3142	0	0	0	0	E_{13}
15	3214	1	1	1	0	E_{23}
16	3241	0	0	0	0	E_{23}
17	3412	1	0	1	1	E_{34}
18	3421	1	0	1	1	E_{34}
19	4123	1	0	0	0	E_{14}
20	4132	1	0	0	0	E_{14}
21	4213	1	1	1	0	E_{24}
22	4231	1	0	0	1	E_{24}
23	4312	1	1	1	1	E_{34}
24	4321	1	1	1	1	E_{34}
		12	4	8	6	

Thus

$$E[C] = \frac{24}{(n+1)(3n+2)}\psi + \frac{(n-2)}{(n+1)(3n+2)}E_{13} + \frac{2(n-2)}{(n+1)(3n+2)}E_{14}$$

$$+ \frac{4(n-2)}{(n+1)(3n+2)}E_{23} + \frac{(n+2)(n-2)}{(n+1)(3n+2)}E_{24} + \frac{2(n-2)}{3n+2}E_{34} ,$$

or $E[C] = \frac{1}{3}E_{24} + \frac{2}{3}E_{34} + 0(\frac{1}{n})$.

Some special cases are as follows:

n	E[C]
2	ψ
3	$\frac{1}{44}\{24\psi + E_{13} + 2E_{14} + 4E_{23} + 5E_{24} + 8E_{34}\}$
4	$\frac{1}{35}\{12\psi + E_{13} + 2E_{14} + 4E_{23} + 6E_{24} + 10E_{34}\}$
5	$\frac{1}{34}\{8\psi + E_{13} + 2E_{14} + 4E_{23} + 7E_{24} + 12E_{34}\}$
6	$\frac{1}{35}\{6\psi + E_{13} + 2E_{14} + 4E_{23} + 8E_{24} + 14E_{34}\}$
7	$\frac{1}{184}\{24\psi + 5E_{13} + 10E_{14} + 20E_{23} + 45E_{24} + 80E_{34}\}$
8	$\frac{1}{39}\{4\psi + E_{13} + 2E_{14} + 4E_{23} + 10E_{24} + 18E_{34}\}$

Note that E[C] depends only on the expected correlations in a sample of 4 blocks: thus, to find E[C] for any n by simulation one needs only to simulate in sets of 4 blocks at a time, if linear block weights are being used.

VII. SOME RELATED PROBLEMS FOR RESEARCH

Recalling the notation introduced in Section II, we may ask: what are the statistical properties of the random variable(s) $G(\underset{\sim}{X}_{(i)})$, i=1,...,n, where G is an appropriate transformation defined on R^m? We give some examples to illustrate the idea and to open some specific questions for investigation.

A. To attach a meaning to the functions D and G we consider the following examples:

(A.1): Let $\underset{\sim}{X}_i = (X_i, Y_i)$, i=1,...,n, be n independent observations from a distribution function F(X,Y). Let D(G) be the projection map on the first (second) coordinate of R^2, that is

$$D(X,Y) = P_1(X,Y) = X ,$$

and

$$G(X,Y) = P_2(X,Y) = Y \quad .$$

Then $G(X_{(i)}) = Y_{[i,n]}$, is the concomitant of the i-th order statistic, as defined by David (1973).

(A.2): Let $\underline{X}_i = (x_{i1},\ldots,x_{im})$, $i=1,\ldots,n$, be n independent observations from a distribution function $F(\underline{X})$. Let D be a measure of variability defined on $R^m(D(x_1,\ldots,x_m) = \Sigma(x_i-\bar{x})^2/m-1$ or $D(x_1,\ldots,x_m) = \max(x_j) - \min(x_j) =$ $\text{Range}(x_1,\ldots,x_m))$. Let $G(\underline{X}) = G(x_1,\ldots,x_m) = (R_1,\ldots,R_m)$, where R_j is the rank of x_j among $x_1,\ldots,x_m$. Random variables of the form $G(\underline{X}_{(i)})$ and $(P_j o G)(\underline{X}_{(i)})$, where P_j is the projection map on the j-th coordinate of R^m, have appeared naturally in studying weighted rankings analysis.

(A.3): In the previous examples we gave two meaningful choices of D. For other choices of G we may consider

$G_1(x_1,\ldots,x_m) = \bar{X}$,

$G_2(x_1,\ldots,x_m) = x_{(j)}$, the j-th order statistic of $(x_1,\ldots,x_m)$,

$G_3(x_1,\ldots,x_m) = R_j$, the rank of x_j among $(x_1,\ldots,x_m)$,

$G_4(x_1,\ldots,x_m) = (x_1,\ldots,x_m)$, the identity map, etc.

While meaningful examples of choices of D are hard to obtain, examples of choices of G are numerous and they lead to interesting statistical and mathematical problems.

