SUB-STRUCTURE ANALYSIS FOR ELASTIC FIELDS WITH STRESS SINGULARITIES

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SUMMARY

In the complex structures of Nuclear Engineering, interfaces between different materials give rise to steep stress gradients under load, thermal or other environments. In an elastic sense, the structural field contains stress singularities. Generally, direct differential equation methods and normal finite element procedures have serious limitations for accurately analysing practical problems of this type, and properly identifying and estimating singular stresses. So continuum-finite element hybrid formulations have been suggested, in which a differential equation solution is used conjointly with a normal finite element breakdown of the structure. There are situations where this concept can be very effectively extended to develop a finite element formulation of the sub-structure type, in which each sub-structure consists of a "single finite element", so that the total structure is represented by only a handful of finite elements.

To illustrate the method, consider the "end problem" of a rectangular strip in which one end is clamped, the opposite end is stressed or stretched, and the two transverse edges are stress free. Under thermal or elastic loading a stress singularity arises at each of the clamped corners. At each corner a biharmonic stress function can be developed as a series of corner functions which satisfy exactly the local set of clamped and free edge conditions. The corresponding displacement functions are used to describe a large element of suitable shape preferably a large sector containing the corner. These two corner elements form two sub-structures.

Similar special displacement descriptions can also be built up to identically satisfy the stress free conditions on the transverse edges of the strip. So, a third element with this description, covering the rest of the strip, is introduced as another sub-structure. The stiffness for each sub-structure is calculated using the strain matrix from the respective stress functions and applying a Gaussian quadrature procedure. The stiffness matrices are assembled and solved for displacements and stresses by a standard finite element displacement procedure. Solution convergence is easily studied by increasing the number of nodes on each interface. The results are cross-checked with a parallel analytical solution.

The special features of this analysis are that complex configurations can be considered, only a few elements are needed, matrix orders are small, stress singularities are properly estimated, solution convergence is through element refinement and, results are highly accurate. The method can be used for three-dimensional situations where suitable displacements descriptions are possible. Symmetries and skew-symmetries in the stress fields, if any, can be taken advantage of, for reducing the number of sub-structures in the problem and thus the size of the matrices.

Introduction

In the complex structures of Nuclear Engineering, there are many structural interfaces between different materials, which give rise to steep stress gradients under load, thermal or other environments. In an elastic sense, the structural field contains stress singularities. Generally, direct differential equation methods and normal finite element procedures have serious limitations for accurately analysing practical problems of this type and properly identifying and estimating the singular stresses therein. So, continuum-finite element hybrid formulations have been suggested [1,2], in which the differential equation solution is used conjointly with a finite element breakdown of the structure. There are situations where this concept can be very effectively extended to develop a finite element formulation of the sub-structure type, in which each sub-structure consists of a single finite element so that the total structure is represented by only a handful of finite elements. This paper presents and discusses this method of sub-structure analysis for elastic fields with stress singularities, considering the 'end problem' of a rectangular strip as the example.

The Problem

To illustrate the method, we consider the 'end problem' of a rectangular strip (Fig.1), in which one end is clamped, the opposite end is stressed or stretched and the transverse edges are stress free. Under thermal or elastic loading, a stress singularity arises at each of the clamped corners and the problem is to predict the stress distribution on this clamped edge.

The Method

Because of the symmetry in the problem, we consider only one half, ABCDE of the structure in Fig.1. The boundary conditions to be satisfied are

on AB : $u = 0,\ v = 0$
on BD : $\sigma_y = 0,\ \sigma_{xy} = 0$
on ED : $\sigma_x = 1.0,\ \sigma_{xy} = 0$
on AE : Symmetry conditions

We divide this half of the structure into two sub-structures (1) and (2). We refer to sub-structure (1) as the CF-element and substructure (2) as HT-element.

