THE COMPUTATION OF TEMPERATURE DISTRIBUTIONS AND THERMAL STRESSES USING FINITE ELEMENT TECHNIQUES

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ABSTRACT

A description is given of a method for calculating transient temperature distributions within structures of arbitrary shape by means of finite element techniques. Two typical nuclear applications, a cylinder-sphere intersection and a cylinder-cylinder intersection are used to illustrate the method. Thermal stresses in these structures are indicated and compared.

1. INTRODUCTION

The calculation of temperatures and stresses in arbitrarily shaped two-dimensional and axisymmetric structures has been possible for some years. Computer programs performing such calculations have employed either finite difference or finite element techniques and both have yielded valuable results. However, a finite difference grid can be uneconomic in numbers of interior grid points when a boundary follows a devious route whereas the boundary has little effect on internal nodes of a finite element mesh. With a general three-dimensional structure a finite difference method could be very expensive to use. For this reason the latest computer programs written at the C.E.G.B.'s Berkeley Nuclear Laboratories use the finite element method and contain libraries of finite elements, including the three-dimensional iso-parametric elements proposed by Irons [1]. These elements allow curved surfaces to be treated easily and accurately. Together with their equivalents in two-dimensional and cylindrical coordinates, the iso-parametric elements have now been used in many different structures.

B.N.L. programs include the simple triangle (Turner et al [2]) and tetrahedron (Argyris [3]) amongst other finite elements. The user of these programs may choose from some twenty different elements those most suitable for his particular problem or mesh generation facilities. It is largely due to the fact that the generation of meshes of say, 1000 triangles has been comparatively easy with the aid of automated mesh definition devices or automatic mesh generation programs that the triangle has been a popular element. However, there is now a tendency to use a smaller number of more complex iso-parametric elements. In three dimensions the tetrahedron is associated with mesh generation problems for many engineers and has been infrequently used except where large numbers could be automatically generated.

In nuclear plant, pressure and temperature are typical stress inducing effects. Having specified the mesh of elements for a structure and the restraints to be applied, pressure loading involves little more than the naming of the loaded element faces. Temperature induced stresses, however, are often time-dependent and in general do not vary at the same rate throughout a structure. Consequently it is advisable to calculate structure temperatures prior to calculating stresses and select some sets of temperatures at just a limited number of time values.

This paper describes the method by which the temperatures within a structure are obtained with the aid of finite elements and gives typical examples of the method applied to a cylinder-sphere intersection and to a cylinder-cylinder intersection. Some calculation approximations are assessed and some of the final thermal stresses are briefly indicated.

2. CALCULATION OF STEADY-STATE AND TRANSIENT TEMPERATURES

The finite element program used within the C.E.G.B. for the calculation of stresses in arbitrary structures is called BERSAFE (Hellen [4]) and that used for temperature calculation in such structures is called FLHE (Fullard [5]). For convenience these programs will be referred to by name in the remainder of the paper.

FLHE was written so that transient (that is, time-dependent) temperatures could be made available for the calculation of thermal stresses as well as being calculated as an end in themselves. Since a wide range of finite elements is available to the user of BERSAFE, it is convenient to allow a similar choice to the user of FLHE. It is then possible to use the same mesh of elements for both computations and avoid any intermediate interpolation of temperatures because of mesh differences. The two programs are not combined because of the need for a user decision before the stresses are calculated. Nodal temperatures are calculated at a series of time coordinates and although it is quite possible to obtain thermal stresses for each of these times it could be an expensive procedure for a large number of elements. Consequently, a pause is advisable in order that just one or two sets of temperatures can be selected for thermal stress calculations.

In BERSAFE, the degrees of freedom in a structure are nodal displacements, with up to three displacements per node. These degrees of freedom are obtained from the solution of a stiffness matrix equation $[K]\{u\} = \{\lambda\}$. In FLHE, the nodal temperatures are the structural degrees of freedom, with a single temperature value at each node. The matrix equations involved are derived in the Appendix to this report and have the form

$$[K]\{\phi\} = \{F\} \tag{1}$$

for a steady-state problem (that is, independent of time) and

$$([G]_t + \frac{2}{\Delta t}[P]_t)\{\phi\}_t = (-[G]_{t-\Delta t} + \frac{2}{\Delta t}[P]_t)\{\phi\}_{t-\Delta t} + \{F\}_t + \{F\}_{t-\Delta t} \tag{2}$$

for a transient problem. Each of the structural matrices in these equations is formed from the individual element matrices.

The solution of the steady-state equation (1) is straightforward and identical to the solution of the stiffness equation. The sole difficulty might be that for any given element the material properties are required in the formation of its elemental matrices. As these properties are based on the centroidal temperature of the element an estimate of the nodal temperatures must be supplied, the centroidal temperature being a function of nodal temperatures. The implication is that final solution of (1) is achieved through an iterative procedure, but this is generally unnecessary for engineering purposes provided that the effect of an error in temperature estimation can be ignored. A front solution method (Hellen [6]) is used to give the required unknowns.

Solution of the transient equation (2) for successive time values yields as many sets of nodal temperature as are desired, each set being based on the preceding set. At the initial time the nodal temperatures throughout the structure must be defined in order that the computation may begin, but unlike the steady-state problem there is no requirement for an estimate of the unknown temperatures at subsequent times.

Matrices [G], [P] and vector {F} of equation (2) must be formed for every set of $\{\phi\}_t$ required, as they change value with changes in elemental temperature or thermal loading. For this reason the various element matrices are formed for unit values of conductivity, heat transfer etc. and written on to a file. Then, for each solution, the file is read and the element matrices are scaled according to the actual values appropriate to the time or temperature. $\{\phi\}_{t-\Delta t}$, the vector of previously calculated nodal temperatures, together with the other matrices, forms a right hand side of 'loads' and a left hand side of 'stiffness coefficients' so that equation (2), when fully set up, is similar to equation (1) and uses a front method of solution to give $\{\phi\}_t$.

