CONSTITUTIVE RELATIONS FOR NON-ELASTIC DEFORMATION

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SUMMARY

The constitutive relations that are described are based on the use of *deformation state variables* (internal variables). The use of state variables for the description of the non-elastic deformation of solids is neither trivial nor self-evident. The purpose of such a formulation is to account for the effects of prior deformation or processing in terms of a small number of variables whose values can be established by tests at the current state of the material. If such a description is possible, the effects of past history are fully accounted for by the state variables and the description of the deformation can be a time-local description.

The search for a state variable description requires a theoretical framework in which the experimental questions can be posed. Such a phenomenological theory was formulated by Hart, Acta Metallurgica 18(1970)599, in connection with a critique of deformation phenomenology. The critique considered time dependent recoverable strain (anelastic strain) as well as unrecoverable ("plastic") strain. However, the analysis was directed principally to the unrecoverable strain component. More recently a complete model has been described that treats both non-elastic components. This current model describes the full range of deformation phenomena including loading transient behavior and Bauschinger effect. The formulation is fully three-dimensional and provision is made for describing texture anisotropy if necessary.

The constitutive equations relate the non-elastic strain rate $\dot{\varepsilon}$, the deviator $\underset{\sim}{\sigma}$ of the stress tensor, and the temperature T according to a set of material parameters and two state variables. The state variables are: σ^*, a scalar, called the hardness, and $\underset{\sim}{a}$, a deviatoric tensor that is equal to the stored anelastic strain. In effect, σ^* represents the current level of isotropic hardening, and $\underset{\sim}{a}$ determines the magnitude and direction of "internal stress" that reflects prior loading directions and determines the anisotropy of subsequent deformation such as Bauschinger effect. The variables σ^* and $\underset{\sim}{a}$ obey time dependent evolution laws. It is postulated that the elastic strain of the material is simply determined at any time by the current value of the stress tensor $\underset{\sim}{\tau}$ and Hooke's Law. The constitutive relations of concern are therefore stated entirely in terms of the non-elastic strain rate. All the relations are incremental and so no finite strain considerations are needed in the constitutive relations.

The constitutive relations are most easily stated with the help of three auxiliary variables that are not directly observable. These are: $\underset{\sim}{\dot{\alpha}}$, the rate of unrecoverable strain ("plastic" strain rate), $\underset{\sim}{\sigma}_a$, the internal stress associated with the stored recoverable strain a, and σ_f, the effective stress for dislocation glide ("friction" stress). In general form the relations are as follows:

$$\underset{\sim}{\dot{\varepsilon}} = \mathrm{d}\underset{\sim}{a}/\mathrm{d}t + \underset{\sim}{\dot{\alpha}}, \qquad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}_a + \underset{\sim}{\sigma}_f .$$

These are the general constraints. Further, we describe the case for texturally isotropic material, noting that deformation induced anisotropy will be explicitly described. In the equations, the light faced symbols represent the second invariant of the corresponding tensor quantities. Thus, for example,

$$\dot{\varepsilon} \equiv +\sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} ,$$

where the summation convention for repeated indices is implied.

Now the isotropy relations are

$$\dot{\varepsilon} = (\dot{\varepsilon}/\sigma_f)\, \underset{\sim}{\sigma}_f , \qquad \underset{\sim}{\dot{\alpha}} = (\dot{\alpha}/\sigma_a)\, \underset{\sim}{\sigma}_a , \qquad a = (a/\sigma_a)\, \underset{\sim}{\sigma}_a .$$

The remaining constitutive equations concern the tensor invariants.

$$\dot{\varepsilon} = f_1(\sigma_f, T), \qquad \dot{\alpha} = f_x(\sigma_a, \sigma^*, T), \qquad a = f_3(\sigma_a), \qquad \dot{\sigma}^* = f_4(\sigma_a, \sigma^*, T).$$

The functions are determined experimentally and are remarkably form insensitive to the metal species.

