LOCAL BOUNDS ON DISSIPATION ENERGY IN SHAKEDOWN THEORY

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Summary

The paper deals with elastic-workhardening discrete structures subjected to quasi-static loads and/or imposed strains, which vary in an arbitrary manner but remain always within a given domain. In particular, it gives a method able to provide bounds on plastic dissipation energy which is performed within any part of the structure during the unknown real loading history. Such a bound has the character of being *optimal* (the most stringent within a class of bounds) as well as of being *local* (the bounding quantity is referred to the same part of the structure, which the bounded plastic work is referred to).

The procedure adopted uses a perturbation technique which inables us to formulate two different , but equivalent problems whose solutions, if any, furnish bounds on plastic work: (a) a boundary value problem which can be viewed as the holonomic description of the original shakedown problem suitably perturbed, and (b) a convex minimization problem which characterizes the above boundary value problem and which directly performs the bound optimization.

Computational aspects are not considered. A simple example concludes the paper.

1. Introduction

In the framework of shakedown theory, the assessment of structural safety is mainly based on the availability of bounding techniques for post-adaptation deformation parameters such as strain, displacement and dissipation energy. Many efforts have been devoted to this aim in recent years and several theorems are now available for bounding strains and displacements, as well as overall dissipation energy (see, e.g., [1-11].) As regards local dissipation energy, only recently have bounding theorems been formulated by the first author of the present paper for elastic-plastic continua [12]. In view of the importance the latter topic may have, it seems to be useful to formulate these theorems with reference to discrete structural models, as will be done in the present paper. This is expected to facilitate numerical applications in the treatment of practical engineering problems.

We thus consider a class of discrete (or discretized) structures, such as trusses, lumped-deformability models, constant strain-constant stress finite element models, whose constituents have, or are supposed to have, a piecewise linear elastic-plastic behaviour. The external actions consist of applied forces and/or imposed strains (which may come, for instance, from temperature changes). The real loading history is supposed to be unknown, but at every time all the actions belong by hypothesis to a given loading domain. We consider dynamic effects as being negligible and we treat displacements as infinitesimal.

Symbology is as usual in matrix methods. Vectors and matrices are indicated by underlined small or capitol letters. The symbol $(\ldots)^T$ means transpose of (...), while a superimposed dot means time derivative.

2. Fundamentals and definitions

Let the typical structural element be elastic-plastic with a polyhedral yield surface, whose faces can only translate as a result of plastic flow, the workhardening being piecewise linear and symmetric with respect to the yielding modes. Denoting the (generalized) stress components of all the elements by $\underline{Q}$ and the associated elastic and plastic strain components by $\underline{e}$ and $\underline{p}$ respectively, the resulting elastic-plastic behaviour of the overall structure can be described by the following matrix equations:

$$\underline{e} = \underline{A}\,\underline{Q}, \tag{1}$$

$$\underline{\varphi} = \underline{N}^T \underline{Q} - \underline{H}\,\underline{\lambda} - \underline{k}, \qquad \underline{p} = \underline{N}\,\underline{\lambda} \tag{2}$$

$$\underline{\varphi} \le \underline{0}, \qquad \dot{\underline{\lambda}} \ge \underline{0}, \qquad \underline{\varphi}^T \dot{\underline{\lambda}} = \dot{\underline{\varphi}}^T \dot{\underline{\lambda}} = 0 \tag{3}$$

where $\underline{A}$ is the elastic coefficient matrix; $\underline{N}$ is the matrix of the unit external normals of the yield faces; $\underline{H}$ is the workhardening coefficient matrix (by hypothesis symmetric and positive semidefinite (psd)); $\underline{k}$ is the vector of the plastic resistances, i.e., of the distances of the yield faces from the origin; $\underline{\varphi}$ is the plastic potential vector, and $\dot{\underline{\lambda}}$ is the plastic activation coefficient vector. Eqs.(3) account for the usual incremental plastic flow laws.

If there are n elements, the vector quantities in Eqs.(1)-(3) are supervectors, as, for instance,

$$\underline{k} = [\, \underline{k}^{1T},\ \underline{k}^{2T},\ \dots,\ \underline{k}^{nT}\,]^T, \tag{4}$$

while the matrix quantities are supermatrices, as, for instance,

$$\underline{H} = \text{diag}\,(\underline{H}^1,\ \underline{H}^2,\ \dots,\ \underline{H}^n). \tag{5}$$

Therefore, Eqs.(1)-(3) can also be thought of as referred to the i-th element, provided the index $i = (1,2,\dots,n)$ is appended to all the relevant vector and matrix ingredients.

