SEISMIC ANALYSIS OF CRACKED REACTOR VESSELS

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SUMMARY

Dynamic analysis of a steel reactor vessel under seismically induced forces is of particular importance, since it is known that the natural frequency of a reactor vessel is usually in the region of that of earthquake motion. Moreover the existence of a circumferential or axial crack in a steel reactor vessel wall may change considerably free and forced vibration characteristics of the vessel. This situation in turn may cause propagation of the crack. For any given material under a specified stress field, there is a crack length of a certain critical value in the material for which the crack will become self-propagating.

In this paper the dynamic response of a finite circular cylindrical elastic shell, representing a nuclear reactor vessel, due to seismically induced forces is investigated. A longitudinal through crack of finite length exists in the vessel wall. While recently a few papers on the elastostatic stress analysis of pressurized cylindrical and spherical shells with cracks appeared in the literature, there is not yet any publication, to the best of author's knowledge, on the investigation of dynamical analysis of finite cylindrical shells with circumferential and longitudinal cracks. Donnell's shallow shell equations were used in those studies. In fracture mechanics it is known that if the crack is only part-through but the net section under the crack is yielded, one may replace this section by tensile tractions and still use the through crack analysis. Therefore, in this paper the analysis is carried for the "through crack" case. Seismically induced force consists of any time dependent symmetrical surface and/or edge loads expressible with a Fourier series expansion. The governing equations of the analysis are the so called Morley equations (L.S.D. Morley "An Improvement on Donnell's Approximation for Thin-Walled Circular Cylinders," *Q. Jl. Mech. Appl. Math.*, **12**, 89, 1959) which are extended by present authors into the dynamical case. These equations have nearly the same simple form as the well-known Donnell equations but include no further assumptions than those basic to linear thin-shell theory.

The linear analysis of the problem is based on the method of superposition. The actual dynamical stresses in the shell are considered as the sum of the following two parts: (a) Stresses caused by prescribed external dynamical loads in a similar shell of finite length without a crack (nominal solution); (b) Stresses in the shell caused by applied edge loads along the crack otherwise free of any other loads (residual solution). These loads are equal in magnitude but opposite in sign to those present in the uncracked shell at the crack location.

Extensive numerical results are presented and some specific and axisymmetrical seismically induced forces and comparisons with those of a non cracked similar shell are discussed. The effect of dynamic loading on the stability of the crack and the relations between critical loads and crack lengths are also investigated.

Furthermore free vibration problem of the cracked cylindrical shell is analyzed. Since free vibrations occur in the absence of all external loads, the homogeneous system of governing equations with homogeneous boundary conditions is considered. The effect of the crack on the natural frequency of the vessel is clearly demonstrated.

1. Introduction

Since their inception, nuclear power plants have been designed and constructed with public safety as a paramount concern. Excessively conservative design practices were used in many areas where detailed knowledge of the system behavior was not available. Since such practices are costly, there is great incentive for gaining additional knowledge about the effects of seismically induced vibrations on nuclear systems. This field has been investigated extensively in recent years [1-8] and is still important with considerable work currently being done in several areas. Among them, the dynamic analysis of a reactor vessel under seismically induced forces is attracting particular attention. It is known that the natural frequencies of reactor vessels are usually in the region of that of earthquake motion.

Moreover, the existence of a circumferential or axial crack in a reactor vessel wall may cause a considerable change in the free and forced vibration characteristics of the vessel. This situation in turn may cause propagation of the crack. For a given material under a specified stress field, there is a certain critical crack length for which the crack will become self-propagating. If the length is ever reached, either by penetration or by the growth of a small crack, the complete loss of the vessel may occur.