The solutions, $\{\phi\}_t$ to equation (2) tend to oscillate about some mean temperature/time curve (Wilson and Nickell [7]) as a result of the linearising of the partial differential heat flow equation. A transient problem is considered as several sets of nodal temperatures between pairs of time points with loading being defined at the time points and linear interpolation being used for intermediate loading values. If solutions are required at the time points only, the oscillations will be apparent. If two or more solutions per pair of time points are required, then FLHE carries out a procedure designed to reduce the level of oscillation. If $\{\phi\}_o$ is the initial set of temperatures and $\{\phi\}_1$ and $\{\phi\}_2$ the next sets calculated, an averaged $\{\phi\}_1^*$ is calculated as $0.25(\{\phi\}_o + 2\{\phi\}_1 + \{\phi\}_2)$. The set $\{\phi\}_2$ is recalculated using $\{\phi\}_1^*$ on the right hand side of equation (2). Subsequent sets of temperature are calculated without further averaging until the next pair of time points is considered. The procedure is then repeated.

In many cases the averaging procedure results in an acceptably smooth transient solution, but because it is based on an assumption of linear variation of temperature with time it follows that a small oscillation will still be detectable. Figure 1 shows the effect of averaging at the start of each pair of time points in the case of heat loss across the boundary of a circular hole in an 'infinite' plate. The plate was initially at unit temperature and fluid passed through the hole at zero temperature. When the period between pairs of time points was divided into five intervals, Δt, agreement between the finite element and theoretical surface temperatures was good. However, when calculations were made only at the time points (one interval between pairs of time points) the oscillations of the finite element solution were obvious. It was only necessary to consider a sector of plate as the heat flow is radial in this problem, but as the radial dimensions cannot extend to infinity, the whole plate began to cool at times greater than 5. The surface temperature given by the finite element mesh then fell away from the theoretical curve as a result of this cooling.

3. TYPICAL PROBLEMS

A typical cylinder-cylinder intersection is shown in Figure 2. Such an intersection might be subjected to internal pressure and to a gradual increase in the temperature of a fluid passing along the nozzle and into the main vessel. The flow would not normally divide in the main vessel but would move unidirectionally. For this reason, it is necessary to consider half the intersection as shown. Under internal pressure loading, only a quarter of the intersection would have to be considered.

Figure 3 shows a variation of fluid temperature together with a typical heat transfer coefficient variation. The positional variation of heat transfer coefficient is shown as contours on the developed inner surfaces of the main vessel and one of the inlet nozzles (Figures 4 and 5).

The typical axisymmetric problem chosen as an example is a cylinder-sphere intersection. To illustrate the differences between the two intersections, the cylinder-sphere intersection has been given the dimensions of the cross-section of the cylinder-cylinder intersection. Heat transfer coefficients corresponding to the cylinder-cylinder cross-section have been used for the cylinder-sphere.

3.1 The Cylinder-Sphere Intersection

The cylinder-sphere was studied by means of axisymmetric iso-parametric elements having four corner and four edge nodes. A mesh of 52 elements was arranged so that there were two elements through the vessel wall, giving a maximum of five nodal temperatures in this direction.

Heat transfer coefficients normally vary with time but the effect of using a constant mean value of heat transfer coefficient per exposed element surface has been examined. Figures 6 and 7 show temperature contours plotted at a time of 900 sec. for cases of variable heat transfer and constant heat transfer. At this time the difference between inside vessel and outside vessel temperatures was a maximum. The general pattern of contours is similar in both cases but the greatest temperature difference through the vessel wall was 90 deg.C for variable heat transfer and only 79 deg.C for constant heat transfer. The constant heat transfer coefficients were values appropriate to a time of 600 sec. and at this time the temperature difference was 80 deg.C for variable heat transfer and 79 deg.C for constant heat transfer. Evidently constant heat transfer coefficients may be used successfully but the actual choice of coefficients is important if maximum temperature differences are being sought.

Thermal stresses in any assemblage of finite elements are calculated by the usual methods (Zienkiewicz [8]) using the BERSAFE program. Figure 8 shows hoop thermal stresses as contours for temperatures at 900 sec. obtained with variable heat transfer coefficients.

3.2 The Cylinder-Cylinder Intersection

Three-dimensional iso-parametric elements were used to study the cylinder-cylinder intersection. The elements had eight corner and 24 edge nodes. The use of the three dimensional equivalents of the axisymmetric elements would have involved too large a mesh for the program. The elements used for the cylinder-cylinder allow a pseudo cubic variation of temperature as compared with a pseudo quadratic variation of temperature allowed by the elements used for the nozzle-sphere. A single solid element of the cubic type gave four nodal temperatures through the vessel wall, and there were 80 elements in the complete mesh. Variable heat transfer coefficients were used for the cylinder-cylinder mesh.

Temperature contours appropriate to a time of 900 sec. are shown on Figure 9 for the cross-section of the cylinder-cylinder. These contours are very similar to those on Figure 6, the main difference being that the absolute temperatures at the intersection are lower in the three-dimensional case. Comparison of the two models is possible because of the geometric identity. The maximum temperature differences are virtually identical (89 deg.C for the cylinder-cylinder cross-section and 90 deg.C for the cylinder-sphere). In the nozzle, away from the intersection, the inner wall temperatures are much closer for the two meshes, but the contours are wider apart for the cylinder-sphere than for the cylinder-cylinder, indicating a lower temperature gradient through the nozzle wall. There is a maximum of 9 deg.C between the temperature differences across the nozzle wall obtained from the two meshes.