G. We now consider a specific example. Let

$$f(x,y) = \begin{cases} (e^{-x})(\frac{1}{a}\, e^{-y/a}) \ , & x,y,a\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\underset{\sim}{X}_1 = (X_1,Y_1)$ and $\underset{\sim}{X}_2 = (X_2,Y_2)$ be two independent observations from the distribution function $F(X,Y)$. Let $D(X,Y) = |X-Y|$ (the range), and $G(X,Y) =$ (R_1,R_2) (again, $R_1(R_2)$ is the rank of $X(Y)$ among (X,Y)). Then it can be shown (based on Section III) that

$$E((P_1 oG)(\underset{\sim}{X}_{(1)})) = 1 + \frac{2a^2+a+1}{(1+a)^3} ,$$

and

$$E((P_2 oG)(\underset{\sim}{X}_{(2)})) = 1 + \frac{3a+1}{(1+a)^3}$$

Noting that $(P_1 oG)(\underset{\sim}{X}_{(i)}) + (P_2 oG)(\underset{\sim}{X}_{(i)}) = 3$, $i=1,2$, we can easily obtain $E((P_2 oG)(\underset{\sim}{X}_{(i)}))$, $i=1,2$. Out of this simple example we may pose the following questions.

1. With D defined above, what is the distribution function of the random variables $P_j(\underline{X}_{(i)})$, $i,j = 1,2$?
2. Let $G_1(X,Y) = \min(X,Y)$, $G_2(X,Y) = \max(X,Y)$ and D as defined earlier. What is the distribution function of the random variables $G_j(\underline{X}_{(i)})$, $i,j = 1,2$? We would also want to investigate the previously mentioned points for $n \geq 3$.

C. Let $\underset{\sim}{X}_i = (x_{i1},\ldots,x_{im})$, $i=1,\ldots,n$ be n independent observations from a distribution function $F(\underline{X})$. Let D be an order function defined on R^m, and consider the corresponding ordered random vectors $\underset{\sim}{X}_{(i)}$, $i=1,\ldots,n$. Let $x(i,j)$ be the j-th order statistic within the i-th ordered vector of observations, namely $\underset{\sim}{X}_{(i)}$. Then for all $i=1,\ldots,n$ we have

$$x_{(i,1)} \leq x_{(i,2)} \leq \cdots \leq x_{(i,m)} \quad .$$

Define

$$\underline{X}^*_{(i)} = (x_{(i,1)},\ldots,x_{(i,m)}) \;,\quad i=1,\ldots,n \quad .$$

Then, a natural question presents itself: What is the distribution function of $x_{(i,j)}$, $i=1,\ldots,n$ and $j=1,\ldots,m$? The sequence $\{x_{(i,j)}\}$ may be called a doubly ordered sequence, and the element $x_{(i,j)}$ may be called a double-order statistic. At this point we note that problems in the theory of order statistics may be formulated as special cases of this setting by appropriate choices of the functions D and G discussed earlier.

NOTE: Section VI of this paper is entirely due to Quade. The mathematical work and the initial drafts of the remaining sections are due to Salama, with Quade having edited them into final form.

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APPENDIX I: Normal Data

m = 2 treatments

1	1.332	1.668
2	1.155	1.845

1	1.358	1.642
2	1.245	1.755
3	1.128	1.872

1	1.377	1.623
2	1.274	1.726
3	1.194	1.806
4	1.101	1.899

1	1.413	1.587
2	1.306	1.694
3	1.193	1.807
4	1.150	1.850
5	1.081	1.919

1	1.406	1.594
2	1.344	1.656
3	1.269	1.731
4	1.198	1.802
5	1.132	1.868
6	1.078	1.922

1	1.403	1.597
2	1.369	1.631
3	1.316	1.684
4	1.221	1.779
5	1.205	1.795
6	1.125	1.875
7	1.061	1.939

1	1.437	1.563
2	1.358	1.642
3	1.331	1.669
4	1.260	1.740
5	1.207	1.793
6	1.145	1.855
7	1.105	1.895
8	1.055	1.945

1	1.441	1.559
2	1.390	1.610
3	1.342	1.658
4	1.284	1.716
5	1.224	1.776
6	1.209	1.791
7	1.132	1.868
8	1.082	1.918
9	1.050	1.950

1	1.460	1.540
2	1.404	1.596
3	1.375	1.625
4	1.307	1.693
5	1.226	1.774
6	1.197	1.803
7	1.168	1.832
8	1.129	1.871
9	1.103	1.897
10	1.065	1.935

m = 3 treatments

1	1.421	2.005	2.575
2	1.223	2.007	2.771

1	1.464	1.987	2.549
2	1.265	2.025	2.710
3	1.216	1.981	2.803

1	1.527	1.976	2.497
2	1.341	1.979	2.679
3	1.224	2.028	2.748
4	1.178	2.009	2.813

1	1.528	1.985	2.487
2	1.381	1.991	2.628
3	1.278	2.019	2.703
4	1.209	2.039	2.751
5	1.165	2.009	2.826

1	1.529	2.011	2.460
2	1.417	1.991	2.591
3	1.317	1.985	2.698
4	1.259	1.991	2.750
5	1.211	1.993	2.796
6	1.163	1.990	2.847

1	1.604	2.017	2.379
2	1.416	2.018	2.566
3	1.339	2.033	2.628
4	1.253	2.013	2.734
5	1.237	1.982	2.781
6	1.186	2.054	2.760
7	1.159	1.983	2.859

1	1.595	2.001	2.405
2	1.461	2.003	2.537
3	1.387	1.993	2.620
4	1.306	2.003	2.691
5	1.239	2.012	2.749
6	1.229	1.999	2.773
7	1.189	2.007	2.804
8	1.139	2.002	2.859

1	1.630	1.983	2.387
2	1.463	1.983	2.554
3	1.396	1.969	2.635
4	1.333	1.991	2.677
5	1.296	1.984	2.720
6	1.241	1.995	2.764
7	1.202	2.011	2.787
8	1.169	2.020	2.811
9	1.142	2.001	2.857