In this paper, the dynamic response of a finite circular cylindrical elastic shell, representing a nuclear reactor vessel, due to seismically induced forces is investigated. Centrally located longitudinal through crack of finite length exists in the vessel wall. The crack is modeled by a line discontinuity of length 2c across which all stress resultants are zero. While a few papers on the elastostatic stress analysis of pressurized cylindrical and spherical shells with cracks have recently appeared in the literature [9-14], there is not yet any publication, to the best of the author's knowledge, on the dynamical analysis of finite cylindrical shells with circumferential and longitudinal cracks. In fracture mechanics, it is known that if the crack is only part-through but the net section under the crack is yielded, one may replace this section by tensile tractions and still use the through crack analysis [11]. Therefore, in this paper, the analysis is carried out for the "through crack" case. Seismically induced force consists of any time dependent symmetrical surface and/or edge loads expressible with a Fourier series expansion. Donnell's shallow shell equations are used in this study.

The linear analysis of the problem is based on the method of superposition. The actual dynamical stresses in the shell are considered as the sum of the following two parts:

a) Stresses caused by prescribed external dynamical loads in a similar shell of finite length without a crack (nominal solution).

b) Stresses in the shell caused by applied edge loads along the crack otherwise free of any other loads (residual solution). These loads are equal in magnitude but opposite in sign to those present in the uncracked shell at the crack location.

2. Outline of the Analytical Procedure

The governing equations of a shallow circular cylindrical shell, resulting from the usual Donnell assumptions, can be written as two coupled fourth-order equations in the normal displacement $\overline{W}$ (X, Y, t) and a stress function $\overline{F}$ (X, Y, t).

$$\nabla^4 \overline{F} - \frac{Ehc^2}{a^3} \frac{\partial^2 \overline{W}}{\partial X^2} = 0 \qquad (1)$$

$$\nabla^4 \overline{W} + \frac{ac^2}{D} \frac{\partial^2 \overline{F}}{\partial X^2} + \frac{c^4 \rho h}{D} \frac{\partial^2 \overline{W}}{\partial t^2} = \frac{c^4}{D} \overline{p}_z \tag{2}$$

In these and in subsequent expressions the following definitions are used

E = Young's modulus
h = Shell thickness
a = Shell radius

$D = \dfrac{Eh^3}{12(1-\nu^2)}$

ν = Poisson's ratio

$\nabla^2 = \dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial Y^2}$

$X = \frac{x}{c}$, non-dimensional axial distance.

$Y = \frac{y}{c}$, non-dimensional circumferential distance.

Considering periodic vibrations of the form

$$\overline{p}_z\,(X, Y, t) = q\,(X, Y) \cos(\omega t + \phi)$$

$\overline{W}$ and $\overline{F}$ can be written in the form:

$$\overline{W}\,(X, Y, t) = W\,(X, Y) \cos(\omega t + \phi)$$
$$\overline{F}\,(X, Y, t) = F\,(X, Y) \cos(\omega t + \phi)$$

Substituting the above into equations (1) and (2) yield:

$$\nabla^4 F - \frac{Ehc^2}{a^3} \frac{\partial^2 W}{\partial X^2} = 0 \tag{3}$$

$$\nabla^4 W - \frac{c^4 \rho h \omega^2}{D} W + \frac{ac^2}{D} \frac{\partial^2 F}{\partial X^2} = \frac{c^4}{D} q \tag{4}$$

Along the crack, the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses must vanish. Suppose, however, that a particular solution has already been found that satisfies eqs. (3) and (4), but that there is a residual moment M_Y, equivalent vertical shear V_Y, normal in-plane stress N_Y, and tangential in-plane stress resultants N_{XY} along the axis $|X| < 1$ of the form

$$M_Y^p = -\frac{D}{c^2} m_o$$

$$V_Y^p = 0$$

$$N_Y^p = -\frac{a^2}{c^2} n_o$$

$$N_{XY}^p = 0$$

Two functions, W (X, Y) and F (X, Y), must now be found which satisfy the homogeneous egs. (3) and (4) and the following boundary conditions:

at $Y = 0$ and $|X| < 1$

$$M_Y^c(X, 0) = \lim_{|Y| \to 0} \frac{-D}{c^2} \left(\frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2} \right) = \frac{D}{c^2} m_o \tag{5}$$

