Optimization of Vibrating Systems with Random Parameters

S.F. Jozwiak
Politechnika Warszawska, Instytut Drog i Mostow, Al. Armii Ludowaj 16, PL-00-637 Warsaw, Poland

Abstract

The minimum weight design of free vibrating structures with random parameters is considered. Formulation of the optimization problem presented in the paper is based on the concept of the expected value. Solution of the corresponding mathematical programming problem has been obtained by applying the chance constrained technique. The basic idea of the method is to convert the probabilistic problem into an equivalent deterministic one. The design problem is formulated as the minimization of weight /volume/ of the structure with the constraints concerning the values of natural frequences. The numerical results of optimization of simple structures are presented.

1. Introduction

Structural design problems can be usually formulated in the form of corresponding optimization problems. Generally recognized nondeterministic nature of structural properties as well as load and arbitrary decisions made in the process of idealization suggest that such problems should be formulated as stochastic optimization problems. Formulation of the problem presented in the paper is based on the concept of the expected value. Solution of the corresponding programming problem has been obtained by means of an indirect method. The basic idea of the method is to convert the probabilistic problem into an equivalent deterministic one. In the paper the chance constrained programming technique has been applied. The technique was oryginally developed by Charnes and Cooper [1] and has been adopted to solve engineering problems, among others, by Rao [2], Davidson, Felton and Hart [3].

2. Formulation of the stochastic optimization problems

Stochastic optimization problem can be stated in the form of the following programming problem:

minimize $\quad Ef(X,Y)$ (1)

subjected to

$$P_j(X,Y) = P\left[g_j(X,Y) \geq 0\right] \geq p_j, \qquad j=1,2,\ldots,k \tag{2}$$

where $f(X,Y)$ - objective function, X- vector of N random variables in which

x_i, $i=1,2,\ldots,n$ $n\leq N$ can be decision variables, Y-vector of m deterministic decision variables, k-number of constraints.

The present analysis assumes that all random variables are statistically independent and follow normal distribution.

Equation (2) denotes that the probability of realizing $g_j(X,Y)$ greater or equal to zero must be greater than or equal to specified probability p_j. The stochastic programming problem stated in eqs. (1),(2) can be converted into an equivalent deterministic nonlinear programming problem by applying the chance constrained programming technique. For this purpose constraint functions $g_j(X,Y)$ are expandet about the mean value of $\overline{X}$:

$$g_j(X,Y) = g_j(\overline{X},Y) + \sum_{i=1}^{N} \frac{\partial g_j}{\partial x_i}\bigg|_{X=\overline{X}} (x_i - \overline{x}_i) + \text{higher order derivative terms}, \qquad (3)$$

where $\overline{X}$-vector of mean values of random variables $\overline{x}_i$, $i=1,2,\ldots,N$.

If the standard deviations σ_{x_i} of x_i are small, $g_j(X,Y)$ can be approximated by the first two terms of equation (3), thus the constraint is linearized with respect to random variables. The mean and standard deviation of $g_j(X,Y)$ are given by :

$$\overline{g}_j = g_j(\overline{X},Y), \qquad \sigma_{g_j} = \left[\sum_{i=1}^{N} \left(\frac{\partial g_j}{\partial x_i}\right)^2 \bigg|_{X=\overline{X}} \sigma_{x_i}^2 \right]^{1/2}. \qquad (4)$$

Carrying out transformations, as shown in [2], the inequality constraints (2) can be finally written in the form

$$\overline{g}_j - \Phi^{-1}(p_j)\sqrt{\sigma_{g_j}^2} \geqslant 0, \qquad (5)$$

where $\Phi^{-1}(p_j)$ is the value of the standard normal variate corresponding to the probability p_j.

Similarly to constraint functions, the objective function $f(X,Y)$ can be expanded about the mean value of $\overline{X}$. Neglecting the terms of order higher than two, the objective function can be wrtten as

$$f(X,Y) = f(\overline{X},Y) + \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\bigg|_{X=\overline{X}} (x_i - \overline{x}_i) = \Psi(X,Y). \qquad (6)$$

The mean $\overline{\Psi}$ and the standard deviation σ_Ψ are :

$$\overline{\Psi} = \Psi(\overline{X},Y), \qquad \sigma_\Psi = \left[\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \bigg|_{X=\overline{X}} \sigma_{x_i}^2 \right]^{1/2}. \qquad (7)$$