1	1.599	2.016	2.385
2	1.455	2.024	2.521
3	1.393	1.994	2.613
4	1.342	1.983	2.675
5	1.307	2.014	2.679
6	1.269	1.991	2.739
7	1.213	2.017	2.770
8	1.218	1.974	2.808
9	1.183	1.989	2.828
10	1.153	1.994	2.853

m = 4 treatments

1	1.409	2.125	2.880	3.585
2	1.273	2.066	2.924	3.736

1	1.432	2.099	2.933	3.535
2	1.328	2.042	2.936	3.693
3	1.245	2.049	2.913	3.793

1	1.471	2.114	2.889	3.525
2	1.356	2.057	2.926	3.660
3	1.290	2.068	2.937	3.704
4	1.226	2.059	2.930	3.784

1	1.464	2.134	2.899	3.502
2	1.392	2.073	2.900	3.635
3	1.299	2.057	2.962	3.681
4	1.241	2.113	2.905	3.741
5	1.214	2.065	2.940	3.781

1	1.552	2.142	2.825	3.480
2	1.374	2.087	2.930	3.608
3	1.333	2.084	2.898	3.684
4	1.306	2.050	2.927	3.716
5	1.263	2.060	2.945	3.731
6	1.200	2.086	2.906	3.807

1	1.506	2.131	2.877	3.485
2	1.430	2.057	2.924	3.578
3	1.371	2.038	2.982	3.609
4	1.332	2.056	2.929	3.682
5	1.278	2.060	2.955	3.707
6	1.227	2.095	2.937	3.741
7	1.173	2.088	2.921	3.817

1	1.552	2.170	2.828	3.449
2	1.427	2.118	2.868	3.586
3	1.373	2.065	2.926	3.635
4	1.330	2.071	2.904	3.694
5	1.298	2.039	2.964	3.698
6	1.262	2.053	2.967	3.717
7	1.220	2.077	2.933	3.769
8	1.191	2.095	2.910	3.803

1	1.571	2.135	2.871	3.422
2	1.462	2.097	2.882	3.558
3	1.382	2.087	2.907	3.623
4	1.360	2.051	2.943	3.644
5	1.326	2.052	2.954	3.667
6	1.298	2.024	2.967	3.710
7	1.250	2.068	2.934	3.747
8	1.242	2.067	2.916	3.774
9	1.200	2.067	2.929	3.802

1	1.584	2.182	2.822	3.411
2	1.469	2.114	2.905	3.511
3	1.369	2.077	2.917	3.636
4	1.369	2.084	2.896	3.650
5	1.333	2.059	2.928	3.679
6	1.307	2.061	2.927	3.704
7	1.271	2.039	2.985	3.704
8	1.243	2.053	2.950	3.753
9	1.218	2.046	2.948	3.787
10	1.181	2.077	2.925	3.815

m = 5 treatments

1	1.399	2.073	3.026	3.898	4.604
2	1.256	2.112	3.007	3.898	4.728

1	1.422	2.102	3.013	3.905	4.558
2	1.341	2.040	3.007	3.926	4.685
3	1.257	2.085	3.000	3.910	4.747
4					

1	1.457	2.124	2.970	3.924	4.524
2	1.360	2.080	3.010	3.920	4.630
3	1.306	2.055	3.054	3.899	4.686
4	1.218	2.154	2.983	3.878	4.766
5					

1	1.488	2.088	3.014	3.903	4.506
2	1.353	2.104	3.026	3.908	4.609
3	1.328	2.073	3.026	3.910	4.664
4	1.288	2.092	2.991	3.904	4.726
5	1.196	2.143	2.994	3.892	4.776

1	1.507	2.129	2.983	3.865	4.515
2	1.377	2.125	2.950	3.937	4.611
3	1.355	2.080	2.978	3.957	4.629
4	1.316	2.084	3.015	3.896	4.690
5	1.265	2.071	3.036	3.884	4.743
6	1.227	2.108	2.994	3.895	4.776

1	1.507	2.114	3.067	3.815	4.497
2	1.399	2.095	3.003	3.934	4.568
3	1.369	2.090	2.965	3.937	4.639
4	1.322	2.095	3.007	3.927	4.649
5	1.299	2.064	3.012	3.939	4.686
6	1.266	2.100	3.011	3.886	4.736
7	1.197	2.150	3.001	3.859	4.792

1	1.523	2.118	2.986	3.882	4.492
2	1.406	2.120	2.996	3.914	4.564
3	1.380	2.067	3.008	3.931	4.614
4	1.355	2.065	2.996	3.894	4.690
5	1.333	2.073	2.977	3.935	4.682
6	1.298	2.042	3.007	3.930	4.723
7	1.250	2.108	2.984	3.899	4.759
8	1.198	2.143	2.988	3.869	4.801
9					

1	1.503	2.122	3.040	3.864	4.471
2	1.432	2.108	2.998	3.909	4.554
3	1.402	2.084	2.998	3.896	4.620
4	1.370	2.072	2.996	3.914	4.648
5	1.336	2.097	2.993	3.922	4.652
6	1.308	2.069	2.994	3.922	4.708
7	1.283	2.094	3.002	3.923	4.698
8	1.246	2.100	2.995	3.894	4.764
9	1.201	2.113	3.010	3.862	4.814
10					