Indicating the element total strain by $\underline{q}$ and the nodal displacements by $\underline{u}$, we write the compatibility equations

$$\underline{q} = \underline{e} + \underline{p} + \underline{\vartheta}\,, \qquad \underline{q} = \underline{C}\,\underline{u}\,, \tag{6}$$

where $\underline{\vartheta}$ is the vector of the imposed strains and $\underline{C}$ is the so-called compatibility matrix. The associated equilibrium equations thus read as

$$\underline{C}^T\,\underline{Q} = \underline{F}\,, \tag{7}$$

$\underline{F}$ being the vector of the nodal applied forces.

Eqs.(1)-(3), (6) and (7) describe a boundary value problem in which the loads $\underline{F}$ and $\underline{\vartheta}$ are to be considered as unknown time functions which, at any time $t \geq 0$, belong to a given loading domain. We suppose that the structure is able to adapt to such loads by shaking down to a purely elastic state at the end of an initial phase during which some finite amount of plastic work is produced within every structural element. Furthermore we expect this amount to be small enough to satisfy given safety criteria accounting for fatigue resistance and/or other failure requirements. In the present context, the only way to verify this expectation is providing *a priori* bounds on plastic work.

3. A fictitious structure and bounds

Let us consider a (fictitious) structure which is identical to the previous one, except for its plastic properties which are defined by the workhardening matrix $\tilde{\underline{H}}$ (also symmetric and psd) and by the plastic resistance vector $\tilde{\underline{k}}$, and let the same loads of Sec.2 act upon it during times $t \geq 0$. Viewing $\tilde{\underline{H}}$ and $\tilde{\underline{k}}$ as results of a perturbation on $\underline{H}$ and $\underline{k}$ respectively, we may say that a perturbed boundary value problem has been so generated.

Suppose we can find two time-independent vectors, say $\underline{Q}^*$ describing a self-stress distribution and $\underline{p}^* = \underline{N}\,\underline{\lambda}^*$, $\underline{\lambda}^* \geq \underline{0}$, describing a plastic strain distribution, which are not necessarily linked with one another but are such that

$$\tilde{\underline{\varphi}} = \underline{N}^T(\underline{Q}^E + \underline{Q}^*) - \tilde{\underline{H}}\,\underline{\lambda}^* - \tilde{\underline{k}} \leq \underline{0}\,, \qquad t \geq 0, \tag{8}$$

where $(\dots)^E$ indicates the elastic response to the given loads for both the real and the fictitious structures. Let us set

$$\tilde{\underline{H}} = \underline{H} - \hat{\underline{H}}\,, \qquad \tilde{\underline{k}} = \underline{k} - \hat{\underline{k}}\,, \tag{9}$$

and let $\hat{\underline{H}}$ and $\hat{\underline{k}}$ be some arbitrary time-independent perturbing terms. Since, by Eqs.(2),(3) and (8), $\underline{\varphi}^T\dot{\underline{\lambda}} = 0$ and $\tilde{\underline{\varphi}}^T\dot{\underline{\lambda}} \leq \underline{0}$, after some easy transformations we get the inequality

$$\underline{\lambda}^T\hat{\underline{H}}\,\dot{\underline{\lambda}} + \hat{\underline{k}}^T\dot{\underline{\lambda}} \leq -(\tilde{\underline{Q}} - \underline{Q})^T\dot{\underline{p}} - (\underline{\lambda} - \underline{\lambda}^*)^T\tilde{\underline{H}}\,\dot{\underline{\lambda}} \,, \tag{10}$$

whre by definition

$$\tilde{\underline{Q}} = \underline{Q}^E + \underline{Q}^*, \qquad \forall t \geq 0. \tag{11}$$

