ABSTRACT

CHEN, YEN-MING. Risk Assessment Model with Integrated-Information of Copula and Choqeut Fuzzy Integral. (Under the direction of Dr. Simon M. Hsiang).

Risk assessment, as a critical step in a risk management, is the determination of quantitative or qualitative value related to a well-defined situation and a recognized hazard. Quantitative risk assessment requires estimation of the magnitude of the potential loss, and the probability that the loss will occur. As such, risk is defined as a measure of the perceptual degree of the probability that harm will occur and the severity of the harm threatened. Risk cannot be assessed without psychophysical measurement and judgmental heuristics.

There is long history of risk assessment in military operation, nuclear, aerospace, oil, airlines and railroad industries. More recently, the focus is on healthcare, environmental and global climate change. Current methods based risk matrices face many shortcomings, such as unrealistic assumption of independence, ambiguous inputs and outputs, and lack of accommodation for human irrationality. To overcome these challenges two techniques are proposed to investigate the interdependent structure in probability distributions, and the perceptual aggregation of risk severities.

1. Copula can be used to describe the dependence between random variables. The marginal distribution functions describe the marginal distribution of each component of the random vector and the copula describes the dependence structure between the components.
2. Choquet fuzzy integral, based on a risk priority-scoring system, has advantages in estimating the dependency among preferences in the human mind.

Both Choquet fuzzy integral and Copula can be used to aggregate risks. Coquet fuzzy integral, based on sequential order, provides a local view of priority for satisfaction. Copula as a joint distribution presents a macro view of parallel interactions of risks. By integrating Choquet fuzzy integral and Copula through sharing complimentary information, a better understanding of risk dependency could be obtained and therefore the reliability of risk assessment will be enhanced and the optimality could be achieved.

Risk Assessment Model with Integrated-Information of Copula and Choqeut Fuzzy Integral

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DEDICATION

Dedicated to

my parents Shue-Cheng Chen and Jin-Hua Lu, for their love, sacrifice, and support.

BIOGRAPHY

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Chapter 1

Introduction

1.1 Motivation

In both public and private sectors, there is a continuous drive searching for an efficient and effective way to assess risks, especially for extremely large-scale ones, such as terrorist attacks and natural catastrophes. For example, Hurricane Katrina, one of the deadliest natural disasters in the history of the United States, caused 1.2 million people under some type of evacuation order, at least 1,836 lost lives and property damages around $81 billion (Knabb, 2005). In the report of Hurricane Katrina 2006, the Select Committee of US House of Representatives reported a conclusion that the most of damages could be imputed to inadequate preparation in lots of aspects, such as un-built levees, incomplete evacuation, unreliable communication, and insufficient emergency shelters. We learned a lesson from such a painful experience. As a result, in order to reduce possible future losses in the risk of hurricanes, a great deal of resources has been allocated to build up a higher level of preparedness in broad aspects. However, the entire resources we can utilize are quite limited. If one takes more, others have to share less. Thus, a decision problem comes up: to what level of preparedness we will feel ourselves comfortable due to "fully preparing for the risk". In practice it is not possible to have a null probability of having a loss in a risk. In other words, we could just put efforts to lower the probability of losses to some point at which we decide the risk is being prepared. In addition, no matter big as a nation or small as an individual is always under the threat of multiple risks in daily life. Though we will to get “fully prepared” for all the risks, however the fact is that we sometimes have to sacrifice some risks for others due to insufficiency of available resources. However, in many situations, we do not have a suitable unit to compare the losses of different kind of risks. For example, it is hard to compare the loss of human lives and properties. Intuitively, to properly solve the resource allocation problem, our first goal would try to reduce the most “painful” losses, but meanwhile keep other risks still in an acceptable level. Therefore, this kind of risk management problems typically include probability distributions to model the possible losses, proper risk measures to rescale the probability and loss by human perception, pre-decided thresholds of satisfaction, and a method to find optimality. The techniques needed for this problem are developed separately in fields of statistics, economics, psychophysics, and operation research. This dissertation will be doing a comprehensive research across above fields to enhance the understanding of this problem and develop a method utilizing advantages of different techniques.

1.2 Background

In reality, human judgments are an important part in decision-making process because human beings are rather the arbitrators of a decision problem or the final customers the decision would like to please. In other words, human satisfaction is usually the critical criterion to evaluate a final decision, especially with respect to risk problems. By definition, risk is based on the nature of uncertainty, which means no one can precisely predict what events or outcomes will definitely happen in the future. Thus, all the decisions and actions taken before events occur are basically only to mitigate risks to an acceptable level which the decision makers feel satisfied with. However, do human determine satisfaction or, to put it in another way, perceive risks purely by the physical facts? Some characteristics of human judgment regarding uncertain outcomes can be seen in several famous paradoxes.

The St. Petersburg paradox, proposed by D. Bernoulli in 1738, is a paradox which is based on a lottery game (one will earn a reward of 2^n dollars if s/he flips a fair coin and obtains the first head on the n^{th} trial) that leads to an expected reward of infinite value, but would however be considered to be worth only a small amount of money. Bernoulli further solved this paradox by presenting the expected utility theory, among which utility is defined as a measure of relative satisfaction. The St. Petersburg paradox is a good example to explain that humans have decreasing marginal utilities over economic values, like money in this case. The increments of utilities are getting smaller and smaller so that the expected utilities would converge to a relatively small value rather than infinity. Thus, a rational person would be willing to only pay a little amount for the lottery even though the literal expected payoff goes infinity. Bernoulli's work initiated an important idea that human distort what they perceive in minds while making decision. This idea inspired a number of researches and had great impacts in various areas. The most important one is John von

Neumann and Oskar Morgenstern’s work in 1947. They defined axioms of rationality such that any person satisfying the axioms has a real-valued utility function, and rationality should be modeled as maximizing an expected utility. Conversely, any agent acting to maximize the expectation of a utility function will obey Von Neumann-Morgan’s rationality axioms. According to them, the reason the expected utility hypothesis works is that it is human nature to make decisions by maximizing what they “expect”.

Von Neumann and Morgenstern’s work complemented the theoretical foundation of expected utility theory and also attracted more researchers’ attention. Besides St. Petersburg paradox, another phenomenon of mental effects can also be seen in Allais paradox (Allais, 1953). In each of his experiments, participants were asked to make a choice between two gambles. He intended to design the gambles as shown in table 1.2.1 to compare participants’ choices for a specific purpose.

Table 1.2.1 – The payoffs of gambles in Allais paradox

Experiment 1				Experiment 2			
Gamble 1A		Gamble 1B		Gamble 2A		Gamble 2B	
Payoff	Chance	Payoff	Chance	Payoff	Chance	Payoff	Chance
$1 million	100%	$1 million	89%	Nothing	89%	Nothing	90%
		Nothing	1%	$1 million	11%		
		$5 million	10%			$5 million	10%

According to the results of Allais’ studies, he asserted that most people would choose gamble 1A in experiment 1 and gamble 2B in experiment 2. This result shows two interesting facts. First, it provides an example that the decision is not always made by the choice of maximal expected payoff. The less preferred gamble 1B contrarily has a higher expected payoff of $1.39 million. Second, the choices of respective experiments are not consistent to each other. To make it easier to visualize, the same gambles can be rewritten as the payoffs and associated probabilities shown in table 1.2.2.

Table 1.2.2 – The rewritten payoffs of gambles in Allais paradox

<table>
<tr><th colspan="4">Experiment 1</th><th colspan="4">Experiment 2</th></tr>
<tr><th colspan="2">Gamble 1A</th><th colspan="2">Gamble 1B</th><th colspan="2">Gamble 2A</th><th colspan="2">Gamble 2B</th></tr>
<tr><th>Payoff</th><th>Chance</th><th>Payoff</th><th>Chance</th><th>Payoff</th><th>Chance</th><th>Payoff</th><th>Chance</th></tr>
<tr><td>$1 million</td><td>89%</td><td>$1 million</td><td>89%</td><td>Nothing</td><td>89%</td><td>Nothing</td><td>89%</td></tr>
<tr><td rowspan="2">$1 million</td><td rowspan="2">11%</td><td>Nothing</td><td>1%</td><td rowspan="2">$1 million</td><td rowspan="2">11%</td><td>Nothing</td><td>1%</td></tr>
<tr><td>$5 million</td><td>10%</td><td>$5 million</td><td>10%</td></tr>
</table>

One may note that the yellow row denotes the common outcomes and could be omitted under the assumption that equal outcomes added to each of the two choices have no effect on the relative desirability of one gamble over the other. Thus, a rational person who prefers 1A than 1B in the first experiment should consistently choose 2A in the second experiment because s/he is about comparing the same outcomes. However, the result of experiments disagrees with this argument. Allais' experiments provide an instance to show that it may not be valid to evaluate uncertain outcomes by the independence axiom of expected utility theory. The axiom states that outcomes within a gamble are considered independently and hence identical outcomes between gambles could be cancelled when comparing gambles in whole. Allais paradox highlights the fact that conventional theories, e.g., expected utility theory (Von Neumann and Morgenstern, 1947), overlook the notion of complementarities.

In other words, the value of one part in a gamble may depend on the possible outcome in the other part. In the above gamble 1B, there is a notable 1% chance of getting nothing. Though one percent is only a slight chance to loose, however it carries a great sense of disappointment since the other choice 1A ensure a certain win. On the other hand, the feelings of disappointment do not arise in experiment 2 because gamble 2A does not guarantee a certainty to win. Thus, the participants actually do act rationally when choosing 1A and 2B in the experiments, and it points out that expected utility theory is not sensitive enough to capture the influences of the common complementary parts.

To amend the incompleteness of expected utility theory, the famous prospect theory was proposed by Daniel Kahneman and Amos Tversky in 1979 as a more accurate description of how human make decisions between alternatives with uncertain outcomes. The theory basically describes the decision processes in two stages, editing and evaluation. In the first stage, outcomes are framed by following some judgmental heuristics. Also, people set a reference point and then consider lesser outcomes as losses and greater ones as gains. Human have stronger feelings on losses, i.e., a gain cannot compensate for the same amount of loss. In other words, people treat gains and losses by different value (utility) functions. In the following evaluation phase, people behave as if they would compute expected value, based on the assigned values (utilities) of potential outcomes and their respective “weights”. Prospect theory assumes that people use “decision weights” instead of probabilities to calculate expected value. Weights here can be seen as a set of distorted probabilities in decision-maker’s mind. According to a wide range of empirical findings, typical decision weights are generally lower than the corresponding probabilities, except in the range of small probabilities (Tversky and Fox, 1994). In the last step, similar to expected utility theory, the final decision will be made by choosing the alternative with a higher expected value.

Kahneman was awarded the 2002 Nobel Memorial Prize in Economics for his great contributions in prospect theory and other works in behavioral economics. Prospect theory and the related studies point out two important ideas: the first, utility might be reference based, in contrast with additive utility functions, and the reference is obtained by a series of mental heuristics; the second, probabilities might be distorted in calculation of expected value so that the outcomes of smaller probabilities are over-weighted and the ones of higher probabilities are under-weighted. The interaction of overestimation-underestimation of probabilities and concavity-convexity (gain-loss) of the value function, i.e. utility function, leads to so-called fourfold pattern of risk attitudes (Tversky and Fox, 1994): risk-averse behavior in gains involving moderate probabilities and in losses with small probabilities; risk-seeking behavior in losses involving moderate probabilities and gains with probabilities. These findings of prospect theory imply that risk is actually a concept of both physics and psychophysics. The skeleton of a risk is constructed by its physical factors, such as possible outcomes, potential severities, probability distributions, etc.; however, the flesh, which gives the real appearance, is shaped from psychophysical judgments. In other words, the true essence of risk could be more properly interpreted as a reflected image of future uncertain events onto human perception.

Following the steps of expected utility theory and prospect theory, a number of researches in risk assessment theories have had great interests in how to properly include human behavioral patterns into decision models. In the previous paragraphs, those examples have illustrated that a decision is not realistic if it is made without considering human judgments. However, it is also proved that human judgments are following some heuristics and could be altered by encountered situations, personal experiences or received information. Specifically speaking, human judgment is a critical factor in the decision-making process; however the judgments sometimes lack awareness of important information. One main function of human judgment is to reframe and prioritize risks so that the consequent decision can fulfill people's expected emphases. Since the resources are usually limited, a decision allocating resources to the risks that people think more "important" or "urgent" will be considered more effective and satisfied. As mentioned before, risk is about uncertainty, i.e. events may or may not happen, so the optimality is based on a level of preparation which the decision makers feel satisfied with. However, one satisfied decision is not necessarily as good as prospected. Judgment-based priority means it is basically a result of mental distortions so it could have problems of coherence and could undervalue or even ignore important facts. If optimality is defined as the level of satisfaction, from the view of the decision maker, a preferred decision should be able to cope with the highly-prioritized risks while still keep other risks in a tolerant level. For example, little wine could be good to heart disease but harmful to alimentary canal. A man with family history of heart disease may put much more emphasis on the risk of heartache so that he drinks a glass of wine every night. He may successfully get rid of heartache but finally end up in stomachache, which is not expected by him. To improve this problem, there should be a mechanism which can take advantages by letting human judgments and un-distorted statistical information properly inform each other. Hence, this research will do systematical analysis and seek contributions to a mutually-informed decision model of human judgments and statistics-based information.

1.3 Problem Statement and Approach Frameworks

Making decisions or taking actions in time based on insufficient information and unreliable assumptions is a typical example of risk problems. In real operation, due to limited resources and reacting time to an event, there are many challenging issues involved in risk assessment, for examples:

1. Is it reasonable to treat every risk equally weighted for the purpose of preparation?
2. If it is not true, which risks should be prioritized first?
3. How to model preferences to prioritize these risks?
4. Is this preference structure reliable and robust?
5. Given the preference structure, to what level will the preparations be considered sufficient?
6. How the preparation for one risk could affect other risks?
7. Could I mitigate some risks but exaggerate others?
8. How likely could “surprising” situations occur after the planed preparation is done?
9. Are those “surprising” situations sustainable?

These questions are actually not easy for the decision makers, especially in the case that the budgets, labors and resources are limited and constrained by space and time, yet no risk can be easily disregarded. After searching in the literature, a systematical analysis and a comprehensive solution to these issues are still under development. Thus, by using existent methods, the decision makers could face a situation that they may focus on one risk, reviewing every aspect and develop tunnel vision, and neglect the other more threatening factors. How to find the reasonable trade-off among risks always trouble the decision makers.

As a result, a more comprehensive risk assessment model is now broadly catching attentions in many research areas, such as statistics, operations research, economics, psychology, and management. In most risk assessment problems, decision makers usually face risks from different categories which cannot directly compare to each other in quantities; for example, it is hard to compare the life losses to properties damages. In these cases, decision makers commonly have to adopt a judgmental way, e.g. weighting system, to integrate evaluations of qualitatively different risks. On the other hand, utility theory and prospect theory both point out that human perception plays an essential role in proper evaluation of uncertain events. According to them, both severity and probability of a risk are distorted in human mind with a referencing system which is altered by personal characteristics and outside reference factors. Therefore, a decision without considering human judgments in perceptions and preferences are not appropriate and demanded and hence not ‘optimal’. However on the reverse side, human judgments can help yield the most demanded decision but cannot guarantee a “rational” or “reasonable” solution. It is really a common case that one may put emphasis on the most unwanted risk but ignore the potential severe hazards.

In conclusion, to provide solutions to the issues in the previous paragraphs, there are several key tasks to overcome, which are ignored or only partially considered in existent techniques:

1. obtaining human judgments (preferences) structure,
2. investigating additivity properties of risks in human judgment structure,
3. finding probability distributions and dependence structure of risks,
4. evaluating possible human irrationality, and
5. determining optimality or sub-optimality by a model fully investigating information of probability and human judgments.

In this research, we are going to thoroughly investigate and systematically analyze current methods and integrate their advantages to develop a conceptually new approach in solving risk assessment problems. Through this approach, one should be able to analyze a risk assessment problem comprehensively and realistically; the challenging issues have answers as well as the key tasks can be accomplished.

In the following sections, initial assumptions and frameworks are discussed. Briefly, risks are assumed to have the property of competing-risks structure and will be modeled by survival analysis (section 1.3.1 and 1.3.2). To quantify risks, additive risk measures are considered (section 1.3.3). Risk matrix (section 1.3.4) is the typical example of existent method and good comparison to see the improvement of proposed approach. Furthermore, decision making process will be based on a framework of multi-objective optimization (section 1.3.5). And, two aggregating methods (Choquet fuzzy integral and copula, section 1.3.6 and 1.3.7) will be compared for more insight.

1.3.1 Survival Analysis

Risk is an abstract concept which is difficult to define, although many examples (e.g., NASA, banking, FAA, 911) can be used to elaborate the multifaceted nature of risk. Informally, risk is often expressed as a product of two components: a probabilistic term regarding the possibility of losses due to a critical event, and a magnitude term gives the severity of losses caused by the event. It is assumed that there is an embedded distribution of losses for every considered risk.

The two Chinese characters used to represent risk are "危機" where 危 means hazard, and 機 opportunity. It is plausible to argue that in many real world situations the price to seek opportunity is the exposure to hazards. In this research, the works will concentrate on the type of risks commonly evaluated by survival analysis.

Survival analysis is a branch of statistics dealing with death in biological organisms and failure in engineering systems. More generally, survival analysis is based on the time of event data; in the context, death or failure is considered as an "event". The basic logging mechanism of survival analysis is a counting process – the number (of death or failures) is counted when an event happens and the occurring time or time between events is recorded. Note that recurrent failures or repairable events are not considered in the proposed research. That is, the death or failure event could happen only once for each subject or agent of interest. Subsequently, the stochastic analysis would involve the following estimations:

1. the fraction of the population that could survive beyond a certain time threshold,
2. the hazard rate about the potential failure could occur – this estimation is only based on those who live or remain functional after the given time threshold,
3. the embedded probability distribution of deaths or failures, and
4. the particular covariates or circumstances that affect the odds of survival.

Survival analysis can be expanded to more general risk problems, in which one can replace the survival time with other loss measures. For example, if one would like to know the rainfall of a category-three hurricane, he can replace the dependent variable survival time (e.g., 24 hr) with the amount of rainfall and estimate the likelihood beyond certain gauge threshold (e.g., 800mm), although the covariates such as location, month, temperature, atmospheric pressure, may be the same for both survival time and rainfall. With sufficient number of events recorded, the risk of rainfall can be modeled by using the tool of survival analysis.

1.3.2 Competing Risks

This research will assume the case: misfortunes never come singly. In survival analysis, it is assumed that each individual is at risk of only one event at any moment of time. In reality, multiple factors could contribute to death or failure of one individual, yet the individual can die for only one cause. In other words, the risks actually 'compete' to each other. Suppose we take a machine system as an instance. Malfunction of one component results in the straining of other components and eventually one of them causes the whole machine to break down. In the literature, when formulating the competing risk model, it is often assumed that the survival time of individual component is statistically independent to others (Molife, 2003). This research is intending to discard this unrealistic assumption and therefore will suppose a property of interdependence in the competing-risks structure.

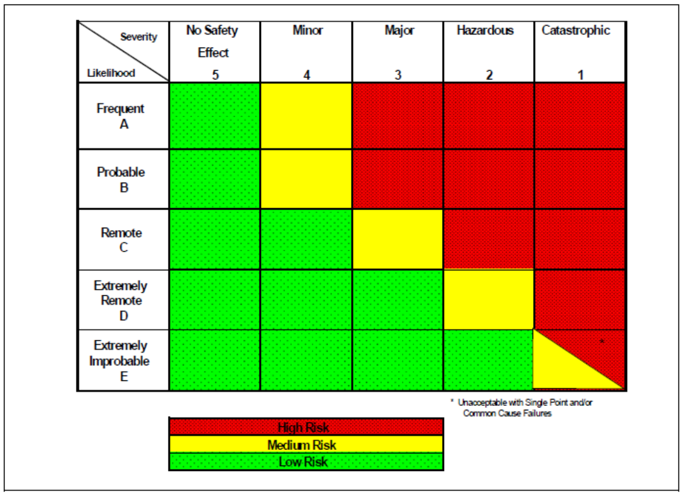
1.3.3 Risk Measure

Quantifying the value of uncertain risk is one of the key tasks of risk management. A risk must be quantified in order to model and analyze it. This quantification is usually achieved by modeling the uncertain loss as a random variable, to which then a certain function is applied. Such functions are usually called risk measures. Risk measures are divided in two types: non-additive and additive. A good example of non-additive risk measure is value-at-risk, also called VaR (see chapter 2.4.3). Basically, it is defined as a threshold value and the probability that loss exceeds this value. VaR is a widely used risk measure in loss of in financial assets since the composited information of threshold and probability is more useful for the purposes of financial control, financial reporting and computing regulatory capital. However, VaR is non-additive and not a coherent risk measure to evaluate a combination of risks. It is not informative about the magnitude of the losses larger than the threshold. On the contrary, an alternate risk measure, CVaR (see chapter 2.4.3) is an additive and coherent risk measure. CVaR is the expected loss, given the condition that the loss exceeds the threshold. It satisfies properties of monotonicity, sub-additivity, homogeneity, and translational invariance. Thus, the CVaR of individual risk can be ordered, added, weighted, or averaged to provide coherent information and an evaluation of the combined set of risks. In chapter 2, both VaR and CVaR are reviewed but CVaR is selected to perform the objectives of this research.

1.3.4 Risk Matrix

The shortage of compressive approaches in risk analysis results in potential problems for many real complex, large-scale applications. For example, Federal Aviation Administration (2006) proposed a guidance of safety risk management. This official government document proposed a two dimensional risk matrix to classify risks into three levels of High, Medium and Low by severity and probability. Table 1.3.1 shows a safety order of precedence for design for minimal risk is based on the risk matrix.

Table 1.3.1 – Risk Matrix (FAA 2006)



At the first glance, the table looks reasonable and applicable. However, the tabulation gives a false sense of limited (if any) interdependence of risks, ignoring the fact that the judgment of low and high are relative depending on the individual experience of risk creep. In many situations, no one can be certain that ignoring low level risks will not accrue, procrastinate and increase the probability of a severe risk which is original considered unlikely to happen. Consider two examples from NASA. The Columbia space shuttle disaster was a result of damage sustained during launch when a piece of foam insulation the size of a small briefcase broke off the Space Shuttle external tank under the aerodynamic forces of launch. In the Challenger space shuttle disaster, disintegration of the entire vehicle began after an O-ring seal in its solid rocket booster failed at liftoff. The O-ring failure caused a breach in the rocket booster joint it sealed, allowing pressurized hot gas from within the solid rocket motor to reach the outside and impinge upon the adjacent SRB attachment hardware and external fuel tank. In both cases, many senior engineers and project managers under estimate the “low level” risks, and the potential to interact with the chained interactions of other risks.

Another challenging example of risk assessment is in financial area, where the risk management is based on maximizing the expected returns yet minimizing variation of a portfolio. However, these two respects are intrinsically oxymoron. The financial crisis of 2007-2010 was triggered by a liquidity shortfall in the US banking system, causing the collapse of large financial institutions, the bailout of banks by national governments, and downturns in stock markets around the world. In hindsight, many expert opinions are involved in the problem of trading off profits and variations. Even with support of most detailed banking and accounting documentation, it is still illusive to distinguish an expert’s technical judgment from his gut feeling.

From engineering to management, from design to clinical triage, current approaches for risk assessment problems are somehow fragmented or lacking holistic analytics. A rigorous research, supported by modern multivariate database and statistical model based optimization, is needed to overcome the difficulties with comprehensive understanding of all respects in risk assessment problem.

1.3.5 Multi-objective Optimization

The decision maker's job is to aggregate separate objectives to find the optimal solution. Since the measures may be qualitative or quantitative and may include absolute targets allowing no deviation or targets which may have scope for limited tolerable deviation, there are a number of methods to combine several objectives into a single form; however, many of them are not easy to solve. In the literature, multi-objective optimization is defined as the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints (Steuer, 1986). If competing risks are well formed as a multi-objective problem, there should not be a single solution that simultaneously minimizes each objective to its extreme.

When an objective has reached the optimal point of itself, other objectives suffer as a result. The optimum solution must be acquired through trade-offs between conflicting objectives. Thus, the decision maker has to give up some objectives a bit to trade off the others. In the proposed research, the decision-making process of competing risks problems is assumed to have the property of multi-objective optimization.

1.3.6 Choquet Fuzzy Integral

Since risks can be seen as one kind of utility or human perception, the proposed research will emphasize on how people do 'trade-off' between competing risks in the multi-objective optimization. In competing risks problems, one should be optimizing a measurement of human sensations rather than an objective of only statistical values. In the literature, there are a number of methods that use human preferences to aggregate individual objectives and then determine the decision. Choquet fuzzy integral, a fuzzy integral created by Gustave Choquet (1953), is applied as a powerful aggregation operator over a finite set of element (Grabisch, 2000). Thus, Choquet fuzzy integral method can involve human judgments as well as the underlying information to evaluate the 'optimum' in multi-objective problems. Through human evaluations, i.e. score system, in Choquet fuzzy integral method, incommensurable elements are then able to be compared to one another in a rational and consistent way. Choquet fuzzy integral satisfies a more realistic assumption that there are uncertain interactions between human judgments. Thus, a more accurate index of priority can be obtained in Choquet fuzzy integral method by adding interaction terms over risks.

Unlike methods that assume concrete measurements, preferences differ from person to person, so the result depends on who is making the decision and what their goals and preferences are (Satty, 2005). The extreme preferences of decision makers may result in an unreliable decision. The potential problems will be investigated and improved in this research.

1.3.7 Copula

Most traditional fully-parametric survival models in the competing risks framework assume the events to be mutually independent (Spivey, 1991), so that the relationships between survival times are not identifiable for most dependent models. With this assumption, the probability distribution of joint survival time is the product of the marginal probability of each survival time. This assumption simplifies a competing risks problem to a combination of single-risk assessment problems. However, as described in 1.3.3, the assumption of independence is not practical. A precise parametric specification of a dependent competing risks model would be very helpful for a more concise summary of the data. Therefore, more and more recent researchers put efforts on developing a model with dependence between competing risks. Copula models have been showed it is well equipped for handling competing risks of survival times (Frees 1996). A copula is a function that joints univariate marginal distributions to a full multivariate distribution with a variety of dependence structure. Unlike simpler models such as an independent competing risks approach, copula can provide more accurate inferences that a decision maker can take the advantage from. Copula models will be studied in the proposed research and the emphasis will be on how the information obtained from copula models can inform human decisions.