In the paper we describe the testing procedures, the computational procedures for using the equations, and the prospects for extension to radiation environment behavior.

1. Introduction

The design and service prediction of metallic structural elements for use in power reactors depends critically on the reliability of the constitutive relations used to describe the mechanical response of the materials of which the structural elements are fabricated. Since the structural elements may experience a very broad range of operating conditions, the testing needed to establish the relations can be costly in time and money. This is especially true for those elements that are subject to radiation environment. It is highly desirable, then, that optimum use be made of test data that can be generated cheaply.

The constitutive relations that will be described here are based on the concept of deformation state variables (internal variables). The testing experience thus far with these constitutive relations has shown them to be reliable over very broad ranges of stress and temperature. There has lately been strong indication that they can be extended to describe the effects of nuclear radiation flux and fluence as well. In the latter case considerable use can be made of thermal environment test results in the prediction of radiation environment behavior.

In the following we shall discuss briefly the theoretical bases for the constitutive relations. The proposed equations will then be presented somewhat prescriptively. The experimental bases will be described with special reference to the load relaxation test. Calculation methods will then be considered, and finally we shall discuss the prospects for radiation environment application and the experimentation that remains to be done.

2. Theoretical Basis for the Constitutive Relations

2.1 Critique of Classical Constitutive Relations

The critique on which the subsequent development is based was given by Hart[1] in a study of prior attempts at the formulation of a mechanical equation of state. The principal object of that study was the plastic strain. In one form or another, the accumulated plastic strain appears as a material state variable in most of the current constitutive equations including those that take account of strain rate effects. It was shown in the paper that only one component, a minor one, of the non-elastic strain could be considered to be a state variable. That component is the recoverable, anelastic strain. The major non-elastic strain component is not a state variable but, rather, describes past deformation history. There is no test that can determine the quantity of plastic strain that a specimen has undergone.

It was shown that there was good reason to believe that the current mechanical state of the material could be described by a single deformation state variable (in addition to the current anelastic strain), and that such a variable should satisfy an incremental evolution equation with respect to plastic straining. Experimental criteria were stated for testing this hypothesis.

2.2 Formulation of the Problem

2.2.1 The Strain Components

We are concerned in this paper only with the non-elastic strain. It is assumed that at any state of stress, the material exhibits elastic strain according to Hooke's Law in addition to any non-elastic strain effects.

Our entire formulation is incremental and all rates are real time rates. In the following we represent tensor quantities either by index notation or by bold face symbols (indicated here by wavy underscoring).

The observable total non-elastic strain rate $\dot{\boldsymbol{\varepsilon}}$ is considered to be the sum of the time rate of change of a recoverable anelastic strain $\boldsymbol{a}$ and an unrecoverable plastic strain rate $\dot{\boldsymbol{\alpha}}$. Thus we write

$$\dot{\boldsymbol{\varepsilon}} = \frac{d}{dt}\boldsymbol{a} + \dot{\boldsymbol{\alpha}}. \tag{1}$$

By this form of writing we emphasize that $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\alpha}}$ are not the time derivatives of state quantities but are simply rates. In the following we shall use the dot notation for the time derivative of $\boldsymbol{a}$ for simplicity.

2.2.2 The State Variable Concept

Since the anelastic strain is a stored strain in the material, it is not surprising that it will be considered as a state variable characterizing the current state of the material. This state variable will be responsible for the mechanical anisotropy due to prior straining (Bauschinger Effect).

Since the accumulated plastic strain is not a state variable, the question arises whether there is an internal variable whose value can be always determined that can account for the effects of prior straining. There is ample experimental evidence now that such a state variable does exist, and that it fully determines the plastic strain rate response of metals over quite broad conditions of loading history and temperature. This evidence is reviewed in an article by Hart, et al.[2].

2.3 Micro-Mechanical Background

Under this heading we note briefly that the formulation of the detailed constitutive relations described below was guided in part by considerations of deformation micro-mechanism.