For the purpose of optimization the new objective function F_d can be constructed as:

$$F_d = k_1\overline{\Psi} + k_2\sigma_\Psi, \qquad (8)$$

where $k_1 \geqslant 0$, $k_2 \geqslant 0$ indicate the relative importance of $\overline{\Psi}$ and σ_Ψ for minimization. Thus the stochastic optimization problem stated in the form of stochastic programming problem eqs (1),(2) has been converted into an equivalent deterministic nonlinear programming problem stated in eqs (8),(5). An inconsistence must be mentioned here between the definition of the objective function given by eq. (1) and equivalent deterministic function (8). Eq. (8) is consistent with the eq. (1) only for $k_1=1$ and $k_2=0$. For $k_1=0$ and $k_2=1$ F_d is equal to σ_Ψ and the problem is formulated as optimization with the criteria concerning standard deviation of objective function.

3. Transformation of frequency constraints

To avoid resonance it is prescribed that the lowest fundamental frequency ω_1 must be greater than the excitation frequency Ω. System fundamental frequency ω_j, j=1,2,...,L is given by

$$\omega_j = \sqrt{\lambda_j} \quad . \tag{9}$$

Eigenvalues λ_j are solutions of eigenvalue problem

$$(K - \lambda_j M) \Lambda_j = 0 \, , \tag{10}$$

in which K- system stiffness matrix, M- system mass matrix, Λ_j- j-th eigenvector. For structure with random parameters, the constraint of natural frequency ω_1 being grater than the excitation frequency can be written in a form

$$P\left[\omega_1 - \Omega' \geq 0\right] \geq p \, , \tag{11}$$

where $\Omega' = (1+\alpha)\Omega$ and parameter $\alpha \cong 0.25$ [4].

The inequality (11) states that probability of ω_1 being grater or equal to Ω' must be grater or equal to specified value p. In case when K and M are functions of random variables x_i, i=1,2,...,N λ_j, ω_j and Λ_j are also functions of random variables. Carrying out transformatios as shown in previous chapter, the condition (11) can be written in a form

$$\overline{\omega}_1 - \Phi^{-1}(p)\, \sigma_{\omega_1} - \Omega' \geq 0 \, , \tag{12}$$

where $\overline{\omega}_1$ - mean value and σ_{ω_1}-standard deviation of ω_1.

The mean $\overline{\omega}_j$ and standard deviation σ_{ω_j} of j-th frequency are

$$\overline{\omega}_j = \sqrt{\overline{\lambda}_j} \, , \qquad \sigma_{\omega_j} = \frac{1}{2\sqrt{\overline{\lambda}_j}} \left[\sum_{i=1}^{N} \left(\frac{\partial \lambda_j}{\partial x_i}\right)^2 \Big|_{X=\overline{X}} \sigma_{x_i}^2 \right]^{1/2} . \tag{13}$$

Mean value of $\overline{\lambda}_j$ can be obtained solving eq. (10) for mean values $\overline{x}_i$ of random variables. Derivatives $\frac{\partial \lambda_j}{\partial x_i}$ are given by formula [5]:

$$\frac{\partial \lambda_j}{\partial x_i} = \frac{\Lambda_j^T \left(\partial K/\partial x_i - \lambda_j\, \partial M/\partial x_i\right) \Lambda_j}{M_j^*} \, ; \qquad M_j^* = \Lambda_j^T M \Lambda_j \, . \tag{14}$$

Other formulations of optimization problems with constraints concerning fundamental frequences can be given. For example, as the structural weight increases rapidly with the increasing fundamental frequency, sometimes it may be more economical to place excitation frequency between two consecutive natural frequences, demanding $\omega_i < \Omega < \omega_{i+1}$.

4. Examples

To illustrate the techique, two numerical examples are presented. Inequality-constrained problems in the form of equation (12) were solved by application of Mathematical Programming Library, using Powell's method. In both cases it was assumed that random variables are statistically independent and follow normal distribution. Structure volume and weight are taken as objective functions and constraints are placed on the structure's dimentions and natural frequences.