1.4 Specific Aim and Contributions

Management of risks requires a proactive, systematic approach applied when decisions are made across the life cycle. The goal of risk assessment is to determine the characteristics of considered risks, to see how these risks can be altered, and to mitigate risks at least to an acceptable level of the decision maker. Thus, this dissertation will have specific aims and contributions in the following:

1. **Investigation of existent methods** – This dissertation is devoted to developing a more comprehensive view point of risk assessment. The shortcomings of existent risk assessment methods are our major breakthrough points. A thorough discuss of the knowledge base of existent risk is done and their strengths and weaknesses are fully investigated. It is observed that existent methods commonly fail to assess the effects of human judgments, ignore correlation among risks and lack proper tool to aggregate risks. Thus, the solutions to this issues and further analysis are provided in this dissertation.

2. **Applications of human judgments** – Prospect theory and psychophysical rules all suggest that risk is not just a statistical problem. What may be deemed as risky decision for one may be regarded as tolerable to others since the comparison and relevant ranking is meaningless without specifying the frame of reference based on the level of risk acceptance, the potential risk trade-off, and the personal preference. As such, how to include human judgment and properly apply it in risk assessment is the first task to be done in this dissertation.

3. **Correlation structure of probability** – Existent risks assessment methods usually assume risks are mutually independent for the purpose of easy operation. In reality, however, this assumption is not appropriate. Therefore, this dissertation has aim on searching a tool to cope with the correlation structure in risk assessment. The method of copula is proposed and a full model is developed and demonstrated. The strengths and the most valuable information carried by copula model are documented as well.

4. **Human preference structure** – Preference is an expression of human judgments and mental effects. According to judgmental heuristics, human have preference due to a mental mechanism. Thus, human preference structure reveals how human perceive risks and what they demand. In addition, as suggested by psychophysical rules, the degree of preference is not additive, i.e., the sum of individual risks is not equal to their union. This dissertation has focus on this issue. A scoring system and the tool of Choquet fuzzy integral are proposed and a complete model is developed an demonstrated. The valuable information and its applications are also documented.

5. **Integrated information model** – Both copula and Choquet fuzzy integral improve existent methods; however, they are still a partial solution to the issues. Copula has strength on linking risks through their joint probability and thus is useful for tracking security. On the other hand, Choqeut has advantage on taking care of higher-prioritized risks, and thus is useful for taking emphases. This dissertation has major contributions on systematically analyzing and documenting useful information and implements of these two tools, as well as developing an integrated information model which provides a brand

new idea to make copula and Choquet fuzzy integral complement each other for a demanded and robust solution.

Chapter 2

Literature Review

The goal of risk assessment is to solve management problems, finding a strategy to mitigate the possible losses to a satisfactory level. Beyond the estimation of risk consequences, to more appropriately evaluate the underlying human decision process, the role of the human perception must be clarified. Therefore, risk, as we assumed, is evaluated by perceptions of severity and likelihood of an uncertain event. In other words, the first step of risk assessment is to determine its probability and the potential magnitude of loss, as well as how these two factors are perceived in the process of decision-making. The goal of this research is to develop a realistic yet robust risk assessment model. The decision should be made under the assumption of rationality, the condition of optimality and the consideration of robustness. A realistic and robust decision means it conforms to human preferences and remains within the rational barrier. For this purpose, it is crucial to have more understanding in how mental effects work in risk perceptions. We pay attention not only to the effects in single dimension, i.e. mental distortion to severity probability of a single risk, but also to the characteristics when human must aggregate multiple risks for an evaluation.

To review the current state of knowledge, the definitions of elemental factors of risk are reviewed in chapter 2.1. Prospect theory and its challenge to the conventional utility theory are discussed in chapter 2.2. Prospect theory provides the descriptive of the real-life choices, rather than optimal utility, where the key argument is that both the probabilities and outcomes would be distorted (i.e., nonlinearly transformed) in human mind. The prospect theory demonstrates the mental influence in probabilities and magnitude of a risk. To provide more experimental and theoretical supports to prospect theory, several basic psychological rules are review in section 2.3. In addition, psychological experimental findings on judgmental heuristics, discussed in section 2.4, also provide essential rules of how human make decisions. The sections 2.3 and 2.4 are the background theories of human nature in decision-making and also demonstrate the necessity of including human into a decision-making model.

2.1 Elemental Risk Factors

In Oxford English Dictionary, risk is defined as exposure to the possibility of loss, injury, or other adverse or unwelcome circumstance. Usually, risk can be seen as relating to the probability of uncertain future events. For instance, in information security risk is defined as "the potential that a given threat will exploit vulnerabilities of an asset or group of assets and thereby cause harm to the organization". In finance, risk is often defined as the unexpected variability or volatility of returns and thus includes both potential worse-than-expected as well as better-than-expected returns. More specifically speaking, according to Factor Analysis of Information Risk (Jones, 2006), risk is defined as the probable frequency and probable magnitude of future loss. An intuitive point is that you can’t have significant risk without the potential for significant loss. In other words, it doesn’t matter how exposed to harm an asset is; if the asset is not worth much, the risk will not be high. This is because risk always includes a value component and a probability. There are three important things to recognize from this definition. First and most obvious – risk is a probability issue. Second – risk has both a frequency and a magnitude component. And third –this definition for risk applies well regardless of whether we're talking about investment, market, credit, legal, insurance, or any of the other risk domains that are commonly dealt with in business, government, and life. In other words, the fundamental nature of risk is universal, regardless of context.

The purpose of risk analysis is to provide the decision-maker with the best possible information about loss probabilities. Consequently, it's crucial that decision-makers accept the risk analysis methodology being used, and that the information resulting from the analysis is in a form that's useful to them. In this regard, the limitations of traditional risk analysis methods will become clear as we progress through this research.

2.2 Utility Theory and Prospect Theory

As stated in section 2.1, risk is an uncertain event which consists of two main elements: possibility and severity. Actually, decision-making of risk is a process through serial activities of perceiving, judging, and comparing. According to the psychophysical rules and mental heuristics, it can be inferred that both dimensions of a risk, i.e. possibility and severity, could be altered or distorted in each step of these mental activities. For example, the decision maker would have diminishing marginal utility in his/her perception of severity; or s/he may magnify or underrate possibility and severity due to some judgmental heuristics. These mental behaviors have become a critical part to study a real case of risk-decision problem and form an essential reference of late decision theories. In this study, expected utility theory and prospect theory will be used to provide descriptive frameworks and mathematical supports to accommodate these mental behaviors in the decision-making process.

2.2.1 Utility Function and Expected Utility Theory

The famous St. Petersburg paradox arises when there is no upper bound on the potential rewards from very low probability events (see section 1.2). Because the expected value goes infinity in some probability distribution, an expected-wealth-maximizing person would will to pay an infinite amount to take this gamble. This scenario obviously contradicts our common sense. A solution was proposed by Bernoulli in 1738: in real life, decision is made on the basis of utility so the expected utility of the gamble is finite, even if its expected value is infinite.

There are several definitions of utility in the dictionary: (1) fitness for some purpose; (2) something useful or designed for use; (3) a service (as light, power, or water) provided by a public utility; and (4) equipment or a piece of equipment to provide such service or a comparable service. As such, the meaning of utility relevant to this study could be a measure to represent an individual's satisfaction (or dissatisfaction) to the level of fitness, the service, or the functionality of hardware or software. By estimating this measure, one can mathematically model satisfaction and then describe people's psychological behavior in terms of attempts to increase or decrease the utility of interest. The utility function, expressing utility as a function of entities, is to quantify conceptual utility as a numerical value.

By fundamentally different assumptions, utility is distinguished in two types of indices: (1) cardinal – mathematical indices that preserve preference orderings uniquely up to positive linear transformations, and (2) ordinal – pairs of options can be ordered such that one is considered to be worse than, equal to, or better than the other. The more complete explanations are showed in the following:

Cardinal Utility –

Early economists considered utility as a physical property that can be measured and compared across persons. For example, person A and B have two expected utility indices that preserve preference orderings; then these two indices are related by a linear transformation, i.e. for every point x on A's index, the corresponding point y on B's index satisfies the form: $y = ax + b$, where a and b are constant. When cardinal utility is used, the magnitude of utility differences is considered as a behaviorally significant quantity.

Ordinal Utility –

Ordinal utility assumes that one can only give a rank of preference to different outcomes. Taking risk as example, the utility function only assigns a ranking number to every possible outcome. An outcome with a higher utility (ranking) number only means it is preferred to another one with a lower number. Thus, for a comparison of ordinal utility, the magnitude of the utility numbers has no meaning, only the ranking. One cannot say that one bundle gives twice as much utility as another, or one bundle is preferred by how much.

However, utility cannot be measured directly from subjected entities, so researchers suggested a way to observe actual behavior and assume that, in a perfectly competitive equilibrium, the underlying relative utilities will be revealed by this behavior. This approach is known as "*revealed preference theory*", pioneered by Samuelson (1938, 1948) and further developed by many researchers (see Varian, 2005). Under this theory, if a person chooses a certain bundle of outcomes while another bundle of outcomes is available, then it is assumed that the utility of first bundle is always greater than the utility of the second. Thus, for a particular consumption set, if the utilities of the most-preferred and the least-preferred bundle are determined, one can use these two values to estimate the utility of all other bundles in the set (Afriat, 1967). A utility function can also be determined by this approach.

In economics, a utility function is usually depicted as a twice-differentiable function of wealth $U(W)$ defined for $W > 0$, which has the properties of non-satiation (the first derivative $U'(W) > 0$) and risk aversion (the second derivative $U''(W) < 0$) (Merton, 1990). As such, a utility function measures a person's relative preference for different levels of total wealth (e.g., assets, physical property, health levels). The non-satiation property states that utility increases with wealth, i.e., that more wealth is preferred to less wealth. Furthermore, concavity of the utility function in first quadrant states the property of risk aversion, i.e. that the marginal utility of wealth decreases as wealth increases. To see why utility functions are concave, consider the extra utility obtained by the obtainment of one additional cup of water. For someone who is thirsty and only has one cup of water, obtaining one more cup is very important. For someone who has already drunk ten cups of water, obtaining one more cup is nearly meaningless. However, the property of diminishing marginal utility does not guarantee risk-averse if both gains and losses are taken into consideration. Thus, for the attitude toward risks, it is not necessarily assumed risk-aversion (concavity) of utility function. A decision maker with a utility function $U(W)$ is

1. risk-averse if and only if $U(W)$ is strictly concave;
2. risk-neutral if and only if $U(W)$ is a linear function;
3. risk-seeking if and only if $U(W)$ is a strictly convex.

In the presence of different situations, one may take risk-aversion toward gains, yet take risk-seeking toward losses. In decision theories, utility functions are commonly used to quantify subject's utility for the purpose of optimization. It is intuitive to assume that people will always try to maximize utility in any decision. In the presence of risky outcomes, a decision maker could use the expected utility criterion as a rule of choice: higher expected utility are simply preferred.

The expected utility theory is based on the hypothesis in which preferences of people with regard to uncertain outcomes are determined by the payoff of outcomes, the probabilities of occurrence, and the utility of the payoff. Until the mid-twentieth century, expected utility was mainly referring to a conceptual term, contrasted with "statistical expectation" for the expected value. In 1947, John von Neumann and Oskar Morgenstern presented four relatively explicit axioms of "rationality" such that any person satisfying the axioms has a utility function and expected utility criterion is relevant to choice between options. A person who follows von Neumann-Morgenstern rationality will satisfy the axioms:

1. Completeness: for any two simple options *A* and *B*, either $B \prec A$, $A \prec B$ or $A = B$ (denoting $B \prec A$ to mean 'A is preferred to B').
2. Transitivity: for any three options *A*, *B* and *C*, if $C \prec B$ and $B \prec A$, then $C \prec A$.
3. Convexity/continuity (Archimedean property): If $C \prec B \prec A$, then there is a probability P between 0 and 1 such that the option $P \cdot C + (1-P) \cdot A$ is equally preferable to *B*.
4. Independence: for any three options *A*, *B* and *C*, $B \prec A$ if and only if $P \cdot B + (1-P) \cdot C \prec P \cdot A + (1-P) \cdot C$

2.2.2 Prospect Theory

Expected utility theory has been proved useful to explain some decisions that seem to contradict the expected value. However, like most elementary mathematical models, expected utility theory is an abstraction of reality. The mathematical model of expected utility theory and its primitive concepts do not guarantee a reliable guide to human behaviors in decision-making. For example, Allais paradox is a counterexample to the independence axiom (see section 1.2). In 1979, two psychologists, Kahneman and Tversky, proposed a critique of expected utility theory and presented prospect theory as an alternative model of decision-making of risky outcomes. Expected utility theory is a prescriptive approach which assumes people follow von Neumann-Morgenstern rationality to maximize their expected utility and concerns how decisions "should" be made. On the contrary, prospect theory is a descriptive approach and concerns how decisions are "actually" made in psychological reality.

In general, people behave to choose the alternative of a higher expected value (utility), which is based on the payoff and the associated probability of potential outcomes. In expected utility theory, only payoff is transformed to utility through individual's utility function. However, in prospect theory, Kahneman and Tversky (1979) proposed that for an uncertain event, probabilities are also transformed in decision-making process. Thus, the "expected utility" is evaluated by the following formula:

$$U = w(P_1)v(x_1) + w(P_2)v(x_2) + \cdots \tag{2.2.1}$$

where $x_1, x_2, \cdots$ are potential outcomes and $P_1, P_2, \cdots$ are associated probabilities. The function w and v are respectively called weighting function and value function. Kahneman and Tversky further observed several properties of these two functions in their survey. These properties outline human mental behaviors in the decision-making process of risky events. Detailed explanations of weighting function and value function are described below:

Weighting function – prospect theory treats possibilities as "decision weights," and it assumes that these weights do not always correspond to associated probabilities of outcomes. Therefore, for each probability P, there is a decision weight $w(P)$, which meets the requirement that $w(0)=0$ and $w(1)=1$. Furthermore, Kahneman and Tversky (1979) stated that decision weights are generally lower than the corresponding probabilities, except in the range of low probabilities, i.e. $w(P)<P$ for moderate and high values of P and $w(P)>P$ for small values of P (see Figure 2.2.1). Besides, it needs not to be true that $w(P)+w(1-P)=1$ due to unsymmetrical distortion. The reflection point of a weighting function is usually assumed anchored around $1/e \approx 0.36$, depending on how much over-estimation of low probability and under-estimation of high probability.

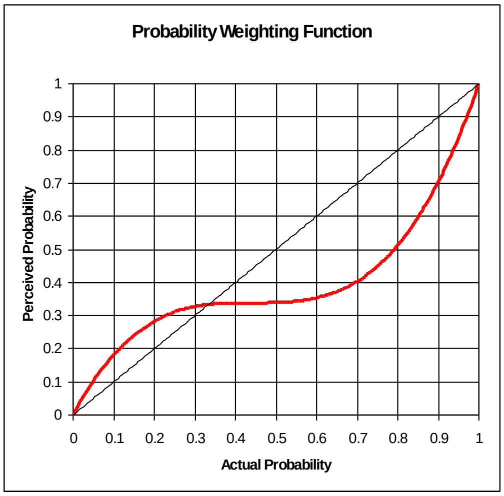


Figure 2.2.1 – Weighting Function

Asymmetric value function – Kahneman and Tversky (1979, 1992) found empirically that, in risky gambles, people do not rely on the levels of final wealth they can obtain but on gains and losses relative to some reference points, which may vary from situation to situation. Thus, in prospect theory, value is assigned to gains and losses respectively rather than to a final asset. Let $\overline{\mathbb{R}}$ be the set of extended real numbers and $\Omega = [a,b] \subset \overline{\mathbb{R}}$ in which $a < 0$ and $b > 0$. The value function v is defined over gains and losses relative to a reference point (*status quo*) $x_0 \in \Omega$ with $a < x_0 < b$, satisfying $(-1)^i v^{(i)}(x) \le 0$ for any $x \in [x_0, b]$, and $v^{(i)} \ge 0$ for any $x \in [a, x_0)$, $i = 1, 2$, where $v^{(i)}(x)$ is the i^{th} derivative of v.

Kahneman and Tversky (1979) also suggested the following value function:

$$v(x) = \begin{cases} x^{\gamma G} & if\, x \ge 0 \ \ and \ \ \gamma G \in (0,1) \\ -\lambda(-x)^{\gamma L} & if\, x < 0, \lambda > 1, \ and \ \gamma L \in (0,1) \end{cases} \tag{2.2.2}$$

Without loss of generality, the *status quo* can be assumed zero such that positive outcomes are referred as gains and negative outcomes as losses. The parameter $\lambda > 1$ describes the degree of loss aversion and $\gamma G,\ \gamma L \in (0,1)$ measure the degree of diminishing sensitivity. As result, the value function is normally concave for gains (risk aversion), commonly convex for losses (risk seeking) and is generally steeper for losses than for gains (loss aversion) (see Figure 2.2.2). Declining sensitivity holds in both gains and losses.

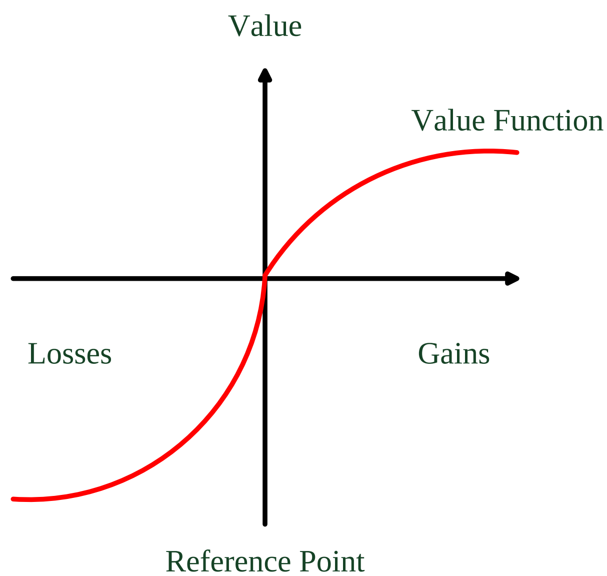


Figure 2.2.2 – Value function for gains and losses

Kahneman and Tversky obtained the properties of value function through several experimental researches so it does reflect major human mental behaviors in decision-making. The value function can be thought as a representation of some central components of the human perceptive mechanism (see section 2.3 and 2.4). In summary, it has three important features, each of which captures an essential element of mental behaviors (Thaler, 1999):

1. The value function is defined over gains and losses relative to some reference point – anchoring heuristics (see section 2.4);
2. Both the gain and loss functions display diminishing sensitivity. That is, the gain function is concave and the loss function is convex – psychophysical rules (see section 2.3);

3. Loss aversion refers to people's tendency to strongly prefer avoiding losses to acquiring gains.

These three features principally characterize the shape of the value function in prospect theory. However, the value function and weighting function are altered from person to person and from situation to situation. Basically, prospect theory is similar to expected utility theory (von Neumann and Morgenstern, 1947) in assuming a decision maker would assign a value (utility) to each outcome. The major difference is that in prospect theory, a decision maker is assumed to "edits" options prior to assign values to outcomes. Editing operations, including framing of outcomes as gains and losses as well as framing of segregating or integrating, are based on theories of mental accounting (Kahneman and Tversky, 1984) described below:

Outcome framing –

In order to clarify the heuristics and biases that influence the way of decision-making, through experimental designs and observations, Kahneman and Tversky (1984) propose three ways that outcomes might be framed: (1) in terms of a minimal account, (2) a topical account, or (3) a comprehensive account. These key findings are based on the psychophysical foundation that the human brain can only compare things relatively, and cannot measure anything absolutely (see section 2.3). As such, a person who uses minimal account to compare two options will only consider the differences between these two options, disregarding all common components. On the contrary, a topical account compares options by relating the outcomes of possible options to a reference level that is determined by the context of the decision. In the paper of Kahneman and Tversky (1984), they presented an example that most people would like to travel 20 minutes to save \$5 while the item costs \$15 but will not do so while it costs \$125. This example exhibits topical framing because the decision is to relate \$5 saving to a reference of item cost. For a \$15 purchase, five dollars seems like a significant saving, but not so for a \$125 purchase. The third one, a comprehensive account includes wide range of mental adaptions and contextual factors, such as current wealth, future earnings, and possible outcomes of other probabilistic holdings. The mental heuristics in section 2.4 are good examples to show mental adaptions of comprehensive account. For example, a decision maker may judge outcomes using the availability heuristic, i.e. outcomes may be assumed to be associated if they are easy to retrieve from memory.

Hedonic framing –

The theory of outcome framing basically deals with evaluation of options under the assumption of no interactive effects between outcomes; however, it may not always true. In the outcome framing phase, the severity (value) of having flu or measles may be individually evaluated by taking references of current wellness and personal experience. However, the severity of simultaneously having flu and measles would be much exaggerated, comparing to the severity of individual disease. To resolve this problem, Thaler (1985) presented the hedonic editing hypothesis to describe how people with a prospect theory value function could mentally integrate or segregate two outcomes before they are evaluated in order to achieve the highest perceived value. Let $(x; y)$ denote a combined outcome of x and y and

$v(\cdot)$ denote the subjective value function. Then, the two outcomes x and y are *mentally integrated*, i.e. evaluated by their addition, if $v(x;y) = v(x+y)$; on the other hand, they are *mentally segregated*, i.e. evaluated separately, if $v(x;y) = v(x) + v(y)$. In other words, under the assumption of maximal value, people will integrate outcomes before evaluation when $v(x+y) > v(x) + v(y)$, or will segregate outcomes when $v(x+y) < v(x) + v(y)$.

Synergy effects could play an important role in risk problems. One may want to suffer back-to-back risks rather than they all come together. Like the flu and measles example in the previous paragraph, the synergy effect of combined diseases may cause extra severity to the total of two individual diseases. Thus, for any risk-decision problem, in order to characterize the property of mental distortion, it will be valuable to investigate whether the possible outcomes are mentally integrated or segregated. Egozcue and Wong (2010) examined Thaler's (1985) work and presented the brief conclusions:

For any two values of outcomes $\mathrm{x},\ \mathrm{y} \in \Omega = [a,b] \subset \overline{\mathbb{R}}$ in which $a < 0$ and $b > 0$

1. If x, y > 0, then value-maximizers who are loss averse, loss tolerant, or loss neutral will prefer to segregate.
2. If x, y < 0, then investors who are loss averse, loss tolerant, or loss neutral will prefer to integrate.
3. If $y < 0 < x$ with $x + y > 0$, then loss averters and persons who are loss neutral will prefer to integrate; whereas persons who are loss-tolerant will sometimes prefer to integrate, sometimes prefer to segregate, and will be neutral between integration and segregation in other circumstances.

4. If $y < 0 < x$ with $x+y < 0$, then persons who are loss neutral or loss tolerant will prefer to segregate; whereas loss averters will sometimes prefer to segregate, sometimes prefer to integrate, and will be neutral between integration and segregation in other circumstances.
5. If $y < 0 < x$ with $x + y = 0$, then loss averters will prefer to segregate; investors who are loss-tolerant will prefer to integrate, and persons who are loss neutral will be neutral between integration and segregation.

Furthermore, Jarnebrant and Toubia (2009) provide mathematical analysis to support Egozcue and Wong's work. They characterize regions under which segregation of gains and losses is preferable to integration, and identify how the tradeoff between integration and segregation is influenced by the magnitude of the gain, the magnitude of the loss, and the degree of loss aversion of the decision maker. Suppose for an outcome x, a person has a value function $v(x)$ such that

$$v(x) = \begin{cases} g(x) & x \geq 0 \\ -\lambda g(-x) & x < 0 \end{cases} \tag{2.2.3}$$

Then, for any fixed loss L, there exists a loss aversion coefficient $\lambda^* = \frac{g'(0)}{g'(L)}$, such that:

1. For all $\lambda > \lambda^*$, the value derived from integration is greater than that derived from segregation for any gain $G \leq L$.

2. For all $\lambda < \lambda^*$, there exists a gain $G^* \in]0, L[$ such that the value derived from segregation is greater than that derived from integration for any gain $G < G^*$, and the reverse is true for any gain $G > G^*$.

2.3 Psychophysical Rules

As mentioned in chapter 1, St. Petersburg paradox, Allais paradox and prospect theory all imply that human follow some rules in their perception of uncertain outcomes, or in other words, human have a specific way to distort the external world through their mental judgments when making decision. Psychophysics, by definition, is a discipline within psychology that quantitatively investigates the relationship between physical stimuli and the sensations as well as perceptions they affect. The rules of mental judgments can be referred to the psychophysical measurement theories, which provide axioms of perceptual processes by studies of the effects on a participant's subjective experience or behavioral changes reacting to systematically varying stimuli (Stevens, 1957). These psychophysical rules clearly describe the behavioral patterns of human sensations and therefore can be taken as theoretical explanation of human mental judgments on risks, such as decreasing marginal utility, one major mental effect in expected utility theory and prospect theory. These are also the most basic rules regulating the mental effects in decision-making process in the latter sessions.

2.3.1 Weber's Law

The psychophysical rules can be traced back to the beginning of Weber's Law. Investigators usually used absolute thresholds and difference thresholds to specify the perceptual domain in classic psychophysics. The increment threshold (or called just noticeable difference) is determined as the minimum amount by which stimulus intensity must altered to produce a noticeable variation in sensory experience. German psychologist E.W. Weber (1834) observed that heavier weights are more difficult to discriminate and are associated with larger increment threshold. The relationship, known as *Weber's Law*, can be expressed as:

$$\frac{\Delta\phi}{\phi} = constant \qquad (2.3.1)$$

where $\Delta\phi$ represents the difference threshold and ϕ represents the initial stimulus intensity in physical domain.

The fraction, $\frac{\Delta\phi}{\phi}$, consisting of the ratio between the just noticeable increment and the base magnitude, is called *Weber fraction* and could vary for different sensory types and dimensions. The fraction tends to have a constant value, except at very low stimulus values, where the estimation rapidly increases. See Figure 2.3.1 for example of Weber fractions.