The plastic deformation of metals is accomplished by the motion of crystal dislocations in lattice slip planes under the action of the stress operating on the dislocations. The non-elastic strain rate corresponds to a net dislocation flux rate in the material. The dislocations encounter in the crystals both regions in which glide is limited only by a moderate friction and regions where obstacles to glide (precipitates or dislocations of other slip systems) must be overcome by higher localized stresses or by thermal activation. Where dislocations are held up by obstacles, dislocation pile-ups result. These pile-ups exert additional force on the lead blocked dislocations and also exert back stresses (internal stresses) on the dislocations gliding under friction control. The mechanical and kinematical relations implied by this are well represented by the model described in the next section.

3. The Proposed Constitutive Relations

3.1 The Variables

The proposed equations are relationships among the current values of the applied stress deviator $\underset{\sim}{\sigma}$, the total non-elastic strain rate $\dot{\underset{\sim}{\varepsilon}}$, a scalar state variable σ^*, a tensor state variable $\underset{\sim}{a}$, the time rates of change of σ^* and $\underset{\sim}{a}$, and the absolute temperature T. For convenience in formulating the equations we also introduce two auxiliary stresses $\underset{\sim}{\sigma}_a$ and $\underset{\sim}{\sigma}_f$ that satisfy the constraint condition

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_a + \underset{\sim}{\sigma}_f . \tag{2}$$

For completeness we again note that we defined $\dot{\underset{\sim}{\alpha}}$ above such that

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{a}} + \dot{\underset{\sim}{\alpha}} . \tag{1'}$$

Our resultant equations will be such that, given the current values of the state variables σ^* and $\underset{\sim}{a}$ (or $\underset{\sim}{\sigma}_a$), the total inelastic strain rate $\dot{\underset{\sim}{\varepsilon}}$ will be uniquely determined by $\underset{\sim}{\sigma}$ and T, and the time rates of change of σ^* and $\underset{\sim}{a}$ will also be determined.

3.2 The Isotropy Relations

We shall formulate the equations for the case of isotropic materials. This isotropy refers only to the crystalline texture of the material. Provision for textural anisotropy can be readily made by a method described by Hart[3] in the paper that first presented these full equations. The flow anisotropy that results from deformation history is explicitly described by the equations presented here.

We introduce, for each deviatoric tensor, the second invariant denoted by light face symbol. Thus, for example,

$$\dot{\varepsilon} \equiv + \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} \;, \tag{3}$$

where the summation convention is employed for repeated indices.

Now the isotropy relations for each component are

$$\dot{\underset{\sim}{\varepsilon}} = (\dot{\varepsilon}/\sigma_f)\underset{\sim}{\sigma}_f, \tag{4}$$

$$\dot{\underset{\sim}{\alpha}} = (\dot{\alpha}/\sigma_a)\underset{\sim}{\sigma}_a, \tag{5}$$

$$\underset{\sim}{a} = (a/\sigma_a)\underset{\sim}{\sigma}_a. \tag{6}$$

3.3 The Scalar Equations

The remaining equations are among the scalar invariants and σ^* and T. They are

$$\ell n(\sigma^*/\sigma_a) = (\dot{\varepsilon}^*/\dot{\alpha})^\lambda, \tag{7}$$

$$\dot{\varepsilon}^* \equiv (\sigma^*/G)^m f \exp(-Q/RT), \tag{8}$$

$$\frac{d}{dt}\,\ell n\sigma^* = \dot{\alpha}\Gamma(\sigma^*,\sigma_a) - \mathcal{R}(\sigma^*,T). \tag{9}$$

$$\sigma_a = \mathcal{M}a, \tag{10}$$

$$\dot{\varepsilon} = \dot{a}^*(T)(\sigma_f/\mathcal{M})^M. \tag{11}$$

In these equations $\mathcal{M}$, M, m, λ, f, and Q are constants to be measured for each material, G is the modulus of rigidity at temperature T, and R is the gas constant. The functions $\dot{a}^*$, Γ, and $\mathcal{R}$ are determined experimentally.