Figure 10 shows axial thermal stresses in the cross-section of the cylinder-cylinder caused by temperature variation at a time of 900 sec. The important region is the intersection area. A 100 MN/m^2 tensile stress appears near the weld-nozzle junction for both problems considered (see figure 8) and the compressive stresses on the inner surface of both nozzles are very similar.

More intensive investigations would be needed to assess completely the value of axisymmetric idealisations and these were outside the scope of the work reported here. With regard to pressure loading of a cylinder-cylinder, Money [9] has carried out such investigations. It is possible that temperature may be found to be only a minor effect in certain high pressure cases. However, from the work so far, it would appear that the cylinder-sphere can provide a valid indication of the axial thermal stresses in the cross-section of a cylinder-cylinder intersection.

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APPENDIX

Time Dependent Heat Flow Solved by Means of Finite Elements

The following is an expansion of the description given by Zienkiewicz [8] of a method for dealing with field problems.

The equation of conduction of heat in a homogeneous, isotropic solid takes the form:-

$$k\nabla^2\phi + Q = \rho c \frac{\partial\phi}{\partial t} \tag{1}$$

OR

$$k\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right\} + Q - \rho c \frac{\partial\phi}{\partial t} = 0 \tag{1a}$$

where k is a conductivity coefficient (W/m deg.C),

ϕ is temperature (deg.C),

Q is rate of heat generation (W/m^3),

ρ is mass density (kg/m^3),

c is specific heat (J/kg deg.C),

t is time (s),

and x, y, z are coordinates.

At the surface of the solid, the following equation specifies boundary conditions:-

$$k \frac{\partial\phi}{\partial n} = q + H(\psi - \phi) \tag{2}$$

OR

$$k\left\{l \frac{\partial\phi}{\partial x} + \frac{m\partial\phi}{\partial y} + \frac{n\partial\phi}{\partial z}\right\} - q - H(\psi - \phi) = 0 \tag{2a}$$

where n of eq. (2) is the outward normal direction,

q is flux (W/m^2), positive inwards,

H is surface heat transfer coefficient (W/m^2 deg.C),

ψ is ambient temperature (deg.C),

and l, m, n are direction cosines of the outward normal.

At a particular instant of time, $\partial\phi/\partial t$ is constant and eq. (1) reduces to Poisson's equation $[\nabla^2\phi = f(x,y,z)]$. Eq. (1), together with the boundary conditions, specifies the heat flow problem in a unique manner. However, an alternative formulation is possible with the aid of the calculus of variations. The Euler theorem states that if the integral

$$I(u) = \iiint f(x,y,z,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})\, dx\, dy\, dz \tag{3}$$

is to be minimized, then the necessary and sufficient condition for this minimum to be reached is that the unknown function u(x,y,z) should satisfy the following differential equation

$$\frac{\partial}{\partial x}\left\{\frac{f}{\partial(\partial u/\partial x)}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial f}{\partial(\partial u/\partial y)}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial f}{\partial(\partial u/\partial z)}\right\} - \frac{\partial f}{\partial u} = 0 \tag{4}$$

within the same region, provided u satisfies the same boundary conditions in both cases.

If $f(x,y,z,\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$ is given as

$$f = \frac{k}{2} \{(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2\} - (Q - \rho c \frac{\partial\phi}{\partial t}) \phi \tag{5}$$

then eq. (4) is

$$\frac{\partial}{\partial x} \{k \frac{\partial\phi}{\partial x}\} + \frac{\partial}{\partial y} \{k \frac{\partial\phi}{\partial y}\} + \frac{\partial}{\partial y} \{k \frac{\partial\phi}{\partial z}\} + Q - \rho c \frac{\partial\phi}{\partial t} = 0 \tag{4a}$$

which is identical to eq. (1a).

The problem is thus one of minimising the functional

$$\chi(\phi) = \iiint \{\frac{k}{2} [(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2] - (Q - \rho c \frac{\partial\phi}{\partial t}) \phi\} \, dx \, dy \, dz \tag{6}$$

in which $\partial\phi/\partial t$ is an invariant.

Using finite elements, ϕ at any point within a given element is expressible in terms of the nodal temperatures of the element:-

$$\phi = \sum_i N_i \, (x,y,z) \, \phi_i$$

or

$$\phi = [N]\{\phi\}^e \tag{7}$$

Superscript e specifies consideration of a given element.

The process of minimising the 'functional' χ is best accomplished with respect to the nodal values of ϕ. The contributions to each differential, such as $\partial\chi/\partial\phi$, from a typical element, should be evaluated, then added to all other contributions and equated to zero.

If the value of χ associated with a given element is called χ^e (implying an integration limited to the volume of the element) we can write, by differentiating eq. (6):-

$$\frac{\partial\chi^e}{\partial\phi_i} = \iiint \{k \, [\frac{\partial\phi}{\partial x} \, \frac{\partial}{\partial\phi_i} \, (\frac{\partial\phi}{\partial x}) + \frac{\partial\phi}{\partial y} \, \frac{\partial}{\partial\phi_i} \, (\frac{\partial\phi}{\partial y}) + \frac{\partial\phi}{\partial z} \, \frac{\partial}{\partial\varphi_i} \, (\frac{\partial\phi}{\partial z})]$$

$$- (Q - \rho c \frac{\partial\phi}{\partial t}) \frac{\partial\phi}{\partial\phi_i}\} \, dx \, dy \, dz \tag{8}$$

From eq. (7) we have:

$$\frac{\partial\phi}{\partial x} = \sum_i (\frac{\partial N_i}{\partial x} \cdot \phi_i) \quad \text{etc.}$$

or

$$\begin{Bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \cdots\cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \cdots\cdots & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial z} & \cdots\cdots & \frac{\partial N_n}{\partial z} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{Bmatrix}^e = [A]\{\phi\}^e \tag{9}$$

and assuming that the element has n nodes, [A] is a 3xn matrix.