From Eqs.(1) and $(6)_1$, recalling that $\underline{q}^E = \underline{e}^E + \underline{\vartheta}$, we deduce for $\dot{\underline{p}}$:

$$\dot{\underline{p}} = \dot{\underline{q}} - \dot{\underline{q}}^E + \dot{\underline{e}}^E - \dot{\underline{e}} = \dot{\underline{q}} - \dot{\underline{q}}^E + \underline{A}(\dot{\tilde{\underline{Q}}} - \dot{\underline{Q}})\,; \tag{12}$$

hince , since $(\tilde{\underline{Q}} - \underline{Q})^T(\dot{\underline{q}} - \dot{\underline{q}}^E) = 0$, Eq.(10) becomes

$$\underline{\lambda}^T\hat{\underline{H}}\,\dot{\underline{\lambda}} + \hat{\underline{k}}^T\dot{\underline{\lambda}} \leq -\dot{B}(t), \qquad \forall t \geq 0, \tag{13}$$

where $B(t)$ is the psd quadratic form

$$B = \tfrac{1}{2}(\tilde{\underline{Q}} - \underline{Q})^T\underline{A}(\tilde{\underline{Q}} - \underline{Q}) + \tfrac{1}{2}(\underline{\lambda} - \underline{\lambda}^*)^T\tilde{\underline{H}}(\underline{\lambda} - \underline{\lambda}^*). \tag{14}$$

Then , an integration of Eq.(13) over the time interval $(0,t_1)$, dropping the subtractive term $B(t_1)$, gives

$$\left[\tfrac{1}{2}\underline{\lambda}^T\hat{\underline{H}}\,\underline{\lambda} + \hat{\underline{k}}^T\underline{\lambda}\right]_{t=t_1} \leq B(0), \qquad (\underline{\lambda}(0) = \underline{0}) \tag{15}$$

where $B(0)$ is the psd quadratic form

$$B(0) = \tfrac{1}{2}\underline{Q}^{*T}\underline{A}\,\underline{Q}^* + \tfrac{1}{2}\underline{\lambda}^{*T}\tilde{\underline{H}}\,\underline{\lambda}^* \,. \tag{16}$$

Let us now introduce n arbitrary nonnegative numbers γ_i and the diagonal matrix

$$\underline{\Gamma} = \text{diag}\,(\gamma_1\underline{I}_e^1, \gamma_2\underline{I}_e^2, \ldots, \gamma_n\underline{I}_e^n) \tag{17}$$

where $\underline{I}_e^i$ is the identity matrix of as many dimensions as there are faces in the yield polyhedron of the i-th element. Then,let it be

$$\hat{\underline{H}} = \omega\,\underline{\Gamma}\,\underline{H} = \text{diag}\,(\omega\gamma_1\underline{H}^1, \omega\gamma_2\underline{H}^2, \ldots, \omega\gamma_n\underline{H}^n), \tag{18}$$

$$\hat{\underline{k}} = \omega\,\underline{\Gamma}\,\underline{k} = [\,\omega\gamma_1\underline{k}^{1T}, \omega\gamma_2\underline{k}^{2T}, \ldots, \omega\gamma_n\underline{k}^{nT}\,]^T, \tag{19}$$

w here $\omega > 0$ is an unknown perturbance multiplier. Eq.(15) thus takes the form:

$$\sum_{i=1}^{n}\gamma_i W_i(t_1) \leq \frac{1}{\omega}B(0) \tag{20}$$

where $W_i(t_1)$ is plastic work dissipated in the i-th element during time interval $(0,t_1)$, i.e.,

$$W_i(t_1) = [\,\tfrac{1}{2}\underline{\lambda}^{iT}\underline{H}^i\underline{\lambda}^i + \underline{k}^{iT}\underline{\lambda}^i\,]_{t=t_1}\,. \tag{21}$$

Finally, if we take $\gamma_i = 1$ for some i and $\gamma_j = 0$ for every $j \neq i$, $i,j \in (1,2,..,n)$, Eq. (20) becomes

$$W_i(t_1) \leq \frac{1}{\omega}B(0), \qquad \text{for some } i \in (1,2,\ldots,n), \tag{22}$$

which is valid for any subsequent time t_1.