Typical values for some of the constants are: $M \simeq 8$, $m \simeq 4\text{-}5$, $\lambda \simeq 0.15$, $\mathcal{M}$ is of the order of G, and Q is commonly the activation energy for atomic self-diffusion of the metallic species.

The function $\mathcal{R}$, which provides for static recovery of the <u>hardness</u> σ^*, has not yet been investigated systematically. It is substantially zero for temperatures up to one-half the melting temperature, and we shall ignore it in our discussion.

The function $\dot{a}^*$ has received only slight investigation so far.

The function Γ will be discussed further below in connection with the experiments.

The state variable σ^* which we have called the hardness represents the isotropic strain hardening. It seems to be highly stable against recovery.

The state variable $\underset{\sim}{a}$ reflects the direction and magnitude of prior straining. It undergoes time dependent change at any abrupt stress change.

The schematic rheological diagram in Fig. 1 is illustrative of the constitutive equations. Tensor relations are implied for all the elements.

4. The Experimental Basis

4.1 The Basic Experiments

We shall discuss the experimental basis in terms of the component elements of the model presented in Section 3 and Fig. 1. The three components will be called the $\dot{\alpha}$-element, the a-element, and the $\dot{\varepsilon}$-element.

The $\dot{\alpha}$-element was investigated by means of an improved load relaxation technique that is described by Hart and Solomon[4]. The analysis of those tests and of many others is described in ref. [2]. The results of the analysis established the functional relationships of eqs. (7) and (8) above. Of special interest in those equations is the scaling of the $\dot{\alpha}$-σ_a relationship with respect to the state variable σ^*.

The a-element, as is expressed in eq. (10), is a linear elastic one. This relationship is most easily investigated at values of σ that are very small compared to σ^*. Under that condition $\dot{\alpha}$ is negligibly small compared to $\dot{\varepsilon}$ and the material behaves entirely as an anelastic material. Investigations at a single temperature for high purity Al were reported by Nir et al.[5] that resulted in a verification of eq. (10) and also of eq. (11) for moderate ranges of σ_a. It is distinctly possible that a departure from linearity may occur for the a-element at higher values of σ_a.

The $\dot{\varepsilon}$-element was also investigated in the study reported in ref. [5]. The explicit functional form of eq. (11) was found to hold well over about five decades of $\dot{\varepsilon}$ values. It is possible that this is only an approximation to a more accurate function. That is being investigated by extending the anelasticity measurements to other temperatures. Further verification of the form of eq. (11) was provided by a study of 316 Stainless Steel at room temperature at both low and high stress levels by Nir et al.[6].

The strain hardening relationship shown in eq. (9) was first investigated in tensile tests of 1100 Al and 316 Stainless Steel by Wire et al.[7]. The principal conclusion from that study was that R was negligibly small in the range of temperature investigated and that Γ depended only on σ_a and σ^* with no further dependence on T. It was possible to make a fair fit to the data with

$$\Gamma = C(\sigma_a/\sigma^*)^k(\sigma^*/G)^{-m}, \qquad (12)$$

where k and C are constants. For 1100 Al k ≃ 8. The reliability of this representation is subject to some question, and further investigation is in progress.

4.2 Further Experiment

The basic experimental studies have established the form of the $\dot{\alpha}$-element quite well. Moreover, the value of λ of 0.15 seems to fit all of a very broad range of metals and alloys. Because of that, it is possible to utilize data from relatively complex test configurations to further investigate Γ and the a- and $\dot{\varepsilon}$-elements.

So far all the testing has been one-dimensional. The three-dimensional form of the equations is still prescriptive. Explicit three-dimensional testing is needed.