CF-Element

Referring to Fig.2, at corner B we develop a biharmonic stress function as a series of corner functions [3], which satisfy exactly the set of clamped and free edge conditions. The corner function is

$$\Phi_{CF} = \sum \alpha_n r^{\lambda_n+1}\left(A_1 \sin(\lambda_n+1)\theta + A_2 \cos(\lambda_n+1)\theta + A_3 \sin(\lambda_n-1)\theta + A_4 \cos(\lambda_n-1)\theta\right)$$

Where α_n are free constants.

$$A_4 = 1.0\,, \quad ANR = ((1+\nu)(\lambda+1)\cos(\lambda-1)\tfrac{\pi}{2} + (4-(1+\nu)(\lambda+1)\cos(\lambda+1)\tfrac{\pi}{2})$$

$$ADR = ((1+\nu)(\lambda+1)\sin(\lambda-1)\tfrac{\pi}{2} - (4+(1+\nu)(\lambda-1)\sin(\lambda+1)\tfrac{\pi}{2})$$

$$A_3 = -ANR/ADR\,, \quad A_2 = (4-(1+\nu)(\lambda+1))/(1+\nu)(\lambda+1)$$

$$A_1 = -A_3(4+(1+\nu)(\lambda-1))/(1+\nu)(\lambda+1)$$

When the clamped and free edge conditions ($u_r = u_\theta = 0$ on $\theta = 0$, clamped edge, and $\sigma_\theta = \sigma_{r\theta} = 0$ on $\theta = \pi/2$, corresponding to the free edge) are satisfied exactly, the stress function φ_{CF} yields an eigenequation with a set of eigenvalues, the first of which is realand the rest are complex. The corresponding displacement functions are used to describe a large element of a suitable shape, in this case a large sector containing the corner B. This defines the CF-element.

<u>HT-Element</u>

The second (HT) element in the problem then takes a shape shown in Fig.3. Considering the symmetry of the stress field about AB and the free edge conditions on CD, we write for this element a stress function in biharmonic hyberbolic-trigonometric series as:

$$\phi_{HT} = \sigma_0 y^2/2 + \sum \beta_n\left(\sin\mu_n \cos\mu_n\tfrac{y}{b} - \tfrac{y}{b}\cos\mu_n \sin\mu_n\tfrac{y}{b}\right)\cosh\mu_n\tfrac{x}{b}$$

$$+ \sum \gamma_n\left(\sin\mu_n \cos\mu_n\tfrac{y}{b} - \tfrac{y}{b}\cos\mu_n \sin\mu_n\tfrac{y}{b}\right)\sinh\mu_n\tfrac{x}{b}$$

where β_n and γ_n are free constants.

φ_{HT} satisfies exactly the symmetry conditions and the free edge conditions on $y = b$ by acquiring a set of complex eigenvalues μ_n from the characteristic equation

$$\sin 2\mu + 2\mu = 0$$

<u>Basic Finite Element Equations</u>

The stresses and displacements in the two elements are obtained from the relationships in the Appendix. The displacements in an element can be represented by

$$\{u\} = \lfloor u_n \rfloor \{\alpha_n\} \qquad \ldots \quad (1a)$$

It is sometimes possible to separate out some components of $\{u\}$, generally those arising from a simple stress system statically equivalent to the load passing through the element, as explicit functions $\{u_0\}$. Thus we can re-write equation (1a) for each sub-structure element as

$$\{u\}_i = \{u_0\}_i + [u_n]_i \{\alpha_n\}_i \qquad \ldots \quad (1b)$$

In the conventional finite element procedures the $[u_n]_i$ are expressed in simple functional sequences like polynomials (power series, Hermité, Chebychev, etc.) or fourier series. In the present method, the $[u_n]_i$ are chosen such that at any stage of truncation, they correspond to the terms in a stress function identically satisfying the differential equation as well as the boundary conditions on the specified edges. Then, integrating successively we have, with a termwise correspondence with eq.(1a), the element stresses

$$\{\sigma\}_i = \{\sigma_o\}_i + [D_n]_i \{\alpha_n\}_i \qquad \ldots (2)$$

the element stress function

$$\phi_i = \phi_{oi} + \sum \phi_{ni} \alpha_{ni}$$
$$= \phi_{oi} + \{\phi_n\}_i^T \{\alpha_n\}_i$$

and the auxiliary displacement function [ ] for the element

$$\psi_i = \psi_{oi} + \{\psi_n\}_i^T \{\alpha_n\}_i$$

By introducing the proper nodal coordinates in eq. (1b), the nodal displacement matrix $\{U\}_i$ for all nodes of the i^{th} sub-structure element, can be written as

$$\{U\}_i = \{U_o\}_i + [U_n]_i \{\alpha_n\}_i \qquad (3)$$

<u>The Flexibility Matrix for the i^{th} sub-structure element</u>

By the principle of virtual displacements, we write the total potential energy of the i^{th} element as

$$(\Pi_p)_i = V_i - W_i = \frac{1}{2}\int_{v_i} \{\epsilon\}_i^T [E] \{\epsilon\}_i \, dv - \{U\}_i^T \{F\}_i$$