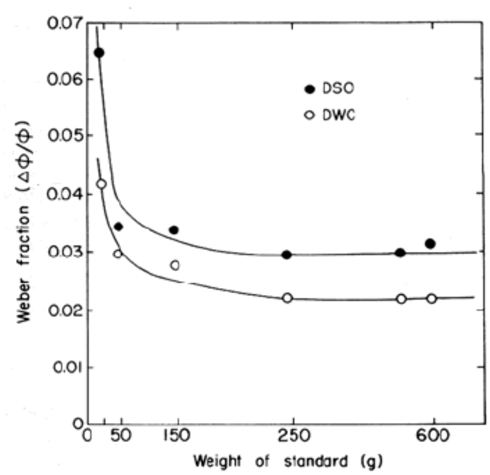


Figure 2.3.1 – Weber fraction of lifted weights by two observers

Weber's Law, simply stated, says that the size of the just noticeable difference is a constant proportion of the original stimulus value. In other words, when measuring increment thresholds on various intensity backgrounds, the thresholds increase in proportion to the background. It is easy to realize from the common experience that you must yell to be heard when you are in a noisy bar while just a whisper works in a quiet classroom. According to Weber's Law, most human sensations are based on a ratio scale of the intensities between stimuli. Though Weber's work was on basis of sensorial experiments, the result can be extended to the case of human mental judgments. For example, one may think the discount of $5 is a good bargain for a $10 merchandise but may not feel the same if it costs $1,000. This mental behavior also follows Weber's Law that the just noticeable difference is a constant proportion of the original prices.

The most important contribution of Weber's Law is that it provides an experimental evidence and theoretical analysis to explain the property of decreasing marginal utility. This property is one generally-used fundamental assumption of nowadays economic analysis and decision theories, such as utility theory (see section 2.2.1) and prospect theory (see section 2.2.2). Weber's Law also supports the assumption that decision-making of risks is actually a result of mental judgments. The property of diminishing marginal utility suggests a regulation of human attitudes towards the outcomes of risks. Therefore, human may show different attitude of risk-aversion, risk-seeking or risk-neutral towards various combinations of gains and losses. Furthermore, Weber fraction determines the relation between the discrimination threshold and the base intensity. In general, it is reasonable to assume that Weber's Law holds true for many different dimensions and the Weber fraction of each dimension may differ from each other. In other words, one may have different sensitivity on each dimension. A Weber fraction of 1% shows a high sensitivity to increments, while a Weber fraction of 20% is rather indicates lower sensitivity to increments. These mental judgments, not only within one dimension but also across dimensions, form the base of human decision-making process. However, following these mental judgments could be beneficial to achieve satisfaction while it is also possibly irrational in some cases.

2.3.2 Modified Weber's Law

Weber's Law is good as a baseline to compare performance and as a rule-of-thumb, but it is not always true. Though Weber's Law is generally valid across a broad range of stimulus intensities, however the Weber fraction tends to increase in the region of very small intensities (see Figure 2.3.1). To solve this problem, a better fit to the empirical data can be obtained by adding a small constant offset to the reference stimulus term. The modified Weber's Law is expressed as:

$$\frac{\Delta\phi}{\phi + a} = constant \qquad (2.3.2)$$

where a is a constant which represents the sensory noise existing when $\phi = 0$.

The modified Weber's Law points that the difference threshold $\Delta\phi$ is actually determined by not just the initial stimulus ϕ, rather ϕ plus continuous background noise a. See Figure 2.3.2 for the example of modified Weber's Law:

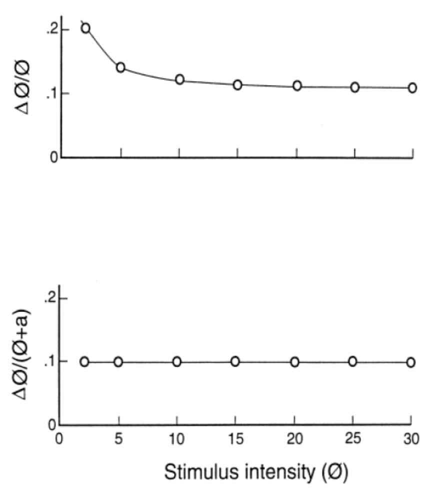


Figure 2.3.2 – Weber Fraction in Weber's Law and modified Weber's Law

In addition to the original one, one important finding of modified Weber's Law is the background noise in human sensation. With respect to decision-making, this psychophysical rule of background noise can be interpreted as a prejudiced attitude towards different type of risk. In facts, human could have biased views from one risk to another. These prejudices would alter the shapes of the utility functions and could lead to a considerable deviation in final decision.

2.3.3 Fechner's Law

To fully describe the relationship between the physical strength of a stimulus and its strength as perceived by humans, in 1860, Gustav T. Fechner presented an analytical interpretation of Weber's findings. Fechner postulated that sensation increases as the log of the stimulus. The relationship is formulated by a differential equation:

$$dp = k_1 \cdot \frac{d\phi}{\phi} \tag{2.3.3}$$

where dp is the differential difference in perception; $d\phi$ is the differential increase in the stimulus and ϕ is the stimulus at that instant. k_1 is a constant determined by experiments. Integrating the above equation gives

$$p = k_1 \ln \phi + C \tag{2.3.4}$$

where C is the constant of integration. C can be determined by setting $p = 0$, i.e. no perception, and it gives

$$C = -k_1 \ln \phi_0 \tag{2.3.5}$$

where ϕ_0 is the absolute threshold below which the stimulus is not perceived.

On the other hand, C also presents an absolute threshold in perception and only the portion above this threshold is interesting.

In addition, Fechner argued that two stimuli can be discriminated if they generate a perceptual response which exceeds some specific threshold. Thus, he used "the number of just noticeable differences", which represent equal increments of perception, as the agents for a perceptual scale of sensation magnitude. Then, the following relationship, known as *Fechner's Law*, can be derived:

$$\psi = k \ln \phi \tag{2.3.6}$$

where k is a constant; ψ is the sensation magnitude and ϕ here is the intensity of the stimulus above the absolute threshold.

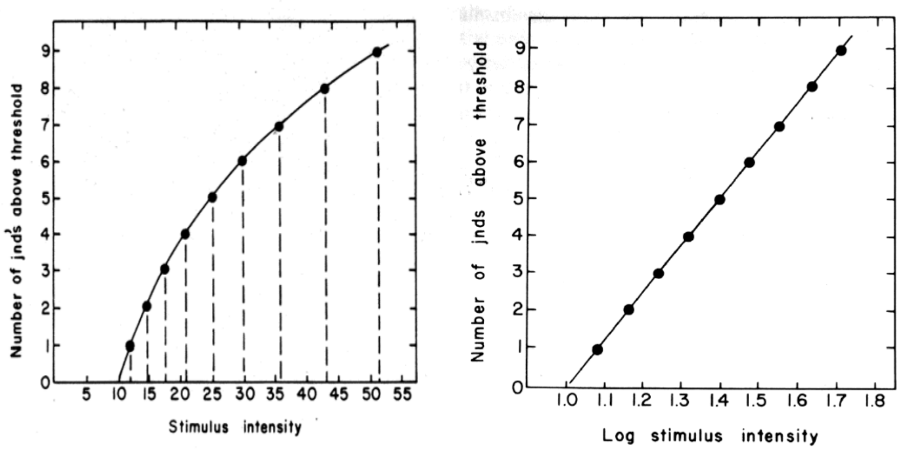


Figure 2.3.3 – Fechners's Law

Since Fechner's work is on the basis of Weber's findings, so theoretically Fechner's Law holds as long as Weber's Law also holds.

Let $\Delta\phi$ represents discrimination threshold; then

$$\Delta\psi = k \cdot \left[\ln(\phi + \Delta\phi) - \ln\phi\right] = k \cdot \ln\frac{\phi + \Delta\phi}{\phi} \qquad (2.3.7)$$

Raising the constant e to both sides gives

$$e^{\frac{\Delta\psi}{k}} = K = \frac{\phi + \Delta\phi}{\phi} \qquad (2.3.8)$$

Rearrange the equation to have

$$\left(K - 1\right)\phi = \Delta\phi \qquad (2.3.9)$$

If we define $k_w = K - 1$, then the Weber's Law is derived:

$$\frac{\Delta\phi}{\phi} = constant = k_w \qquad (2.3.10)$$

Fechner's theorem provides a broader and deeper analysis of the relationship between sensation magnitude and physical stimuli. Not only comprehending the findings of Weber's Law, it further introduces the proportionality of logarithm transformation between the threshold of just-notice-difference and the stimulus magnitude. Whatever the risk is, it can only be perceived according to that proportionality. The logarithm formation of Fechner's Law also defines the mathematical expression of possible utility function. This logarithm shape conforms to the property of decreasing marginal utility and is adopted in expected utility theory and prospect theory.

2.4 Judgmental Heuristics

In section 2.1, the relationship between sensation and stimuli is explained and formulated by Weber's Law and Fechner's Law; however, in addition to proportionality, human also have judgments on relativity in the process of decision-making. For example, in Fechner's Law, only the stimuli above the absolute threshold are effective and considerable. However, how to find the absolute threshold or how to determine the coefficient constant is still left unknown. It is hard to find the complete mathematical expressions to describe the mental judgments on relativity, but some researches in the literature still provide helpful findings to predict behavioral patterns of these judgments. Kahneman and Tversky (1973) stated that when making decisions, people do not absolutely follow the calculus of chance or statistical theory of prediction; instead, they tend to rely on a limited number of heuristics which sometimes yield reasonable judgments yet sometimes lead to systematic errors. In summary, they defined three cognitive heuristics of judgments: availability, representativeness, and anchoring-and-adjustment.

2.4.1 Availability

The availability heuristic is a mental rule of thumb whereby judgments are based on the instances or associations which easily come to mind. Availability has been reported to be affected by imaginability, familiarity, and vividness, and has been supported by evidence of stereotypic and scenario thinking (Kahneman and Tversky, 1979).

2.4.2 Representativeness

The representativeness heuristic is a mental shortcut by which people classify an event according to the degree of similarity in essential properties to a typical case. In contrast to availability heuristic which reflects assessments of associative distance, representativeness involves assessments of similarity distance. Kahneman & Tversky gave supporting evidence from the facts that people ignore base rates, neglect sample size, overlook regression toward the mean, and misestimate conjunctive probabilities, where base rate is information about the frequency of different categories in the population.

2.4.3 Anchoring-and-Adjustment

The anchoring-and-adjustment heuristic is a mental shortcut by which people start from an initial value, and then adjust from this anchor to a final answer. The initial value may be obtained from the formulation of the problem or a partial computation. One example of anchoring-and-adjustment is biased sampling, whereby people make generalizations from samples of information they know are biased or atypical.

Chapter 3

Shortcomings of Existent Risk Assessment Methods

As defined in ISO 31000 (ISO, 2009), risk management is the identification, assessment and prioritization of risks, followed by coordinated actions or application of resources to mitigate the probability and impact of unwanted events. Among which, assessment is the most crucial step since all the intentions of risk management must rely on it. Risk assessment can be seen in various fields. For example, in systems engineering, risk assessments can be seen within safety engineering and reliability engineering when it concerns threats to life, environment or machine functioning. Insurance companies perform risk assessments to evaluate premiums and to provide strategies for property, financial or operational risks. Furthermore, medical, hospital, and food industries also perform risk assessments on a continual basis to control risks. Even for an individual, risk assessment is practiced to decide whether to carry an umbrella in a cloudy morning. Essentially, risk assessment is everywhere whenever a person or a manager attempts to quantify potential losses so as to come up with appropriate actions to accomplish his/her objectives and risk tolerance. However, the water that bears the boat is the same that swallows it. Inadequate risk assessment can result in severe consequences of unexpected and unsustainable huge losses for companies as well as individuals. For example, the recession that began in 2008 was largely caused by the loose credit risk management of financial firms. The report issued by the United States Senate (2011) found that “the crisis was not a natural disaster, but the result of high risk, complex financial products, undisclosed conflicts of interest, and the failure of regulators, the credit rating agencies, and the market itself to rein in the excesses of Wall Street". High risky loans were granted because banks could bundle them with low risky assets and insure the designed packages to insurance companies. The same also happened to insurance companies and larger reinsurance companies. In those transactions, risks should have been thoroughly analyzed and assessed to settle the decisions. However, those efforts still failed to sound the early alarm. Though there were also other causes of the financial crisis in 2008, current approaches of risk assessment still ought to be investigated to ameliorate shortcomings.

Simply speaking, risk assessment is to determine quantitative or qualitative value of a risk related to an uncertain situation and a recognized threat. Quantity and quality are two different, however interactive concepts in risk assessment. By definition, quantitative risk assessment only takes into consideration what are nominally in quantity values such as statistical probabilities, number of lost lives or monetary losses. The commonly-used quantitative (probabilistic) risk assessments are based on calculations of two components: R, the magnitude of the potential loss S, and the probability F, that the loss will occur. By way of manipulating S and F, the degree of danger is translated into a number (index) and this number becomes the basis to make decision. Quantitative risk assessment may be overly reductive since it ignores qualitative differences among risks. In other words, quantitative assessment may drop out important non-quantifiable information, such as human mental judgment, historical knowledge, ethical perspectives, variation in preferences, etc. (see examples in judgmental heuristics and outcome framing in section 2.3 and 2.4). In brief, quantitative risk assessment has a weakness in reflecting judgmental priorities so that it may make people doubt its effectiveness, or divert the decision from the principles of an open society. On the other hand, relying on qualitative assessment could also be problematic. Qualitative measures are usually highly dependent on human judgments and preferences. From the cognitive perspective, preferences are reference-based measures and may vary from person to person, time to time, or case to case. Therefore, qualitative assessment is a reasoning or emotional process which can be rational or irrational and can be based on explicit assumptions or tacit assumptions. Radical judgments and preferences could take place in some situations and those mental effects lead to extremely biased assessments of risks. A good decision must be based on reliable and adequate risk assessments. However, both quantitative and qualitative risk assessments stand discredited in practice. A proper risk assessment is concerned with precision, rationality, satisfaction and the robust choice it leads to. We are seeking a way to integrate both quantitative and qualitative assessments to ameliorate shortcomings of using each sole method. The goal is to fulfill judgment-based qualitative satisfaction as well as retain statistics-based quantitative rationality.

In section 3.1, the essential factors of a risk and the criticisms of conventional probabilistic risk assessment are introduced. How human judgments play a role in risk assessment and the shortcomings are discussed in section 3.2. Section 3.3 points out the principles of robustness to give an idea of potential improvements. Two candidate solutions copula and Choquet fuzzy integral methods are analyzed and their advantages and limitations are described as well.

3.1 Conventional Risk Assessment Methods

Once risks have been identified, they must then be assessed as to their potential severity of impact (generally a negative impact, such as damage or loss) and to the possibility of occurrence. These quantities can be either simple to measure, in the case of the value of a lost building, or impossible to know for sure in the case of the probability of an unlikely event occurring. Therefore, in the assessment process it is critical to make the best information in order to properly prioritize the implementation of the risk management plan. Risk assessment usually answers three basic questions:

1. What can go wrong with the studied entity, or what are the covariates that lead to adverse consequences?

2. What and how severe are the potential detriments as a result of the occurrence of the covariates?

3. How likely to occur are these undesirable consequences, or what are their probabilities or frequencies?

Simply speaking, risk assessment is a systematic and comprehensive methodology to evaluate risks. Risk is defined as a possible detrimental outcome of an activity or action. In general, risk is characterized by two major components:

1. the severity (magnitude) of the adverse consequences, and

2. the frequency (probability) of occurrence of each consequence.

In usual quantitative risk assessment, the severity of consequences is expressed numerically, e.g., monetary unit, and their frequency of occurrence is expressed as probability. The whole process of risk assessment basically can be seen as a function with a two-dimensional input (severity and probability). The output of this process is a risk index for decision-making. Thus, the most intuitive way of risk assessment is to look at these two dimensions respectively. In other words, it is assumed that severity and probability are orthogonal to each other. As such, the process of conventional risk assessment can divide into two phases: the first phase is to evaluate each dimension respectively, and the second phase is to combine these two evaluations as a risk index. This concept of assessment procedure is not complicated but according to the results of literature review, there are still a number of shortcomings or improvement opportunities in the assumptions and the operations of conventional risk assessments. For examples, the effects of human judgments and mental distortions are not relevantly considered and evaluated; aggregation of risks is not under appropriate assumptions and the two dimensions of severity and possibility may not be rally orthogonal. The diagnosis of conventional risk assessments and the criticisms are documented in the following sections.

3.1.1 Risk Matrix

The risk matrix is a simple and easily visualized tool for classifying system risk. Every risk identified in a system is assigned to a cell in the matrix according to the respective evaluation of severity and possibility. The impact of the risky event is assessed on a scale of minimum to maximum possible impact of an occurrence of the risk. However, the scale can be arbitrary and need not be on a linear scale. The possibility of occurrence is likewise commonly assessed on a scale from low to high. This axis may be expressed in either mathematical terms or may be expressed in English expressions. The scale can be arbitrary or non-linear, depending on decisions by subject-matter experts. As shown below, Figure 3.1.1 is an example of such a matrix created for the system risk assessment.

Category of frequency	Category of consequence (Loss)			
	I: Low	II: Moderate	III: Medium	IV: High
A: Frequent	LT	IT	IT	IT
B: Probable	T	LT	LT	IT
C: Occasional	T	LT	LT	IT
D: Remote	T	T	LT	IT
E: Improbable	T	T	T	LT

IT: Intolerable risk LT: Limited tolerable risk T: Tolerable risk

Figure 3.1.1 – A risk valuation matrix used to evaluate the system risk for a system
Source: MIL-STD-882D

As seen in the risk matrix of Figure 3.1.1, it is made up from a set of 20 risk value categories. The value of each category is defined from a qualitative evaluation of every single risk. The limits for intolerable (IT) risks are set by the decision maker, the scaling of levels beneath IT are also done by the decision maker. The category between the two, limited tolerable has floating limits of what is safe enough. A system is not considered safe at a satisfactory level if a risk is defined into one of the IT categories. The same is true if a risk is defined into one of the LT categories in addition to that the decision maker is not willing to tolerate the risk level. In these cases further risk reducing action is needed. In the case of risks in the T-category no risk reducing actions need to be done.

The matrix system is a combination of both qualitative and quantitative handling. The quantitative values are categorized by qualitative evaluations. The use of the matrix makes it possible to compare different risks with each other. For example, a risk in the IT category is a higher risk than one in the T category. However, this quantitative feature does not compile for comparison of all system risks. It is not obvious which one of two risks in the same category is the greatest risk just from reading the matrix; for example, a risk classed in the probability category C and consequence category II compared to a risk classed in the categories D and III respectively. The qualitative categories that the matrix is created from do not endorse total quantitative information. Therefore, the difference is not precise when an upper-left risk is compared to a lower-right one.

In addition, the intervals between high and low categories are not equal spaced since those categories are decided by qualitative judgments. Figure 3.1.2 shows two possible cases of qualitative classifications. Thus, if only the risk matrix is provided, a consequence from this is an uncertainty of how much larger a risk with frequency category A is compared to a risk with frequency category B.

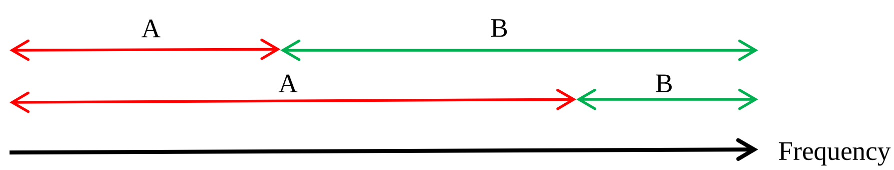


Figure 3.1.2 – Categories of frequency due to different qualitative classification

It is the same thing in consequence categories. For example, the injury and environmental damage categories under the consequence identification have the same problem as the identification of frequency. It is not possible to read a quantitative relationship between the categories. This problem indicates inconsistent scaling of the risk matrix. The current method for presenting the risk matrix has an inconsistent visual presentation. Qualitative definitions of the intervals in the axes of a matrix may be of different sizes. An uncontrolled variation of the matrix intervals will result in that each cell covers an uncontrolled large part of the defined space for a risk. This problem may be solved by keeping all cells to the same size. Equally sized cells can give the impression that all risk categories and all jumps between categories are equally large. However, the number of unnecessary cells may largely increase in order to get enough resolution. This offsets the purpose of quick and easy visualization in the first place.

With regard to the problem of system management, risk matrix does not provide sufficient information to handle the total system risk of a system. The system is defined safe as long as all risks meet the tolerable risk criterion. There is no consideration made on the total amount of risks and the possible interactions between risks. By applying this methodology it is possible to extend a system with an infinite number of tolerable risks and it is still considered safe. The chance for a system safety related accident to happen could in reality increase when adding risks. However, if all the new risks are categorized in the tolerable risk level, the new modified system is classed as tolerable by the present method. According to the equation $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, Prob(one event occurs) $\rightarrow 1$, when the system expands with an infinite number of risks. The probability that two or more risks concurrently happen is also much higher when the number of risks increase. The only exception to this increase is when all the risks at question are completely positively dependent. This conflict between the methodology and reality needs to be handled to get a better evaluation of the safety in a system.

Risk matrix has the advantages of easy construction and qualitative (judgmental) handling. However, it lacks sufficient resolution to compare risks in different cells, and the corresponding management policy does not consider the effects of possible correlations in severity or frequency.

3.1.2 Probabilistic Risk Index

In the method of risk matrix, risk is analyzed and classed by severity and possibility respectively. The risk index, e.g. (occasional, moderate), is expressed qualitatively in two dimensions. However, due to its low resolution this kind of two-dimensional index is not applicable in some cases. A decision maker, for instance, may prefer a quantitative risk index for a continuously distributed monetary loss. This sort of quantitative index is basically obtained from numerical operations of severity and possibility, e.g. the expected loss: severity multiplies probability and sum the products up. In financial field, there are two similar ways commonly used to express risk: one is value at risk (VaR), i.e., a threshold point and the probability that the loss exceeds that threshold; the other is conditional value at risk

(CVaR), i.e., the expected loss given that the loss already exceeds some value. These indices are designed for different purpose of usage and have their own advantages to highlight some characteristics of a risk. However, they are actually doing the same task, jointing severity and possibility as a numerical risk index.

Qualitative risk index usually refers to so-called probabilistic risk assessment. In general, a probabilistic risk index is in the form:

Risk Index = Impact of risk event $\otimes$ Probability of occurrence, or expressed in notations

$$R = S \otimes F \tag{3.1.1}$$

where $\otimes$ means a complex operation of multiplication.

As mentioned above, the primary advantage of probabilistic risk assessment is that it can provide a quantitative description of the degree of variability or uncertainty (or both) in risk estimates. The quantitative analysis can provide a more comprehensive characterization of risk than is possible in the point estimate approach (e.g. risk matrix). Another significant advantage is the additional information and potential flexibility it affords decision maker. Risk management decisions are often based on an evaluation of high-end risk. When using probabilistic risk assessment, the risk manager can select a specific upper-bound level from the high-end range of percentiles of risk.

However, probabilistic risk assessment may not be appropriate for every analysis. The primary disadvantages are that it generally requires more time, resources, and expertise on the part of the assessor, reviewer, and risk manager than a point estimate approach. Note that the probability of risk occurrence is sometimes difficult to estimate, since the past data on frequencies are not readily available. The main obstacle to using probabilistic risk assessment is usually the lack of well-documented frequency distributions of variables. Likewise, the impact of the risk is not easy to estimate since it is often difficult to estimate the potential loss of risk occurrence. As the saying goes, "garbage in, garbage out". Inaccurate or unreliable estimation in probability distributions would cause strong doubts and may result in severe consequences.

In addition, in the case of multiple risks, probabilistic risk assessment models do not account for unexpected common-cause failure modes. For example, when it comes to future safety, nuclear designers and operators often assume that they know what is likely to happen, which is what allows them to assert that they have planned for all possible contingencies. Yet there is one weakness of the probabilistic risk assessment method that has been demonstrated with the Fukushima I nuclear accident (Japan, 11 March 2011). This accident shows the difficulty of modeling common-cause or common-mode failures. From most reports it seems clear that a single event, the tsunami, resulted in a number of failures that set the stage for the accidents. These failures included the loss of offsite electrical power to the reactor complex, the loss of oil tanks and replacement fuel for diesel generators, the flooding of the electrical switchyard, and perhaps damage to the inlets that brought in cooling water from the ocean. As a result, even though there were multiple ways of removing heat from the core, all of them failed. The risk manager did not see beforehand the common-cause and correlations of possible risks; as a result, when the risks occurred together, the contingent plan designed for respective risk was totally in vain. The possibility that multiple risks happen concurrently in the complex system was under-estimated so that the disaster took place totally out of expectation. This example gives a lesson that the results of probabilistic risk assessment are not reliable if the common causes and probability correlations between risks are carefully considered and investigated.

Probabilistic risk assessment model also has a concern of application. The assessment is not complete if human judgments are not involved in the model. According to the analyses of Kahneman and Tversky’s prospect theory (see section 2.2), human judgments are indispensable factors for a realistic decision. On the other hand, expert opinions can provide useful information for forecasting, making decision, and assessing risks. Expert judgment can be considered as an informed assessment or estimate, bases on the expert's training and experience, about an uncertain quantity or quality of interest. Expert judgments are required in most steps of risk assessments: hazard identification, risk estimation, risk evaluation and analysis of options. In the probabilistic risk assessment, expert opinion is used in two ways:

1. To structure a problem. Experts determine which data and variables are relevant for analysis, which analytical methods are appropriate and which assumptions are valid.
2. To provide estimates. For example, experts may estimate failure or incidence rates, determine weighting for combining data sources, or characterize uncertainty.