$$V_Y^c(X, 0) = \lim_{|Y| \to 0} \frac{-D}{c^3} \left((2-\nu) \frac{\partial^3 W}{\partial X^2 \partial Y} + \frac{\partial^3 W}{\partial Y^3} \right) = 0 \tag{6}$$

$$N_Y^c(X, 0) = \lim_{|Y| \to 0} \frac{a^2}{c^2} \frac{\partial^2 F}{\partial X^2} = \frac{a^2}{c^2} n_o \tag{7}$$

$$N^{c}_{XY}(X,0) = \lim_{|Y|\to 0} \frac{-a^2}{c^2} \frac{\partial^2 F}{\partial X \partial Y} = 0 \tag{8}$$

The following continuity conditions must be satisfied at $Y = 0$ and $|X| > L$

$$\lim_{|Y|\to 0} \left[\frac{\partial^n}{\partial Y^n} W^c(X,0^+) - \frac{\partial^n}{\partial Y^n} W^c(X,0^-) \right] = 0 \tag{9}$$

$$\lim_{|Y|\to 0} \left[\frac{\partial^n}{\partial Y^n} F^c(X,0^+) - \frac{\partial^n}{\partial Y^n} F^c(X,0^-) \right] = 0 \tag{10}$$

$$n = 0, 1, 2, 3$$

Solutions of the following form are now assumed:

$$W^c(X,Y) = \int_0^\infty P(s)\, e^{b(s)Y} \cos Xs \, ds \tag{11}$$

$$F^c(X,Y) = \int_0^\infty Q(s)\, e^{b(s)Y} \cos Xs \, ds \tag{12}$$

Substitution of (11) and (12) into eqs. (3) and (4) yields:

$$\int_0^\infty \left\{ \frac{Ehc^2 s^2}{a^3} P + (b^2 - s^2)^2 Q \right\} e^{bY} \cos Xs \, ds = 0 \tag{13}$$

$$\int_0^\infty \left\{ \left[(b^2 - s^2)^2 - \frac{c^4 \rho h \omega^2}{D} \right] P - \frac{ac^2 s^2}{D} Q \right\} e^{bY} \cos Xs \, ds = 0 \tag{14}$$

For the above integrals to vanish, it is sufficient to let their integrands vanish and this yields the following equation in $(b^2 - s^2)^2$,

$$(b^2 - s^2)^2 \left[(b^2 - s^2)^2 - \frac{c^4 \rho h \omega^2}{D} \right] + \frac{Ehc^4 s^4}{a^2 D} = 0 \tag{15}$$

Equation (15) yields eight values of b, but only the negative roots have physical meaning.

$$b_1 = -\left[s^2 + k_1 A_1 \right]^{\frac{1}{2}}; \qquad b_3 = -\left[s^2 + k_1 A_2 \right]^{\frac{1}{2}}$$

$$b_2 = -\left[s^2 - k_1 A_1 \right]^{\frac{1}{2}}; \qquad b_4 = -\left[s^2 - k_1 A_2 \right]^{\frac{1}{2}} \tag{16}$$

where $A_1^2 = \lambda^2 + \sqrt{\lambda^4 - s^4}$ $\qquad A_2^2 = \lambda^2 - \sqrt{\lambda^4 - s^4}$

and $$\lambda^2 = \frac{\rho \omega^2 a c^2}{Eh} \sqrt{3(1-\nu^2)} \qquad k_1^2 = \frac{2c^2}{ah} \sqrt{3(1-\nu^2)} \tag{17}$$

Equations (11) and (12) now become:

$$W(X,Y) = \int_0^\infty \left\{ P_1 e^{b_1|Y|} + P_2 e^{b_2|Y|} + P_3 e^{b_3|Y|} + P_4 e^{b_4|Y|} \right\} \cos Xs \, ds \tag{18}$$

$$F(X,Y) = \frac{-Ehc^2}{a^3 k_1^2} \int_0^\infty \left\{ A_2^2 P_1 e^{b_1|Y|} + A_2^2 P_2 e^{b_2|Y|} + A_1^2 P_3 e^{b_3|Y|} + A_1^2 P_4 e^{b_4|Y|} \right\} \frac{\cos Xs}{s^2} \, ds \tag{19}$$