where [E] is the elasticity matrix defined by

$$\{\sigma\}_i = [E]\{\epsilon\}_i$$

Minimizng this energy with respect to generalised coordinates $\{\alpha_n\}_i$

$$\{F_o\}_i + [H]_i\{\alpha_n\}_i - [U_n]_i^T\{F\}_i = 0 \qquad (4)$$

where

$$[H]_i = \int_A [D_n]_i^T [E]^{-1} [D_n]_i \, t \, dA$$

and

$$\{F_o\}_i = \int_A [D_n]_i^T [E]^{-1} \{\sigma_o\}_i \, t \, dA$$

If $[u_n]_i$ is a rectangular matrix which occurs in problems like the 'end problem', inversion for $[u_n]_i$ is non-unique. So we rearrange eq.(4) as follows:

$$\{\alpha_n\}_i = [H]_i^{-1}[U_n]_i^T\{F\}_i - [H]_i^{-1}\{F_o\}_i \tag{5}$$

Now the nodal displacements can be expressed as

$$\{U\}_i = \{U_{no}\}_i + [U_n]_i[H]_i^{-1}[U_n]_i^T\{F\} \tag{6}$$

where

$$\{U_{no}\}_i = \{U_o\}_i - [U_n]_i[H]_i^{-1}\{F_o\}_i$$

Using eq.(6), the stress functions given for the CF-element and HT-elements and the relations given in the appendix, we write

$$\{U\}_{CF} = [U_n]_{CF}[H]_{CF}^{-1}[U_n]_{CF}^T\{F\}_{CF} \tag{7}$$

$$\{U\}_{HT} = \{U_{no}\}_{HT} + [U_n]_{HT}[H]_{HT}^{-1}[U_n]_{HT}^T\{F\}_{HT} \tag{8}$$

Eq.(7) defines the relationship of the nodal displacements and nodal forces in the CF-element through a flexibility matrix. $[U_n]_{CF}\ [H]_{CF}^{-1}\ [U_n]_{CF}^T$. Similarly for the HT-element, they are related through $[U_n]_{HT}\ [H]_{HT}^{-1}\ [U_n]_{HT}^T$.

Assembly of the elements

The assembly of the final flexibility matrix is done by matching the displacements at the nodal points on the common boundary AC (Fig.4).

For example, if we connect the two elements through the node point 2, we match the conditions at node point 2' in CF-element and 2 in HT-element. The conditions are

$$F_{CF} = -F_{HT}$$

$$U_{CF} = U_{HT}$$

This physically implies that at node point 2, since no external force is applied and no slip is allowed both the elements should yield equal, but opposite nodal forces, and exhibit the same displacements.

For the CF-element

$$\begin{bmatrix} U_1' \\ V_1' \\ U_2' \\ V_2' \\ U_3' \\ V_3' \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{16} \\ P_{21} & \cdots & \cdots & P_{26} \\ \vdots & & & \vdots \\ P_{61} & \cdots & \cdots & P_{66} \end{bmatrix} \begin{bmatrix} F_1' \\ F_2' \\ F_3' \\ F_4' \\ F_5' \\ F_6' \end{bmatrix}$$

For the HT-element

$$\begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \\ U_5 \\ V_5 \\ U_6 \\ V_6 \end{bmatrix} = \begin{bmatrix} U_{no1} \\ U_{no2} \\ U_{no3} \\ U_{no4} \\ U_{no5} \\ U_{no6} \\ U_{no7} \\ U_{no8} \\ U_{no9} \\ U_{no10} \\ U_{no11} \\ U_{no12} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{16} & X_{17} & \cdots & X_{112} \\ X_{21} & & & \vdots & \vdots & & \vdots \\ \vdots & & & & & & \\ X_{61} & \cdots & & X_{66} & X_{67} & \cdots & X_{612} \\ X_{71} & \cdots & & X_{76} & X_{77} & \cdots & X_{712} \\ X_{81} & & & \vdots & \vdots & & \vdots \\ \vdots & & & & & & \\ X_{121} & \cdots & & X_{126} & X_{127} & \cdots & X_{1212} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{bmatrix}$$

and from the matching conditions we write

$$U_1' = U_1 \,, \quad V_1' = V_1 \,, \quad U_2' = U_2 \,, \quad V_2' = V_2 \,, \quad U_3' = U_3 \,, \quad V_3' = V_3$$