5. Application to Structural Calculations

5.1 Methods

The constitutive equations presented above are directly applicable to problems of prediction of deformation response in mechanical structural elements. The calculations will yield strain results as a function of time. Since the equations are incremental in real time, the application in numerical routines is especially straightforward. There has already been some application of the equations both in full and modified form to the analytical solution of problems of the flow of thick-walled cylinders and spheres under various combinations of loading. Some of these have been reported recently by Kumar and Mukherjee[8], [9]. Further discussion of that work and of more current applications are to be presented by the same authors in the present symposium[10].

5.2 Material Characterization

The results of any predictive calculation depend on the initial values of the state variables at different points of the structure. These values may vary from point to point because of the mechanical history acquired during fabrication. This may be a difficult problem to settle, and would be for any constitutive relations. One way of solving this is to compute the fabrication history in forming the structure from the initial stock. Another way is to try a range of initial values until a trial initial behavior is represented. Subsequent deformation behavior should then follow from those initial values.

6. Application to Radiation Environment

An extension of the state variable theory has been proposed recently by Hart and Li[11] to account for the effects of nuclear radiation fluence and flux. In that proposal it is noted that the effect of accumulated fluence cannot be much different than the effect of structure change that might result from aging or other micro-structural metallurgical transformations. The effects of aging concern principally the $\dot{\alpha}$-element of the model. The principal effect is to cause a change of σ^* and a change of $\dot{\varepsilon}^*$ (mainly by a change of f). The other parameters in eqs. 7 - 11 are not expected to change. The test of this proposal is most simply carried out by tests out-of-reactor following irradiation to various fluence levels.

The effect of flux is expected to be described by a flux dependence of $\dot{\varepsilon}^*$, perhaps as an effective temperature shift.

There is some evidence for the proposed relations from in-reactor creep behavior and from post-irradiation load relaxation studies. This evidence is discussed in ref. 11.

The advantage of this description, if it survives crucial test, is that considerable use can be made of out-of-reactor test data to predict in-reactor behavior.

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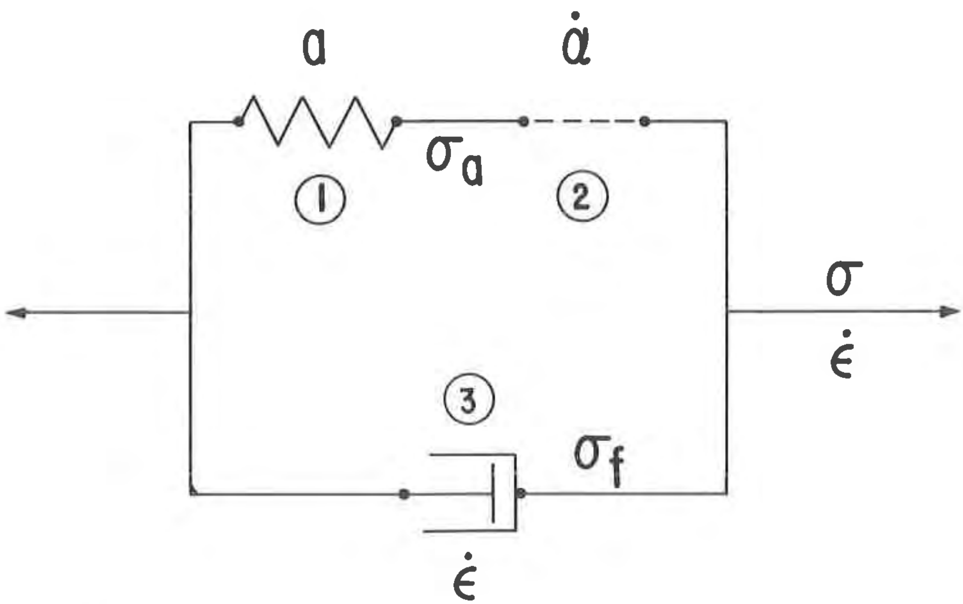

Figure 1. A diagram representing the constitutive relations for metal grain matrix non-elastic flow; the element denoted by 1 is the a-element; 2 is the $\dot{\alpha}$-element; 3 is the non-linear $\dot{\varepsilon}$-element.

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