Example 1. The structure treated in the first example is the truss shown in Fig 1. The volume of the structure is taken as objective function. Design variables are joint coordinates y_i i=1,2,3 /$y_1=x_2$, $y_2=z_2$, $y_3=x_3$/. To simplify calculations it is assumed that the only random variable is element length. Design variables are restricted to lie between the prescribed values and constraint is placed on natural frequency. The stochastic programming problem can be stated as follows:

minimize EV (15)

with constraints

$$y_i^{min} \leqslant y_i \leqslant y_i^{max}, \qquad i=1,2,3 \qquad (16)$$

$$P\left[\omega_1 - \Omega' \geqslant 0\right] \geqslant p, \qquad (17)$$

where ω_1 can be calculated according to formula (9), y_i^{min}, y_i^{max} i=1,2,3, p, Ω' are given values.

Corresponding deterministic programming problem has the form:

$$\text{minimize} \quad \overline{V} = \sum_{i=1}^{3} a_i \, \overline{l}_i \qquad (18)$$

with constraints

$$y_i^{min} \leqslant y_i \leqslant y_i^{max}, \qquad (19)$$

$$\overline{\omega}_1 - \Phi^{-1}(p)\sigma_{\omega_1} - \Omega' \geqslant 0, \qquad (20)$$

where l_i - i-th element length, a_i - cross section area of i-th member, $\overline{\omega}_1$ can be calculated according to formula (13).

The numerical results presented in Figs 2 and 3 were obtained for E=210000MPa a_i=1cm^2 /i=1,2,3/, material density ϱ=7.8 10^{-2}N/cm^3, Ω'=3125 rad/sek, $\sigma_i = \sigma_1$ for i=1,2,3. In the Fig. 2, the influence of standard deviation σ_1 on optimal volume $\overline{V}$ is presented. The optimum volume $\overline{V}$ v/s probability p is shown in Figure 3.

Example 2. The example concerns the minimum weight design of homogeneous plane strain structure presented in Fig. 4. Typical element of unit thickness is analysed. Structure width y_1 and angle y_2 are design variables and material density is the only random variable. Stochastic programming problem is stated as follows,

minimize element weight

with constraints

$$y_i^{min} \leqslant y_i \leqslant y_i^{max} \qquad (21)$$

$$P\left[\omega_1 - \Omega' \geqslant 0\right] \geqslant p. \qquad (22)$$

Performing transformations as in previous example, the corresponding deterministic programming problem can be formulated. The mean value $\overline{W}$ and standard deviation σ_W are given by formulae:

$$\overline{W} = \overline{\varrho}\, l \sum_{i=1}^{7} a_i, \qquad \sigma_W = \sigma_\varrho l \sum_{i=1}^{7} a_i, \qquad (23)$$

where l=1 cm, a_i- i-th finite element area, ϱ, σ_ϱ - mean and standard deviation of material density. The numerical results were obtained for: E=30000MPa, $\overline{\varrho}$=2.5 10^{-2}N/cm^3, Ω'=12500 rad/sek. The influence of σ_ϱ on mean value of $\overline{W}$ and standard deviation σ_W of element weight is shown in Fig. 5. The optimal values of decision variables and the lowest natural frequency ω_1 of the structure are also given in the figure.

5. Conclusion

The application of chance constrained programming technique in the optimization of vibrating systems has been presented. The technique enables the evaluation of the effect of random character of structural parameters on the optimum design. Two numerical examples concerning plane truss and plane strain structure are presented to demonstrate the effectiveness of the approach.

References

/1/ CHARNES, A., COOPER, W.W., "Chance constrained programming", Management Science, 6, 73-78, 1959.

/2/ RAO, S.S., "Optimization-Theory and Application", Wiley Eastern Ltd, 1978.

/3/ DAVIDSON, J.W., FELTON, L.P., HART, G.C., "Optimum design of structures with random parameters", J.Comp.Struct., 7, 481-486, 1977.

/4/ THERMANN, K., "Optimum design criteria of dynamically loaded elastic structures", Proc. of IUTAM Symposium on Optimization in Structural Design, ed. by Z. MROŻ and A.SAWCZUK, Springer V., Berlin 1975.

/5/ HASSELMAN, T.K., HART, G.C., "Modal analysis of random structural systems", J.Eng.Mech.Div., Proc. ASCE EM 3, 561-579, June 1972.