and

$$F_1' = -F_1 \,, \quad F_2' = -F_2 \,, \quad F_3' = -F_3 \,, \quad F_4' = -F_4 \,, \quad F_5' = -F_5 \,, \quad F_6' = -F_6$$

using the above conditions, we assemble the final relations for the complete structure as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ U_4 \\ V_4 \\ U_5 \\ V_5 \\ U_6 \\ V_6 \end{bmatrix} = \begin{bmatrix} U_{no1} \\ U_{no2} \\ U_{no3} \\ U_{no4} \\ U_{no5} \\ U_{no6} \\ U_{no7} \\ U_{no8} \\ U_{no9} \\ U_{no10} \\ U_{no11} \\ U_{no12} \end{bmatrix} + \begin{bmatrix} -(X_{11}+P_{11}) & -(X_{12}+P_{12}) & \cdots & -(X_{16}+P_{16}) & X_{17} & \cdots & X_{112} \\ -(X_{21}+P_{21}) & -(X_{22}+P_{22}) & \cdots & -(X_{26}+P_{26}) & X_{27} & \cdots & X_{212} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -(X_{61}+P_{61}) & -(X_{62}+P_{62}) & \cdots & -(X_{66}+P_{66}) & X_{67} & \cdots & X_{612} \\ -X_{71} & -X_{72} & \cdots & -X_{76} & X_{77} & \cdots & X_{712} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -X_{121} & -X_{122} & \cdots & -X_{126} & X_{127} & \cdots & X_{1212} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{bmatrix}$$

With proper boundary conditions, if any, we partition the matrix and obtain the solution

$$\begin{bmatrix} \{p\} \\ \{q\} \end{bmatrix} = \begin{bmatrix} [X_{11}] & [X_{12}] \\ [X_{21}] & [X_{22}] \end{bmatrix} \begin{bmatrix} \{f_1\} \\ \{f_2\} \end{bmatrix}$$

In the present problem, we know $\{p\}$, $\{f_2\}$, $[x_{11}]$, $[x_{12}]$, $[x_{21}]$, $[x_{22}]$. So we can determine $\{f_1\}$ and $\{q\}$. Using $\{f_1\}$ and the corresponding relations eq. (5,2) for the CF-element, we can determine the stress distribution on the clamped edge.

Results and Discussion

Fig.5 and 6 present the normal stresses and the shear stresses on the clamped edge of the 'end problem' using the two-element breakdown indicated in Fig.1. The CF-element has five nodes and the stiffness is calculated using 6 eigenvalues with 11 free real constants. The HT-element has ten nodes requiring 5 eigenvalues with 10 real free constants. The inter-element boundary carries five nodes.

The results indicate the singular nature of the stress distribution on the clamped edge very well. More results and discussion will be presented later. The coefficient α_1 of the term yielding the corner singular stress for the short strips (length/width = 0.75) is 0.57857. This is to be related to the value of 0.57817 obtained by Benthem [5] for an infinite strip.

References

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[4] Coker, E.G., Filon, L.N.G., 'A Treatise on Photoelasticity', Cambridge University Press, 1931.

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APPENDIX

Basic Relationships

It is convenient to use Coker and Filon's Auxiliary Displacement function [4] to define the stresses and displacements from a stress function.

Function	Symbol	Coordinates: Polar	Coordinates: Cartesian
Stress function	ϕ	$\varphi(r,\theta)$	$\varphi(x,y)$
Auxiliary displ. function	ψ	$(r\psi_\theta)_r = \nabla^2\phi,\ \nabla^2\psi = 0$	$\psi_{xy} = \nabla^2\phi,\ \nabla^2\psi = 0$
Normal stress		$\sigma_r = \phi_r/r + \phi_{\theta\theta}/r^2$ $\sigma_\theta = \phi_{rr}$	$\sigma_x = \phi_{yy}$ $\sigma_y = \phi_{xx}$
Shear stress		$\sigma_{r\theta} = -(\phi_\theta/r)_r$	$\sigma_{xy} = -\phi_{xy}$
Normal displacement		$EU_r = -(1+\nu)\phi_r + r\psi_\theta$	$EU = -(1+\nu)\phi_x + \psi_y$
Tangential displacement		$EU_\theta = -(1+\nu)\phi_\theta + r^2\psi_r$	$EV = -(1+\nu)\phi_y + \psi_x$