1	1.529	2.118	2.992	3.896	4.465
2	1.440	2.082	3.009	3.914	4.555
3	1.390	2.094	3.006	3.928	4.582
4	1.366	2.088	2.995	3.940	4.612
5	1.347	2.082	3.014	3.889	4.669
6	1.311	2.070	3.014	3.946	4.659
7	1.291	2.107	2.992	3.898	4.712
8	1.273	2.070	3.023	3.896	4.737
9	1.235	2.088	3.013	3.904	4.760
10	1.183	2.134	3.004	3.874	4.804

APPENDIX II: Uniform Data

m = 2 treatments

1	1.409	1.591
2	1.238	1.762

1	1.410	1.590
2	1.351	1.649
3	1.209	1.791

1	1.432	1.568
2	1.353	1.647
3	1.273	1.727
4	1.192	1.808

1	1.451	1.549
2	1.392	1.608
3	1.380	1.620
4	1.303	1.697
5	1.155	1.845

1	1.468	1.532
2	1.401	1.599
3	1.365	1.635
4	1.317	1.683
5	1.275	1.725
6	1.129	1.871

1	1.453	1.547
2	1.408	1.592
3	1.347	1.653
4	1.363	1.637
5	1.314	1.686
6	1.236	1.764
7	1.123	1.877

1	1.442	1.558
2	1.443	1.557
3	1.384	1.616
4	1.362	1.638
5	1.326	1.674
6	1.304	1.696
7	1.226	1.774
8	1.096	1.904

1	1.463	1.537
2	1.430	1.570
3	1.420	1.580
4	1.391	1.609
5	1.330	1.670
6	1.300	1.700
7	1.274	1.726
8	1.196	1.804
9	1.091	1.909

1	1.457	1.543
2	1.440	1.560
3	1.421	1.579
4	1.363	1.637
5	1.326	1.674
6	1.336	1.664
7	1.312	1.688
8	1.256	1.744
9	1.208	1.792
10	1.079	1.921

m = 3 treatments

1	1.645	1.981	2.374
2	1.344	2.022	2.634

1	1.717	1.983	2.301
2	1.495	2.013	2.492
3	1.299	2.021	2.679

1	1.749	1.968	2.283
2	1.595	2.009	2.396
3	1.417	2.005	2.577
4	1.229	2.029	2.741

1	1.807	1.960	2.233
2	1.633	2.012	2.355
3	1.515	2.003	2.481
4	1.380	1.962	2.658
5	1.231	2.013	2.756

1	1.793	2.033	2.173
2	1.636	2.035	2.329
3	1.534	2.048	2.418
4	1.425	2.019	2.555
5	1.332	2.012	2.656
6	1.218	1.979	2.803

1	1.827	1.981	2.191
2	1.679	1.987	2.334
3	1.618	1.995	2.387
4	1.511	1.984	2.505
5	1.402	1.986	2.612
6	1.303	2.016	2.681
7	1.192	1.979	2.829

1	1.817	2.023	2.159
2	1.692	1.969	2.339
3	1.684	1.967	2.349
4	1.550	2.009	2.441
5	1.450	2.004	2.546
6	1.366	1.985	2.649
7	1.249	2.030	2.721
8	1.179	2.000	2.821

1	1.841	1.986	2.173
2	1.725	2.011	2.265
3	1.675	2.023	2.303
4	1.565	2.003	2.433
5	1.489	1.986	2.525
6	1.440	2.003	2.557
7	1.319	2.006	2.675
8	1.259	2.027	2.714
9	1.177	1.981	2.841

1	1.865	1.973	2.162
2	1.736	1.997	2.267
3	1.709	1.972	2.319
4	1.634	2.014	2.352
5	1.543	2.020	2.437
6	1.486	1.972	2.542
7	1.412	1.983	2.605
8	1.307	2.015	2.679
9	1.259	1.975	2.765
10	1.173	2.001	2.825

m = 4 treatments

1	1.731	2.212	2.766	3.290
2	1.440	2.107	2.877	3.575

1	1.831	2.290	2.682	3.196
2	1.571	2.172	2.794	3.462
3	1.367	2.110	2.901	3.621

1	1.867	2.296	2.691	3.145
2	1.648	2.200	2.819	3.333
3	1.492	2.110	2.903	3.495
4	1.362	2.088	2.868	3.681

1	1.907	2.348	2.597	3.147
2	1.721	2.146	2.795	3.338
3	1.589	2.124	2.847	3.439
4	1.467	2.092	2.915	3.525
5	1.302	2.110	2.904	3.682

1	1.940	2.345	2.650	3.064
2	1.726	2.227	2.783	3.263
3	1.586	2.133	2.867	3.413
4	1.488	2.139	2.855	3.517
5	1.429	2.076	2.904	3.590
6	1.287	2.131	2.872	3.709

1	1.945	2.377	2.615	3.062
2	1.759	2.275	2.762	3.203
3	1.680	2.173	2.794	3.352
4	1.575	2.118	2.864	3.442
5	1.492	2.122	2.854	3.530
6	1.421	2.087	2.891	3.600
7	1.265	2.133	2.858	3.744

1	1.942	2.373	2.666	3.018
2	1.811	2.251	2.729	3.209
3	1.677	2.189	2.811	3.321
4	1.567	2.170	2.876	3.386
5	1.525	2.109	2.876	3.489
6	1.481	2.059	2.920	3.540
7	1.374	2.070	2.943	3.612
8	1.255	2.105	2.885	3.755