Science-based professions are an important part of logical decision-making, where specialists apply their knowledge and judgments in a given area to make informed decisions. For example, medical decision-making often involves making a diagnosis and selecting an appropriate treatment. However, in situations with higher time pressure, higher stakes, or increased ambiguities, experts use intuitive decision-making rather than structured approaches, following a judgments-based decision approach to fit a set of indicators into the experts' experience and immediately arrive at a satisfactory course of action. It is particularly true if rare but high consequence events were found to dominate the overall risk, especially as these risk assessment is very sensitive to assumptions. This inherent limitation means that any model without involving human judgments is always subject to revision and to doubt as to its completeness.

In summary, even when statistical estimates are available, in many cases risk is associated with rare failures of some kind, and data may be sparse. Often, the probability of a negative event is estimated by using the frequency of past similar events or by event tree methods, but probabilities for rare failures may be difficult to estimate. This makes risk assessment difficult in hazardous industries, for example nuclear energy, where the frequency of failures is rare and harmful consequences of failure are numerous and severe. In addition, prospect theory states that a realistic decision is affected by mental effects. Thus, probabilistic risk assessment studies require important ameliorative tools like human judgment analysis and probability correlation analysis. Human judgments analysis deals with methods for modeling human mental effects while correlation analysis deals with methods for evaluating the effect of inter-risks dependencies which tend to cause simultaneous failures and thus significant increases in overall risk. The latter task can be done by applying copula functions (see section 5.1) to estimate correlations and to find joint distributions. The demonstration of copula method in risk assessment model and the diagnosis are documented in chapter 5.

3.2 Distortion in Human Judgments

As discussed in section 3.1, quantitative risk assessments may give a green light to "safe" or "acceptable" or "insignificant" risks that are actually potential for severe losses. The Fukushima nuclear accident tells that quantitative risk assessment models may disregard some matter factors, which are not explicit or quantifiable. To solve the, the judgments of experience experts can play an important. In literature, human judgments in risk assessment have attracted great interests in many disciplines. Judgments can be seen in critical steps of decision-making such as risk identification, perception, interpretation, evaluation, and optimization. A risk assessment model is basically a model of uncertainties related to an indeterminate system. Thus, how to address these uncertainties qualitatively and quantitatively affects the completeness and the credibility of the risk assessment. The judgmental heuristics reviewed in section 2.4 give a good analysis. The rule of thumb is availability heuristic. Decision makers assess the probability of an event by the ease with which instances or occurrences can be brought to mind. For example, project managers sometimes estimate the chance of risk occurrence based on similar tasks that have been previously completed. If they are making their judgment based on risks they remember, it can cause inaccurate estimation. The anchoring heuristic refers to the human tendency to remain close to the initial estimate. For example, during brainstorming meeting engineers estimated the chance of the risks equal 10%. During a discussion they said that actual chance will be between 8 and 12 percent. So they always remain close to the original estimate. Judgments concerning the probability of a scenario are influenced by amount and nature of details in the scenario in a way that is unrelated to the actual likelihood of the scenario. It is called the representativeness heuristic. For example the project has two potential risks. One of them is very well documented, but another one has a very limited descriptions. Decision maker sometimes may assume that chance of occurrence of the first one will be higher than second, which in reality may not be the case. Decision makers can be exposed to many cognitive and motivational factors that can lead to biases in perceptions. This effect is often referred to as selective perception. For example, estimation of a task's cost can be influenced by the intention to fit the task into the project's budget. As a result, some of the project parameters can be overestimated. Another type of biases is related to management push for a better project performance. Such management biases may cause underestimation of certain risks.

Therefore, the judgment should be obtained through an elicitation process that seeks to minimize biases caused by judgmental heuristics such as availability, anchoring and adjustment, representation, etc. (see section 2.4) and to help the expert construct the subjective probability distribution. The use of expert judgment is subject to a developing set of rules which include:

1. Experts are capable of expressing useful opinions as probability distributions. Usually these are what we would call uncertainty rather than variability, although sometimes both are confused.
2. Effort must be made to reduce or account for the biases of experts. Overconfidence seems to be the most important of these for risk analysis.

3. Expert opinions are almost surely dependent, in contrast to the mistaken notion of identical, independently distributed observations. It is arguable that experts should be dependent to some extent.
4. It should be noticed we want experts to look at the problem from very different point of views.

However, there are still criticisms on the effects of human judgments. For example, there are some objections among some risk analysts associated with using the subjective probability, provided by expert, as a measure of uncertainty in risk analysis. They believe that these judgments are superficial and imprecision. Experts' ability to express their uncertainty in terms of probabilities and whether such statements constitute a sufficiently credible basis in the elicitation and decision-making contexts are discussed extensively in the literature. A large portion of this work is basis on so-called heuristics and biases. Risk analysts may practically express these biases as mechanisms that: first, lead to inconsistency between the expert's system knowledge and his/her assessment of uncertainty, or introduce a disparity between the perceived uncertainty and the probability figure which is eventually obtained from expert.

As for the cases of multiple risks, an important application of human judgments is the preference structure, which is usually presented by a score system or priority order. The use of priority scoring and rating systems is widespread and is becoming even more prevalent. Thus, it is necessary to examine some intrinsic limitations in the performance of priority setting rules and scoring systems, evaluated as guides to rational action. It is of great practical importance to understand limitations of risk-scoring methods and develop improved approaches to risk management. Two commonly recommended principles for allocating risk management resources to remediate uncertain hazards are: (1) select a subset to maximize risk-reduction benefits (e.g., maximize the expected utility of the selected risk-reducing activities), and (2) assign priority weights to risk-reducing opportunities and then select activities from the top of the priority list down until no more can be afforded (Cox, 2009). When different activities create uncertain but correlated risk reductions, as is often the case in practice, then these principles are inconsistent: priority scoring and ranking fails to maximize risk-reduction benefits. In some cases, risk-scoring methods are not appropriate for probability-correlated risks. More constructively, when risk-reducing opportunities have correlated consequences, due to uncertainties about common elements, then methods for optimizing selection of a portfolio of risk-reducing opportunities (i.e. maximal expected utility) can achieve significantly greater risk reductions for resources spent than can priority-scoring rules. Real-world risk priority scoring systems used in homeland security and terrorism risk assessment, environmental risk management, information system vulnerability rating, business risk matrices, and many other important applications do not exploit correlations among risk-reducing opportunities or optimally diversify risk-reducing investments. As a result, they generally make suboptimal risk management recommendations. Applying portfolio optimization methods instead of risk prioritization ranking, rating, or scoring methods can achieve greater risk reduction value for resources spent.

Here is an example to illustrate why risk-scoring method may not be appropriate. Any risk-averse decision maker prefers a single random draw from a normal distribution with mean 1 and variance 1, denoted N(1, 1), to a single draw from normal distribution, N(1, 2), having mean 1 but variance 2. Therefore, a scoring rule would assign a higher priority to draws from N(1, 1) than to draws from N(1, 2). But suppose that X and Y are two N(1, 2) random variables that are perfectly negatively correlated with $Y = 2 - X$. (This might happen, for example, if effects depend only on the sum of X and Y, which has a known value of 2, but the relative contributions of X and Y to their sum are uncertain.) Then, drawing once from X and once from Y (each of which is N(1, 2)) would yield a sure gain of 2. Any risk-averse decision maker prefers this sure gain to two draws from N(1, 1). Unfortunately, any priority rule that ignores correlations among opportunities would miss this possibility of constructing a risk-free gain by putting X and Y in the same portfolio, as it would always assign draws from N(1, 1) a higher priority than draws from N(1, 2). This example shows that priority-setting rules can recommend dominated portfolios, such as allocating all resources to risk reductions drawn from N(1, 1) instead of pairing negatively correlated N(1, 2) risk reductions, because they cannot describe optimal portfolios that depend on correlations among risk-reducing opportunities, rather than on the attributes of the individual opportunities. In general, the best choice of a subset of risk-reducing activities (in quantitative benefits) is not obtained by priority scores. Instead, optimization techniques that consider interdependencies among the consequences of different risk-reducing activities are essential.

3.3 Robust Selection

Probability distributions affected by parameter uncertainty have long been a focus of the academic community. Solutions to probabilistic optimization problems can exhibit remarkable sensitivity to perturbations in the parameters of the problem, thus often rendering a computed solution highly infeasible. Risk problems are not exceptional. For example, applying expected value and expected utility for decision-making requires knowing the probability of various outcomes; however, this is usually unknown in practice. Statistical regression models are commonly used to determine these unknown parameters but they are selected under uncertainly. Besides, the magnitude of impacts and the parameters of probability distributions are often estimated from dated, incomplete or even analogous database. It is also difficult to test the credibility of these parameters since the concerned risks are usually infrequent but severe. Thus one must make assumptions, but then the expected value of various decisions is very sensitive to the assumptions. This is particularly a problem when the expectation is dominated by rare extreme events, as in a long-tailed distribution. The black swan theory concludes that there may be no relevant guide in the past data in cases where the event of interest is very different from existing experience. The frequency and impact of totally unexpected events is generally underestimated. With hindsight, they can be explained, but there is no prospect of predicting them.

In section 3.2, we have discussed that human judgments can provide intellectual information to quantitative risk assessment model. The experts' valuable experience and judgments can help adjust probability distributions, reflect perceptional reality, and set up priorities. These judgmental behaviors could be beneficial to prevent extreme events occurring due to biased estimation. However, these human judgments, affected by mental heuristics, are very subjective and arbitrary (see section 2.4 and 3.2). The experts may make biased judgments due to environmental factors and personal risk-adverse or risk-seeking attitude. Thus, the judgmental actions should have a guiding range to increase stability and prevent absurd decisions.

One important goal of risk management is to prevent unexpected consequences happening. In other words, a successful risk assessment model should lead to a robust decision. A robust decision is the best possible choice, one found by eliminating all the conceivable uncertainty within available resources, and then choosing, with known and acceptable levels of satisfaction and risk. We have known both probabilistic assessment and human judgments are potentially biased and not robust. Two principles exist to promote a robust decision: the principle of optimality and the principle of avoiding catastrophes.

- Principle of optimality – risks should be reduced to an optimal level subject to fixed amount of available resources. In real life, optimization is necessary since the amount of resource is limited.
- Principle of avoiding catastrophes – risks with catastrophically consequences should have higher priority, irrespective to their probability estimates. One may be able to tolerate occurrence of many minor risks but cannot afford the loss of a severe one.

The first principle is based on the operations of statistical measures. The second principle is handled by human judgments. Both principles are to be seen as a guiding framework for decision-making, because the principles cannot be met to one hundred percent in practice. Furthermore, it is also possible the principles are actually in conflict with each other. To be more specific, for single risk only the first principle needs to be considered; the conflict happens in the process aggregating multiple risks to a single risk index. Setting priority to risks means the losses are not equal weighted and therefore may lead to a sub-optimal solution.

In conclusion, a robust decision is based on three tasks: accurate estimates of the probability distributions, human judgments in guided range, and balance of principle of optimality and principle of avoiding catastrophes. Both statistical and judgmental information are essential to accomplish all three tasks. A robust decision is made by the mechanism that statistical model and human judgments can inform each other.

3.4 Partial Solution – Copula and Choquet Fuzzy Integral

By investigating the criticisms of previous sections, the current approaches of risk assessments have two major shortcomings: failing to involve human judgment and improvement-needed risk aggregation techniques. The importance of human judgment and its functions have been discussed in previous chapters. This section will focus on the problems of aggregation techniques.

As defined in section 3.1, risk has two components: severity and possibility. Conventional approaches of risk assessment are based on assumption of independence in both dimensions, or at least in one dimension. For example, the famous von Neumann-Morgenstern utility theorem suggests decisions based on total expected utility, where both severity and probability are assumed independent. However, in general, risks are not independent in either severity or possibility. First, judgmental heuristics suggest that the magnitude of aggregated risks is perceived by personal mental heuristics and references. As a result, the severity is not additive and should not be treated independent. Second, risks may have common causes or inexplicit correlations in possibility (i.e. probability in probabilistic assessment). The assumption of independence in possibility must be carefully tested.

A comprehensive method which can handle correlations in both severity and possibility is still undeveloped. However, two aggregation methods can partially solve the problem: copula and Choquet fuzzy integral. The probability dependence can be handled by copula. In probability theory and statistics, a copula can be used to describe the dependence between random variables. The cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula. Among all sorts of copula family, Archimedean copula is less sensitive to the scale or location-based correlation, since they are based on Kendall Tau or rank correlation (see section 4.2.2). On the other hand, Choquet fuzzy integral deals with dependence in severity. In decision theory, Choquet fuzzy integral is a way of measuring the expected utility of an uncertain event. The preference (dependence) structure of risks is obtained by scoring system and a fuzzy measure in

Choquet fuzzy integral model. Choquet fuzzy integral is less sensitive to the score variation than weighted utility function, since the aggregation is based on rank or ordered preference.

Choquet fuzzy integral and copula both provide some robustness properties and solve partial correlation problem; thus they are our candidates for aggregation methods. Their properties and utilizations are described in chapter 5. The improved mechanism, which integrates information of both methods to provide improved decision model is illustrated in chapter 6.

Chapter 4

Construction of Risk Assessment Model

In this research, it is assumed the case of multiple risks. Thus, there must be a method to aggregate the measures of individual risks, in order to provide the broad view of any considered action. In section 4.2, Choquet fuzzy integral and copula are reviewed to provide two ways of aggregation methods. Using the Choquet fuzzy integral to denote the expected utility of belief functions is a sequential way, from high to low risk priority, to reconcile human heuristic biases and prioritization conflicts. On the contrary to Choquet fuzzy integral, copula, as an aggregation method of formulating a multivariate distribution, is in such a way that various interdependence of the risks can be evaluated. By assuming that each transformed marginal risk has a uniform distribution, the interdependence of the all risks can be expressed as a multivariate distribution on the obtained uniforms. As such, a copula, a joint distribution on marginally uniform random variables, provides a different view to aggregate risks for decision making. A good decision about the trade-off among risks should be agreeable with human preference or satisfy human perspective of risk priority; however, human heuristics could be biased, based on wrong information, or loaded with conflicting anecdotes. Copula is extensively seen in financial area, but not used in human-based risk assessment problems. The application and its usage is an opportunity for the research.

One objective of this research is to integrate and use the advantages of Choquet fuzzy integral method and copula method. Thus, in section 4.3, three types and their properties of risk measures are reviewed to help understand and determine the advantages of both methods.

4.1 Competing Risks Model

Mark Twain once said: “History doesn't repeat itself, but it does rhyme”. Some particular risk problems are an analogous situation. For example, one man could only die from one cause; however there actually coexist several possible threats of death and they all act as potential risks. Before the event of death occurs, the actual cause could be anyone of the potential risks. Hence, these risks are said to “compete” to each other for the survival time of the man and therefore are called “competing risks”. Thus, a model of survival time in the presence of such competing risks is known as a competing risks model. In general, competing risks models satisfy the following assumptions:

1. each failure mechanism leads to a particular type of risk;
2. the event of failure is counted when the first of all the failure mechanism occurs;
3. each failure mechanism has a probability distribution model of its life time.

In one word, a competing risks model consists of several failure mechanisms, each of which follows a probability distribution and compete to each other to happen. To analyze and characterize a competing risks model for the purpose of decision-making, three areas of interests in the analysis are discussed in following sections:

1. the embedded probability distributions of a risk (section 4.1.1);
2. the rate that failure will occur in the next moment (section 4.1.2);
3. the conditional expected survival time (section 4.1.3).

4.1.1 Survival Function

The survival function is a function of time which presents the probability that an agent will survive beyond a specified time point. The term *reliability function* is commonly used in engineering while the term *survival function* is used in a broader range of applications. In general, the survival function describes the property of a set of events (usually associated with casualty or failure) onto time. See the following for the mathematical definition:

The survival function, denoted as *S*, is defined as

$$S(t) = \Pr(T > t),\ t \geq 0 \tag{4.1.1}$$

where T is a random variable denoting the time of casualty or failure, t is some specified time (age), and Pr stands for probability. By the definition, the survival function is the probability that the occurring time of an event is latter than some specified time. Usually, unless immediate death and eternal life are allowed, it can be assumed that $S(0) = 1$ and $S(t) = 0$ as $t \to \infty$.

To obtain the survival function, one approach is to consider the complementary distribution:

$$F(t) = 1 - S(t) = \Pr(T < t) \tag{4.1.2}$$

$F(t)$ is a cumulative distribution function (c.d.f.) which yields the probability that the agent has an event before age *t*. Furthermore, $f(t)$, the p.d.f. of $F(t)$, is the probability density that the system will have an event at exact time *t*. It can be expressed as:

$$f(t) = \lim_{\Delta t \to 0} \frac{\Pr(t \le T \le t + \Delta t)}{\Delta t} \tag{4.1.3}$$

where *T* denotes the event happening time.

Since $F(t)$ is defined as an integral of a non-negative function, i.e. $f(t)$, it is implied that survival function $S(t)$ must be non-increasing with time *t*, i.e. $S(t_2) \le S(t_1)$ if $t_2 > t_1$. This property can also be interpreted by the fact that survival at later time is only possible if surviving at earlier time.

4.1.2 Hazard Function

The hazard function, also known as *the failure rate* or *hazard rate*, is a function of time *t*. One can interpret the hazard function as a rate of failure of an agent right after the agent reaches age *t*. In mathematical expression, hazard function, denoted as $h(t)$, is the ratio of the probability density function $f(t)$ to the survival function $S(t)$, given by

$$\begin{aligned}
h(t) &= \lim_{\Delta t \to 0} \frac{\Pr\left[t < T < t + \Delta t \mid T > t\right]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\Pr\left[t < T < t + \Delta t\right]}{\Pr(T > t) \cdot \Delta t} \\
&= \lim_{\Delta t \to 0} \frac{S(t) - S(t + \Delta t)}{S(t) \cdot \Delta t} \\
&= \frac{1}{S(t)} \lim_{\Delta t \to 0} \frac{S(t) - S(t + \Delta t)}{\Delta t} \\
&= \frac{1}{S(t)} \left(-S^{'}(t)\right) = \frac{f(t)}{S(t)}
\end{aligned} \tag{4.1.4}$$

Furthermore, the hazard function, survival function and density function can be transformed from each other (see Figure 4.1.1).

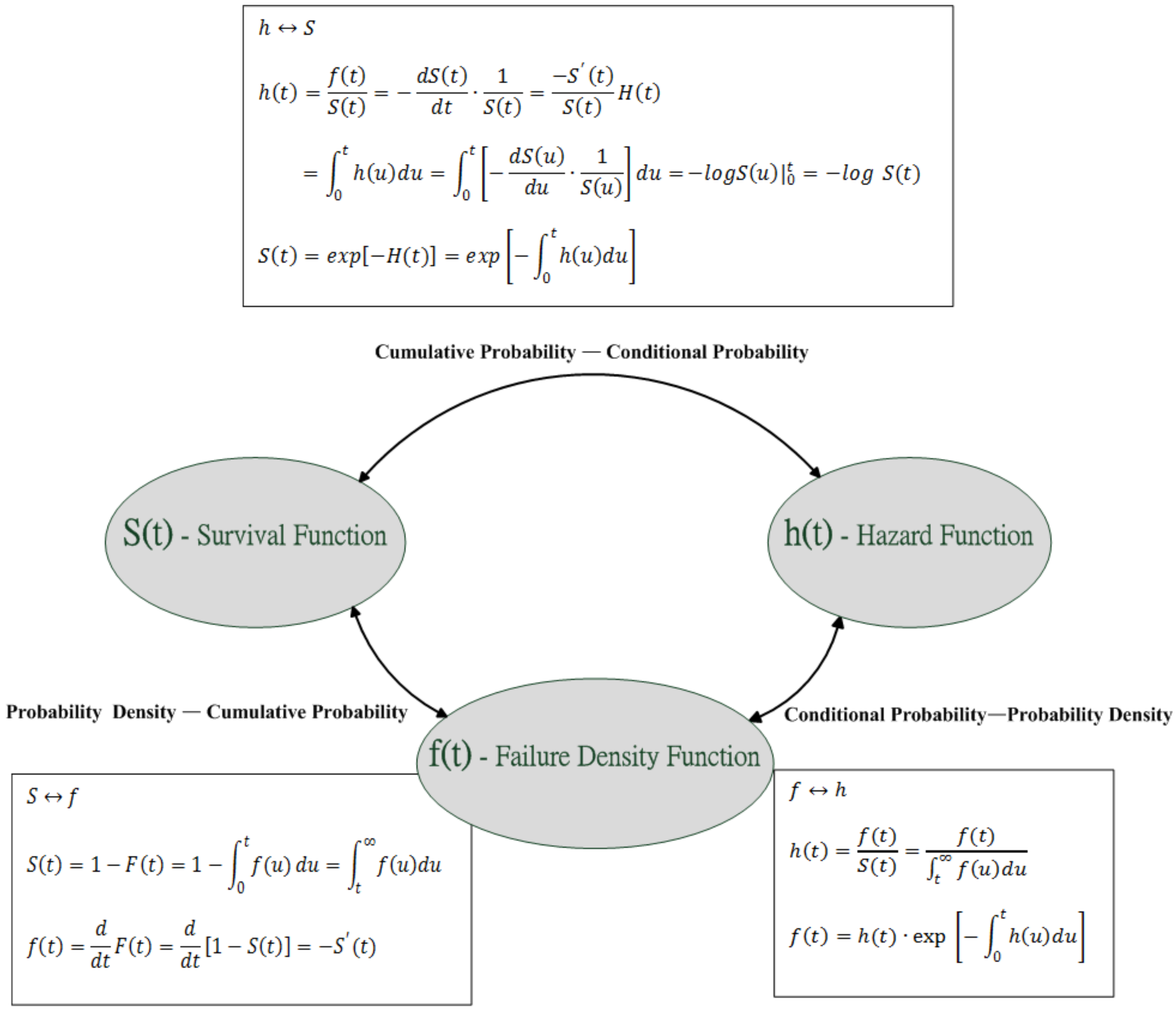


Figure 4.1.1 – Transformations between Survival, Density and Hazard Function

4.1.3 VaR and CVaR

In the last decade, many new risk measures have been introduced; however, risk, a measure of the probability and severity of adverse effects, remains a notion that many find hard to follow, and its quantification has challenged and confused many disciplines. Why? One is that risk is a complex composition of two parts – one is the psychophysical judgment part (the severity of potential damages, or unfavorable hazardous effects), the other is a way of expressing knowledge or belief that an event will occur or has occurred.

To provide an operator among these two imagined manmade constructs, a risk functional measure is critical as a map which assigns a real risk value to a probability measure. Two risk functional measures will be introduced in this study: the value at risk, and conditional value at risk. The purpose of a risk functional measure is to provide a critical link between the probability functions and the management of risk trade-off.

Value at risk (VaR) is a statistical measure of the risk that estimates the maximum loss that may be experienced on an option of action with a given level of confidence. VaR comes with a probability that says how likely it is that losses will be smaller than the amount given if bigger value is preferred. Since it is described by a confidence level with a given value, it is difficult to compare VaR's. Thus, the concept of an expected value was proposed. Conditional value at risk (CVaR) is the expected loss given the condition that the loss is beyond VaR.

VaR

Value at Risk (VaR) conventionally is used most often in financial industries to capture the potential losses in value of their traded portfolios from market movements. VaR is a measure of the risk exposure in losses on a specific portfolio of financial assets. VaR is typically calculated with 95% confidence, which means that there is (on average) a 95% chance of the loss on the portfolio being lower than the VaR calculated. There are three key elements of VaR – a specified loss in value, a fixed time period over which risk is assessed and a confidence level. Given a risk, e.g., a portfolio of assets, and its probability distribution, VaR is defined as the threshold (or a cut) that the probability of having a loss exceeding this value is equal to a pre-determined level of confidence. The mathematic expression is as following:

Given a risk L and some level $\alpha \in (0,1)$, then

$$VaR = inf\left\{l \in \Re : P\left(L > l\right) \le 1-\alpha\right\} = inf\left\{l \in \Re : F_L\left(l\right) \ge \alpha\right\} \tag{4.1.5}$$

where F_L is the distribution function of risk L.

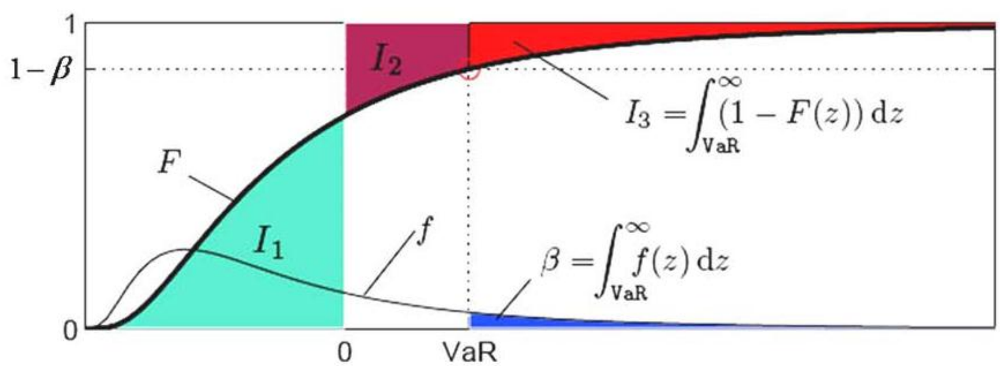
One notable shortage of VaR is that its output is not a standard derivation or an overall risk measure but in terms of an absolute vale and the probability that the losses will exceed that vale. This shortage makes VaR an inappropriate measure of risk when one intends to compare investments with very different scales and returns; in other words, VaR measures look at only a small slice of the risk that an asset is exposed to but a great deal of valuable information in the distribution is ignored.