Now, from eq. (9):- $$\frac{\partial}{\partial \phi_i}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial N_i}{\partial x} \quad \text{etc.}$$

or

$$\frac{\partial}{\partial \phi_i}\begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \{A_i\} \tag{10}$$

From eq. (7), $$\frac{\partial \phi}{\partial \phi_i} = N_i \tag{11}$$

Similarly, $$\frac{\partial \phi}{\partial t} = \sum_i \frac{\partial}{\partial t}(N_i\ \phi_i) = \sum_i N_i \frac{\partial \phi_i}{\partial t}$$

or $$\frac{\partial \phi}{\partial t} = [N]\{\partial\phi/\partial t\}^e \tag{12}$$

Eq. (8) can now be rewritten:-

$$\frac{\partial \chi^e}{\partial \phi_i} = [\iiint \{k\ \{A_i\}^T[A]\}\ dx\ dy\ dz]\ \{\phi\}^e - Q \iiint N_i\ dx\ dy\ dz$$
$$+ [\rho c \iiint Ni[N]dx\ dy\ dz]\ \{\partial\phi/\partial t\}^e \tag{13}$$

k, Q and ρc are assumed constant throughout the element.

When all nodes on the element are considered, eq. (13) becomes

$$\{\frac{\partial \chi}{\partial \phi}\}^e = k\ [\iiint [A]^T[A]\ dx\ dy\ dz]\ \{\phi\}^e - Q\ \{\iiint [N]^T\ dx\ dy\ dz\}^e$$
$$+ \rho c\ [\iiint [N]^T[N]\ dx\ dy\ dz]\ \{\frac{\partial \phi}{\partial t}\}^e \tag{14}$$

or $$\{\frac{\partial \chi}{\partial \phi}\}^e = k[g_b]^e\{\phi\}^e - Q\ \{f_b\}^e + \rho c\ [p_b]^e\{\partial\phi/\partial t\}^e \tag{14a}$$

Where subscript b implies 'within an element'.

Now, considering surface behaviour, the functional given by eq. (6) has to be modified by the addition to the right hand side of:-

$$-(q + H\psi) \int_A \phi\ dA + H \int_A \frac{\phi^2}{2}\ dA \tag{6a}$$

in which the integrals are over the area of the face in question.

The additional terms on the right hand side of eq. (8) are:-

$$-\ (q + H\psi) \int_A \frac{\partial \phi}{\partial \phi_i}\ dA + H \int_A \phi\ \frac{\partial \phi}{\partial \phi_i}\ dA \tag{8a}$$

and eq. (13) is modified by the addition of:-

$$-(q + H\psi) \int_A N_i \, dA + H \, [\int_A N_i [N] dA] \{\phi\}^e \tag{13a}$$

Finally, eq. (14) becomes modified by

$$-(q + H\psi)\{ \int_A [N]^T dA\} + H \, [\int_A [N]^T [N] dA [\{\phi\}^e \tag{14b}$$

or

$$-(q + H\psi)\{f_s\}^e + H \, [g_s]^e \{\phi\}^e \tag{14c}$$

where subscript s implies 'at a surface'.

Eq. (14a) and Eq. (14c), for all elements, when added together are equated to zero. For the complete system we have:-

$$\{\partial\chi/\partial\phi\} = [G]\{\phi\} - \{F\} + [P]\{\partial\phi/\partial t\} = \{0\} \tag{15}$$

where $[G]$ is formed by summing all $k[g_b]^e$ and $H[g_s]^e$,

$\{F\}$ is formed by summing all $Q\{f_b\}^e$ and $(q + H)\{f_s\}^e$,

$[P]$ is formed by summing all $\rho c[p_b]^e$,

$\{\phi\}$ is a vector of all nodal temperatures,

and $\{\partial\phi/\partial t\}$ is a vector of all nodal temperature gradients.

Solution of Equation (15)

If the problem is steady state, (i.e. independent of time), eq. (15) is readily solved in the form:

$$[G]\{\phi\} = \{F\} \tag{16}$$

However, solution of the transient, (i.e. time-dependent) problem requires eq. (15) to be modified. It is assumed that in the interval of time Δt, the values of $\{\partial\phi/\partial t\}$ vary linearly with time.

Then

$$\{\phi\}_t = \{\phi\}_{t-\Delta t} + \frac{\Delta t}{2} \left(\{\frac{\partial\phi}{\partial t}\}_t + \{\frac{\partial\phi}{\partial t}\}_{t-\Delta t}\right) \tag{17}$$

This procedure was suggested by Wilson and Clough [10] in the context of dynamics problems. Alternatively, eq. (17) may be expressed in the form:-

$$\{\frac{\partial\phi}{\partial t}\}_t = -\{\frac{\partial\phi}{\partial t}\}_{t-\Delta t} + \frac{2}{\Delta t} \left(\{\phi\}_t - \{\phi\}_{t-\Delta t}\right) \tag{17a}$$

Now consider eq. (15) at time $t-\Delta t$ and time t:-

$$[G]_t\{\phi\}_t - \{F\}_t + [P]_t\{\partial\phi/\phi t\}_t = \{0\} \tag{18}$$

$$[G]_{t-\Delta t}\{\phi\}_{t-\Delta t} - \{F\}_{t-\Delta t} + [P]_{t-\Delta t}\{\partial\phi/\partial t\}_{t-\Delta t} = \{0\} \tag{19}$$