Substitution of (18) and (19) into eqs. (5-9) yields:

$$\lim_{|Y|\to 0} \int_o^\infty \Big\{ (b_1^2-\nu s^2) P_1 e^{b_1|Y|} + (b_2^2-\nu s^2) P_2 e^{b_2|Y|} + (b_3^2-\nu s^2) P_3 e^{b_3|Y|}$$

$$+ (b_4^2-\nu s^2) P_4 e^{b_4|Y|} \Big\} \cos Xs \, ds = -m_o \tag{20}$$

$$\lim_{|Y|\to 0} \int_o^\infty \Big\{ (b_1^2-(2-\nu)s^2) b_1 P_1 e^{b_1|Y|} + (b_2^2-(2-\nu)s^2) b_2 P_2 e^{b_2|Y|}$$

$$+ (b_3^2-(2-\nu)s^2) b_3 P_3 e^{b_3|Y|} + (b_4^2-(2-\nu)s^2) b_4 P_4 e^{b_4|Y|} \Big\} \cos Xs \, ds = 0 \tag{21}$$

$$\lim_{|Y|\to 0} \int_o^\infty \Big\{ A_2^2 P_1 e^{b_1|Y|} + A_2^2 P_2 e^{b_2|Y|} + A_1^2 P_3 e^{b_3|Y|} + A_1^2 P_4 e^{b_4|Y|} \Big\} \cos Xs \, ds = \frac{a^3 k_1^2 n_o}{Ehc^2} \tag{22}$$

$$\lim_{|Y|\to 0} \int_o^\infty \Big\{ A_2^2 b_1 P_1 e^{b_1|Y|} + A_2^2 b_2 P_2 e^{b_2|Y|} + A_1^2 b_3 P_3 e^{b_3|Y|} + A_1^2 b_4 P_4 e^{b_4|Y|} \Big\} \frac{\sin Xs}{s} \, ds = 0 \tag{23}$$

Eqs. (21) and (23) are satisfied if the integrands vanish which lead to

$$b_4 P_4 = \frac{G_1 - G_2}{2k_1 A_1^2 A_2} b_1 P_1 - \frac{G_3 + G_2}{2k_1 A_1^2 A_2} b_2 P_2 \tag{24}$$

$$b_3 P_3 = \frac{G_2 - G_3}{2k_1 A_1^2 A_2} b_1 P_1 + \frac{G_1 + G_2}{2k_1 A_1^2 A_2} b_2 P_2 \tag{25}$$

where

$$G_1 = k_1 (A_1^3 - A_2^3)$$
$$G_2 = \nu_o s^2 (A_1^2 - A_2^2) \tag{26}$$
$$G_3 = k_1 (A_1^3 + A_2^3)$$

and

$$\nu_o = 1-\nu \tag{27}$$

It can be shown that the continuity conditions are satisfied if the following combinations vanish:

$$\int_o^\infty \Big\{ (c_1+\lambda) L_1' + (c_2+\lambda) L_2' \Big\} \cos Xs \, ds = 0 \quad |X| > 1 \tag{28}$$

$$\int_o^\infty \Big\{ (c_1-\lambda) L_1' + (c_2-\lambda) L_2' \Big\} \cos Xs \, ds = 0 \quad |X| > 1 \tag{29}$$

where $$c_1 = \frac{A_1^2}{s^4} (\nu_o s^2 - k_1 A_1), \quad c_2 = \frac{A_1^2}{s^4} (\nu_o s^2 + k_1 A_1) \tag{30}$$

$$L_1'(s) = \frac{b_1 P_1 (A_1^2 - A_2^2)}{A_1^2}, \quad L_2'(s) = \frac{b_2 P_2 (A_1^2 - A_2^2)}{A_1^2} \tag{31}$$