CVaR:

Conditional Value-at-Risk (CVaR) is proposed as an alternative risk measure to overcome the shortages of VaR. CVaR is defined as the expected loss given that the loss is greater or equal to VaR, or it is also equal to the expected losses that are strictly greater than VaR. Rockafellar and Uryasev (2002) shows CVaR is a coherent risk measure and furthermore CVaR can be minimized using linear programming techniques for portfolio-optimization problems. The mathematical definition is as following:

$$CVaR_q = E\left[X|A\right] \tag{4.1.6}$$

where X is a continuous random variable and A denotes the sets $X \le q$ or $X \ge q$ for negative and positive tails respectively. Calculations and interchanging equations of CVaR are shown in figure 4.1.2.



$$VaR_{1-\beta} = F^{-1}(1-\beta) \qquad CVaR = VaR + \frac{1}{\beta} I_3$$

$$\beta\left(CVaR - VaR\right) = I_3 \Rightarrow if \begin{cases} VaR = 0 \\ \beta = 1 \end{cases} \Rightarrow \quad CVaR = I_3$$

$$I_1 = \int_{-\infty}^{E(t)} F(t)dt \qquad I_2 = \int_{E(t)}^{VaR} (1 - F(t))\, dt = \int_{E(t)}^{VaR} R(t)dt$$

Figure 4.1.2 – Calculations and Formulas of CVaR

4.2 Aggregation of Multi-risks

Broadly speaking, risk aggregation refers to efforts to develop quantitative risk measures that incorporate multiple types or sources of risk. This is typically accomplished via mathematical or statistical techniques designed to assess the probability distributions of potential adverse outcomes. The most common, however basic, approach is to estimate the potential losses, e.g. expected loss, associated with each of the considered risk and then add them up. This approach ignores two important concerns, which are also the challenges of proper aggregation methods. The first, in most cases, not all the risks are equally prioritized. It is assumed that human may have preferences for the risks by following a series of mental activities and environmental references. Thus, the priority sequence from human judgments should be taken into consideration when aggregating risks. The second concern is that the probability distributions of risks may not be mutually independent. Thus, it is inappropriate to aggregate the measurement of individual risk without knowing the correlation structure (joint distribution). For example, it is meaningless to tell the total expected loss by just adding up the CVaR's of two risks since the expected value could vary with different correlation.

In the literature, two kinds of aggregation functions can take care of aforementioned challenges. However, both of them only partially solve one concern. The method of Choquet fuzzy integral can be used to perform human preferences in some fuzzy measure to do the aggregation; however it does not deal with distribution correlations. On the other hand, the method of copula is a way to aggregate risks through the joint distribution, but there is no human judgment involved. Though both methods have weaknesses, they can still provide the baseline knowledge which the possible improvement must rely on.

4.2.1 Choquet Fuzzy Integral

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set and let $P(X)$ denote the power set of X or set of all subsets of X. A fuzzy measure over a set X is a function (Sugeno, 1977).

$g : P(X) \rightarrow [0,1]$ such that

1. $g(\phi) = 0, g(X) = 1$;
2. If $A, B \subset P(X)$ and $A \subset B$, then $g(A) < g(B)$.

λ fuzzy measure satisfies the following property: for all $A, B \subset X$ with $A \bigcap B = \phi$,

$$g(A \bigcup B) = g(A) + g(B) + \lambda g(A) g(B) \tag{4.2.1}$$

for some fixed $\lambda > -1$ (Sugeno, 1977).

The value of λ can be calculated from $g(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i) \tag{4.2.2}$$

Supposing $A_i = \{x_i, x_{i+1}, \cdots, x_n\}$, when g is a λ fuzzy measure, the value of $g(A_i)$ can be calculated recursively from

$$g(A_n) = g(\{x_n\}) = g_n \tag{4.2.3}$$

$$g(A_i) = g_i + g(A_{i+1}) + \lambda g_i g(A_{i+1}) \text{, for } 1 \le i < n \text{ (Sugeno, 1977)} \tag{4.2.4}$$

When a fuzzy measure is available on a finite set X, one can express the fuzzy integral as a computational scheme to integrate all values from the individual subset nonlinearly (Chiang, 1999).

Choquet fuzzy integral is a fuzzy integral based on any fuzzy measure that provides alternative computational scheme for aggregating information. Assume $h(x_1), h(x_2), \cdots, h(x_n)$ are the evidence provided by the input sources $x_1, x_2, \cdots, x_n$ respectively, and g is a λ fuzzy measure, then a Choquet fuzzy integral has the form

$$\int_x h(\cdot) \otimes g(\cdot) \tag{4.2.5}$$

For a finite set of X, the Choquet fuzzy integral can be computed by:

$$E_g(h) = \sum_{i=1}^{n} [h(x_i) - h(x_{i-1})] g(A_i) \tag{4.2.6}$$

where $h(x_1) \le h(x_2) \le \cdots \le h(x_n)$ and $h(x_0) = 0$ (Chiang, 1999). Another computation formula for the finite set case can also be expressed as

$$E_g(h) = \sum_{i=1}^{n} h(x_i) [g(A_i) - g(A_{i+1})] \tag{4.2.7}$$

where we take $g(A_{n+1})$ to be zero (Chiang, 1999). The calculation of the Choquet fuzzy integral with respect to fuzzy measure requires the knowledge of the fuzzy density and the input value. Figure 4.2.1 shows an illustration of two formula of Choquet fuzzy integral (vertical and horizontal).

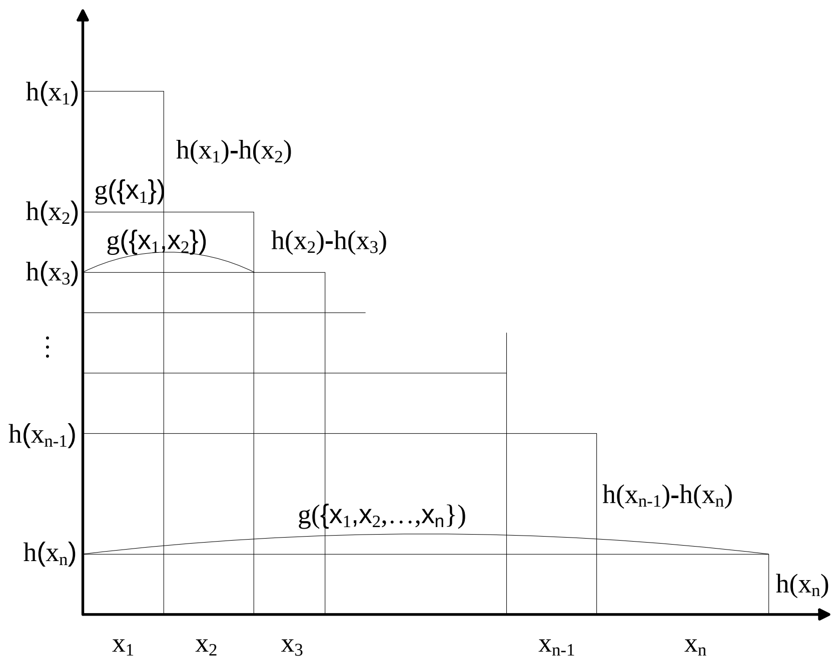


Figure 4.2.1 – An illustration of Choquet fuzzy integral

Since the assumption of equal priority is not realistic in many cases of multiple risks, Choquet fuzzy integral with respect to a fuzzy measure is useful to include human preferences. Here, take an example. Let $X = \{x_1, x_2, \cdots, x_n\}$ denote a set of evaluating risks which are concerned by the decision maker. Then $g(A_i)$, where $A_i = \{x_i, x_{i+1}, \cdots, x_n\}$, is a λ fuzzy measure of the weight of A_i in the decision makers' mind (a.k.a. preferences).

Assume $h(x_1), h(x_2), \cdots, h(x_n)$ are the score measures of input risks $x_1, x_2, \cdots, x_n$ respectively. The score measure can be some kind of risks measure, e.g. CVaR. Then, the total score of X can be obtained from the Choquet fuzzy integral

$$E_g(h) = \sum_{i=1}^{n} h(x_i)[g(A_i) - g(A_{i+1})] \tag{4.2.8}$$

One can compare the scores of $X's$ to find the best strategy.

4.2.2 Copula

Conventionally, researchers relied on the assumption of independent competing risks. However, they realize the shortcomings of this assumption and recognize the need to analyze the dependence structure. Unlike correlation, copula does not depend on whether the distribution of variables is elliptical or not. Hence copula has been a tool some researchers have used to understand relationships in competing causes were variables are not always elliptically distributed. Copula can simultaneously estimate the dependence parameter and the parametric marginal survival distributions. The technique allows separate statistical models to be specified for the marginal model of each type of death.

Copula is a function to formulate a joint multivariate distribution with input of the marginal distributions in various levels of dependence. The idea of copula based on Sklar's theorem (Sklar, 1959):

Let $F \in \mathcal{F}(F_1, F_2, \cdots F_n)$ be an n-dimensional distribution function with marginal distributions $F_1, F_2, \cdots F_n$. Then there exists a copula C such that

$$F(x_1, x_2, \cdots x_n) = C(F_1(x_1), F_2(x_2), \cdots F_n(x_n)) \tag{4.2.9}$$

In the theorem, copula is a n-dimensional distribution function on $[0,1]^n$ with uniform marginal distributions. Let X_i denotes a random variable with continuous distribution function F_i. Considering the distribution transformation $U_i := F_i(X_i), \quad i = 1, 2, \cdots n$, we can obtain $U_i \sim U(0,1)$ and $X_i = F^{-1}(U_i), \quad i = 1, 2, \cdots n$. Thus, the copula C is to define the multivariate distribution of $U = (U_1, U_2, \cdots, U_n)$ and the joint distribution is obtained from:

$$F(x) = P(X \le x) = P\left(F_i^{-1}(U_i) \le x_i\right) = P\left(U_i \le F_i(x_i),\right) = C(F_1(x_1), F_2(x_2), \cdots F_n(x_n))$$

where $X = [X_1, X_2, \cdots, X_n], \quad i = 1, 2, \cdots n$ (4.2.10)

Sklar's theorem shows that in assessing multivariate distributions, one can separate the dependence structure from the marginal distributions. Conversely, one can construct a multivariate joint distribution from two parts (1) a set of marginal distributions, and (2) a selected copula. This is very useful in evaluating operational risk since the dependence structure is captured in the copula function and is independent of the form of the marginal distributions. Typically in practice, distributions of losses of various types are identified and modeled separately using competing risk models. There is often very little understanding of possible interdependencies among the competing risks, although it is commonly acknowledged the linkages exist. As such, Sklar's theorem and the derived copulas allow one to experiment with different copulas while retaining identical marginal distributions.

In addition, it can also be shown that there is an unique copula associated to the joint distribution, F , which is given as

$$C(u_1, u_2, \cdots u_n) = F(F_1^{-1}(x_1), F_2^{-1}(x_2), \cdots, F_n^{-1}(x_n)) \tag{4.2.11}$$

Also, the density of the joint distribution can be obtained from

$$f(x_1, x_2, \cdots x_n) = c(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i) \tag{4.2.12}$$

where f_i denotes the marginal density functions and c is the density function of the copula. Thus, conversely, c can be obtained from

$$c(u_1, u_2, \cdots u_n) = \frac{f(F_1^{-1}(u_1), F_2^{-1}(u_2), \cdots, F_n^{-1}(u_n))}{\prod_{i=1}^{n} f_i(F_i^{-1}(u_i))} \tag{4.2.13}$$

There are many families of copulas in the literature. They differ in the methods of formulation and in the dependence structure they represent. Some families of copulas are listed below.

1. Gaussian Copula:

One common-used and basic example is the Gaussian copula, which is formulated from the multivariate normal distribution with correlation matrix, R and is given by

$$C_R^{Ga}(u_1, u_2, \cdots u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{\sqrt{(2\pi)^n |R|}} exp\left\{\frac{-u' R^{-1} u}{2}\right\} du \qquad (4.2.14)$$

where $u = (u_1, u_2, \cdots u_n)$ and Φ^{-1} is the inverse of the cumulative distribution function of the univariate standard normal distribution.

2. Archimedean Copulas:

Archimedean copulas are a particular family which is capable of capturing a various range of dependence and thus are useful in empirical modeling. The advantage of Archimedean copulas is that they are able to model dependence in arbitrarily high dimensions with only one parameter. Archimedean is formed by so-called "generator function". The generator function, denoted as φ, could be any function satisfying the properties: $\varphi : [0,1] \to [0, \infty]$ with continuous derivatives on (0,1) and $\varphi(1) = 0$, $\varphi'(t) < 0$, $\varphi''(t) > 0$ for all $0 < t < 1$.

Bivariate Archimedean copulas without a singular component, i.e. $\lim_{t \to 0+} \varphi(t) = \infty$ and $\varphi(t)$ is strictly decreasing, take the form:

$$C(u_1, u_2; \theta) = \varphi^{-1}\left(\varphi(u_1; \theta) + \varphi(u_2; \theta)\right) \qquad (4.2.15)$$

where θ is the dependence parameter imbedded in the functional form of the generator.

In bivariate Archimedean copulas, the dependence parameter θ between tow random variables is estimated straightforward by using *Kendall's tau rank correlation*, which is defined as

$$\tau = \frac{(number\, of\ concordant\ pairs) - (number\, of\ disconcordant\ pairs)}{\frac{1}{2}n(n-1)} \qquad (4.2.16)$$

Acting as estimation of the dependence parameter, Kandall's tau directly determine a copula and it is also unique to that copula (Genest and Mackay, 1986). The copula function and Kandall's tau function satisfy the rule of one-to-one mapping.

In this research, we will choose Archimedean copulas to obtain numerical joint distributions since we can choose different generator functions to reach wide ranges of dependence properties. Theoretically, infinite number of Archimedean copulas could be developed; however, table 4.2.1 lists four most popular and well-developed copulas in the literature.

Table 4.2.1 – Formulations of popular Copulas

	$C(u_1, u_2, \theta)$	$\varphi(t)$	Range of θ
Clayton	$(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$	$\theta^{-1}(t^{-\theta} - 1)$ (strict)	$(0, \infty)$
Frank	$-\frac{1}{\theta}\log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right)$	$-log\left[\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right]$	$(-\infty, \infty)$
Gumbel	$\exp\left\{-\left[(-logu_1)^{\theta} + (-logu_2)^{\theta}\right]^{1/\theta}\right\}$	$(-logt)^{\theta}$	$[1, \infty]$

Source: Trivedi (2005)

4.3 Classification of Risk Measure

How to measure risks is always the first and critical task for risk management. Risk itself actually is a subjective concept because different decision makers could take different viewpoints to realize their objectives of risk management. Generally speaking, a risk measure is to transform a potential loss to a numerical value that can summarize the characteristics of the analyzed risk for easier comparisons and classification. This technique is broadly used in practice of industries of insurance, banking, finance and government etc. For example, insurance companies apply risk measures as an index of the level of potential losses when calculating insurance premiums. However, there are still shortcomings of usage of risk measures. The most obvious, one may easily question that detailed information would be lost if only using one number to outline the whole losses. It is true but in many cases we do not have alternative choices, especially when we need a standard index to make regulations (Heyde et al. 2006). Or, for the purpose of management, a simple and effective method is much easier for decision makers to handle to achieve their objectives. Therefore, in the literature, it attracts a lot of attention of researchers on how to design a good risk measure which contains essential information and has good properties for analysis as well.

In general, a risk measure has the following mathematic expression. Let Ω be the set of all possible states and X be the set of all the random losses defined on Ω. Then a risk measure is a mapping from X to $\mathbb{R}$. The risk measure associated with a random variable X is commonly denoted as $\rho[X]$. A risk measure is assumed to have certain properties listed below:

property 1 – Normalized: $\rho[0]=0$ is finite;

property 2 – Translative: If $a \in \mathbb{R}$ and $X \in \Omega$, then $\rho(X+a)=\rho(X)+a$;

property 3 – Monotone: If $X_1, X_2 \in \Omega$ and $X_1 \le X_2$, then $\rho(X_1) \ge \rho(X_2)$.

Once having a risk measure in hand, one can determine an acceptable set by choosing a threshold. For example, if ρ is a risk measure, then $A_\rho = \{X \in \Omega : \rho(X) \le 0\}$ is an acceptable set.

In addition to aforementioned properties, a risk measure may have other useful properties. These specific properties determine the usages and strengths of a risk measure. In this research, one objective is to integrate information of Choquet-integral and copula. One approach to achieve this goal is through the investigatioin of the properties of these two risk measures. Three kinds of typical risk measures and their key properties are reviewed in the following sections.

4.3.1 Coherent Risk Measure

In the paper of Artzner et al. (1999), they outline the desirable properties that an ideal coherent measure should have. According to them, a risk measure $\rho : X \to \mathbb{R}$ is coherent if it satisfies the following axioms:

A1 – Subadditivity: $\rho(x+y) \le \rho(x)+\rho(y)$ for all $x, y \in X$;

A2 – Positive homogeneity: $\rho(\lambda x)=\lambda\rho(x)$ for all $x \in X$ and $\lambda > 0$ and $\lambda \in \mathbb{R}$;

A3 – Monotonicity: if $x \le y$, then $\rho(x) \le \rho(y)$ for all $x, y \in X$;

A4 – Transitional invariance: $\rho(x+\alpha)=\rho(x)+\alpha$ for all $x \in X$.

If ρ satisfies the first two conditions, it also has the property of convexity. Subadditivity ensures that risk cannot increase from a merger; it is a crucial property related to convexity and rules out the sort of quantile risk measures such as VaR. The positive homogeneity axiom implies that risk scales linearly with the size of a random variable. The monotonicity axiom simply states the reasonable requirement that the risk of a random variable should always be less than the corresponding risk of a random variable which always results in a greater loss (this also rules out standard deviation). Finally, translation invariance implies that there is an absolute scale for measuring risks; note that it immediately rules out the commonly used risk measures of standard deviation or variance.

Artzner et al. (1999) also show the correspondence between acceptance sets and coherent risk measures. The proposition is stated as following. For a closed acceptance set satisfying axiom A1 - A4, the induced risk measure ρ_A is coherent. Conversely, for a coherent risk measure ρ , the induced acceptance set A_ρ is closed and satisfies axiom A1 - A4. The intuition behind this proposition is that coherent risk measures correspond exactly to a set of acceptable random variables, and this acceptance set satisfies the axiom A1 - A4. In brief, any risk measure violating these axioms could lead to serious inconsistencies.

4.3.2 Distorted Risk Measure

As discussed in chapter 2.2 and 2.3, in reality risk measures have a close connection with the psychological theory of people's preference of uncertainties and risk. Psychologists and economists have discovered many preference anomalies in which rational people may systematically violate the axioms of the expected utility theory. In particular, Kahneman and Tversky proposed a model of choice under uncertainty called the "prospect theory". The prospect theory states that people evaluate uncertain prospects using "decision weights" that may be viewed as distorted probabilities of outcomes. The theory can explain a variety of preference anomalies including the Allais paradoxes (see section 1.2).

The prospect theory has been further developed by many people, including the contributions from Kahneman and Tversky (1992), Tversky and Wakker (1993), Schmeidler (1986), and Yaari (1987) on using comonotonic random variables. Their models are also referred to as "anticipated utility", "rank-dependent models" and "Choquet expected utility". Therefore, it is of interests to examine the particular class of risk measures – Choquet fuzzy integrals (see section 4.2.2). These measures are expectations of a random variable under an appropriate distortion of the original distribution. It turns out that the Choquet fuzzy integral is a fairly general risk measure with quite a bit of modeling freedom; in fact, many commonly used risk measures may be cast as a Choquet fuzzy integral. By examining its functions, Choquet fuzzy integral is indeed the expected value under the distorted distribution. For any distortion function g (e.g. the weighting function of prospect theory), the Choquet fuzzy integral μ_g satisfies translation invariance, monotonicity, and positive homogeneity (Brown, 2006). In addition, μ_g satisfies subadditivity if and only if g is concave. Thus, μ_g is coherent if and only if g is concave.

4.3.3 Co-monotonic Risk Measure

It is not obvious whether or not all coherent risk measures can be represented as a Choquet fuzzy integral under a concave distortion function. Brown (2006) examined this question and it turns out that the key property is a risk measure's behavior under sums of comonotonic random variables. He begins with some definitions. The set $A \in \mathbb{R}^n$ is a comonotonic set if for all $x \in A, y \in A$, we have either $x \leq y$ or $y \leq x$. Clearly any one-dimensional set is comonotonic. It is also not difficult to see that any set in $\mathbb{R}^n$ with $n > 1$ cannot be full-dimensional and also be comonotonic. We can extend this idea naturally to random variables. A random variable $X = (X_1, \cdots, X_n)$ is comonotonic if its support $A \in \mathbb{R}^n$ is a comonotonic set.

Comonotonicity is an important property when considering sums of random variables with arbitrary dependencies. Comonotonic random variables have "worst-case" summation properties among all dependence structures. In our case, comonotonicity has a very specific relationship to coherent risk measures and Choquet fuzzy integrals. In particular, the following property is of interests.

A risk measure $\rho : X \to \mathbb{R}$ is comonotonically additive if, for any comonotonic random variables x and y, we have

$$\rho(x+y) = \rho(x) + \rho(y)\,. \tag{2.6.1}$$

If a coherent risk measure is comonotonically additive, it is said the risk measure is comonotone. The following result is due to Schmeidler (1986).

A risk measure $\rho : X \to \mathbb{R}$ can be represented as the Choquet fuzzy integral with a concave distortion function if and only if ρ is comonotone.

The subadditivity property says we can do no worse by aggregating risk when dealing with a coherent risk measure; Schmeidler's result implies that we will not benefit from diversifying risk when our risk measure is a Choquet fuzzy integral and the underlying random variables are comonotonic. The theorem also allows us to answer the question of whether all coherent risk measure can be represented in the Choquet fuzzy integral form. The answer is no (Brown, 2006), and using Schmeidler's theorem, this can be done by constructing a coherent risk measure which violates comonotonic additivity.

Chapter 5

Copula and Choquet Fuzzy Integral Decision Modles

The interdependence among multiple risks of interest pervade all aspects of decision making, from effects of pollutants on health and the environment, reliabilities of weaponry, failures of nuclear power plants, and to consequences of investment. Such interdependency generally requires values for parameters that cannot be measured or assessed with certainty:

1 After obtaining a simulated distribution for the model input and output, how can the human experts characterize the importance of various model inputs with respect to model output?

2 How can one perform probabilistic inversion for generalization of the conclusion?

3 The standard product moment, or Pearson correlation cannot be assessed independently of the marginal distributions, whereas the Kendall's correlation can.

Furthermore, several key issues set the approach apart from other risk analysis fields. These include:

1. Causal of exposures and treatment – Decision maker's opinions and beliefs about the plausible consequences are expressed as explicit quantitative models of exposure (treatment) and of exposure response relations and stratifications. These models usually represent aspects of causation and are intended to describe how changes in actions propagate through one or more causal trees to change treatment and effects.

2. Inferred risks and consequences – Risks cannot be directly observed or measured. They refer to potentials. Therefore, they must be inferred from models, knowledge, assumptions, and data. Inferences about risks are often drawn from laboratory studies, experimental results, historical data, or individual experiences. This calls for conclusions based on forms of inference that are often challenging compared to routine statistical calculations of conditional probabilities or expected values.
3. Psychophysical Judgment – The potential perception-action coupling due to the comprehension, internalization and action of risk. Decision maker's understanding of risk and their resulting behaviors may be affected by attempts to describe the results of formal risk assessments. Even if risk information is communicated accurately, behaviors may not change in the ways that models of rational behavior predict.
4. Statistical inferences and expert opinions may be very different – Consequences of risk management decisions may include failures or deaths. Such consequences require special evaluation techniques, since the extreme outcomes cannot be fully compensated. Furthermore, elicits responses that may reflect political, psychophysical, or emotional attitudes more strongly than economic tradeoffs and values, for example, prospect theory (Kahneman, 1979) shows that preferences revealed by practical behaviors and choices, including willingness to accept compensation for hazardous occupations are not optimized based on the conventional utility theory.

In this chapter, possible solutions to the issues listed above are introduced. Copula model utilizes a rank coefficient to generate joint distribution to cope the issues of probability correlation. On the other hand, Choquet fuzzy integral gives a way to apply fuzzy measures to perform human judgments. They both provided improved approaches to analyze risks and reveal critical information for decision-making.

5.1 Copula Decision Model

5.1.1 Introduction

In this chapter we will mention the usefulness of correlation as a measure of interdependence of random variables. For instance if we have the risks $x_1, x_2, \ldots, x_n$ being multivariate distributed, the use of correlation as a measure of dependence can be considered strongly, with the further assumption that each of the risks under investigation has a probability distribution. Typical case is that of the elliptical distributions. The copula, a function to find a joint distribution, is used as a tool for understanding dependence in risks. The main theme of this section is to introduce the copula modeling in empirically estimating the likely dependence of the various causes of risks. In common risk assessment, to avoid the complication of dependence of risks, independence is usually assumed between risks. However, to avoid the complexity of dealing with the multivariate distribution we will confine research to just two risks.

5.1.2 Copula Model

The full review of copula functions is written in section 4.2.2. Here is an abstract to show the model. According to Sklar's theorem (Sklar, 1959), the multivariate distribution function can be split into two parts, the marginal distribution function and the dependence function which is the copula. A copula is a function that links (couples) univariate marginals to their full multivariate distribution.

Theorem 5.1.1. *Let* $F \in \mathcal{F}(F_1, F_2, \cdots F_n)$ *be an n-dimensional distribution function (c.d.f.) with marginal distributions* $F_1, F_2, \cdots F_n$. *Then there exists a copula* C *such that*

$$F(x_1, x_2, \cdots x_n) = C(F_1(x_1), F_2(x_2), \cdots F_n(x_n)) \tag{5.1.1}$$

Proof: Sklar's theorem (Sklar, 1959)

Theoretically, infinite number of copulas could be developed. In this research, Archimedean copulas are selected to obtain joint distributions since we can choose different generator functions to reach wide ranges of dependence properties.

Definition 5.1.1. A copula C is called Archimedean if it admits the representation

$$C(u_1, u_2, \cdots u_n) = \psi(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots \psi^{-1}(u_n)) \tag{5.1.2}$$

where ψ is the so called generator. The above formula yields a copula if and only if the k^{th} derivatives of ψ satisfy $(-1)^k \psi^{(k)}(x) \geq 0$, and for all $x \geq 0$

and $k = 0, 1, \cdots, d-2$ and $(-1)^{d-2} \psi^{d-2}(x)$ is non-increasing and convex.

In bivariate Archimedean copulas, the dependence parameter between tow random variables is estimated straightforward by using *Kendall's tau rank correlation*.