1	1.976	2.387	2.659	2.977
2	1.799	2.302	2.688	3.210
3	1.709	2.234	2.773	3.283
4	1.607	2.196	2.822	3.374
5	1.565	2.092	2.920	3.422
6	1.470	2.127	2.869	3.533
7	1.436	2.089	2.916	3.558
8	1.357	2.101	2.924	3.617
9	1.224	2.140	2.874	3.760

1	2.017	2.414	2.578	2.990
2	1.832	2.338	2.670	3.159
3	1.717	2.267	2.763	3.252
4	1.629	2.210	2.792	3.368
5	1.578	2.126	2.870	3.426
6	1.523	2.113	2.886	3.477
7	1.484	2.075	2.925	3.514
8	1.441	2.059	2.914	3.585
9	1.342	2.082	2.920	3.654
10	1.227	2.116	2.868	3.788

m = 5 treatments

1	1.695	2.366	3.000	3.655	4.284
2	1.484	2.220	2.992	3.778	4.526

1	1.782	2.391	3.016	3.585	4.226
2	1.590	2.224	3.004	3.792	4.389
3	1.413	2.229	2.986	3.789	4.584

1	1.832	2.397	3.025	3.580	4.166
2	1.707	2.217	2.998	3.761	4.317
3	1.560	2.170	2.996	3.798	4.474
4	1.385	2.185	2.993	3.800	4.636

1	1.872	2.449	3.006	3.522	4.150
2	1.730	2.234	2.994	3.734	4.308
3	1.594	2.221	2.963	3.839	4.383
4	1.511	2.170	3.019	3.826	4.475
5	1.331	2.220	3.024	3.789	4.636

1	1.864	2.486	3.001	3.540	4.109
2	1.734	2.304	2.986	3.727	4.249
3	1.613	2.220	3.016	3.786	4.365
4	1.573	2.174	2.997	3.846	4.410
5	1.470	2.129	3.008	3.883	4.510
6	1.326	2.220	2.977	3.799	4.678

1	1.903	2.500	2.997	3.481	4.118
2	1.754	2.310	3.014	3.636	4.285
3	1.684	2.190	3.041	3.759	4.326
4	1.604	2.212	2.975	3.846	4.363
5	1.556	2.174	2.988	3.806	4.476
6	1.456	2.170	3.011	3.788	4.575
7	1.290	2.254	2.986	3.757	4.714

1	1.928	2.516	3.060	3.458	4.038
2	1.765	2.384	2.988	3.650	4.213
3	1.687	2.271	3.031	3.717	4.294
4	1.644	2.211	2.991	3.781	4.373
5	1.553	2.190	3.020	3.817	4.420
6	1.518	2.182	2.984	3.834	4.483
7	1.394	2.194	3.002	3.853	4.557
8	1.270	2.262	3.004	3.741	4.722

1	1.884	2.609	3.013	3.403	4.091
2	1.802	2.320	3.031	3.648	4.199
3	1.674	2.308	3.012	3.719	4.286
4	1.678	2.210	2.975	3.783	4.354
5	1.631	2.169	2.982	3.793	4.424
6	1.550	2.162	3.007	3.823	4.458
7	1.499	2.166	3.002	3.824	4.509
8	1.399	2.197	3.002	3.794	4.608
9	1.240	2.277	3.000	3.751	4.732

1	1.935	2.565	2.982	3.440	4.077
2	1.824	2.368	3.005	3.620	4.183
3	1.716	2.297	3.000	3.726	4.261
4	1.668	2.206	3.030	3.788	4.307
5	1.657	2.162	2.980	3.809	4.393
6	1.599	2.150	3.016	3.851	4.384
7	1.519	2.161	3.045	3.825	4.450
8	1.494	2.144	2.985	3.829	4.548
9	1.398	2.146	3.016	3.838	4.602
10	1.221	2.274	3.000	3.763	4.742

APPENDIX III: Exponential Data

m = 2 treatments

1	1.400	1.600
2	1.260	1.740

1	1.431	1.569
2	1.349	1.651
3	1.231	1.769

1	1.456	1.544
2	1.390	1.610
3	1.308	1.692
4	1.191	1.809

1	1.477	1.523
2	1.409	1.591
3	1.328	1.672
4	1.252	1.748
5	1.154	1.846

1	1.479	1.521
2	1.430	1.570
3	1.373	1.627
4	1.322	1.678
5	1.217	1.783
6	1.148	1.852

1	1.497	1.503
2	1.441	1.559
3	1.415	1.585
4	1.376	1.624
5	1.305	1.695
6	1.226	1.774
7	1.121	1.879

1	1.487	1.513
2	1.458	1.542
3	1.403	1.597
4	1.346	1.654
5	1.339	1.661
6	1.278	1.722
7	1.217	1.783
8	1.122	1.878

1	1.488	1.512
2	1.448	1.552
3	1.428	1.572
4	1.408	1.592
5	1.352	1.648
6	1.321	1.679
7	1.260	1.740
8	1.215	1.785
9	1.111	1.889