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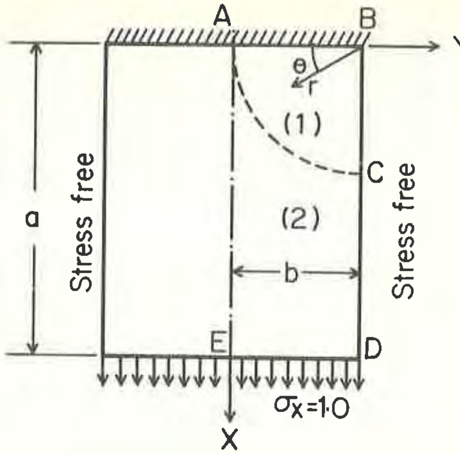

FIG. 1. THE PROBLEM AND THE COORDINATE SYSTEM

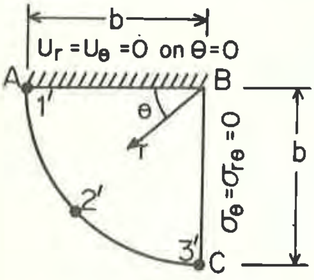

FIG. 2. CF - ELEMENT

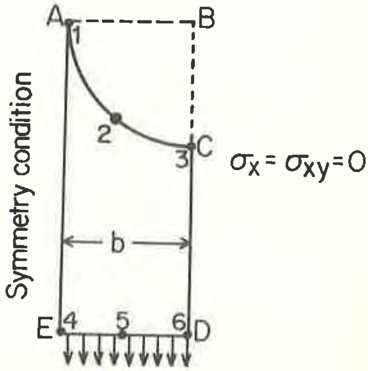

FIG. 3. HT - ELEMENT

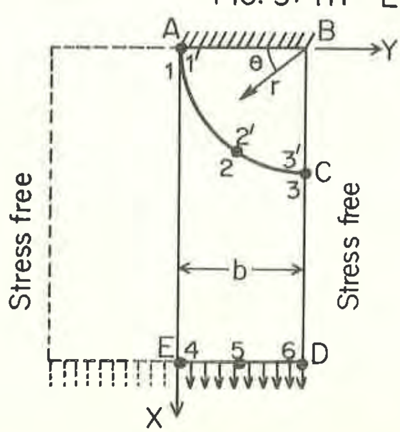

FIG. 4. ASSEMBLY OF ELEMENTS

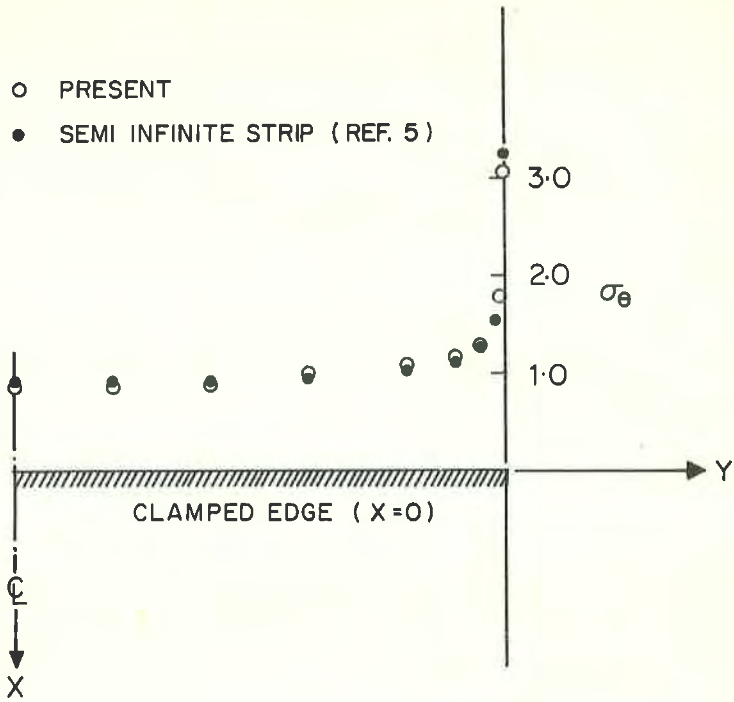

FIG. 5 NORMAL STRESS DISTRIBUTION AT X=0

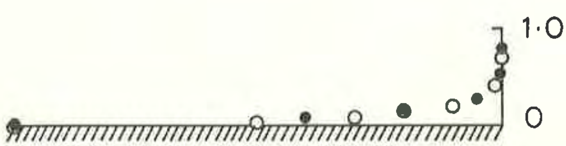

FIG. 6 SHEAR STRESS DISTRIBUTION AT X=0