Definition 5.1.2. Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be a set of joint observations from two random variables X and Y respectively, such that all the values of x_i and y_i are unique. Any pair of observations (x_i, y_i) and (x_j, y_j) are said to be concordant if the ranks for both elements agree: that is, if both $x_i > x_j$ and $y_i > y_j$ or if both $x_i < x_j$ and $y_i < y_j$. They are said to be discordant, if $x_i > x_j$ and $y_i < y_j$ or if $x_i < x_j$ and $y_i > y_j$. If $x_i = x_j$ or $y_i = y_j$, the pair is neither concordant nor discordant. The *Kendall τ coefficient* is defined as:

$$\tau = \frac{(number\, of\ concordant\ pairs) - (number\, of\ disconcordant\ pairs)}{\frac{1}{2}n(n-1)} \tag{5.1.3}$$

Based on the Kendall's tau coefficient, four most popular and well-developed copulas are shown in table 5.1.1:

Table 5.1.1 – Copulas and Kendall's tau coefficient

	$C(u_1, u_2, \theta)$	$\varphi(t)$	θ
Clayton	$(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$	$\theta^{-1}(t^{-\theta} - 1)$	$\frac{2\tau}{1-\tau}$
Frank	$-\frac{1}{\theta}\log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right)$	$-log\left[\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right]$	$\frac{D_1(\theta) - 1}{\theta} = \frac{1-\tau}{4}$
Gumbel	$\exp\left\{-\left[(-logu_1)^{\theta} + (-logu_2)^{\theta}\right]^{1/\theta}\right\}$	$(-logt)^{\theta}$	$\frac{1}{1-\tau}$

Note: τ is kandall's tau rank correlation; $D_1(\theta)$ is a Debye function of the first kind.

5.1.3 Numerical Demonstration

A brief introduction to the example data (FEMA, 2010)**:**

The data used to demonstrate the models is obtained from the website of Federal Emergency Management Agency (FEMA). It describes an emergency situation that hurricanes struck the gulf area and resulted in severe casualty and property damage. The major defense is to maintain and coordinate the pump operations during a critical 48 hour period or over 4 high tides. Data was collected from 770 pump station. There are five potential failure (risk) conditions during a major hurricane hit:

1. No failure
2. Flood – overflow or accumulation of an expanse of water that submerges the pump station.
3. Motor – mechanical failure.
4. Surge – onshore gush of water usually associated with levee or structural failure.
5. Jammed – accumulation of trash or landslide materials.

The covariates for the competing risks include the following:

1. Backup pump – a redundant system used to protect the station from flooding when the main pump is not operating.
2. Bridge Crane – allow vertical access to equipment and protecting materials.
3. Servo – servomechanism is used to provide control of a desired operation through the supervisory control and data acquisition systems.

4. Gear – an overdrive used to allow the choice of an extra-high overall gear ratio for high speed run.
5. Trash-rack cleaner – protecting hydraulic structures against the inlet of debris.
6. Elevation – elevation of the pump station.
7. Slope – surrounding ravine slope of the pump station.
8. Age – difference between the installation and the current date.

Step 1 – Calculating marginal distributions

The marginal distributions here are referred to survival function defined in survival analysis (see section 4.4.1). The survival function is a function of time which presents the probability that an agent will survive beyond a specified time point.

Definition 5.1.3. The survival function, denoted as *S*, is defined as

$$S(t) = \Pr(T > t),\ t \geq 0 \tag{5.1.4}$$

where *T* is a random variable denoting the time of casualty or failure, *t* is some specified time (age), and Pr stands for probability.

Definition 5.1.4. The failure function, denoted as *F* , is defined as

$$F(t) = 1 - S(t) = \Pr(T < t) \tag{5.1.5}$$

$F(t)$ is a cumulative distribution function (c.d.f.) which yields the probability that the agent has an event before age *t*.

Survival data have two common features that are difficult to handle with conventional statistical methods: censoring and time-dependent variables (Allison 2010). For example, one possibility is to perform a logistic regression analysis with a dichotomous dependent variable: failed or not failed. However, this analysis ignores information on the timing of failure. It's reasonable to suspect that pumps which are failed in two days when a hurricane arrives have a higher potential to be failed than those which are not failed until two weeks. Thus, regressions which can deal with the above difficulties should be used in further analysis of competing risks problem. These regression models should be able to predict the survival functions, underlying density functions and hazard functions for any values of covariates.

According to Allison (2010), SAS currently has the most comprehensive set of full-featured procedures for doing survival analysis. Therefore, SAS procedures, such as LIFEREG and PHREG will be used as the calculating too. Both LIFEREG and PHREG are based on linear regression models and the parameters are estimated by maximal likelihood method. In the regression model, survival time is defined from a time origin to the occurrence of some given event (end-point). The models for the response variable consist of a linear effect composed of the covariates and a random term. The review of the SAS procedure, including assumptions and numerical results can be seen in Appendix A. In this section, because we focus on the demonstration of the dependence between two risks, we obtain the distributions by setting all covariates equal to zero. We have the distributions of Flood and Jammed as bellow:

$$F_{Flood}(t) = 1 - \exp(-(0.0084t)^{1.4853}) \quad (5.1.6)$$

$$F_{Jammed}(t) = 1 - \exp(-(0.0209t)^{1.5585}) \quad (5.1.7)$$

Figure 4.1.1 shows the probability distributions of failure time of risk Flood and Jammed.

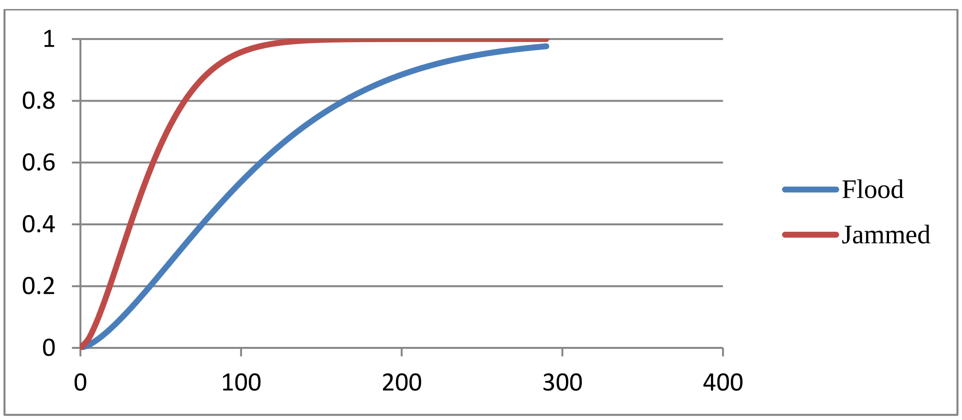


Figure 5.1.1 – Probability distributions of failure time of risk Flood and Jammed

Step 2 – Identifying a copula

Due to the nature of competing risks, the data itself contains no information of correlations. Let $Y_1, Y_2, \cdots, Y_k$ denote the variables of the potential survival times of a set of competing risks, $R_1, R_2, \cdots, R_k$. By definition of competing risks (see chapter 2.4), only one risk, say R_i, is the cause of death. One exceptional characteristic of competing risks problem is that we can only observe $\min(Y_1, Y_2, \cdots, Y_k)$ while other elements in $Y_1, Y_2, \cdots, Y_k$ are not observable. Peterson (1976) argued that there is no way of knowing from the data in the competing risks problem whether or not a wrong assumption of independence is being made since the data contain no information on whether or not the risks are independent. Also, Kaishev et al. (2006) argue "The fact that estimation of the copula parameters is not possible imposes the necessity of considering them as free parameters. They could be set according to a priori available information, about the degree of pair-wise dependence between the two competing risks, expressed through Kendall's τ or Spearman's ρ"

Thus, in practice the kendall's τ needs to be determined by experts or the decision maker according to their professions and prior information. Here, we will arbitrarily select Clayton copula function and three values of kendall's τ to demonstrate copulas with different degree of association. Figure 5.1.2, 5.1.3 and 5.1.4 show the Clayton copula, $C(u,v)$ and $c(u,v)$, of τ =0.2, 0.5, and 0.8.

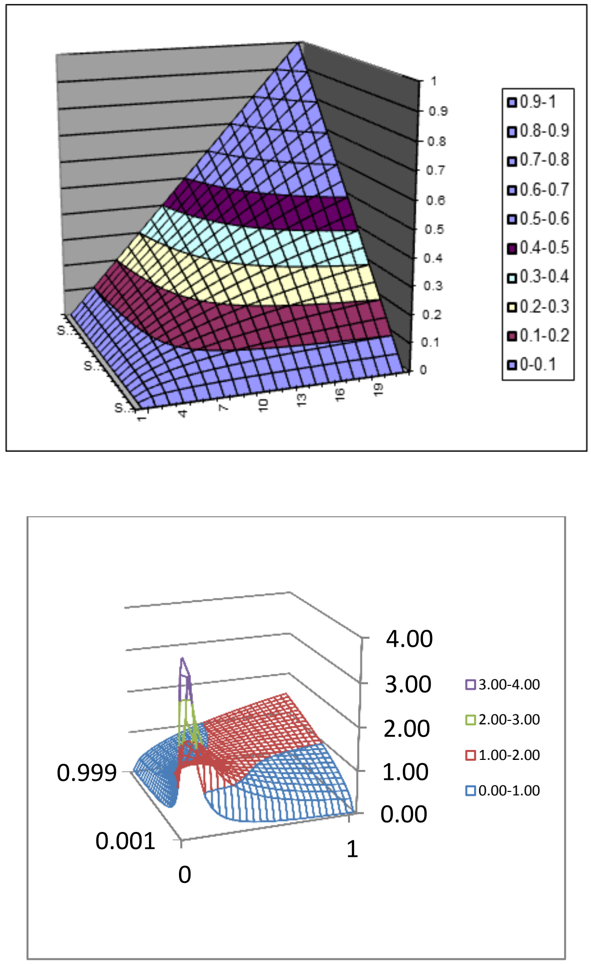


Figure 5.1.2 – Clayton copula $C(u,v)$ and $c(u,v)$ when τ =0.2

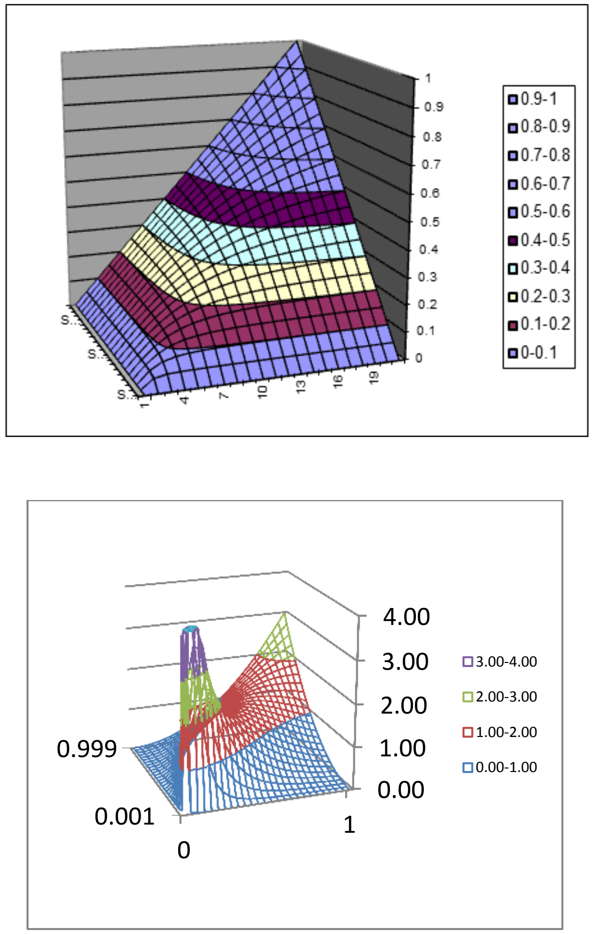


Figure 5.1.3 – Clayton copula $C(u,v)$ and $c(u,v)$ when τ =0.5

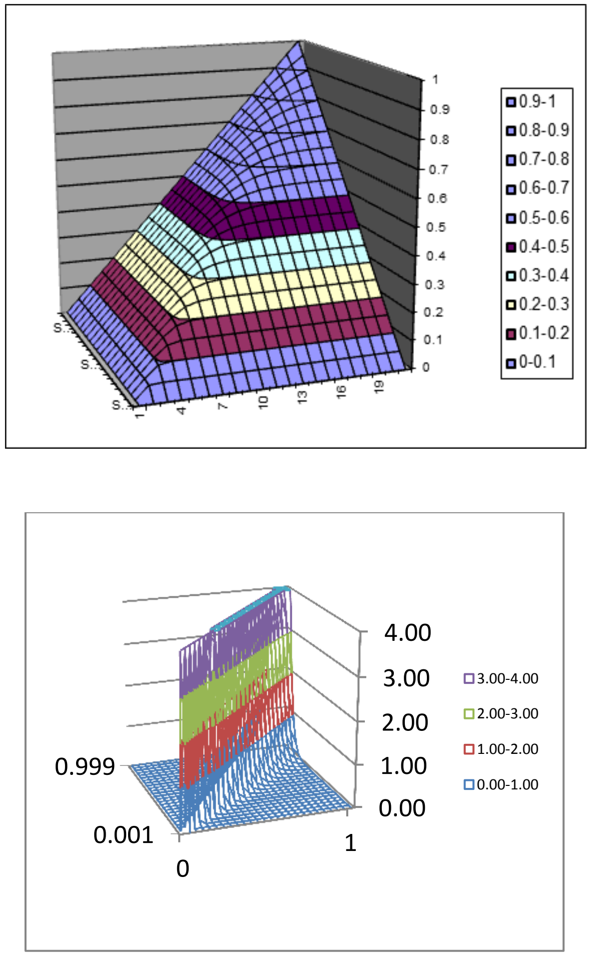


Figure 5.1.4 – Clayton copula $C(u,v)$ and $c(u,v)$ when τ =0.8

Step 3 – Computing risk index

From the statistical perspective, risk management is a procedure for shaping a survival distribution based on the acceptable level of risk. Risk measures are used to characterize the risks for assessment. As discussed in chapter 4.1.3, the VaR and CVaR have been popular risk measures in the last decade. These two risk measures originate from one dimensional case, but the concept can be extended to higher dimensions. The interests are of analyzing whether the agent at high risk of one type of failure is also at high risk or oppositely at low risk at another (i.e. interdependency of risks).

Back to our numerical example, suppose the critical time is 24 hours. The CVaR of individual risk can be estimate as following.

The CVaR of risk Flood is computed as following: (see Figure 4.1.2 for illustration)

$$\boldsymbol{I}_3 = \int_{24}^{\infty} \boldsymbol{exp}(-(0.0084\boldsymbol{t})^{1.4853})\boldsymbol{dt} = 84.47$$

$$\boldsymbol{\beta} = 1 - \boldsymbol{F_T}(24) = 0.9115$$

$$\boldsymbol{CVaR}_{Flood} = \boldsymbol{VaR} + \frac{1}{\boldsymbol{\beta}}\boldsymbol{I}_3 = 24 + \frac{1}{0.9115} \times 84.47 = 116.672$$

The CVaR of risk Jammed is computed as following:

$$\boldsymbol{I}_3 = \int_{24}^{\infty} \boldsymbol{exp}(-(0.0209\boldsymbol{t})^{1.5585})\boldsymbol{dt} = 21.896$$

$$\boldsymbol{\beta} = 1 - \boldsymbol{F_T}(24) = 0.7109$$

$$\boldsymbol{CVaR}_{Jammed} = \boldsymbol{VaR} + \frac{1}{\boldsymbol{\beta}}\boldsymbol{I}_3 = 24 + \frac{1}{0.7109} \times 21.896 = 54.7997$$

The dependence between risks and the joint distribution and can be determined by equation 5.1.1 and the copula function. The joint distribution is of the form:

$$F_{joint}(t_1, t_2) = C(F_{Flood}(t_1), F_{Jammed}(t_2)) \tag{5.1.8}$$

$$\boldsymbol{I}_3 = \int_{24}^{\infty}\int_{24}^{\infty} C(F_{flood}(t_1), F_{Jammed}(t_2))dt_1 dt_2$$

$$\boldsymbol{\beta} = 1 - \boldsymbol{F}_{Joint}(24, 24) = C(F_{Flood}(24), F_{Jammed}(24))$$

$$\boldsymbol{CVaR}_{Jammed} = \boldsymbol{VaR} + \frac{1}{\boldsymbol{\beta}}\boldsymbol{I}_3$$

5.2 Choquet Fuzzy Integral Decision Model

5.2.1 Introduction

A Choquet fuzzy integral-based approach implementation is investigated in this chapter. The concept here is to use a fuzzy measure to capture the judgmental preferences over risks. Modeling human perception and determining the preference index for risks (i.e. interactions of weights in human mind) is a crucial to a decision-making process. In reality, priorities and opinions of human experts, sometimes are required to be handled by a robust technique to handle uncertainties, ambiguities, and contradictions. An overwhelmingly important fact about human reasoning is that it is not a static process. In the approach of Choquet fuzzy integral, human preferences are elicited and preliminary priorities are advanced. We generalized the fuzzy integral as an excellent component for decision analysis. The generalization involves applying a fuzzy integral-based operator in risk aggregation, resulting in increased applications.

5.2.2 Choquet Fuzzy Integral Model

The Choquet fuzzy integral is a fuzzy integral based on a fuzzy measure that provides alternative computational scheme for aggregating.

Definition 5.2.1. Assume $h(x_1), h(x_2), \cdots h(x_n)$ are the input evidence provided by the sources $x_1, x_2, \cdots, x_n$ respectively, and g is a λ fuzzy measure, then we can construct a Choquet fuzzy integral in the form of

$$\int_x h(\bullet) \circ g(\bullet) \tag{5.2.1}$$

If interpreted by notations of risk assessment, $x_1, x_2, \cdots, x_n$ denote the set of interested risks, and $h(x_1), h(x_2), \cdots h(x_n)$ are some sort of risk measures, such as CVaR or expected value, to present the magnitudes of according risks. $g(\bullet)$ here is our fuzzy measure, determined by a process of human judgments, presenting the weights of importance when aggregating risks. Finally, λ is the parameter to determine those weights. $g(\bullet)$ is shaped by Definition 5.2.2 and Porperty 5.2.1.

Definition 5.2.2. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set and let $P(X)$ denotes the power set of X or set of all subsets of X. A fuzzy measure over a set X is a function $g: P(X) \to [0,1]$ such that

1. $g(\phi) = 0, g(X) = 1$;
2. If $A, B \subset P(X)$ and $A \subset B$, then $g(A) < g(B)$.

The λ fuzzy measure satisfies the following property:

Property 5.2.1. For all $A, B \subset X$ with $A \cap B = \phi$,

$$g(A \cup B) = g(A) + g(B) + \lambda g(A) g(B) \tag{5.2.2}$$

for some fixed $\lambda > -1$.

If we define $A_i = \{x_i, x_{i+1}, \cdots, x_n\}$, then according to Property 5.2.1, the value of $g(A_i)$ can be calculated recursively from

$$g(A_n) = g(\{x_n\}) = g_n \tag{5.2.3}$$

$$g(A_i) = g_i + g(A_{i+1}) + \lambda g_i g(A_{i+1}), \text{ for } 1 \leq i < n \tag{5.2.4}$$

On the other hand, the value of λ can be calculated by solving $g(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i) \tag{5.2.5}$$

With respect to risk assessment, since g and λ are the fuzzy measures used to capture human preferences over risks, it is assumed to be from a source of human judgment. In practice, rating or scoring system is used to compute g and the parameter λ. The operating procedure and an example will be given in chapter 5.2.3. After obtaining the inputs risk measure, $h(x_1), h(x_2), \cdots h(x_n)$, and solving the parameter λ and the fuzzy measures g's, the final risk index can be obtained by the Choquet fuzzy integral defined in Definition 5.2.2.

Definition 5.2.2. For a finite set of X, the Choquet fuzzy integral can be computed as follows:

$$\int h \, dg = E_g(h) = \sum_{i=1}^{n} [h(x_i) - h(x_{i-1})] g(A_i) \tag{5.2.6}$$

where $h(x_1) \le h(x_2) \le \cdots \le h(x_n)$, $h(x_0) = 0$ and $A_i = \{x_i, x_{i+1}, \cdots, x_n\}$.

Another computation formula for the finite set case can also be expressed as

$$E_g(h) = \sum_{i=1}^{n} h(x_i)[g(A_i) - g(A_{i+1})] \tag{5.2.7}$$

where we take $g(A_{n+1})$ equal to zero.

It is obvious that the calculation of the Choquet fuzzy integral with respect to λ fuzzy measure requires the knowledge of the fuzzy density g and the input value h. Choquet fuzzy integral has the following important properties.

Property 5.2.1. The Choquet fuzzy integral is a monotonically increasing function with respect to $h(x_i)$. This can be easily obtained by taking partial derivative of the Choquet fuzzy integral

$$\frac{\partial E_g(h)}{\partial h(x_i)} = g(A_i) - g(A_{i+1}) = g_i + \lambda g_i g(A_{i+1}) = g_i(1 + \lambda g(A_{i+1})) \tag{5.2.8}$$

Property 5.2.2. For all $h, g \in [0,1]^n$, the range of the Choquet fuzzy integral is as follows:

$$x_{\min} \le E_g(h) \le x_{\max} \tag{5.2.9}$$

where $x_{\min} = \min(h(x_1), h(x_2), \cdots, h(x_3))$ and $x_{\max} = \max(h(x_1), h(x_2), \cdots, h(x_3))$.

Figure 5.2.1 shows the case of two input variables (i.e., $n = 2$), the Choquet fuzzy integral value represents a point in the unit square. The unit square consists of all possible density values of $\{g_1, g_2\}$. The $x_{\min}$ defines in the left lower vertex whereas the $x_{\max}$ defines in the right upper vertex.

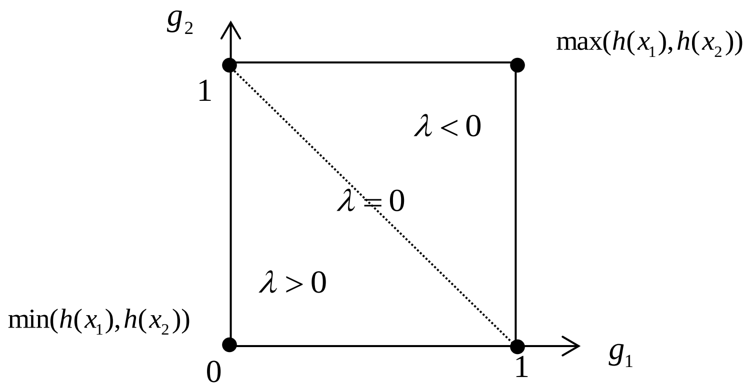


Figure 5.2.1 – The geometry of Choquet fuzzy integral values

Choquet fuzzy integral behaves like parameterized aggregator, it can cover the range between min and max of the input level by assigning the density values appropriately. If all density values are equal and $\sum_i g_i = 1$, the Choquet fuzzy integral is equivalent to arithmetic mean.

Let the inputs be denoted by $h(x_i)$ of dimension n and the corresponding densities be denoted by g_i. The architecture of Choquet fuzzy integral is shown in Figure 5.2.2. Input vector consists of information sources, and an integrating function called fuzzy integral that performs the "weight" function. In our model, the λ fuzzy measures here are used to present experts or the decision maker's judgments of interactions (interdependence) between risks.

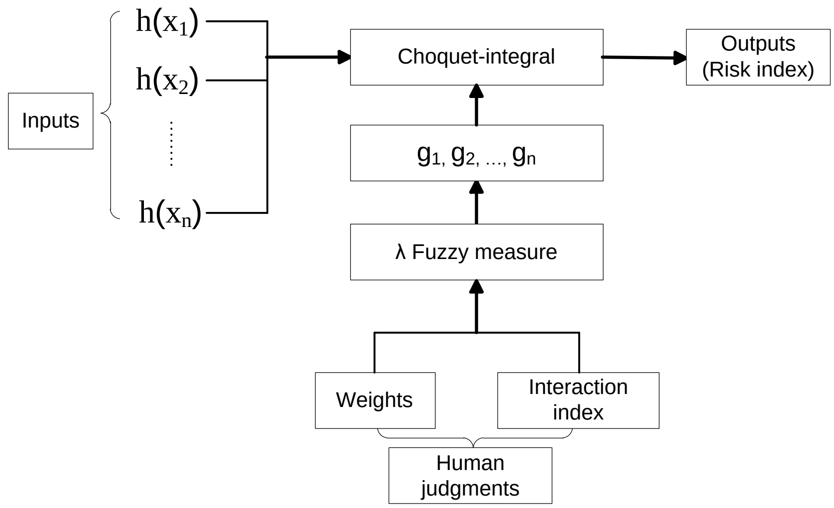


Figure 5.2.2 – The architecture of Choquet fuzzy integral model

5.2.3 Numerical Demonstration

The next content is based on the same example of two risks in section 5.1. The CVaR's of risk Flood and Jammed are used as the inputs risk measures. How to use scoring system to estimate the fuzzy measures is also demonstrated in this chapter.

Step 1 – Scoring risks

In the literature, starting with Kahneman and Tversky (1981,1984) there is a consistent stream showing that utility's transitivity and invariance assumed among the axioms introduced by von Newman and Morgenstern are often not fulfilled and dependent of the context. Dominance in between two alternatives A and B is established through the question

“ Which alternative you prefer out of A and B ? ” Preference’s intensity is established through the question: “ On a scale from…to … by how much you prefer the alternative you chose to the other one ? ”

Also, as assumed the decision maker or the experts must express their preferences to compute the fuzzy measures. The simplest way is to arbitrarily assigned the value of parameter λ ; however, except very experienced person, it is not easy to elicit accurate estimate. Thus, the scoring system, an intuitive and numerical scale, is an excellent tool to help determined the fuzzy measures since one can give the score by comparisons of risk pair. Score here is assumed as the magnitude of importance or level of emergency in perception of the decision maker. There are a number of scoring or rating methods in literature. Here, we adapt the method of 1-9 scale by Satty (1978). The vector of priorities is the derived scale associated with the matrix of comparisons. Suppose a pair-comparison matrix of risks is given by Table 5.2.1. A score to the union of risks must be given so that the fuzzy information can be elicited.