Substitute for $\{\partial\phi/\partial_t\}_t$ in eq. (18) by means of eq. (17a):-

$$[G]_t\{\phi\}_t - \{F\}_t - [P]_t\{\partial\phi/\partial t\}_{t-\Delta t} + \frac{2}{\Delta t}[P]_t\{\phi\}_t - \frac{2}{\Delta t}[P]_t\{\phi\}_{t-\Delta t} = \{0\} \tag{20}$$

Adding eq. (19) to eq. (20) and rearranging yields:-

$$([G]_t + \frac{2}{\Delta t}[P]_t)\{\phi\}_t = (-[G]_{t-\Delta t} + \frac{2}{\Delta t}[P]_t)\{\phi\}_{t-\Delta t} + \{F\}_t + \{F\}_{t-\Delta t} \qquad (21)$$

$\{\phi\}_{t-\Delta t}$ is known from a preceding solution (or as a vector of initial nodal temperatures) and therefore $\{\phi\}_t$ is found from the solution of eq. (21) which is in a form comparable with eq. (16).

<u>Non-Dimensional Coordinates</u>

Suppose that N_i (see eq. (7)) is a function of non-dimensional coordinates ξ, η, ζ (as it is with iso-parametric elements). Then:-

$$\frac{\partial\phi}{\partial\xi} = \sum_i \left(\frac{\partial N_i}{\partial\xi}\phi_i\right) \text{ etc.}$$

or

$$\begin{Bmatrix} \dfrac{\partial\phi}{\partial\xi} \\ \dfrac{\partial\phi}{\partial\eta} \\ \dfrac{\partial\phi}{\partial\zeta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial\xi} & \cdots\cdots & \dfrac{\partial N_n}{\partial\xi} \\ \dfrac{\partial N_1}{\partial\eta} & \cdots\cdots & \dfrac{\partial N_n}{\partial\eta} \\ \dfrac{\partial N_1}{\partial\zeta} & \cdots\cdots & \dfrac{\partial N_n}{\partial\zeta} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{Bmatrix}^e = [\delta]\{\phi\}^e \qquad (22)$$

Now x, y and z are functions of ξ, η and ζ of the same form as ϕ, i.e. $x = [N]\{x\}^e$. Thus for x, y and z there is an equation similar to eq. (22):-

$$\begin{bmatrix} \dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\xi} & \dfrac{\partial z}{\partial\xi} \\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial y}{\partial\eta} & \dfrac{\partial z}{\partial\eta} \\ \dfrac{\partial x}{\partial\zeta} & \dfrac{\partial y}{\partial\zeta} & \dfrac{\partial z}{\partial\zeta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial\xi} & \cdots\cdots & \dfrac{\partial N_n}{\partial\xi} \\ \dfrac{\partial N_1}{\partial\eta} & \cdots\cdots & \dfrac{\partial N_n}{\partial\eta} \\ \dfrac{\partial N_1}{\partial\zeta} & \cdots\cdots & \dfrac{\partial N_n}{\partial\zeta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix}$$

$$= [\delta][x\ y\ z]^e = [J] \qquad (23)$$

[J] is called the Jacobian Matrix.

Now $$\frac{\partial\phi}{\partial\xi} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\xi} + \frac{\partial\phi}{z}\frac{\partial z}{\partial\xi}$$

etc.

Thus, since x, y and z are functions of ξ, η and ζ we have:-

$$\begin{Bmatrix} \dfrac{\partial\phi}{\partial\xi} \\ \dfrac{\partial\phi}{\partial\eta} \\ \dfrac{\partial\phi}{\partial\zeta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\xi} & \dfrac{\partial z}{\partial\xi} \\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial y}{\partial\eta} & \dfrac{\partial z}{\partial\eta} \\ \dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\zeta} & \dfrac{\partial z}{\partial\zeta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\phi}{\partial x} \\ \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\phi}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial\phi}{\partial x} \\ \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\phi}{\partial z} \end{Bmatrix} \qquad (24)$$

Inserting eq. (24) and substituting eq. (22) gives:-

$$\begin{Bmatrix} \frac{\partial\phi}{\partial x} \\ \\ \frac{\partial\phi}{\partial y} \\ \\ \frac{\partial\phi}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial\phi}{\partial\xi} \\ \\ \frac{\partial\phi}{\partial\eta} \\ \\ \frac{\partial\phi}{\partial\zeta} \end{Bmatrix} = [J]^{-1}[\delta]\{\phi\}^e \tag{25}$$

And this is compared with eq. (9) yielding:-

$$[A] = [J]^{-1}[\delta] \tag{26}$$

Transforming from x, y, z coordinates to ξ, η, ζ, changes the integration

$$\iiint dx\, dy\, dz \qquad \text{to} \quad \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\, d\xi\, d\eta\, d\zeta$$

where $\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} = |J|$, the determinant of the Jacobian.