Dual integral equations of the above type are hard to handle, so the problem will be cast to singular integral equations. Let:

$$u_1(X) = \int_0^\infty \left\{(c_1+\lambda)\, L_1' + (c_2+\lambda)\, L_2'\right\} \cos Xs\, ds \qquad |X| < 1 \tag{32}$$

$$u_2(X) = \int_0^\infty \left\{(c_1-\lambda)\, L_1' + (c_2-\lambda)\, L_2'\right\} \cos Xs\, ds \qquad |X| < 1 \tag{33}$$

By a Fourier inversion, it is obtained that

$$P_1 = \frac{s^4}{2\pi k_1 \lambda b_1 A_1 (A_2^2 - A_1^2)} \int_0^1 \left\{\lambda(u_1+u_2) - c_2(u_1-u_2)\right\} \cos \xi\, s d\xi \tag{34}$$

$$P_2 = \frac{s^4}{2\pi k_1 \lambda b_2 A_1 (A_2^2 - A_1^2)} \int_0^1 \left\{c_1(u_1-u_2) - \lambda(u_1+u_2)\right\} \cos \xi\, s d\xi \tag{35}$$

$$P_3 = \frac{s^4}{2\pi k_1 \lambda b_3 A_2 (A_2^2 - A_1^2)} \int_0^1 \left\{c_3(u_1-u_2) - \lambda(u_1+u_2)\right\} \cos \xi\, s d\xi \tag{36}$$

$$P_4 = \frac{s^4}{2\pi k_1 \lambda b_4 A_2 (A_2^2 - A_1^2)} \int_0^1 \left\{\lambda(u_1+u_2) - c_4(u_1-u_2)\right\} \cos \xi\, s d\xi \tag{37}$$

where $$c_3 = \frac{A_2^2}{s^4} (\nu_o s^2 + k_1 A_2) \tag{38}$$

$$c_4 = \frac{A_2^2}{s^4} (\nu_o s^2 - k_1 A_2) \tag{39}$$

Substitution of eqs. (34-37) into eqs. (5) and (7) yield

$$M_Y = \frac{-D}{8\pi k_1 c^2 \lambda} \int_{-1}^{1} \left\{L_1^* u_1 + L_2^* u_2\right\} d\xi \tag{40}$$

$$N_Y = -\frac{Eh}{8\pi k_1^3 a \lambda} \int_{-1}^{1} \left\{L_3^* u_1 + L_4^* u_2\right\} d\xi \tag{41}$$

where

$$L_1^* = \int_0^\infty H_1^*(s, Y) \cos [(X-\xi)s]\, ds \tag{42}$$

$$L_2^* = \int_0^\infty H_2^*(s, Y) \cos [(X-\xi)s]\, ds \tag{43}$$

$$L_3^* = \int_0^\infty H_3^*(s, Y) \cos [(X-\xi)s]\, ds \tag{44}$$

$$L_4^* = \int_0^\infty H_4^*(s, Y) \cos [(X-\xi)s]\, ds \tag{45}$$

where H_1^*, H_2^*, H_3^*, and H_4^* are known functions of s and Y. Eqs. (40) and (41) can now be written

$$\lim_{|Y|\to 0} M_Y = \frac{D}{c^2} m_o = \frac{-D}{8\pi k_1 c^2 \lambda} \frac{d}{dX} \int_{-1}^{1} \left\{ L_1 u_1 + L_2 u_2 \right\} d\xi \quad |X|<1 \tag{46}$$

$$\lim_{|Y|\to 0} N_Y = \frac{a^2}{c^2} n_o = \frac{Eh}{8\pi k_1^3 a\lambda} \frac{d}{dX} \int_{-1}^{1} \left\{ L_3 u_1 + L_4 u_2 \right\} d\xi \quad |X|<1 \tag{47}$$