Table 5.2.1 – Comparison matrix of risk Flood and Jammed

	Flood	Jammed	Flood+Jammed
Flood	1	5	1/2
Jammed	1/5	1	1/7
Flood+Jammed	2	7	1

Note that score 1 means two risks are equally important; score 7 means Flood is seven times important than Jammed. For dimensions more than 2, consistency must be tested. A measure for the departures from the consistency condition is the consistency index CI introduces as being the ratio between $(\lambda_{\max} - n)$ and $(n-1)$: CI $= \dfrac{\lambda_{\max} - n}{n-1}$, where $\lambda_{\max}$ is the largest eigenvalue of the comparison matrix and n is the number of risks in the matrix. The criterion is CI<0.1, suggest by Satty (1978). If the condition of consistency is not met, the matrix should be remade.

Step 2 – Computing eigenvalues and eigenvectors for fuzzy measures

The fuzzy measure, g , is estimated from the corresponding items in the eigenvector with largest eigenvalue. The comparison matrix in Table 5.2.1 gives the eigenvector in Table 5.2.2.

Table 5.2.2 – Eigenvector of comparison matrix of risk Flood and Jammed

	Eigenvector	After standardization
Flood	0.4877	0.56
Jammed	0.1099	0.13
Flood+Jammed	0.8661	1

The first column of Table 5.2.2 is the eigenvector with the largest eigenvalue. Since we assume that $g(X)=1$, i.e., the union of all risks is measured as 1, the second column is the values after standardization. Also, the second column gives the following estimated fuzzy measures: $g_{Flood}=0.56$; $g_{Jammed}=0.13$; $g_{Flood+Jammed}=1$.

According to the results of Table 5.2.2 and equation 5.2.4, the parameter λ is computed as following:

$$\begin{aligned} & g_{Flood+Jammed} = g_{Flood} + g_{Jammed} + \lambda g_{Flood} g_{Jammed} \\ & \Rightarrow 1 = 0.56 + 0.13 + 0.56 \times 0.13\lambda \\ & \Rightarrow \lambda = 4.2582 \end{aligned} \tag{5.2.10}$$

However, for a higher dimensional case, it is require to solve multiple equations of higher-order and may not be able to obtain a consistent value of λ. Thus, λ^* is determined by selecting one of those $\lambda's$ so that λ^* minimizes the absolute value of sum of difference, i.e., to minimize $\left|\sum_i \left(g(A_i) - g_i + g(A_{i+1}) + \lambda g_i g(A_{i+1})\right)\right|$.

Step 3 – Computing Choquet fuzzy integral

Table 5.2.3 shows the inputs of CVaR's and the fuzzy measure obtained in previous steps. Choquet fuzzy integral is used to find the the "fuzzy-weighted" average of the input CVaR's. The fuzzy measure g represents the weight of importance for that risk.

Table 5.2.3 – Inputs of CVaR's and their fuzzy measure

Risk	Flood	Jammed
Input of CVaR, h	116.67	54.8
Fuzzy measure, g	$g_{Flood} = 0.56$	$g_{Jammed} = 0.13$
λ	4.2582	

According to equation 5.2.6, the output of Choquet fuzzy integral is computed as following:

$$\begin{aligned} E_g(h) &= h(Jammed) \times g_{Flood+Jammed} + \left[h(Flood) - h(Jammed)\right] \times g_{Flood} \\ &= 54.8 \times 1 + (116.67 - 54.8) \times 0.56 \\ &= 89.4472 \end{aligned} \tag{5.2.11}$$

Another computation formula for the finite set case can also be expressed as

$$\begin{aligned} E_g(h) &= h(Flood) \times g_{Flood} + h(Jammed) \times \left[g_{Flood+Jammed} - g_{Flood}\right] \\ &= 116.67 \times 0.56 + 54.8 \times (1 - 0.56) \\ &= 89.4472 \end{aligned} \tag{5.2.12}$$

Figure 4.2.3 illustrates two ways to do Choquet fuzzy integral. Another illustration for a general case is shown in Figure 4.2.1.

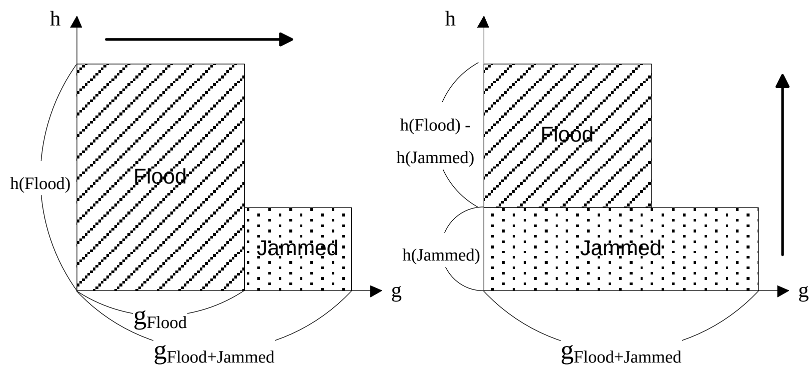


Figure 5.2.3 – Choquet fuzzy integral in vertical and horizontal ways

The equation (5.2.7) or the numerical demonstration of equation (5.2.12) shows an interesting fact of Choquet fuzzy integral. In any subset of the risks, the severer one (larger h) has higher priority to take its part of "weight". The milder risks only share the remaining amount. For example, risk Flood takes weight of g_{Flood} ; risk Jammed takes the remaining weights of the subset (Flood+Jammed). In other words, Choquet fuzzy integral complies with the logic that severer risks should be considered first.

Chapter 6

Integrated Information Model

Previous chapters have shown that making a decision in full considerations of all aspects could be hard. Almost any existent decision technique involves some shortcomings or dissatisfaction. The techniques of calculating marginal probability distributions and risk measures are already well developed in competing risk model and survival analysis. The probability distributions and measures give the most valuable and fundamental information of risk assessment. Most of existent risk assessment methods are basically serial operations or transformations of this fundamental information. For example, method of expected utility calculates the sum of product of probability and distorted value; method of CVaR calculates the conditional expected value on tail probability distributions and risk matrix also uses information of probability and severity to determine the risky levels. However, as characterized in chapter 3, there are two major shortages of these existent methods when they aggregate risks; the first is lack of human preferences in the evaluation process, the second is failure to take consideration of dependence structures or sub-additivity property when aggregating risks. In chapter 4 and 5, we have shown that copula and Choquet fuzzy integral both are useful tools to aggregate individual risk measures and provide a way to solve a part of shortcomings. Copula solves the problem of probability dependence but does not include preference in the model. Choquet fuzzy integral gives a way to model sub-additivity in human preference but ignores the information of probability dependence. Copula utilizes

Kendall's tau, a rank correlation parameter estimated by pairwise comparisons, to give the joint probability from inputs of marginal probabilities. The most valuable information from copula is the changing direction and range in probability when one wants to control one risk by manipulating another. Copula can connect risks via joint probability distributions and thus the decision maker can know if the conditions set to individual risks conflict to each other or a broader target. Besides, the information from copula is free to human judgment so intuitively it is a very useful tool to determine human irrationality of over weighting some risk and set the boundary of safe alternatives. On the other hand, Choquet fuzzy integral model gives weights to risks and aggregates risks to a weighted average. Human preferences can play the key role to determine those weights and the structure of Choquet fuzzy integral also gives a way to deal with the property of sub-additivity. The most valuable information of Choquet fuzzy integral model is the priority sequence of risks and their relative weights which are obtained from the decision maker's preference. This information is useful to point out high-priority risks which the decision maker may want to emphasize and put more resources on. Therefore, Choquet fuzzy integral model can provide a preference-wise realistic and efficient way to manage risks. However, the water that bears the boat is the same that swallows it. Human may be irrational or contradict to selves, so the weights obtained from extra biased human preference may result in a decision which deviated from the preset goal of management.

Neither copula nor Choquet fuzzy integral is a comprehensive approach though they both improve particular shortcomings of conventional decision models. However, these two methods can complement each other to provide the decision maker a more complete view when making decisions. Copula has advantage in looking for a conservative and safe decision, yet Choquet fuzzy integral can give a way to find the most demanded one. Therefore, a significant improvement can be done by integrating information from both models. Section 6.1 and 6.2 respectively describe the information and its applications of copula model and Choquet fuzzy integral model. And the integrated-information model is explained and demonstrated in section 6.3.

6.1 Information from Copula Model

Actuaries, researchers, and academics have been comfortable assuming independence of competing risks due to reduction of complexity in the assessment process, in comparison with the amount of work required when dependence is assumed. However in reality, most analysts recognize the need to use the later assumption, especially in some particular cases. For example, one may be able to sustain some amount of damage due to fire or flood respectively, but cannot afford the loss if both risks happen concurrently. In this case, the first priority for the decision maker is to assess the probability of the concurrent event since it is the most unwanted situation. Any actions to mitigate the risks should also be devoted to reduce concurrence of risks. However without the information of probability dependence, analysts usually assume these two risks are mutually independent, so that the joint probability is just a simple product of two marginals. This inaccurate calculation may result in inefficient or ineffective preparation which sometimes brings a big trouble.

In short, the copula decision model (see section 5.1) applies Kendall's tau and copula function to yield the information of joint probability and dependence structure. For instance, a decision maker faces threat of two risks, x_1 and x_2, and their c.d.f. $F_1(x_1)$ and $F_2(x_2)$, can be obtained from survival analysis (see section 4.1). Then, according to Sklar's theorem, equation (5.1.1) and (5.1.2), the joint distribution $F(x_1,x_2) = C(F_1(x_1), F_2(x_2), \theta) = \psi(\psi^{-1}(F_1(x_1), F_2(x_2))$, where C is the copula, ψ is its generating function and θ, a calculation of Kendall's tau, is the correlation parameter in ψ. Once ψ is selected and Kendall's tau is fixed, each one of $F_1(x_1)$, $F_2(x_2)$ and $F(x_1,x_2)$ can be determined by the other two. Copula works as a bridge to provide an explicit link of two risks. This can be a very useful and practical tool to manage risks. First, the decision maker can be able to compare alternatives by the joint probabilities. Furthermore, the decision maker can control the probability of individual risks and check if the joint probability is within acceptable bound. Or the reverse way, s/he can set a target of joint probability, control the level of the risk with higher priority, and then determine what level the other risk should at least be mitigated to. In the following, the information and the application are illustrated in figures.

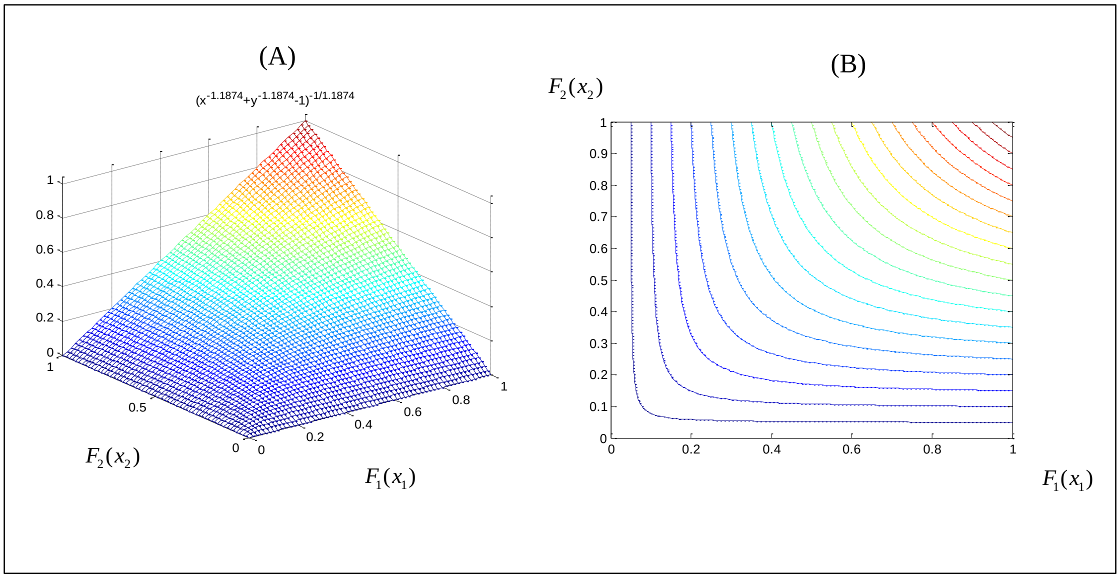


Figure 6.1.1 – Illustration of copula and its contour map

Part (A) of figure 6.1.1 shows a Clayton copula with a Kendall's tau equal to 0.372528 and the right hand side part (B) illustrates the contour lines apart in an interval of 0.05. The map of contour lines is a powerful tool to visualize the big picture of the relations between two marginal probabilities and the joint of probability. See examples in the following.

Example 6.1.1. Assume the decision maker is to evaluate a preparation alternative in which $F_1(x_1^*) = 0.3$ and $F_2(x_2^*) = 0.4$. For the purpose of easy explanation, x_1^* and x_2^* are defined as the value of VaR, i.e., they are the critical values that only the amounts within these values are consider as a loss. In other words, risk x_1 has a probability of loss equal to 0.3 and risk x_2 has a probability of loss equal to 0.4. In figure 6.1.2, one can easily see the joint probability that both losses currently happen is between 0.2 and 0.25. So, the decision maker can accept this alternative if the joint probability is tolerable; otherwise s/he should find a way to improve it. Actually, the copula $C_{Clayton}(0.3, 0.4, \theta) = 0.2167$.

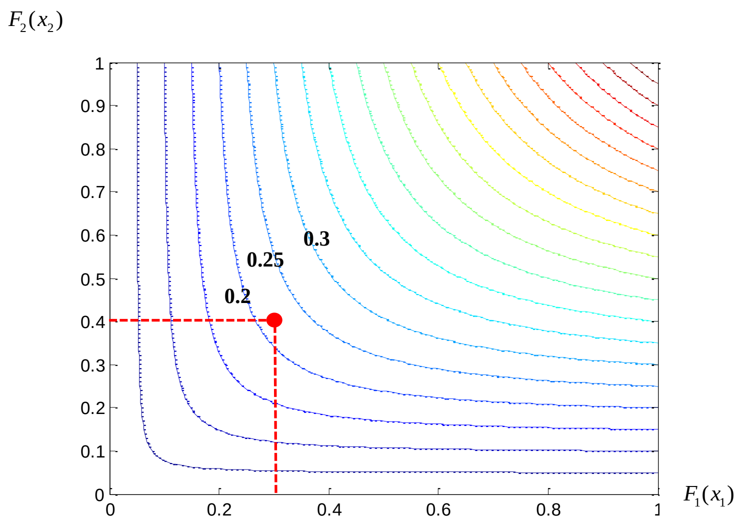


Figure 6.1.2 – Example 1 of how to apply copula information

Example 6.1.2. There may be a reverse case of example 6.1.1. In most practical cases, the first priority for decision maker is usually to satisfy the target level of joint probability, since a situation of concurrent losses is the most unwanted. Suppose target level is set as $C_{Clayton}(F_1(x_1^*), F_2(x_2^*), \theta) = 0.2$ and the current preparation level is the point X in figure 6.1.3. Then the shadow in part (A) shows the region where an acceptable alternative should be able to reach. This can give the decision maker an idea of improvement direction when s/he comes up an alternative; for instance, the direction of green arrow (shorter) may be better than the yellow. If we further assume the risk x_2 was reduced to the level $F_2(x_2^*) = 0.3$ due to some preset action and it is fixed, then the acceptable region of risk x_1 can be easily visualized as shown if part (B).

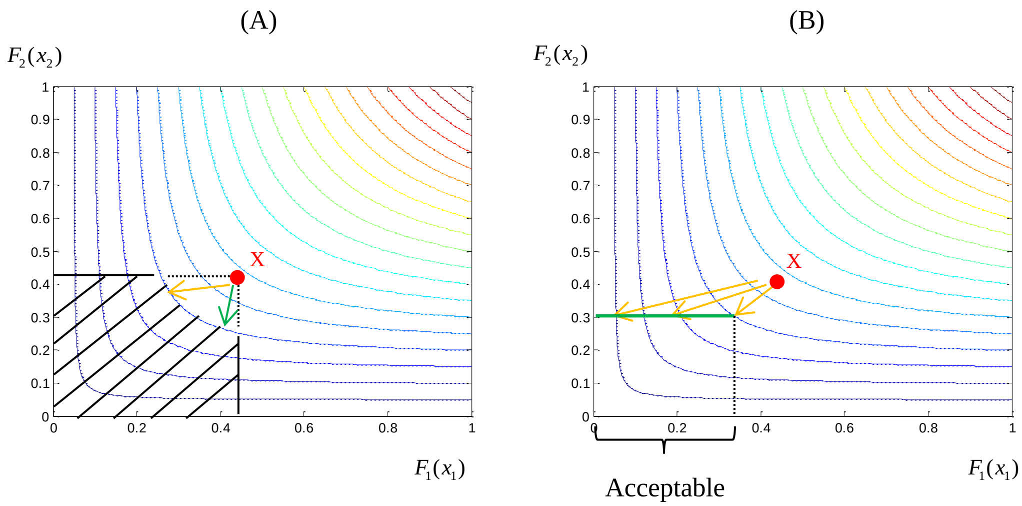


Figure 6.1.3 – Example 2 of how to apply copula information

Without the information of copula, all of these management actions in above examples are not available. In that case, the decision maker can only take actions on respective risks and use his/her experience to arbitrarily judge the joint probability. The decisions made by this kind of lopsided analysis may result in lots of wasted resource, inefficient improvements or even worse, no awareness of unprepared danger. Taking a simple example: under the assumption of independence, the joint probability of two marginal probabilities of 0.3 and 0.2 is only 0.06; however, if these two risks are highly dependent then the joint probability may be closed to 0.2. Therefore, the information of copula makes it possible to manage risks without losing the big picture, and thus copula can play the key role in determining a statistically optimal, macroscopic and secured decision.

6.2 Information from Choquet Fuzzy Integral Model

Choquet fuzzy integral models the effects of human judgments. For example, framing effects arise because one frequently frames a decision using multiple scenarios, wherein one may express benefits either as a relative risk reduction, or as absolute risk reduction. Extrinsic control over the cognitive distinctions adopted by decision makers can be done by altering the presentation of relative risks and absolute benefits. People generally prefer the absolute certainty inherent in a positive framing-effect, which offers an assurance of gains. When decision-options appear framed as a likely gain, risk-averse choices predominate. A shift toward risk-seeking behavior occurs when a decision-maker frames decisions in negative terms, or adopts a negative framing effect. However, the framing process is continuous and easily affected by outside factors. The fuzzy measures in Choquet fuzzy integral model are excellent tool to model these floating effects.

In summary, the most valuable information obtained from Choquet fuzzy integral is the fuzzy parameter λ and the weights associated to each risk. Considering a case of two risks x_1 and x_2, the weight g can be calculated by solving equation (5.2.4): $g(\{x_1, x_2\}) = g_1 + g_2 + \lambda g_1 g_2$ and $g(\{x_1, x_2\}) = 1$. In practical operation, g_1, g_2 and λ are determined through a scoring system and a specific process (see section 5.2.3). Then, the Choquet fuzzy integral can be calculated by

$E_g(h) = h(x_1) \times g\left(\{x_1, x_2\}\right) + \left[h(x_2) - h(x_1)\right] \times g_2$ if $h(x_2) > h(x_1)$, where h represents a measurement of risk, i.e., VaR or CVaR. One can see that Choquet fuzzy integral basically is a weighted average of individual risk measures. The equation can also be rewritten as $E_g(h) = h(x_2) \times g_2 + h(x_1) \times \left[g\left(\{x_1, x_2\}\right) - g_2 \right]$. The severer risk (with bigger h) takes its full weight first and the other one only takes the remaining. In this case, higher severity does not mean higher weight. Specifically speaking, the fuzzy measures (weights) in Choquet fuzzy integral model express the degree of importance of each risk. The "importance" here is totally a human judgment. Due to framing effects, human may mentally exaggerate or diminish the measurements of risks. In other words, human actually perceive risks in different scales. Thus, the fuzzy weights yield the information to outline the map of mental satisfaction. In addition, the fuzzy parameter presents the mental effect when aggregating risk. Three types of effect can be found as bellow:

1. If $\lambda > 0$, then $g\left(\{x_1, x_2\}\right) > g_1 + g_2$, and

   $E_g(h) = h(x_2) \times g_2 + h(x_1) \times \left[g\left(\{x_1, x_2\}\right) - g_2 \right] > h(x_2) \times g_2 + h(x_1) \times g_1$.

   In this case, there is a positive effect in aggregation. The event of concurrent losses is more painful than the event that two losses happen respectively. And we can find that risk x_1, the less severe one, actually plays a more important role than when it stands alone. This implies a phenomenon that given one risk has high probability to occur, the decision maker actually pays more than normal to the other one.

2. If $\lambda < 0$, then $g\left(\{x_1, x_2\}\right) < g(\{x_1\}) + g(\{x_2\})$, and $E_g(h) < h(x_2) \times g_2 + h(x_1) \times g_1$. There is a negative effect in aggregation. The event of concurrent losses is less painful than the event that two losses happen respectively. In this case, risk x_1 instead plays a less important role than when it stands alone. This implies a phenomenon that given one risk has high probability to occur, the decision maker actually cares less to the other one.
3. If $\lambda = 0$, then $g\left(\{x_1, x_2\}\right) = g(\{x_1\}) + g(\{x_2\})$, and $E_g(h) = h(x_2) \times g_2 + h(x_1) \times g_1$. No effect in aggregation. No mental exaggeration or dilution when two risks concurrently happen.

In addition to the fuzzy parameter λ, the weights also g reveals useful information. Suppose here we use $F_1(x_1^*)$ and $F_2(x_2^*)$ defined in example 6.1.1 as the risk measurements of $h(x_1)$ and $h(x_2)$. Then, the Choquet fuzzy integral can be express as equation (6.2.1).

$$E_g(h) = F_2(x_2^*) \times g_2 + F_1(x_1^*) \times \left[g\left(\{x_1, x_2\}\right) - g_2 \right] \qquad (6.2.1)$$

In equation 6.2.1, the fuzzy weights are predetermined constants, so the Choquet fuzzy integral is just a linear combination of two variables $F_2(x_2^*)$ and $F_2(x_2^*)$. And the gradients are determined by partial differentials as shown in (6.2.2).

$$\left[\frac{\partial E_g(h)}{\partial F_1(x_1^*)}, \quad \frac{\partial E_g(h)}{\partial F_2(x_2^*)}\right] = \left[g\left(\{x_1, x_2\}\right) - g_2, \quad g_2\right] \tag{6.2.2}$$

Like the contour lines of copula (see (B) in figure 6.1.1), equation 6.2.1 also provides a linkage of two risks, by which one can have an idea of how the risks can be exchanged with each other. To more specific, one unit of $F_1(x_1^*)$ is exchangeable to $\left(\frac{g\left(\{x_1, x_2\}\right) - g_2}{g_2}\right)$ unit of $F_2(x_2^*)$. As copula can show how tow risks are tradable with respect to joint probability, this exchange rate of fuzzy weights represents the way in aspect of human preference. Figure 6.2.1 gives an illustration to visualize the exchangeability.

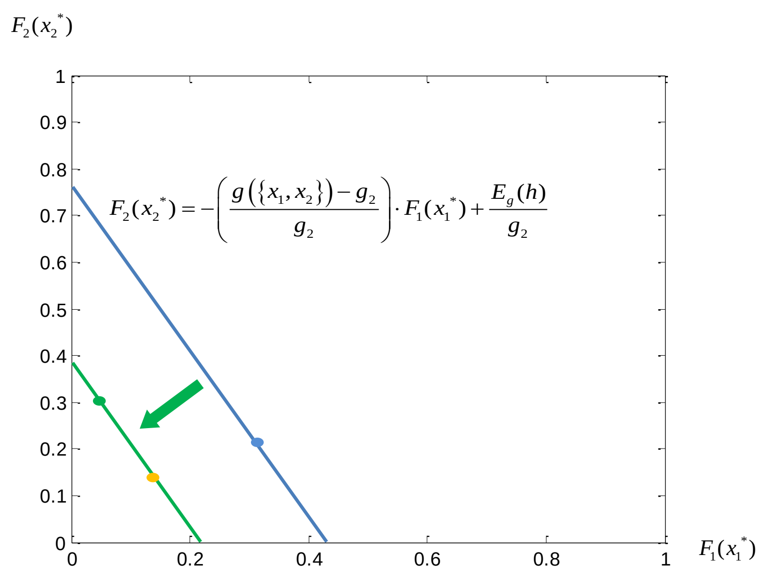


Figure 6.2.1 – An illustration to information of Choquet fuzzy integral

Figure 6.2.1 shows that the subset of points with equal value of Choquet fuzzy integral forms a straight line. Mathematically, all points on the line are exchangeable. One can see the line as an indifferent frontier on which all pair of $\left(F_1(x_1^*), F_2(x_2^*)\right)$ are equally good. The decision maker can easier evaluate the improvement alternatives by checking which can reach the furthest frontier. For example, the frontier is improved along with the green arrow and thus, the green point means a better preparation than the blue one. Furthermore, since all points on the line are indifferent, the yellow point may replace the green point as a better alternative if it costs less to implement.

Unlike the method of copula which treats the risks equally important, method of Choquet fuzzy integral applies human preference and the fuzzy parameter to impose fuzzy weights to risks. Therefore, improvement alternatives which lean to higher-weighted risks can stand on an advantageous position if evaluated by Choquet fuzzy integral. This prioritized characteristic makes Choquet fuzzy integral superior to take emphasis on improving satisfaction. This strength as well could complement the shortage of lacking focal points in the method of copula.

6.3 Integrated Information Model

In section 6.2 and 6.3, we have known that either copula or Choquet fuzzy integral model can contribute certain indispensable yet limited information to decision-making process. However, meaningful improvements can be done by integrating the advantages from both sides because they are mutually complementary. In summary, copula is based on an unbiased view and has strength in restricting risks through proper analysis of their joint probability. Hence, it is powerful tool to guard the decision in a safe domain. In general, the safe domain may require human judgment to define its boundary, e.g. an upper limit of joint probability. However, since it is reasonable to assume later decision should not violate those beforehand judgments, copula can ensure rationality in the integrated information model. On the contrary, Choquet fuzzy integral has strength on finding a solution corresponding to the decision maker's preference and prospect. Risks are prioritized and weighted through a scoring system which effectively reveals the real importance of each risk in the decision maker's perception. This weight structure gives painful risks precedence over other risks in the evaluation process. Therefore, Choquet fuzzy integral can provide information of the preference structure and ensure the level satisfaction in the integrated information model.