Thus, for example, $[g_b^i] = [\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [A]^T[A].|J|\ d\xi\ d\eta\ d\zeta]$

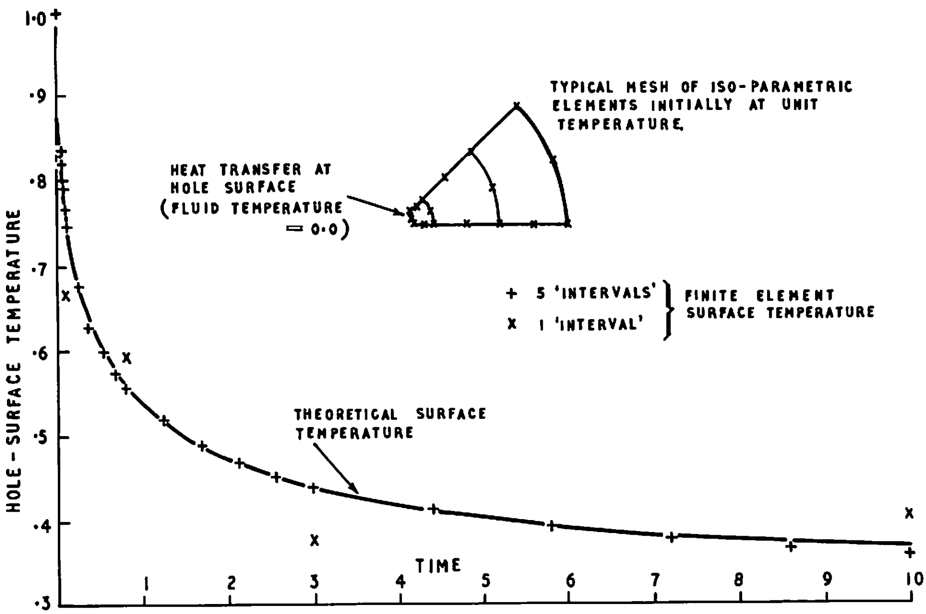


FIG. 1. HEAT LOSS AT A HOLE IN AN 'INFINITE' PLATE.

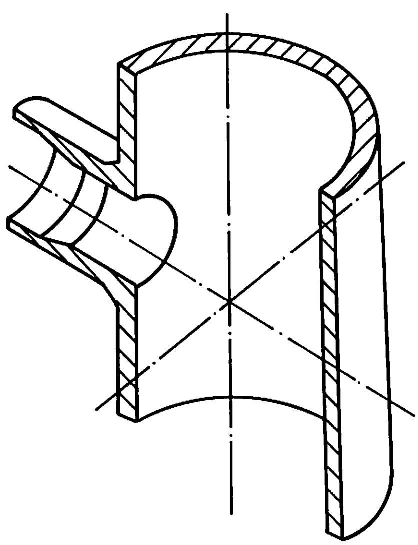


FIG. 2. TYPICAL CYLINDER-CYLINDER INTERSECTION.

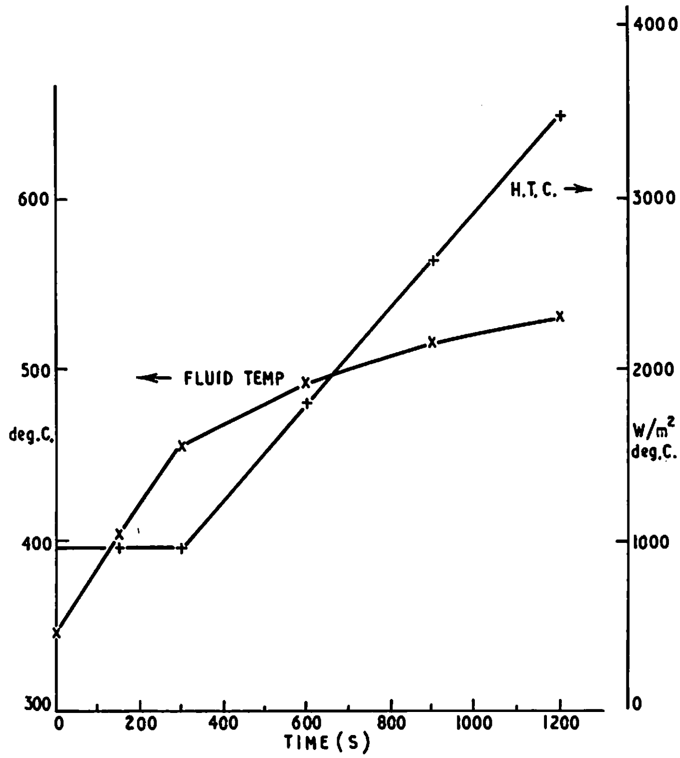


FIG. 3. TYPICAL HEAT TRANSFER COEFFICIENT VARIATION AND FLUID TEMPERATURE VERSUS TIME.

HEAT TRANSFER COEFFICIENTS IN W/m^2 deg.C
TIME = 575 S.

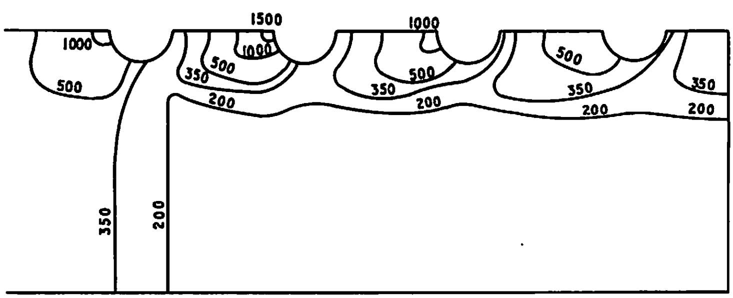


FIG. 4. MAIN VESSEL COEFFICIENTS.