4. Bound optimization

Since in Eq.(20) vectors $\underline{Q}^*$ and $\underline{\lambda}^*$ have only to satisfy Eq.(8), we may make the bound the most stringent by an optimal choice of the above parameters. This, remembering Eqs.(9),(17), (18) and (19), leads to the following (convex) minimization problem:

$$\text{minimize} \qquad \Phi = \frac{1}{\omega}\left[\tfrac{1}{2}\,\underline{S}^T\,\underline{A}\,\underline{S} + \tfrac{1}{2}\,\underline{\mu}^T(\underline{I} - \omega\underline{\Gamma})\,\underline{H}\,\underline{\mu}\right] \tag{23}$$

subject to the conditions $\omega > 0$, $\underline{\mu} \geq \underline{0}$ and

$$\underline{C}^T\,\underline{S} = \underline{0}\,, \tag{24}$$

$$\underline{N}^T(\underline{Q}^E_\nu + \underline{S}) - (\underline{I} - \omega\underline{\Gamma})(\underline{H}\,\underline{\mu} + \underline{k}) \leq \underline{0}\,, \qquad \forall\,\nu \in (1,2,\ldots,m), \tag{25}$$

$$\omega\gamma_i - 1 \leq 0, \qquad \forall\, i \in (1,2,\ldots,n), \tag{26}$$

where the yield conditions are supposed to be verified only at the m vertices of a polyhedral convex hull containing the loading domain, while inequalities (26) assure that the matrix $\underline{\tilde{H}}$ be psd. It is easily seen that any admissible solution $\underline{S},\underline{\mu},\omega$ gives an upper bound by letting $\underline{Q}^* = \underline{S}$ and $\underline{\lambda}^* = \underline{\mu}$ in Eq.(16).

It is worth considering the Kuhn-Tucker conditions associated with the above problem. Following a known procedure we find

$$\underline{C}^T\,\underline{S} = \underline{0} \tag{27}$$

$$\left.\begin{aligned} &\underline{\psi}_\nu = \underline{N}^T(\underline{Q}^E_\nu + \underline{S}) - (\underline{I} - \omega\underline{\Gamma})(\underline{H}\,\underline{\mu} + \underline{k}) \\ &\underline{\psi}_\nu \leq \underline{0}\,, \quad \underline{\ell}_\nu \geq \underline{0}\,, \quad \underline{\psi}^T_\nu\,\underline{\ell}_\nu = 0 \end{aligned}\right\} \forall\,\nu \in (1,2,\ldots,m) \tag{28}$$

$$\underline{C}\,\underline{v} = \underline{A}\,\underline{S} + \underline{N}\,\underline{\ell}_R\,, \qquad \underline{\ell}_R = \sum_{\nu=1}^{m}\underline{\ell}_\nu\,, \tag{29}$$

$$(\underline{I} - \omega\underline{\Gamma})\,\underline{H}\,(\underline{\ell}_R - \underline{\mu}) \leq \underline{0}\,, \qquad \underline{\mu} \geq 0\,, \qquad \underline{\mu}^T(\underline{I} - \omega\underline{\Gamma})\,\underline{H}\,(\underline{\ell}_R - \underline{\mu}) = 0\,, \tag{30}$$

$$\Phi = \tfrac{1}{2}\,\underline{\mu}^T\,\underline{\Gamma}\,\underline{H}\,(\underline{\ell}_R - \underline{\mu}) + \tfrac{1}{2}\,\underline{\mu}^T\,\underline{\Gamma}\,\underline{H}\,\underline{\ell}_R + \underline{k}^T\,\underline{\Gamma}\,\underline{\ell}_R + \sum_{i=1}^{n}\gamma_i\,\xi_i\,, \tag{31}$$

$$\omega\gamma_i - 1 \leq 0, \qquad \xi_i \geq 0, \qquad \xi_i(\omega\gamma_i - 1) = 0, \qquad \forall\, i \in (1,2,\ldots,n) \tag{32}$$

where Φ is given by Eq.(23) and the new symbols are Lagrangian variables with obvious meanings. The following properties are readily deduced as characterizing the solution, if any:

(i) All the variables ξ_i are zero. Supposing in fact that one of them, say ξ_i, be positive, from the resulting equality $\omega\gamma_i = 1$ we find that the yield condition for the i-th element should be $\underline{N}^{iT}(\underline{Q}^{iE}_\nu + \underline{S}^i) \leq \underline{0}, \forall\,\nu \in (1,2,\ldots,m)$, which implies $\underline{Q}^{iE}_\nu + \underline{S}^i = \underline{0}$ for every $\nu \in (1,2,\ldots,m)$, which can be excluded without loss of generality.

(ii) The matrix $\underline{I} - \omega\underline{\Gamma}$ is positive definite. In fact, from property (i), we know that Eq.$(32)_1$ cannot be verified as an equality, and this for every i.