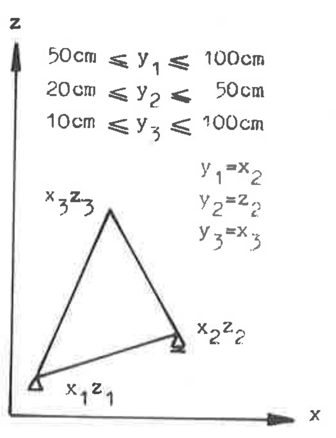

Fig.1 Plane truss, design variables y_i, $i=1,2,3$

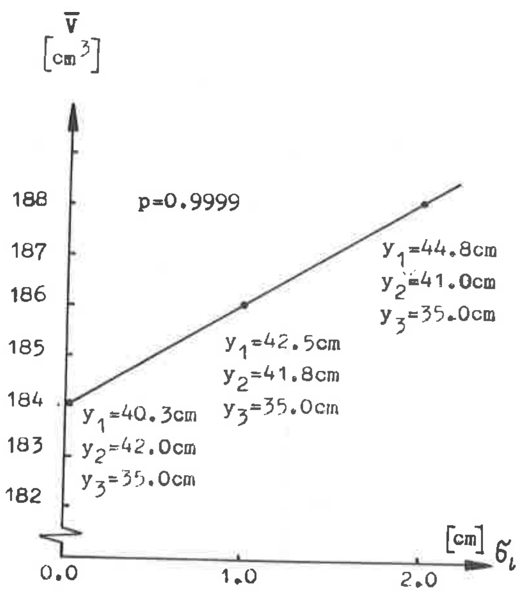

Fig.2 Influence of standard deviation σ_1 on the optimal volume $\overline{V}$ of the structure

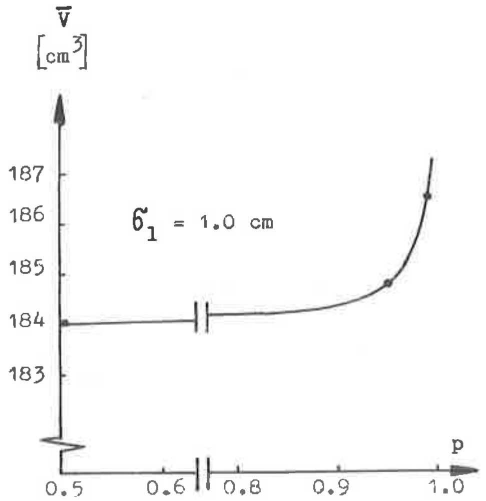

Fig.3 Influence of probability p on the optimal volume of the structure

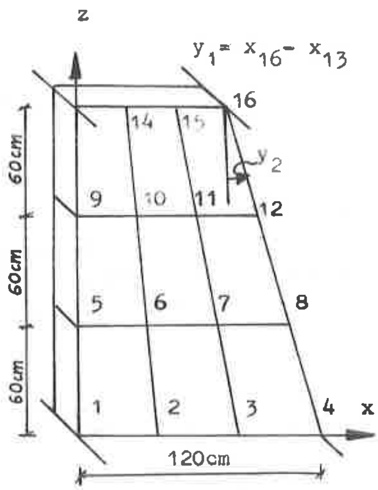

Fig.4 Plane strain structure, design variables y_1, y_2

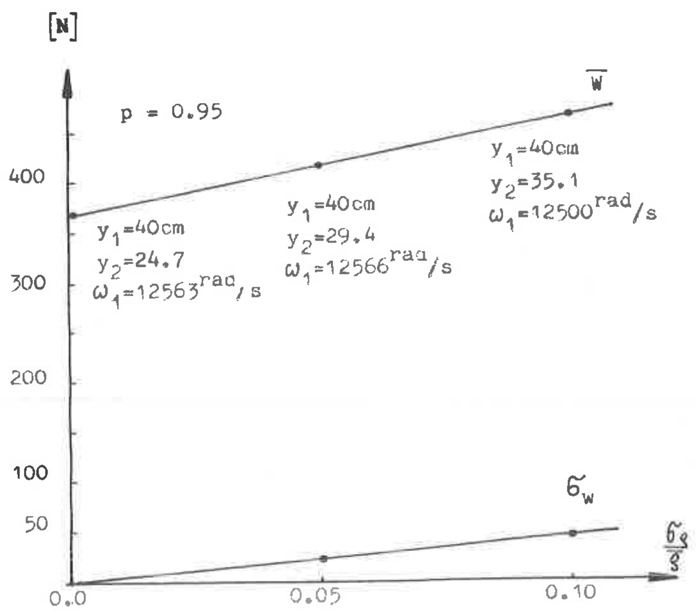

Fig.5 Influence of standard deviation σ_ρ on the optimal weight of the structure