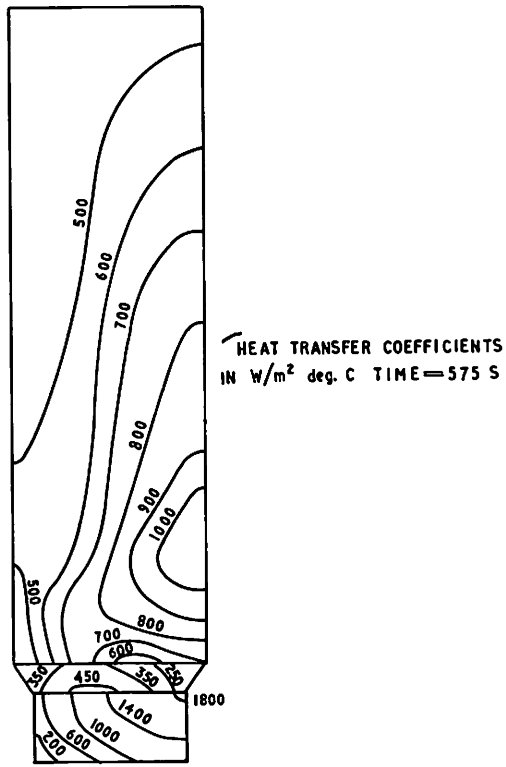


FIG. 5. NOZZLE COEFFICIENTS.

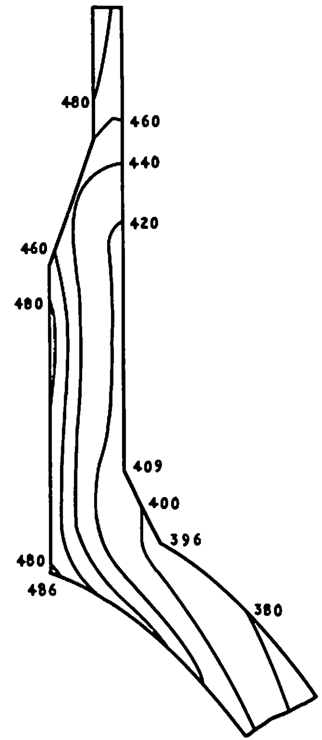


FIG.6. TEMPERATURE CONTOURS - CYLINDER - SPHERE - VARIABLE HEAT TRANSFER

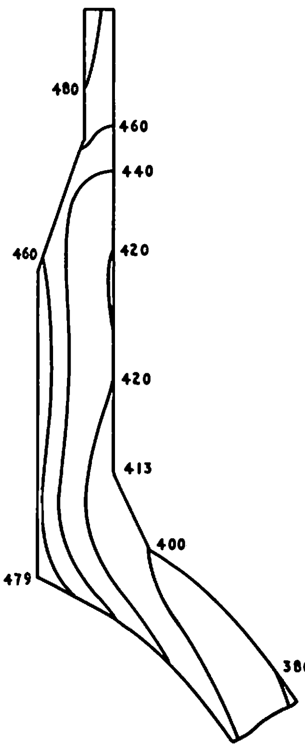


FIG. 7. TEMPERATURE CONTOURS-CYLINDER-SPHERE-
CONSTANT HEAT TRANSFER.

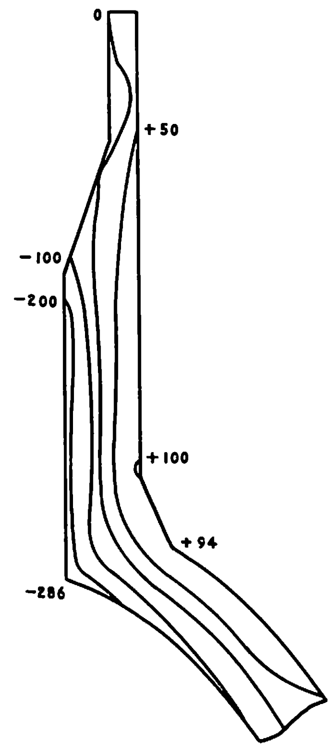


FIG. 8. THERMAL HOOP STRESS CONTOURS-
CYLINDER-SPHERE.

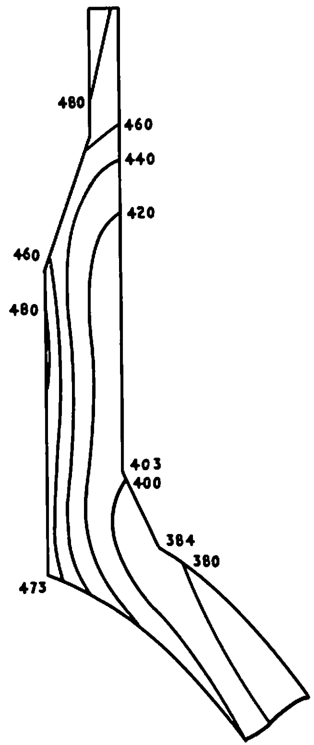


FIG. 9. TEMPERATURE CONTOURS-CYLINDER - CYLINDER-
VARIABLE HEAT TRANSFER.

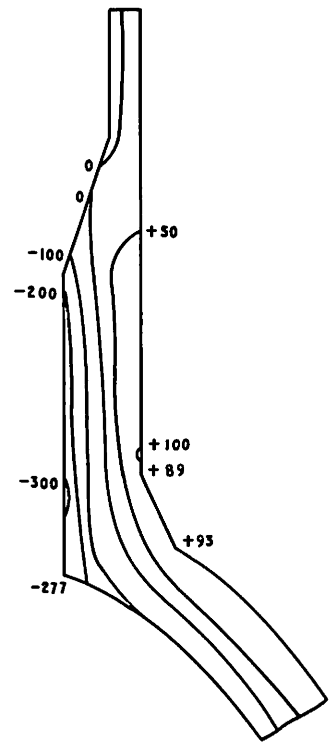


FIG.10. THERMAL AXIAL STRESS CONTOURS -
CYLINDER — CYLINDER.

DISCUSSION

Q Z. ZUDANS, U. S. A.

You have shown that cylinder to sphere approximation for cylinder-to-cylinder problem can be used. Can you give us any information as to how the approximation of cylinder-to-cylinder can be made, by cylinder to sphere as far as geometry is concerned.