where

$$\frac{d}{dX} L_i = \lim_{|Y|\to 0} L_i^* \qquad i = 1,2,3,4$$

$$L_i = \int_0^\infty H_i(s) \sin\left[(x-\xi)\, s\right] ds \tag{48}$$

where $H_i(s) = \lim_{|Y|\to 0} \frac{1}{s} H_i^*(s, Y) \qquad i = 1,2,3,4$

The integrands H_i in eqs. (48) are too complicated to allow L_i to be found in closed form. However, at $s = s_o$, $H_i(s_o) \cong H_i'(s_o)$ and this integrand can be handled. Thus

$$L_i = \int_0^\infty H_i(s) \sin\left[(X-\xi)s\right] ds = \int_0^{s_o} H_i(s) \sin\left[(X-\xi)s\right] ds + \int_{s_o}^\infty H_i'(s) \sin\left[(X-\xi)s\right] ds$$

or

$$L_i = \int_0^{s_o} \left\{ H_i(s) - H_i'(s) \right\} \sin\left[(X-\xi)s\right] ds + \int_0^\infty H_i'(s) \sin\left[(X-\xi)s\right] ds \tag{49}$$

The first integral can be evaluated numerically to yield

$$\int_0^{s_o} \left\{ H_i(s) - H_i'(s) \right\} \sin\left[(X-\xi)s\right] ds = E_i\,(X-\xi) \qquad i = 1,2,3,4 \tag{50}$$

where the E_i are constants in k_1, λ, and s_o. For typical values of shell parameters, $s_o \cong 1.0$.

The second integral in equation (49) can be integrated to yield:

$$L_i = \frac{K_{1i}}{(X-\xi)} + K_{2i}\,(X-\xi) + K_{3i}\,(X-\xi) \ln \frac{k_1 |X-\xi|}{4} \qquad i = 1,2,3,4 \tag{51}$$

for small arguments. In eq.(51), K_{1i}, K_{2i}, K_{3i} are constants. Now $u_1(\xi)$ and $u_2(\xi)$ are assumed to be of the following form:

$$u_1(\xi) = \sqrt{1-\xi^2}\,\left[A_1 + k_1^2 A_2 (1-\xi^2) + \cdots\right] \quad |\xi| < 1 \tag{52}$$

$$u_2(\xi) = \sqrt{1-\xi^2}\,\left[B_1 + k_1^2 B_2 (1-\xi^2) + \cdots\right] \quad |\xi| < 1 \tag{53}$$

Substituting eqs. (52) and (53) into eqs. (46) and (47) and using (50) and (51), two equations in the four unknowns A_1, A_2, B_1, and B_2 result. However, after performing the integrations in eqs. (46) and (47), there will be terms in X and X^3 in both equations.

$$F_1(A_i, B_i)\, X + F_2\,(A_i, B_i)\, X^3 = -\,8\pi k_1 \lambda\, m_o X \tag{54}$$

$$F_3\,(A_i, B_i)\, X + F_4\,(A_i, B_i)\, X^3 = \frac{8\pi a^3 k_1^3 \lambda n_o}{Ehc^2}\, X \tag{55}$$

where the F_i are known functions of A_1, A_2, B_1, and B_2. By equating coefficients of the

X and X^3 terms, four equations result:

$$F_1(A_1, B_1, A_2, B_2) + 8\pi\, k_1 \lambda\, m_o = 0 \tag{56}$$

$$F_2(A_1, B_1, A_2, B_2) = 0 \tag{57}$$

$$F_3(A_1, B_1, A_2, B_2) - \frac{8\pi a^3 k_1^3 \lambda n_o}{Ehc^2} = 0 \tag{58}$$

$$F_4(A_1, B_1, A_2, B_2) = 0 \tag{59}$$

Equations (56-59) can now be solved for A_1, B_1, A_2, and B_2. Substituting the known functions (52) and (53) into (36-39) and the results into (18) and (19), the various stresses near the crack tip can be found. At the time of preparation of this paper, due to the computational difficulties numerical results on stresses were not available. It is expected that these results will be presented at the Conference.

3. References

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