Theoretically both methods of copula and Choquet fuzzy integral can evaluate alternatives and find the optimal one in their own sense. An alternative optimal to either method is dominative and should be taken without questions. However, what should we do if there is no agreement? As stated in the previous paragraph, no method is overall superior to the other one. Therefore, there must be a way to find a compromised yet analytically acceptable solution. We assume that the level of satisfaction is the first priority objective since human is the end customer of the decision. Thus, the goal is set to be maximizing satisfaction in a reasonable and secured domain. Under this assumption, the integrated information model is composed of two major tasks: copula takes responsibility to determine the target domain and then Choquet fuzzy integral is in charge of optimizing satisfaction. The steps of integrated information model are show in the following. We follow the examples of two risks in section 6.1 and 6.2 and use the notations.

Step 1 – Determine feasible domain by copula

The first step is to determine the feasible domain. Here the decision maker can set target boundaries to individual risk and the joint probability as well. These boundaries can be obtained from the survival analysis of each risk and the decision maker's judgment. For example, the decision maker can have an objective value of VaR or CVaR and use them to set the boundaries. Then, let $\left(F_1(x_1)', F_2(x_2)'\right)$ denote the current status, and the feasible domain represented in the copula contour map would look like the shadow area in figure 6.3.1. Mathematically, the feasible domain can be expressed as:

$$S = \left\{\left(F_1(x_1), F_2(x_2)\right) \in [0,1] : F_1(x_1) \le a,\ F_2(x_2) \le b,\ \mathrm{C}\left(F_1(x_1), F_2(x_2)\right) \le c\right\}$$

where a and b are upper limits determined by survival analysis, $\mathrm{C}(\cdot)$ is the copula function and c is its upper limits. (6.3.1)

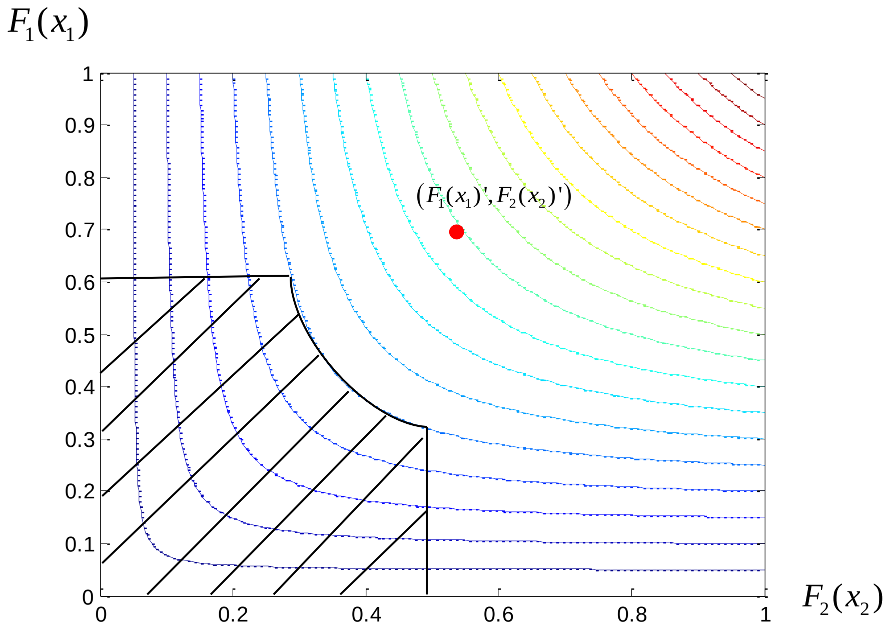


Figure 6.3.1 – Feasible domain determined by survival analysis and copula

Stept 2 – Selecting candidate alternatives by Choquet fuzzy integral

Mitigation of risks is usually a problem with great complexity. It is not easy to find the mathematical optimization, so in practice several initial alternatives are provided by experts in considerations of experience, budget limit, implementation, and other judgments. Further analysis has to be done to decide which one should be selected. After the feasible domain is determined in step 1, the alternatives outside the domain can be excluded immediately and the remaining ones are the candidates. Since these candidates already satisfy the constraint set by copula, the next step should be to maximize human satisfaction according to the preference structure. Choquet fuzzy integral has strength in representing human preference and can take over the task. As shown in section 6.2 and figure 6.2.1, the indifferent frontiers of Choquet fuzzy integral are straight lines with negative slopes. Since the left lower corner has lower value of Choquet fuzzy integral, i.e. higher level of satisfaction in this case, the alternatives below an indifferent frontier should always be selected other than the ones above the line. Figure 6.3.2 shows the illustration of this step.

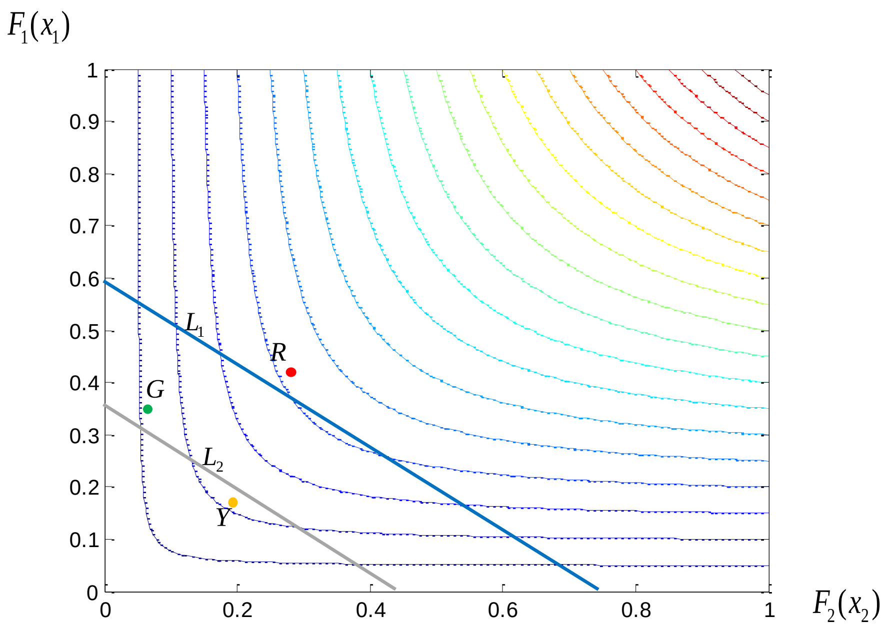


Figure 6.3.2 – An illustration of selecting alternatives using Choquet fuzzy integral

In figure 6.3.2, L_1 and L_2 represent the indifferent frontier of Choquet fuzzy integral. The points R, G, and Y in the feasible domain are our candidate alternatives. It can be seen that G, and Y both dominate R since they are superior no matter in aspect of copula or Choquet fuzzy integral. The interesting part is how to select between G, and Y : Y beats G in

Choquet yet G wins in copula. Under the assumption of maximal level of satisfaction, Y should be selected; however, this is not deterministic. For example, G, and Y may not have big difference in value of Coquet integral but have significant difference in copula. G's outstanding strength in copula may compensate its weakness in Choquet. Decision process is actually a series of judgments. The decision maker may furthermore consider other references to distinguish the alternatives. However, the decision maker still needs the analysis in this step to have comprehensive information and analytical supports.

Step 3 – Improving selected alternative to the equilibrium point

Given an alternative, the decision maker can use the integrated information model to search if there is possible better selection. In our model, an alternative can be improved either in copula or in Choquet fuzzy integral. Therefore, it is valuable to check if there are other choices which can make even with the current selection in one sense but have better performance in the other.

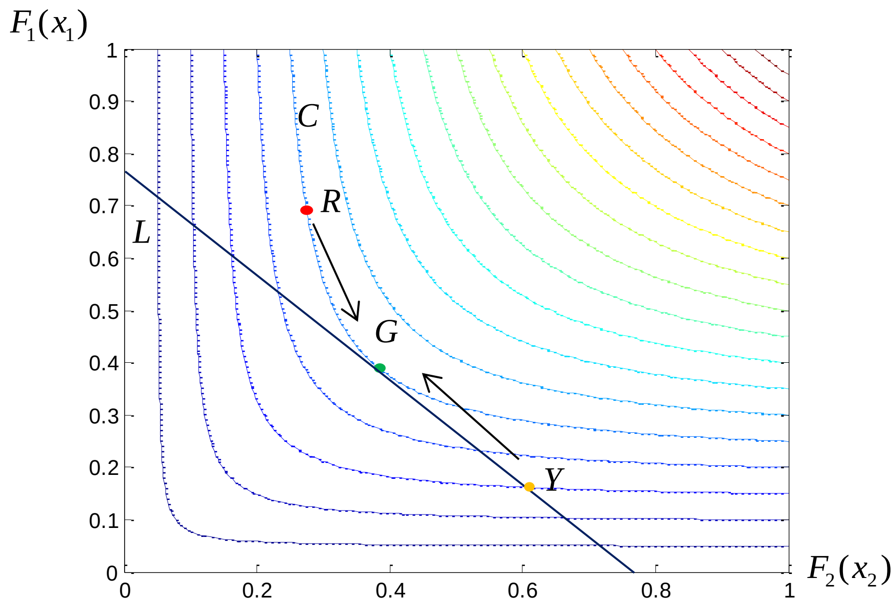


Figure 6.3.3 – An illustration of improving selected alternative via integrated information

Assuming point R in figure 6.3.3 is our current alternative, there must exist a contour line on which all points have the same value of copula, i.e., curve C. Then there also exists a point $G \in C$ where an indifferent frontier of Choquet, i.e. L, which is tangent to C. If $G \neq C$, then G is always better than C since it has the lowest value of Choquet fuzzy integral. The improvement is based on the concept of holding the value of copula and improve the alternative in the sense of Coquet fuzzy integral. One can do it the other way: holding the value of Choquet fuzzy integral and find improvement via copula. For example, supposing Y is our current alternative, then one can find an indifferent frontier of Choquet passing through Y, i.e. the line L. Then there must exist a contour line of copula which is tangent to L. The tangent point G is better than all the points on L. Thus, G is obtained by solve the maximization problem (P) if we start from a fixed value of Choquet fuzzy integral, or its dull problem (P') if we begin from a fixed copula value:

$$
\begin{aligned}
&(P) \\
&\max \quad C\left(F_1(x_1), F_2(x_2)\right) \\
&s.t. \quad F_2(x_2) \times g_2 + F_1(x_1) \times \left[g\left(\{x_1, x_2\}\right) - g_2 \right] = E_g(h) \\
&\qquad 0 \le F_1(x_1) \le 1 \\
&\qquad 0 \le F_2(x_2) \le 1 \\
&\qquad F_1(x_1), F_2(x_2) \in \mathbb{R} \\
&\qquad E_g(h) \text{ is a constant given by Choquet Fuzzy Integral model}
\end{aligned}
$$

$$
\begin{aligned}
&(P') \\
&\min \quad F_2(x_2) \times g_2 + F_1(x_1) \times \left[g\left(\{x_1, x_2\}\right) - g_2 \right] \\
&s.t. \quad C\left(F_1(x_1), F_2(x_2)\right) = c \\
&\qquad 0 \le F_1(x_1) \le 1 \\
&\qquad 0 \le F_2(x_2) \le 1 \\
&\qquad F_1(x_1), F_2(x_2) \in \mathbb{R} \\
&\qquad c \text{ is a constant given by copula model}
\end{aligned}
$$

where $C(\bullet)$ is the used copula function in copula model, g_1, g_2 are the fuzzy weights in Choquet fuzzy integral model.

The tangent point G is the equilibrium point of the Choquet indifferent line L and the copula contour curve C; in other words, G is superior than all other points on L and C. Let Ω denote this set of all the equilibrium points, then Ω forms a frontier line.

$$\Omega = \left\{ \left(F_1(x_1), F_2(x_2)\right) \in [0,1] : \frac{\partial \left[\psi^{-1}\left(\psi(c) - \psi(F_1(x_1))\right) \right]}{\partial F_1(x_1)} = \frac{g_2 - g\left(\{x_1, x_2\}\right)}{g_2}, \ \forall c \,|\, 0 < c < 1 \right\}$$

,where ψ is the generating function of copula.

Theoretically, Ω is the balance between copula and Choquet. It gives the decision maker very useful reference of the set of potential optimal solutions. S/he can start from this foundation and see other conditions, e.g. budgets, regulations, policies, implement ability and etc., to make the final decision.

Chapter 7
Conclusion and Future Research

7.1 Conclusion

In this dissertation, we have taken a psychophysical view to analyze approaches of risk assessment. We introduced human judgments and preference in decision making and investigated how these behaviors affected risks assessment. Moreover, we fully discussed the advantages and disadvantages of conventional approaches, and proposed copula and Choquet fuzzy integral models as improvement. Copula an Choquet fuzzy integral both can solve the problem of correlation when aggregating risks. The difference is copula copes probability correlation yet Choquet fuzzy integral deals with preference correlation. Both methods also can provide complementary information for further analysis. Copula has strength in finding safe solutions yet Chquet fuzzy integral has advantages in looking for demanded ones. Based on their complementary information, the integrated information model takes advantages from both side and provides a brand new idea to determine robust and demanded solutions.

The first job done by this dissertation is to systematically analyze current risk assessment approaches and summarize their advantages and shortcomings. Risk matrix has the advantages of easy construction and qualitative (judgmental) handling. However, it lacks sufficient resolution to compare risks in different cells, and the corresponding management policy does not consider the effects of possible correlations in severity or frequency. Probabilistic risk assessment requires important ameliorative tools like human judgment analysis and probability correlation analysis. Human judgments analysis deals with methods for modeling human mental effects while correlation analysis deals with methods for evaluating the effect of inter-risks dependencies which tend to cause simultaneous failures and thus significant increases in overall risk. The approaches using priority scoring and ranking fail to maximize risk-reduction benefits and are not appropriate for probability-correlated risks. Thus, optimization techniques that consider interdependencies among the consequences of different risk-reducing activities are essential.

Then we propose copula and Choquet fuzzy integral as partial solutions to the shortcomings of current approaches. The probability dependence can be handled by copula. The cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula. Among all copula families, Archimedean copula is less sensitive to the scale or location-based correlation, since they are based on Kendall Tau or rank correlation. On the other hand, Choquet fuzzy integral deals with dependence in severity. The preference structure of risks is obtained by scoring system and a fuzzy measure in Choquet fuzzy integral model.

The third job done is to display the crucial information that is elicited from copula and Choquet fuzzy integral model. With knowing the joint probability distribution from copula model, one can more precisely predict the underlying threat and have a more reliable strategy. Copula function yields the information about probability dependence structure of risks. Any improvement effort with respect to probability could be more effective or efficient if it is directed by joint distribution. The fuzzy measures in Choquet fuzzy integral model express the degree of importance of each risk. Due to framing effects, human may mentally exaggerate or diminish the measurements of risks. The fuzzy weights yield the information to outline the map of mental satisfaction. The fuzzy weights yield the information to outline the map of mental satisfaction.

The last, a mechanism to integrate the information of copula and Chquet fuzzy integral is developed in the integrated information model. In first step, initial alternatives are sieved by a feasible domain determined by the decision maker's judgments. These judgments are made by references of survival analysis and copula. The remaining candidates are then sorted by indifferent frontier of Choquet fuzzy integral. The alternative with optimal value of Choquet fuzzy integral is selected if the assumption of maximal satisfaction is made. However, the decision maker can still make another choice based on this result and other concerns. If there are no other limits, the equilibrium points of copula and Choqeut fuzzy integral form a frontier on which the optimal alternative is located.

7.2 Future Research

There are several areas that we consider for potential future researches. First, we assume that severity and possibility are orthogonal dimensions. However, people tend to overestimate the possibility of extremely severe risk for more conservative preparation. Similarly, people may exaggerate severity of a highly probable risk for the same reason. A further study to clarify the conditions and mechanisms of how severity interacts with possibility is still of interests.

Second, Archimedean copulas are designed to generate bivariate joint distribution. For higher dimensional random vectors, the joint distributions are not easy to obtain through Archimedean copula functions. A potential solution is to use bivariate copulas to approximate the desired information of higher dimensional distribution. Besides, it is also desired to test other copula forms in Archimedean family for better fitness or useful properties.

Third, in this dissertation, we present how to use copula and Choquet fuzzy integral to compute risk index. Also, how to integrate information from both approaches is also proposed. These approaches mainly focus to appropriately evaluate strategies for selection. However, sometimes the decision maker may need a reverse approach. That is, if we want to control the risk index within a target value, then what is the best strategy? This is a concept of two-way management. A better strategy may be obtained after forward and backward improvement.

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APPENDICES

Appendix A

Mathematical Model of LIFEREG and PHREG

Survival data have two common features that are difficult to handle with conventional statistical methods: censoring and time-dependent variables (Allison 2010). For example, one possibility is to perform a logistic regression analysis with a dichotomous dependent variable: failed or not failed. However, this analysis ignores information on the timing of failure. It's reasonable to suspect that pumps which are failed in two days when a hurricane arrives have a higher potential to be failed than those which are not failed until two weeks. Thus, regressions which can deal with the above difficulties should be used in analysis of competing risks problem. These regression models should be able to predict the survival functions, underlying density functions and hazard functions for any values of covariates.

According to Allison (2010), SAS currently has the most comprehensive set of full-featured procedures for doing survival analysis. Therefore, SAS procedures, such as LIFEREG and PHREG will be used as the calculating tool and the associated regression models will be explained in corresponding sections. Both LIFEREG and PHREG are based on linear regression models and the parameters are estimated by maximal likelihood method.

1. PROC LIFEREG

PROC LIFEREG estimates regression models with censored, continuous time data under several alternative distributional assumptions (User Guide, SAS Institute Inc., 2008).

The LIFEREG procedure fits parametric models to failure time. The models for the response variable consist of a linear effect composed of the covariates and a random term. The distribution of the random disturbance term can be taken from a class of distributions. In most of cases, Weibull distribution is a good option due to its simplicity and flexibility. In interpretation, this random distribution is corresponding to the underlying probability distribution of survival time defined in 2.1.

The model assumed for the response y is

$$y = X\beta + \sigma\varepsilon$$

where y is a vector of response values, often the log of the event times, X is a matrix of covariates, β is a vector of unknown regression parameters, σ is an unknown scale parameter, and ε is a vector of errors assumed to come from a known distribution (such as Weibull distribution). The distribution might also depend on additional shape parameters. These models are equivalent to accelerated failure time models when the log of the response is the quantity being modeled. The effect of the covariates in an accelerated failure time model is to change the scale, and not the location, of a baseline distribution of event times.

The model described above is based on an accelerated failure time model (Kalbfleisch and Prentice, 1980). Usually, the scale is $\exp(x_c^{'}\beta_c)$, where x_c is the vector of covariate values and β_c is a vector of unknown parameters. Thus, if T_0 is an event time sampled from the baseline distribution corresponding to values of zero for the covariates, then the accelerated failure time model specifies that, if the vector of covariates is x_c , the event time is $T = \exp(x_c^{'}\beta_c)T_0$. If $y = \log(T)$ and $y_0 = \log(T_0)$, then

$$y = x_c^{'}\beta_c + y_0$$

This is a linear model with y_0 as the error term.

In terms of survival probabilities, this model is

$$\Pr(T > t \mid x_c) = \Pr(T_0 > \exp(-x_c^{'}\beta_c)t)$$

The probability on the left-hand side of the equal sign is evaluated given the value x_c for the covariates, and the right-hand side is computed using the baseline probability distribution but at a scaled value of the argument. The right-hand side of the equation represents the value of the baseline survival function evaluated at $\exp(-x_c^{'}\beta_c)t$.

Models usually have an intercept parameter and a scale parameter. In terms of the original untransformed event times, the effects of the intercept term and the scale term are to scale the event time and to raise the event time to a power, respectively. That is, if

$$\log(T_0) = \mu + \sigma \log(T_\varepsilon)$$

,then

$$T_0 = \exp(\mu)T_\varepsilon^\sigma$$

(This section is edited from User's Guide, SAS Institute Inc., 2008)

2. PROC PHREG

PROC PHREG uses Cox's partial likelihood method (Cox, 1972) to estimate regression models with censored data. The accelerated failure time model used in PROC LIFEREG assumes a parametric form for the effects of the explanatory variables and usually assumes a parametric form for the underlying survival function. Cox's proportional hazards model also assumes a parametric form for the effects of the covariates variables, but it allows an unspecified form for the underlying survival function.

The PHREG procedure performs regression analysis of survival data based on the Cox proportional hazards model. Cox model is widely used in the analysis of survival data to explain the effect of covariates on hazard rates.

The survival time of each observation is assumed to follow its own hazard function, $h_i(t)$, expressed as

$$h_i(t) = h(t; x_i) = h_0(t)\exp(x_i^{'}\beta)$$

where $h_0(t)$ is an arbitrary and unspecified baseline hazard function, x_i is the vector of covariates for the i^th^ observation, and β is the vector of unknown regression parameters associated with the covariates. Then,

$$\log(h_i(t)) = \log(h_0(t)) + x_i^{'}\beta$$

is a linear model with error term $\log(h_0(t))$.

To estimate β, Cox (1972, 1975) introduced the partial likelihood function, which eliminates the unknown baseline hazard $h_0(t)$. Consider the hazard of two observations, $h_i(t)$ and $h_j(t)$, the hazard ratio is defined as

$$HR_{ij} = \frac{h_i(t)}{h_j(t)} = \frac{h_0(t)\exp(x_i^{'}\beta)}{h_0(t)\exp(x_j^{'}\beta)} = \exp\left[(x_i^{'} - x_j^{'})\beta\right]$$

In this equation, $h_0(t)$ cancels out of the numerator and denominator. As a result, the ratio of the hazards is constant over time. If we graph the log hazards for any two individuals, the proportional hazards property implies that the hazard functions should be strictly parallel.

The population can consist of a number of subpopulations, each of which has its own baseline hazard function. PROC PHREG performs a stratified analysis to adjust for such subpopulation differences. Under the stratified model, the hazard function for the j^{th} observation in the i^{th} stratum is expressed as

$$h_{ij}(t) = h_{i0}(t)\exp(x_{ij}^{'}\beta)$$

where $h_{i0}(t)$ is the baseline hazard function for the i^{th} stratum, and x_{ij} is the vector of covariates for the observation. The regression coefficients are assumed to be the same for all observation across all strata.

With the estimated parameters, the survivor function can be expressed as

$$S(t;x_i) = [S_0(t)]^{\exp(x_i^{'}\beta)}$$

where $S_0(t) = \exp(-\int_0^t h_0(u)du)$ is the baseline survivor function.

3. Numerical Demonstration

PROC LIFEREG

There are total four types of risk in the demonstrated case. Here, only the risk of mechanical failure is shown as an example. See Figure A.1 for the result of SAS.

Analysis of Parameter Estimates

Parameter	DF	Estimate	Standard Error	95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept	1	2.2903	0.2284	1.8427	2.7379	100.57	<.0001
age	1	0.2791	0.0372	0.2062	0.3520	56.26	<.0001
servo	1	-0.1118	0.0401	-0.1905	-0.0332	7.77	0.0053
Scale	1	0.1941	0.0161	0.1650	0.2284		
Weibull Shape	**1**	**5.1516**	**0.4279**	**4.3777**	**6.0623**		

Figure A.1 – SAS output of LIFEREG

According the output, the estimated parameters of underlying distribution are obtained by

location parameter $\lambda = \exp\left(intercept\right) = e^{2.2903} = 9.878$ and

shape parameter $\beta = \dfrac{1}{Scale} = 5.1516$.

For this weibull distribution, the survival distribution and density function for untransformed baseline distribution are obtained by corresponding survival function: $S_{T_0}\left(t\right) = \exp(-(\frac{t}{\lambda})^{\beta})$

and its probability density function $s_{T_0}\left(t\right) = -\left(\frac{\beta}{\lambda}\right)(\frac{t}{\lambda})^{\beta-1}\exp(-(\frac{t}{\lambda})^{\beta})$

Then, the corresponding cumulative distribution function of failure time (complementary of survival function) can be written as

$$F_{T_0}\left(t\right) = 1 - S_{T_0}\left(t\right) = 1 - \exp(-(\frac{t}{\lambda})^{\beta})$$

The survival time $T = \exp\left(x^{'}\beta\right)T_0$, β is the vector of parameters and x is the vector of values of covariates. For example, for a pump station with covariates age = 5.3, and servo = 0, then the event time is

$$T = \exp\left(5.3 \times 0.2791\right)T_0 \approx 4.389T_0$$

As well, the distribution of event time can be calculated by

$$F_T\left(t\right) = F_{T_0}\left(\frac{1}{4.3989}t\right) = 1 - S_{T_0}\left(\frac{1}{4.389}t\right) = 1 - \exp\left(-\left(\left(\frac{t}{4.389\lambda}\right)^{\beta}\right)\right) = 1 - exp(-(0.023t)^{1.4066}$$

PROC PHREG

PROC PHREG provides the hazard ratios for competing risks which can be used to easily compare the hazards of different individuals. An example of risk Flood is showing below. See Figure A.2 for the SAS outputs.

```
                              The PHREG Procedure

                    Analysis of Maximum Likelihood Estimates

             Parameter    Standard                            Hazard   Variable
  Variable      DF       Estimate       Error    Chi-Square  Pr > ChiSq   Ratio    Label

      gear      1      -0.85323    0.36749      5.3907      0.0202      0.426    gear
 trashrack      1       0.39118    0.19099      4.1950      0.0405      1.479    trashrack
```

Figure A.2 – SAS output of PHREG

The output presents that the stations with ‘gear’ are associated with 58 percent decrease in the risk of flood; on the contrary, with additional trash rack, the stations are associated with 48 percent increase in flood. In other words, one can reduce the risk of flood either by adding a gear or removing trash rack. PHREG gives a result from which the decision maker can understand how and how much the covariates could alter the hazards of risks. This is important for forming and evaluating the candidate strategies.