(iii) Workhardening effects produced by plastic intensity vectors $\underline{\mu}$ and $\underline{\ell}_R$ are equal. In fact, multiplying Eq.$(30)_1$ by $\underline{\ell}_R \geq \underline{0}$ and then subtracting Eq.$(30)_3$, we obtain the inequality

$(\underline{\ell}_R - \underline{\mu})^T(\underline{I} - \omega\underline{\Gamma})\,\underline{H}\,(\underline{\ell}_R - \underline{\mu}) \leq 0$, which, by property (ii), implies

$$\underline{H}\,\underline{\ell}_R = \underline{H}\,\underline{\mu}, \tag{33}$$

from which it follows that $\underline{\ell}_R = \underline{\mu}$ only if $\underline{H}$ is pd. In the case of kinematic workhardening, Eq.(33) is easily shown to become $\underline{N}\,\underline{\ell}_R = \underline{N}\,\underline{\mu}$.

(iv) As a result of property (i), the third group of constraints, Eq.(26), can be dropped without changes in the solution properties.

(v) Eqs.(27) to (29) describe a holonomic problem for the fictitious structure, which, recalling Eq.(33), can be viewed as the holonomic version of the original shakedown problem suitably perturbed. Self-stresses $\underline{S}$ prove to be the elastic response to plastic strains $\underline{N}\,\underline{\ell}_R$, considered as imposed, and these strains prove to be the sum of analogous strains caused by the m relevant loading conditions.

(vi) Eq.(31), by properties (i) and (ii) and remembering Eqs.(17) and (23), can be given the form

$$\tfrac{1}{2}\sum_{i=1}^{n}\,[\underline{S}^{iT}\,\underline{A}^{i}\,\underline{S}^{i} + (1 - \omega\gamma_i)\,\underline{\ell}_R^{iT}\,\underline{H}^{i}\,\underline{\ell}_R^{i}] = \omega\sum_{i=1}^{n}\gamma_i D_i, \tag{34}$$

where D_i is the holonomic plastic work relative to the i-th element, i.e.

$$D_i = \underline{k}^{iT}\underline{\ell}_R^{i} + \tfrac{1}{2}\,\underline{\ell}_R^{iT}\,\underline{H}^{i}\,\underline{\ell}_R^{i}, \qquad (i = 1,2,\dots,n). \tag{35}$$

(vii) Using Eqs.(17), (23), (31) and (33) along with property (i), the bound of Eq.(20), when optimality is reached, can be given the simple form

$$\sum_{i=1}^{n}\gamma_i W_i(t_1) \leq \sum_{i=1}^{n}\gamma_i D_i. \tag{36}$$

Then, letting $\gamma_i = 1$ for some i and $\gamma_j = 0$ for every $j \neq i$, with $i,j \in (1,2,\dots,n)$, Eq.(36) yields the following "local" bound

$$W_i(t_1) \leq D_i, \qquad \text{for some } i \in (1,2,\dots,n) \tag{37}$$

where both members are referred to the same i-th element.

In order to evaluate the optimal bounds of Eqs.(36) and (37), we have to solve either (a) the holonomic analysis problem governed by the above Kuhn-Tucker conditions, or, (b) the minimization problem of Eqs.(23)-(26). For a given perturbance multiplier ω, problem (a) is, or can be reduced to, a linear complementarity problem, while problem (b) is a quadratic programming problem. Thus, the search for a complete solution to both problems is expected to be possible using known mathematical tools. Computational aspects are not considered here.

5. Example

As an example, let us consider the truss of Fig.1(a). All the bars have the same elastic compliance A, while their yield stress resultant is k for the contour bars and $k\sqrt{2}$ for the diagonal bars. The workhardening coefficient is supposed to be zero for bars 2, 3 and 5, but different from zero, say $c > 0$, for bars 1 and 4, whose workhardening behaviour is assumed to be kinematic (see Fig.1(b)). The load $F = F_0\tau$, (F_0 = elastic limit load of the truss), is allowed to vary in an arbitrary manner, but quasi-statically and with $0 \le \tau \le s$. For $1 \le s \le s_1$ yielding is really possible in bars 1 and 4, while the other bars are always in the elastic range; for $s_1 \le s < s_2$, yielding is really possible for all the bars, s_2 being the adaptation limit load.