1	1.494	1.506
2	1.451	1.549
3	1.421	1.579
4	1.428	1.572
5	1.371	1.629
6	1.350	1.650
7	1.289	1.711
8	1.232	1.768
9	1.207	1.793
10	1.102	1.898

m = 3 treatments

1	1.671	2.071	2.257
2	1.474	2.065	2.461

1	1.721	2.061	2.219
2	1.615	2.061	2.325
3	1.435	2.037	2.528

1	1.811	2.057	2.132
2	1.668	2.069	2.263
3	1.508	2.067	2.425
4	1.393	2.035	2.573

1	1.827	2.064	2.109
2	1.700	2.059	2.241
3	1.580	2.101	2.319
4	1.471	2.109	2.421
5	1.385	2.018	2.597

1	1.833	2.049	2.118
2	1.724	2.070	2.206
3	1.631	2.095	2.274
4	1.527	2.095	2.377
5	1.431	2.049	2.521
6	1.373	2.005	2.621

1	1.853	2.063	2.083
2	1.701	2.125	2.173
3	1.633	2.100	2.267
4	1.570	2.084	2.346
5	1.528	2.099	2.373
6	1.424	2.071	2.505
7	1.351	2.001	2.647

1	1.849	2.046	2.105
2	1.724	2.083	2.193
3	1.674	2.104	2.222
4	1.621	2.090	2.289
5	1.563	2.089	2.349
6	1.495	2.045	2.459
7	1.403	2.086	2.511
8	1.351	1.987	2.663

1	1.847	2.029	2.124
2	1.823	2.035	2.142
3	1.723	2.089	2.188
4	1.650	2.084	2.266
5	1.582	2.081	2.337
6	1.507	2.100	2.393
7	1.451	2.076	2.473
8	1.387	2.059	2.553
9	1.348	1.984	2.668

1	1.904	1.994	2.102
2	1.761	2.079	2.160
3	1.693	2.080	2.227
4	1.634	2.083	2.283
5	1.601	2.100	2.299
6	1.553	2.059	2.388
7	1.487	2.119	2.394
8	1.434	2.072	2.494
9	1.363	2.089	2.548
10	1.326	2.012	2.662

m = 4 treatments

1	1.906	2.458	2.760	2.875
2	1.669	2.367	2.799	3.163

1	1.975	2.476	2.689	2.860
2	1.766	2.403	2.798	3.032
3	1.618	2.311	2.845	3.224

1	2.030	2.493	2.689	2.787
2	1.826	2.451	2.772	2.950
3	1.660	2.417	2.794	3.128
4	1.593	2.268	2.846	3.292

1	2.037	2.491	2.667	2.804
2	1.856	2.477	2.801	2.865
3	1.709	2.436	2.817	3.037
4	1.635	2.367	2.805	3.192
5	1.591	2.249	2.823	3.336

1	2.090	2.521	2.668	2.720
2	1.904	2.487	2.725	2.883
3	1.781	2.475	2.769	2.974
4	1.694	2.436	2.815	3.054
5	1.628	2.357	2.833	3.182
6	1.594	2.219	2.826	3.360

1	2.082	2.514	2.659	2.745
2	1.963	2.478	2.725	2.833
3	1.799	2.447	2.776	2.977
4	1.733	2.412	2.815	3.039
5	1.686	2.353	2.860	3.100
6	1.577	2.322	2.885	3.215
7	1.570	2.221	2.830	3.378

1	2.114	2.500	2.649	2.735
2	1.986	2.471	2.692	2.850
3	1.866	2.485	2.735	2.913
4	1.771	2.469	2.729	3.030
5	1.658	2.483	2.810	3.048
6	1.631	2.396	2.830	3.142
7	1.623	2.309	2.810	3.257
8	1.584	2.199	2.797	3.419

1	2.160	2.515	2.639	2.685
2	1.989	2.503	2.678	2.829
3	1.898	2.502	2.704	2.895
4	1.833	2.471	2.760	2.935
5	1.724	2.430	2.820	3.025
6	1.669	2.337	2.842	3.150
7	1.641	2.308	2.844	3.206
8	1.585	2.293	2.839	3.281
9	1.573	2.161	2.844	3.422

1	2.208	2.472	2.614	2.705
2	2.010	2.516	2.683	2.790
3	1.920	2.514	2.694	2.871
4	1.875	2.482	2.761	2.882
5	1.740	2.470	2.826	2.963
6	1.722	2.418	2.839	3.020
7	1.641	2.412	2.841	3.105
8	1.623	2.347	2.886	3.144
9	1.603	2.267	2.845	3.284
10	1.614	2.153	2.784	3.448

m = 5 treatments

1	2.084	2.736	3.192	3.418	3.569
2	1.846	2.570	3.131	3.539	3.914

1	2.127	2.784	3.166	3.397	3.526
2	1.922	2.653	3.199	3.526	3.700
3	1.823	2.513	3.095	3.589	3.980

1	2.165	2.891	3.121	3.359	3.464
2	1.950	2.783	3.213	3.425	3.629
3	1.823	2.652	3.138	3.567	3.820
4	1.770	2.514	3.078	3.622	4.016

1	2.260	2.876	3.116	3.318	3.430
2	2.007	2.794	3.211	3.431	3.557
3	1.910	2.664	3.208	3.482	3.736
4	1.828	2.586	3.108	3.572	3.905
5	1.783	2.488	3.056	3.562	4.112

1	2.276	2.923	3.153	3.270	3.378
2	2.038	2.873	3.204	3.347	3.538
3	1.904	2.743	3.235	3.448	3.669
4	1.891	2.640	3.247	3.495	3.728
5	1.826	2.587	3.108	3.540	3.938
6	1.807	2.471	3.027	3.616	4.079