A K. FULLARD, U. K.

The approximation of a cylinder-cylinder intersection by an axisymmetric structure (the cylinder-sphere) was strictly only coincidental as far as paper M 5/3 was concerned. The two structures were chosen as typical examples of the application of finite element techniques. However, given that the highest stresses (in the smaller cylinder) were similar in both structures, it would appear that using the cross-sectional geometry of the cylinder-cylinder intersection for the cylinder-sphere, as in the paper, is a reasonable geometrical approximation.

C F. C. WEILER, U. S. A.

In the paper, it was stated that the thermal solution $\{\phi\}_t$ had a tendency to oscillate as a result of linearising the governing partial differential equations in time. Consequently an averaging technique was proposed to eliminate these oscillations.
I would like to make a comment about the reasons why oscillations appear. The transient temperature field solution techniques which have appeared in the literature are all of the "Crank-Nicholson" method of solution type, i. e.,

$$\left[\frac{1}{\Delta t}C+\frac{1}{2}K\right]\{\pi^n\}=\left[\frac{1}{\Delta t}C-\frac{1}{2}K\right]\{\pi^\circ\}+\{Q\}$$

where C = heat capacity matrix, K = conductivity matrix, π^n, π° are the new and old nodal point temperatures, Q = the heat flux vector and Δt = incremental time step. It can be seen in this equation that the weighting of the conductivity matrix K is half on the old temperatures and half on the new ones. In contrast, the "Pure Implicit" method of solution technique is

$$\left[\frac{1}{\Delta t}C+K\right]\{\pi^n\}=\left[\frac{1}{\Delta t}C\right]\{\pi^\circ\}+\{Q\}$$

and the "Pure Explicit" method of solution technique is,

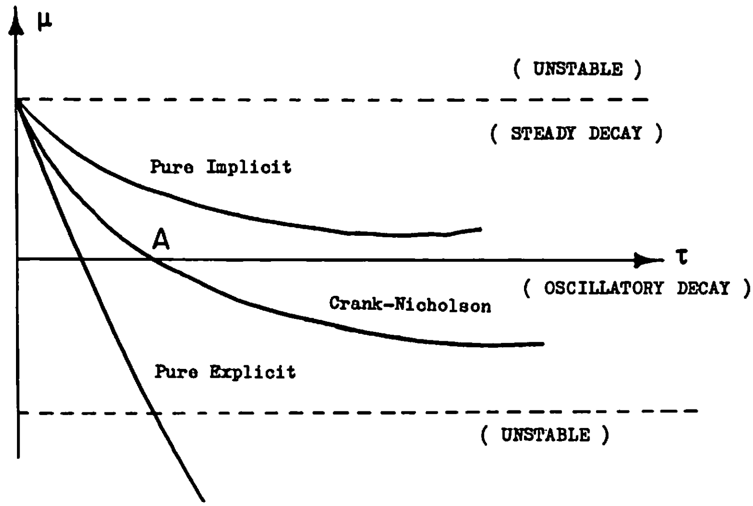
$$\left[\frac{1}{\Delta t}C\right]\{\pi^n\}=\left[\frac{1}{\Delta t}C-K\right]\{\pi^\circ\}+\{Q\}$$

Notice that in the "Pure Implicit" technique, all of the weighting of the conductivity K is on the new temperatures and for the "Pure Explicit" technique, ont the old temperatures. All three of these methods of solution may be characterized by

$$T^n = \quad T^o$$

where μ is some function of K, C and Δt (assuming $Q = 0$). Notice, if

$1 \leq \mu$		unstable growth
$0 \leq \mu$	≤ 1	steady decay, stable
$-1 \leq \mu$	≤ 0	oscillatory decay, stable
μ	≤ -1	unstable, oscillatory growth



Here, τ = some function of K , C and Δ t. Now, if Δ t is small, the Crank-Nicholson technique will produce a solution which will steadily decay in time, with no oscillations. However, if Δ t surpasses some critical value (point A in the above plot of μ versus τ , then the Crank-Nicholson technique will produce an oscillatory decay type of solution. It is interesting to notice that the "Pure Implicit" method is never oscillatory whereas the "Pure Explicit" is first steady decay, secondly, oscillatory decay and finally unstable, all with increasing allowable time increments Δ t.

Hence, it is my opinion that the oscillatory behavior reported in Mr. Fullard's paper is due to the fact of choosing a Δ t too large and not to the linearization of the partial differential equations in time. However, in order to calculate the temperature fields in an "economic" manner, one must chose a time step, Δ t, large enough to render such solutions for as few time steps as accuracy requires. His averaging technique performs this task, however, I personally would feel uneasy using it in that I might "average out" some response which is present in reality.

Numerous other results have been determined, however, these shall be published in a future paper.

A K. FULLARD, U. K.

Dr. Weiler's comment on paper M 5/3 is accepted as a more precise reason for solution oscillation when the Crank-Nicholson technique is used than the author's. (That is, oscillation occurring as a result of linearising the equation). As Dr. Weiler further remarks, 'one must choose a time step, Δ t, large enough to render such solutions for as few time steps as accuracy requires'. However, the author has found that for a given Δ t, in the oscillatory decay regime, the accuracy of solution is dictated by the degree of mesh refinement and by the order of finite element employed. With a smaller Δ t, a coarse mesh simply gave a smoother, invalid temperature versus time curve. So far, our experience is that the mesh is more important than the choice of time step.

The possibility of 'averaging out' some actual response is presumably dependent upon smoothing out the structure thermal loading rather than on the use of an averaging procedure. If a step or near-step change in loading occurs, it would be expected that the initial and final loading conditions would be specified so that the structure could respond approximately.