In order to provide bounds on plastic work performed in bars 1 and 4 (both having equal values of plastic work density), according to the present theory, we have to modify the yield conditions of these two bars by reducing the relevant coefficients k and c to αk and αc respectively, with $\alpha = 1 - \omega$ and $0 < \omega < 1$ (see Fig.1(b)). A holonomic analysis problem has been solved for two relevant loading conditions, i.e., $\tau = 0$ and $\tau = s$, taking into account the disturbance work balance condition, Eq.(34). Making the choice $Ac = 0.1$, $b/Ak = \frac{2}{3} \times 10^3$, (where b is the length of the contour bars), the results are illustrated in Fig.2(a), where the real plastic work W/kb and the holonomic plastic work D/kb are plotted as functions of the maximum load intensity s. We find that $s_1 = 1.16$ and $s_2 = 2$. The bound D/kb diverges for s approaching s_2. Similar considerations can be done with reference to Fig. 2(b), where the case $c = 0$ is considered. In this case it is found that $s_1 = s_2 = \frac{8}{7}$, where s_2 coincides with the plastic load factor.

6. Conclusion

In the present paper we have formulated local bounds on dissipation energy, that is on plastic work produced locally, at any part of the structure, during the adaptation process. The same perturbation procedure previously given by one of the present authors [12] for elastic perfectly plastic continua has been here applied to discrete (or discretized) elastic-work-hardening structures subjected to quasi-static loads and/or imposed strains. This procedure leads to bounds which can be made the most stringent by solving one of two equivalent problems: (a) a holonomic analysis problem for a fictitious structure which is obtained by suitably modifying the plastic properties of the real one, or, (b) a minimization problem which directly produces an optimal bound.

These two problems, described in matrix terms, fall into the framework of complementarity problems and mathematical programming respectively, and so they are expected to be solvable with the use of known mathematical and computational tools. This crucial aspect of the method has not been studied in the present paper.

In order to clarify the method better, in particular from the computational point of view, further research work is hoped for. The effectiveness of the method has been tested by a simple example, but other numerical experiments are necessary. As shown by the example, the bounds furnished by the present method become too big when the load is allowed to approach the failure load, but they are expected to give a useful piece of information especially when the real plastic work is unknown, as generally is the case.

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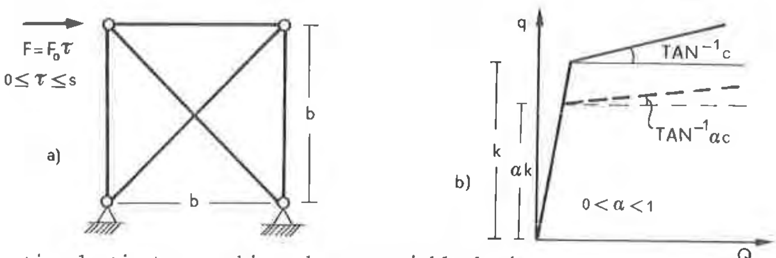

Fig.1 Elastic-plastic truss subjected to a variable load.

(a) Geometrical and loading scheme.

(b) Real (solid line) and perturbed (dashed line) stress-strain diagrams for bar 1.

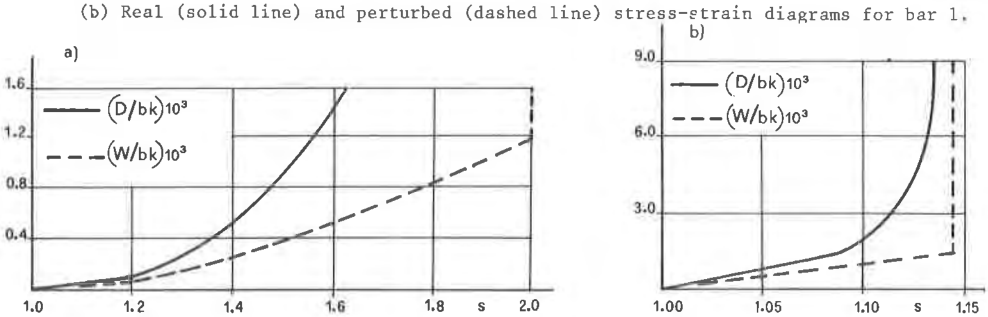

Fig.2 Real plastic work W of bar 1 and upper bound D as functions of the maximum load intensity s.
(a) Workhardening case ($c > 0$ for bars 1 and 4, $c = 0$ for bars 2, 3 and 5).
(b) No-workhardening case ($c = 0$ for all the bars).