1	2.298	2.926	3.143	3.277	3.356
2	2.055	2.835	3.230	3.360	3.520
3	1.964	2.779	3.192	3.484	3.581
4	1.886	2.688	3.144	3.566	3.716
5	1.839	2.582	3.176	3.560	3.843
6	1.786	2.536	3.161	3.582	3.935
7	1.775	2.453	3.030	3.619	4.124

1	2.350	2.917	3.134	3.250	3.350
2	2.122	2.858	3.198	3.359	3.462
3	1.964	2.772	3.255	3.453	3.555
4	1.953	2.731	3.194	3.485	3.636
5	1.877	2.629	3.120	3.574	3.800
6	1.830	2.603	3.136	3.565	3.867
7	1.815	2.521	3.091	3.582	3.992
8	1.764	2.426	3.020	3.615	4.174

1	2.328	2.918	3.154	3.302	3.297
2	2.133	2.883	3.140	3.348	3.497
3	2.064	2.817	3.198	3.426	3.494
4	1.902	2.767	3.240	3.508	3.583
5	1.868	2.676	3.194	3.540	3.724
6	1.840	2.633	3.226	3.519	3.781
7	1.820	2.517	3.184	3.588	3.891
8	1.790	2.464	3.095	3.677	3.974
9	1.806	2.418	3.023	3.620	4.134

1	2.372	2.929	3.144	3.233	3.322
2	2.156	2.909	3.186	3.304	3.444
3	2.035	2.838	3.207	3.356	3.564
4	1.959	2.797	3.188	3.467	3.588
5	1.904	2.746	3.169	3.456	3.726
6	1.889	2.680	3.205	3.486	3.740
7	1.849	2.580	3.205	3.553	3.813
8	1.827	2.586	3.098	3.576	3.914
9	1.784	2.458	3.082	3.615	4.062
10	1.760	2.454	3.016	3.576	4.194

APPENDIX IV: Cauchy Data

m = 2 treatments

1	1.350	1.650
2	1.368	1.632

1	1.369	1.631
2	1.326	1.674
3	1.355	1.645

1	1.382	1.618
2	1.313	1.687
3	1.316	1.684
4	1.355	1.645

1	1.420	1.580
2	1.336	1.664
3	1.340	1.660
4	1.333	1.667
5	1.422	1.578

1	1.420	1.580
2	1.339	1.661
3	1.316	1.684
4	1.310	1.690
5	1.353	1.647
6	1.415	1.585

1	1.418	1.582
2	1.352	1.648
3	1.295	1.705
4	1.305	1.695
5	1.317	1.683
6	1.362	1.638
7	1.436	1.564

1	1.398	1.602
2	1.374	1.626
3	1.317	1.683
4	1.294	1.706
5	1.304	1.696
6	1.312	1.688
7	1.375	1.625
8	1.408	1.592

1	1.441	1.559
2	1.390	1.610
3	1.354	1.646
4	1.294	1.706
5	1.297	1.703
6	1.294	1.706
7	1.309	1.691
8	1.403	1.597
9	1.412	1.588

1	1.423	1.577
2	1.364	1.636
3	1.344	1.656
4	1.340	1.660
5	1.307	1.693
6	1.277	1.723
7	1.305	1.695
8	1.336	1.664
9	1.371	1.629
10	1.433	1.567

m = 3 treatments

1	1.585	1.998	2.417
2	1.654	2.007	2.339

1	1.565	1.989	2.445
2	1.580	1.996	2.424
3	1.673	2.020	2.307

1	1.569	1.998	2.433
2	1.533	1.996	2.471
3	1.573	2.037	2.389
4	1.715	2.021	2.264

1	1.575	1.975	2.451
2	1.535	2.005	2.459
3	1.547	1.975	2.478
4	1.646	1.976	2.378
5	1.713	2.026	2.261

1	1.600	2.022	2.378
2	1.513	2.015	2.472
3	1.501	2.039	2.460
4	1.563	2.025	2.412
5	1.661	1.991	2.348
6	1.727	1.997	2.276

1	1.615	1.983	2.402
2	1.539	1.975	2.485
3	1.533	1.977	2.489
4	1.533	1.997	2.469
5	1.611	1.989	2.399
6	1.658	1.993	2.349
7	1.786	2.000	2.214

1	1.639	1.993	2.367
2	1.519	2.011	2.470
3	1.503	2.011	2.486
4	1.501	1.991	2.507
5	1.550	1.986	2.464
6	1.606	1.987	2.407
7	1.689	2.025	2.286
8	1.769	1.991	2.240

1	1.615	2.013	2.373
2	1.540	1.985	2.475
3	1.508	2.008	2.484
4	1.500	1.994	2.506
5	1.538	2.004	2.458
6	1.559	2.054	2.387
7	1.652	1.975	2.373
8	1.685	2.017	2.298
9	1.821	1.995	2.183

1	1.643	2.019	2.337
2	1.525	2.014	2.461
3	1.503	2.022	2.475
4	1.494	1.988	2.518
5	1.528	1.992	2.480
6	1.582	1.998	2.420
7	1.609	1.966	2.425
8	1.673	1.963	2.364
9	1.758	1.998	2.244
10	1.795	2.019	2.186

m = 4 treatments

1	1.717	2.201	2.782	3.298
2	1.858	2.250	2.731	3.160

1	1.701	2.256	2.736	3.306
2	1.766	2.243	2.738	3.251
3	1.927	2.295	2.706	3.071

1	1.712	2.190	2.791	3.306
2	1.680	2.223	2.792	3.304
3	1.823	2.250	2.727	3.199
4	1.963	2.282	2.692	3.063

1	1.745	2.203	2.773	3.278
2	1.683	2.181	2.792	3.344
3	1.726	2.221	2.748	3.304
4	1.855	2.299	2.721	3.125
5	2.009	2.317	2.674	2.999

1	1.735	2.211	2.813	3.240
2	1.632	2.226	2.801	3.340
3	1.719	2.197	2.755	3.328
4	1.751	2.247	2.745	3.256
5	1.869	2.277	2.714	3.139
6	2.042	2.268	2.679	3.011

1	1.760	2.208	2.780	3.251
2	1.658	2.182	2.794	3.365
3	1.646	2.247	2.761	3.344
4	1.740	2.262	2.759	3.238
5	1.790	2.276	2.727	3.206
6	1.905	2.337	2.670	3.087
7	2.060	2.334	2.633	2.972

1	1.723	2.177	2.827	3.272
2	1.653	2.195	2.787	3.364
3	1.656	2.198	2.806	3.339
4	1.720	2.225	2.787	3.266
5	1.739	2.264	2.744	3.252
6	1.849	2.258	2.735	3.157
7	1.911	2.279	2.724	3.086
8	2.025	2.262	2.751	2.961

1	1.714	2.250	2.800	3.236
2	1.671	2.178	2.823	3.327
3	1.679	2.174	2.801	3.345
4	1.645	2.256	2.752	3.346
5	1.722	2.215	2.756	3.306
6	1.787	2.250	2.770	3.193
7	1.846	2.313	2.705	3.136
8	1.946	2.346	2.681	3.026
9	2.061	2.320	2.685	2.933

1	1.796	2.203	2.784	3.216
2	1.683	2.195	2.767	3.354
3	1.652	2.204	2.795	3.347
4	1.665	2.235	2.773	3.327
5	1.664	2.217	2.774	3.344
6	1.769	2.212	2.733	3.285
7	1.836	2.273	2.711	3.179
8	1.869	2.284	2.720	3.126
9	1.942	2.308	2.708	3.041
10	2.039	2.371	2.677	2.912

m = 5 treatments

1	1.827	2.389	2.998	3.609	4.178
2	1.995	2.574	2.976	3.496	3.958

1	1.824	2.400	2.990	3.589	4.198
2	1.890	2.456	3.002	3.575	4.077
3	2.145	2.569	2.973	3.447	3.866

1	1.794	2.351	2.992	3.672	4.191
2	1.839	2.351	3.018	3.587	4.205
3	2.024	2.389	3.034	3.530	4.024
4	2.163	2.570	2.991	3.426	3.851

1	1.816	2.312	3.001	3.690	4.1[illegible]0
2	1.762	2.381	2.958	3.682	4.217
3	1.874	2.419	3.004	3.553	4.151
4	2.038	2.493	3.036	3.506	3.9[illegible]7
5	2.194	2.612	3.046	3.395	3.753

1	1.787	2.365	2.988	3.721	4.1[illegible]9
2	1.745	2.346	3.030	3.661	4.218
3	1.826	2.409	2.962	3.628	4.174
4	1.946	2.420	2.973	3.584	4.078
5	2.085	2.469	2.999	3.507	3.940
6	2.225	2.574	3.005	3.409	3.788

1	1.824	2.308	3.014	3.671	4.183
2	1.738	2.355	3.030	3.619	4.258
3	1.769	2.353	3.032	3.648	4.198
4	1.922	2.385	3.000	3.575	4.178
5	1.951	2.472	3.037	3.515	4.024
6	2.102	2.522	2.987	3.461	3.928
7	2.282	2.653	2.976	3.349	3.740

1	1.860	2.311	2.992	3.668	4.168
2	1.746	2.343	3.010	3.651	4.250
3	1.768	2.381	3.012	3.627	4.212
4	1.821	2.411	3.046	3.582	4.140
5	1.918	2.449	2.993	3.548	4.092
6	1.990	2.512	3.047	3.508	3.943
7	2.098	2.571	2.996	3.451	3.884
8	2.308	2.602	3.007	3.373	3.710

1	1.849	2.311	3.000	3.710	4.120
2	1.732	2.363	3.029	3.672	4.204
3	1.736	2.374	2.974	3.631	4.286
4	1.851	2.350	2.979	3.581	4.239
5	1.861	2.422	2.963	3.580	4.174
6	1.946	2.504	2.984	3.522	4.043
7	2.046	2.545	2.992	3.452	3.964
8	2.142	2.572	3.012	3.418	3.856
9	2.265	2.612	2.983	3.358	3.782

1	1.904	2.297	3.018	3.652	4.130
2	1.747	2.336	3.003	3.667	4.246
3	1.744	2.321	2.994	3.678	4.264
4	1.748	2.377	3.040	3.653	4.182
5	1.861	2.385	2.970	3.640	4.144
6	1.885	2.428	3.002	3.587	4.098
7	2.021	2.500	3.020	3.484	3.974
8	2.062	2.519	2.978	3.492	3.948
9	2.146	2.574	3.024	3.436	3.851
10	2.237	2.569	3.000	3.438	3.756